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INVERSES OF SEMIGROUP GENERATORS: A SURVEY AND REMARKS

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Abstract. Let X be a Banach space, and let $(e^{tA})_{t\geq 0}$ be a bounded C_0-semigroup on X, with generator A. Suppose that A^{-1} exists as a (closed) densely defined operator. In 1988 deLaubenfels asked whether A^{-1} also generates a C_0-semigroup. The problem is still open in the setting of Hilbert spaces. In this survey we will discuss partial advances obtained so far and mention also related results.

1. Introduction. The theory of C_0-semigroups is a well-established chapter of operator theory with many applications to partial differential equations, mathematical physics, probability theory and other areas of analysis. It contains very few gaps (if any) and the subject has reached its maturity a while ago. In this survey we will discuss a problem which is easy to formulate but probably not so easy to solve, and despite many efforts, especially last years, its solution is still out reach, and it is even difficult to formulate a plausible answer. Namely we will address the following notorious problem formulated in a general framework of Banach spaces.

Problem 1.1. Let A be the generator of a bounded C_0-semigroup on X and assume that there exists a densely defined algebraic inverse A^{-1} of A. Does then A^{-1} also generate a C_0-semigroup?

Note that from the mean ergodic theorem (see e.g. [29, Theorem I.8.20]), it follows that if A generates a bounded C_0-semigroup on X and the range of A is dense in X, then A is injective, and the (algebraic) inverse A^{-1} exists and is densely defined. Moreover, if X is reflexive then the range of A is dense if and only if A is injective.

The problem is of great importance in control theory and numerical analysis, for more details on that see [65]. Moreover, its understanding is crucial in the study of permanence

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properties of functional calculi for A, see [34] and [5]. In what follows, for short-hand, we refer to the problem as "A^{-1}-problem".

Apparently, the problem was first posed by deLaubenfels in [19], where it was proved that if A is the generator of a sectorially bounded holomorphic C_0-semigroup and A^{-1} is well-defined, then A^{-1} generates a sectorially bounded holomorphic semigroup too. The proof of deLaubenfels' result is quite simple. By [44, IX.6], there exists $\theta \in (0, \pi/2)$ such that the sector $\Sigma_{\pi/2+\theta}$ around the positive real half-axis of angle $\pi/2 + \theta$ is contained in the resolvent set of A, and

$$\|R(\lambda, A)\| \le \frac{c_\phi}{|\lambda|}, \qquad \lambda \in \Sigma_{\pi/2+\phi}, \text{ for each } \phi \in (0, \theta). \tag{1.1}$$

Therefore, if A has a dense range, then

$$R(\lambda, A^{-1}) = \lambda^{-1}[I - \lambda^{-1} R(\lambda^{-1}, A)] \tag{1.2}$$

for $\lambda \in \Sigma_{\pi/2+\theta}$, so (1.1) and (1.2) readily imply the estimate

$$\|R(\lambda, A^{-1})\| \le \frac{1 + c_\phi}{|\lambda|}, \qquad \lambda \in \Sigma_{\pi/2+\phi}, \text{ for all } \phi \in (0, \theta),$$

and A^{-1} is the generator of a sectorially bounded holomorphic C_0-semigroup. Another situation when the answer can be obtained easily arises when the space X is Hilbert and A generates a contraction C_0-semigroup on X. In this setting, when the algebraic inverse A^{-1} exists and is densely defined, we have

$$\mathrm{Re}\langle Ax, x\rangle \le 0, \ x \in \mathrm{dom}(A) \qquad \Longleftrightarrow \qquad \mathrm{Re}\langle A^{-1}x, x\rangle \le 0, \ x \in \mathrm{dom}(A^{-1})$$

and A^{-1} generates a C_0-semigroup of contractions on X as well by the Lumer–Phillips theorem.

It is crucial that the A^{-1}-problem can also be formulated in the following manner.

PROBLEM 1.2. Let A be the generator of a bounded C_0-semigroup $(e^{tA})_{t\ge 0}$ on X. Furthermore, assume that the range of A is dense in X. Is then A^{-1} a generator of a *bounded* C_0-semigroup on X?

It has been noted in [64] that Problems 1.1 and 1.2 are essentially equivalent in the following sense. Assume that there exist a Banach space X and a C_0-semigroup $(e^{tA})_{t\ge 0}$ on X such that A^{-1} generates an unbounded C_0-semigroup $(S(t))_{t\ge 0}$ on X. Let

$$\ell_2(\mathbb{N}, X) := \ell_2 - \bigoplus_{n=1}^{\infty} X.$$

Note that the operator $A_0 = \mathrm{diag}(n^{-1}A)$ on $l_2(\mathbb{N}, H)$ generates a bounded C_0-semigroup $(\mathrm{diag}(e^{tn^{-1}A}))_{t\ge 0}$ on $\ell_2(\mathbb{N}, X)$, and its inverse $A_0^{-1} = \mathrm{diag}(nA^{-1})$ is closed and densely defined. If A_0^{-1} generates a C_0-semigroup on $l_2(\mathbb{N}, X)$, then the semigroup should necessarily be given by $(\mathrm{diag}(e^{tnA^{-1}}))_{t\ge 0}$. However,

$$\sup_{t\ge 0} \|S(t)\| = \sup_{t\in[0,1]} \sup_{n\in\mathbb{N}} \|S(nt)\| = \sup_{t\in[0,1]} \|\mathrm{diag}(S(nt))\| = \infty.$$

Thus, if $(e^{tA^{-1}})_{t\ge 0}$ is unbounded, then $(\mathrm{diag}(e^{tnA^{-1}}))_{t\ge 0}$ is not locally bounded, and in particular, is not strongly continuous. Note that $\ell_2(\mathbb{N}, X)$ is a Hilbert space if X is so. In fact, that the direct sum construction above goes back to Chernoff, see [13].

The first counterexample to Problem 1.1, using a nilpotent shift semigroup on $C_0([0,1))$ and the direct sum trick described above, was given by H. Zwart in [65]. It must be pointed out however that a negative answer to Problem 1.1 when $X = c_0(\mathbb{N})$ is contained implicitly in the well-known Komatsu's paper [46] on fractional powers of operators. Komatsu's considerations were put in a broad context of ℓ^p-spaces in [37]. Namely, it was shown in [37] that for every $p \in (1,2) \cup (2,\infty)$ there exists a linear bounded operator A on a Banach space $X = \ell^p(\mathbb{N})$ providing a counterexample to the A^{-1}-problem. The A^{-1}-problem on ℓ^p-spaces is thoroughly discussed in Section 6 below. Recently, a version of this counterexample based on a different argument was given in [27]. Apparently, the simplest counterexample to Problem 1.1 in the setting of Banach spaces was proposed in [21]. It was proved there that the inverse of the differentiation operator $-d/dt$ on the closure of its range in $L_1([0,\infty))$ does not generate a C_0-semigroup. Several sufficient conditions on the resolvent $R(\lambda, A)$, ensuring the generation property of A^{-1} were given in [32]. These conditions are addressed in Section 7. A number of conditions for A^{-1} to generate a C_0-semigroup can be found in the interesting article [21]. Unfortunately, the A^{-1}-problem remains still open if X is a Hilbert space, and it is of primary importance just for that particular class of Banach spaces. Formally, as we will see below, one can give several criteria for A^{-1} to be the generator of a C_0-semigroup. While they can be of certain interest, the criteria do not reveal the impact of the geometry of X to the generation property of A^{-1}, and thus seem to be not very helpful in the framework of Hilbert spaces.

A^{-1}-problem is intimately related to another intriguing and still open problem in operator theory on Hilbert spaces. For any generator A of a bounded C_0-semigroup on a Banach space X define its Cayley transform by

$$V = V(A) := (A+1)(A-1)^{-1}. \tag{1.3}$$

Clearly, V is a bounded operator. Since the spectrum $\sigma(A)$ of A is contained in the closed left half-plane, the spectrum $\sigma(V)$ of V is then contained in the closed unit disk. From the point of view of operator theory and its applications it is of substantial interest to understand the interplay between asymptotic properties of a C_0-semigroup $(e^{tA})_{t\geq 0}$ and the discrete semigroup $(V^n(A))_{n\in\mathbb{N}}$, see e.g. [10], [24], [48], [62], [57], and [58]. For example, it is well-known that A generates a C_0-semigroup of contractions on a Hilbert space, if and only if $V(A)$ is a contraction, and this allows one to transfer H^∞-functional calculus, model theory, etc. from the discrete setting to a continuous one and vice versa. As another instance, recall if a *bounded* operator on a Hilbert space generates a bounded C_0-semigroup, then its Cayley transform is a power bounded operator. This fact was first noted in [4] and [31].

To give one more motivation for the study of power asymptotics for $V(A)$, consider the abstract Cauchy problem

$$u'(t) = Au(t), \quad t \geq 0, \qquad u(0) = u_0 \in \operatorname{dom}(A), \tag{1.4}$$

where A is the generator of a bounded C_0-semigroup. The problem is well-posed and its approximate solutions $U_n \approx u(t_n)$ can be defined as in [9] (see also [65]) by

$$U_{n+1} = r(kA)U_n, \quad n \geq 0, \qquad U_0 = u_0, \tag{1.5}$$

where k is a time step, $t_n = nk$ and r is a rational function with $|r(z)| \le 1$, $\operatorname{Re} z \le 0$, satisfying certain additional assumptions. The scheme (1.5) then provides an approximation $r(kA)^n$ of $e^{t_n A}$, and one of the basic questions in the approximation theory for (1.4) is whether $\{r(kA)^n\}_{n\in N}$ is bounded. Or, in other words, whether the approximation is stable. Choosing

$$r(z) = \frac{1+z/2}{1-z/2}$$

one obtains a well-known Crank–Nicolson approximation scheme, where

$$r(A) = -(A/2+I)(A/2-I)^{-1} = -V(A/2),$$

is in fact the Cayley transform of $A/2$, see [16] and [12] for more details. Various related issues on numerical analysis of abstract differential equations are discussed in the survey paper [38]. Other applications of Cayley's transforms of semigroup generators, e.g. arising in systems theory, are discussed [53] (see also [39]). Thus it is natural to ask the next question.

PROBLEM 1.3. Let A be the generator of a bounded C_0-semigroup on X. Is then the Cayley transform $V(A)$ power bounded?

As for the A^{-1}-problem, it is not so difficult to construct counterexamples for concrete non-Hilbertian Banach spaces (e.g. $X = L_1(0,1)$). It was proved in [31] (see also [39]) that if both A and A^{-1} generate bounded C_0-semigroups on a Hilbert space X, then the answer is positive as well. However, at the moment is not clear whether the assumption on the boundedness of $(e^{tA^{-1}})_{t\ge0}$ can be dropped.

2. Some notation and definitions. It will be convenient to fix some notation for the sequel. Let X be a Banach (and, in particular, Hilbert) space. To underline a specifics of Hilbert spaces, we will sometimes let H stand for a Hilbert space. Let $\mathcal{E} = \mathcal{E}(X)$ be the set of densely defined closed linear operators in X. It will also be convenient to introduce the set $\mathcal{E}_- = \mathcal{E}_-(X)$ of densely defined closed linear operators in X whose spectrum lies in the left half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le 0\}$. For an operator $A \in \mathcal{E}$ we denote its domain by $\operatorname{dom}(A)$, its spectrum by $\sigma(A)$, the identity operator on X by I, and the resolvent of A by

$$R(\lambda, A) := (\lambda I - A)^{-1}$$

defined on the resolvent set $\lambda \in \rho(A) := \mathbb{C} \setminus \sigma(A)$. Let $\mathcal{L}(X)$ be the algebra of bounded linear operators in X.

By $\mathcal{G}_{\mathrm{b}} = \mathcal{G}_{\mathrm{b}}(X)$ we denote the set of generators of bounded C_0-semigroups on X and by $\mathcal{G}_{\exp} = \mathcal{G}_{\exp}(X)$ the set of generators of exponentially stable C_0-semigroups on X. Let $\mathcal{H}(\theta)$, $\theta \in (0,\pi/2]$, be the set of generators of C_0-semigroups on X holomorphic in the sector

$$\Sigma_\theta := \{z = re^{i\phi} : 0 < r < \infty,\ |\phi| < \theta\},$$

bounded and strongly continuous in an arbitrary sub-sector Σ_{θ_0}, $\theta_0 \in (0,\theta)$, and $\mathcal{H}_0$ the set of operators of the form $A = A_0 + \beta I$, where $\beta \ge 0$ and $A_0 \in \mathcal{H}(\theta)$ for some $\theta \in (0,\pi/2]$.

Let finally, for $x \in X$, and $y \in X^*$, $\langle x, y\rangle$ stand for the value of y at x, and for a Hilbert space H let $\langle\cdot,\cdot\rangle$ denote the inner product.

3. General considerations. In this section, we formulate and prove two generation criteria for A^{-1}. Let $A \in \mathcal{G}_{\rm b}(X)$, $A^{-1} \in \mathcal{E}(X)$. Then $\sigma(A^{-1}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le 0\}$ and

$$(I - \lambda A^{-1})^{-1} = A(A - \lambda I)^{-1} = I - \lambda R(\lambda, A), \qquad \lambda > 0.$$

Hence for every $m \ge 0$,

$$(I - \lambda A^{-1})^{-m} = \big[I - \lambda R(\lambda^{-1}, A)\big]^m = I + S_m(\lambda, A),$$

where

$$S_m(\lambda, A) := \sum_{k=1}^{m} \frac{m!(-\lambda)^k}{k!(m-k)!} R^k(\lambda^{-1}, A). \tag{3.1}$$

Recall that

$$R^n(\lambda, A)x = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} e^{tA} x\, dt, \qquad n \in \mathbb{N},\ x \in X, \tag{3.2}$$

for all λ with $\operatorname{Re}\lambda > 0$. On the other hand, inserting (3.2) in (3.1), we obtain

$$\begin{aligned} S_m(\lambda, A) &= -\int_0^\infty e^{-\lambda t} e^{tA} \bigg(\lambda \sum_{k=1}^{m} \frac{m!(-\lambda t)^{k-1}}{k!(m-k)!(k-1)!}\bigg)\, dt \\ &= -\lambda \int_0^\infty e^{-\lambda t} e^{tA} L_{m-1}^{(1)}(\lambda t)\, dt, \end{aligned}$$

where $\{L_m^{(1)} : m \ge 0\}$ are the first order Laguerre polynomials [1, § 6.2]:

$$L_k^{(1)}(t) := \sum_{m=0}^{k} \frac{(k+1)!}{(m+1)!} \, \frac{(-t)^m}{m!(k-m)!}\,. \tag{3.3}$$

Thus,

$$(I - \lambda A^{-1})^{-m} = I - \lambda \int_0^\infty e^{-\lambda t} e^{tA} L_{m-1}^{(1)}(\lambda t)\, dt, \qquad \lambda > 0,\ m \in \mathbb{N}, \tag{3.4}$$

and the Hille–Yosida theorem implies the following proposition.

Proposition 3.1. *Let $A \in \mathcal{G}_{\rm b}(X)$ and $A^{-1} \in \mathcal{E}(X)$. Then A^{-1} generates a C_0-semigroup $(e^{tA^{-1}})_{t\ge 0}$ on X such that $\|e^{tA^{-1}}\| \le Me^{\omega t}$, $t \ge 0$, for some $M > 0$ and $\omega \ge 0$, if and only if*

$$\lambda \bigg\| \int_0^\infty e^{-\lambda t} e^{tA} L_{m-1}^{(1)}(\lambda t)\, dt \bigg\| \le \frac{M}{(1-\omega\lambda)^m}\,, \qquad \lambda \in (0, \omega^{-1}),\ m \in \mathbb{N}.$$

In what follows, let $J_1(\cdot)$ stand for the Bessel function of the first kind and the first order.

Theorem 3.2. *Let A be the generator a bounded C_0-semigroup on X, with dense range. Then A^{-1} is the generator of a C_0-semigroup on X if and only if for some* (*and then for any*) $t_0 > 0$

$$\sup_{n\in\mathbb{N}} \max_{t\in[0,t_0]} \bigg\| \sqrt{t} \int_0^\infty e^{-s/n} \frac{J_1(2\sqrt{ts})}{\sqrt{s}} e^{sA}\, ds \bigg\| < \infty. \tag{3.5}$$

If (3.5) *holds, then*

$$e^{tA^{-1}} x = x - \lim_{n\to\infty} \sqrt{t} \int_0^\infty e^{-s/n} \frac{J_1(2\sqrt{ts})}{\sqrt{s}} e^{sA} x\, ds \tag{3.6}$$

for all $t > 0$ and $x \in X$.

Proof. Suppose that A^{-1} generates a C_0-semigroup $(e^{tA^{-1}})_{t\ge 0}$ on X such that

$$\|e^{tA^{-1}}\| \le M_0 e^{\omega t}, \qquad t \ge 0.$$

Since $A^{-1}(1 - A^{-1}/n)^{-1} = (A - 1/n)^{-1}$, $n \in \mathbb{N}$, we have

$$\|e^{t(A-1/n)^{-1}}\| \le M_0 e^{\omega t}, \qquad t \ge 0, \tag{3.7}$$

and, by Yosida's approximation formula,

$$e^{tA^{-1}} x = \lim_{n\to\infty} e^{t(A-1/n)^{-1}} x, \qquad x \in X, \tag{3.8}$$

(see e.g. the proof of Hille–Yosida's theorem in [22, VII.1]). On the other hand,

$$e^{t(A-1/n)^{-1}} x = \sum_{m=0}^{\infty} \frac{t^m}{m!} (A - 1/n)^{-m} x, \qquad t > 0, \qquad x \in X, \tag{3.9}$$

and from (3.2) it follows that

$$(A - 1/n)^{-m} = \frac{(-1)^m}{(m-1)!} \int_0^\infty e^{-s/n} s^{m-1} e^{sA}\, ds, \qquad m \in \mathbb{N}. \tag{3.10}$$

Hence, substituting (3.10) in (3.9) and interchanging summation and integration (which is legitimate since $(e^{tA})_{t\ge 0}$ is bounded), we have

$$e^{t(A-1/n)^{-1}} x = x - t \int_0^\infty e^{-s/n} \bigg(\sum_{m=0}^{\infty} \frac{(-1)^m (ts)^m}{m!(m+1)!} \bigg) e^{sA} x\, ds.$$

Using the identity

$$\sum_{m=0}^{\infty} \frac{(-1)^m (ts)^m}{m!(m+1)!} = \frac{J_1(2\sqrt{ts})}{\sqrt{ts}}\,,$$

(see [1, § 4.5]), we obtain

$$e^{t(A-1/n)^{-1}} x = x - \sqrt{t} \int_0^\infty e^{-s/n} \frac{J_1(2\sqrt{ts})}{\sqrt{s}} e^{sA} x\, ds. \tag{3.11}$$

Now (3.7) and (3.8) imply (3.5) and (3.6).

Next assume that (3.5) holds. Then by (3.11),

$$\sup_{n\in\mathbb{N}} \max_{t\in[0,t_0]} \|e^{t(A-1/n)^{-1}}\| < \infty,$$

and therefore C_0-semigroups $(e^{t(A-1/n)^{-1}})_{t\ge 0}$ satisfy (3.7) for some $M_0 \ge 1$ and $\omega \ge 0$. Moreover,

$$\lim_{n\to\infty} (A - 1/n)^{-1} x = A^{-1} x, \qquad \text{for any } x \in \operatorname{dom}(A^{-1}),$$

and then, by Trotter–Kato's approximation theorem (see [25, III.4.9]) C_0-semigroups $(e^{t(A-1/n)^{-1}})_{t\ge 0}$, $n \in \mathbb{N}$, converge strongly (and uniformly in $t \in [0, t_1]$) to a C_0-semigroup $(e^{tA^{-1}})_{t\ge 0}$. ■

Remark 3.3. While the paper was in press, Theorem 3.2 was also derived in [27, Th. 2.1].

Similarly, using Euler's approximation formula

$$\lim_{n\to\infty} (I - tA/n)^{-n} x = e^{tA} x, \qquad x \in X,$$

and (3.4), we can prove the next result.

THEOREM 3.4. *Let A be the generator of a bounded C_0-semigroup $(e^{tA})_{t\geq 0}$ on X, with dense range. Then A^{-1} generates a C_0-semigroup on X if and only if for some* (*and then for any*) $t_0 > 0$,

$$\sup_{n\in\mathbb{N}} \max_{t\in[0,t_0]} \left\| \int_0^\infty e^{-s} L_{n-1}^{(1)}(s) e^{snA/t} x\, ds \right\| < \infty, \qquad x \in X. \tag{3.12}$$

If (3.12) *holds, then moreover*

$$e^{tA^{-1}} x = x - \lim_{n\to\infty} \int_0^\infty e^{-s} L_{n-1}^{(1)}(s) e^{snA/t} x\, ds, \qquad t > 0,\ x \in X. \tag{3.13}$$

Observe that if A is the generator of an exponentially stable C_0-semigroup in X, then $A^{-1} \in \mathcal{L}(X)$, so that A^{-1} generates a C_0-semigroup $(e^{tA^{-1}})_{t\geq 0}$ given by

$$e^{tA^{-1}} = \sum_{m=0}^{\infty} \frac{t^m}{m!} A^{-m}, \qquad t \geq 0. \tag{3.14}$$

In this case, we can write an integral representation for $(e^{tA^{-1}})_{t\geq 0}$, see [37, Theorem 1], [64, Lemma 3.2].

COROLLARY 3.5. *Let A be the generator of an exponentially stable C_0-semigroup on X. Then*

$$e^{tA^{-1}} x = x - \sqrt{t} \int_0^\infty \frac{J_1(2\sqrt{ts})}{\sqrt{s}}\, e^{sA} x\, ds \tag{3.15}$$

for all $t > 0$ and $x \in X$.

4. Exponentials of Volterra operators. Recall that the classical integral Volterra operator J acts boundedly on all $L_p = L_p[0,1]$, $1 \leq p \leq \infty$, and is defined by

$$(Jf)(x) = \int_0^x f(s)\, ds, \qquad f \in L_p. \tag{4.1}$$

It is well-known that J is quasinilpotent on L^p, $1 \leq p < \infty$, i.e. the spectrum of J is precisely zero on each of those spaces.

To relate J to the study of A^{-1}-problem, define the differential operator A on L_p, $1 \leq p < \infty$, by

$$(Af)(y) = -f'(y) \text{ with } \operatorname{dom}(A) = \{f \in W_p^1[0,1] : f(0) = 0\}, \tag{4.2}$$

where $W_p^1[0,1]$ stands for a standard Sobolev space. The operator A generates a nilpotent C_0-semigroup $(e^{tA})_{t\geq 0}$ on L_p given by

$$(e^{tA}f)(s) = \begin{cases} f(s-t), & 0 < s-t \leq 1, \\ 0, & \text{otherwise}, \end{cases} \tag{4.3}$$

and $A^{-1} = -J$. If $p = 2$ then

$$\operatorname{Re}\langle Jf, f\rangle_{L_2} = \frac{1}{2} \left| \int_0^1 f(s)\, ds \right|^2 \geq 0, \qquad f \in L_2,$$

and therefore $(e^{-tJ})_{t\geq 0}$ is a semigroup of contractions on L_2. Following [49], let us show that $(e^{-tJ})_{t\geq 0}$ is unbounded on L_p if $p \neq 2$. This will provide a counterexample for Problem 1.2 and then, using the direct sum construction from the introduction, yield concrete counterexamples to A^{-1}-problem on spaces $\ell^2(\mathbb{N}, L_p[0,1])$, $1 \leq p < \infty$, $p \neq 2$.

Suppose that for $p \in [1, +\infty]$, $p \neq 2$, there exists $M_p > 0$ such that $\|e^{-tJ}\|_{L_p} \leq M_p$, $t \geq 0$. Then using (3.2) we obtain

$$\sup_{n\in\mathbb{N}} \|(I+J)^{-n}\|_{L_p} \leq M_p. \tag{4.4}$$

On the other hand, by [52, Theorem 1.1],

$$\|(I+J)^{-n}\|_{L_p} \sim n^{|1/4-1/(2p)|}, \qquad n \in \mathbb{N}, \tag{4.5}$$

(for $p = 1$ this is the result due to Hille [42]), a contradiction to (4.4).

Observe that in fact we have a sharp estimate (see [33]):

$$\limsup_{t\to\infty} t^{-|1/4-1/(2p)|} \|e^{-tJ}\|_{L_p} > 0, \qquad p \in [1, \infty]. \tag{4.6}$$

Below we give a full proof of (4.6) only for $p = 1$ since this case is the least technical. First of all, we need the following formula for the norm of an integral operator on $L_1 = L_1[0, 1]$. Suppose that k is a real-valued continuous function on $[0, 1]$. If a bounded linear operator K on L_1 is given by

$$(Kf)(t) := \int_0^t k(t-s)f(s)\,ds,$$

then

$$\|K\|_{L_1} = \int_0^1 |k(s)|\,ds. \tag{4.7}$$

This result follows from a very general statement on norms of integral operators given in [43, Theorem XI.1.4]. For a particularly simple proof see [63, Lemma 4.5].

By (3.15) and (4.3), we have

$$(e^{-tJ}f)(x) = f(x) - (S(t)f)(x), \qquad t > 0, \quad x \in [0, 1], \tag{4.8}$$

where

$$(S(t)f)(x) = \sqrt{t}\int_0^x \frac{J_1\big(2\sqrt{t(x-s)}\big)}{\sqrt{x-s}}\, f(s)\,ds. \tag{4.9}$$

Now (4.7) and (4.9) yield

$$\|e^{-tJ}\|_{L_1} \asymp t^{1/4}, \qquad \text{as } t \to \infty. \tag{4.10}$$

Indeed, the boundedness of $J_1(s)$, $s > 0$, and the relation

$$J_1(s) = -\frac{\sqrt{2}\cos(s+\pi/4)}{\sqrt{\pi}s^{1/2}} + \mathrm{O}(s^{-3/2}), \qquad s \to \infty. \tag{4.11}$$

see [1, § 4.8], imply that

$$\int_0^t |J_1(s)|\,ds \asymp \sqrt{t}, \qquad t \to \infty. \tag{4.12}$$

So, (4.10) follows from

$$\|S(t)\|_{L_1} = \sqrt{t}\int_0^1 \frac{|J_1(2\sqrt{ts})|}{\sqrt{s}}\,ds = \int_0^{2\sqrt{t}} |J_1(s)|\,ds.$$

Employing (4.8), (4.9) and the asymptotic property (4.11) one can show as in [33] that

$$t_n^{-|1/4-1/(2p)|}\|e^{-t_nJ}\|_{L_p} \geq c > 0, \qquad n \in \mathbb{N}, \tag{4.13}$$

for $c > 0$ and a positive sequence $\{t_n : n \geq 1\}$ such that $t_n \to \infty$, $n \to \infty$. Moreover, from (4.8), (4.9) and (4.12) it follows that

$$\|e^{-tJ}\|_{L_p} \leq 1 + \int_0^{2\sqrt{t}} |J_1(s)|\, ds \leq M t^{1/4}, \qquad t \geq 1,\ 1 \leq p \leq \infty. \tag{4.14}$$

Then, using the contractivity of $(e^{-tJ})_{t\geq 0}$ on L_2, (4.14), and the Riesz-Torin interpolation theorem [45, p. 97] we obtain:

$$\|e^{-tJ}\|_{L_p} \leq M^{|1-2/p|} t^{|1/4-1/(1p)|}, \qquad t \geq 1,\ p \in (1, \infty). \tag{4.15}$$

Thus the bounds (4.13) and (4.15) provide a semigroup counterpart of (4.5) obtained in [52].

Remark also that according to [7, Th. 1.2],

$$\lim_{t\to+\infty} \frac{\ln \|e^{te^{i\theta}J}\|_{L_p}}{t^{1/2}} = \sqrt{2}\cos(\theta/2), \qquad \theta \in (-\pi, \pi], \tag{4.16}$$

where $1 \leq p \leq \infty$. For $|\theta| < \pi$ the right-hand side of (4.16) is positive. On the other hand, if $\theta = \pi$ then the limit in (4.16) is zero.

5. Example of Komatsu. In this section we will present the example due to Komatsu mentioned in the introduction. Consider the right shift operator

$$S(\xi_1, \xi_2, \xi_3, \dots) = (0, \xi_1, \xi_2, \dots),$$

on $c_0 = c_0(\mathbb{N})$, and define

$$C := -I + S. \tag{5.1}$$

Observe that if $a = (a_k)_{k\geq 1} \in l^1(\mathbb{N})$ and a bounded linear operator P on c_0 is given by

$$P = \sum_{k=0}^{\infty} a_k S^k,$$

then

$$\|P\| = \sum_{k=0}^{\infty} |a_k| \tag{5.2}$$

(see [50, Remark 1.3.2]). Therefore C generates an isometric C_0-semigroup $(e^{tC})_{t\geq 0}$ on c_0 given by

$$e^{tC} = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} S^k, \qquad t \geq 0.$$

Moreover, it is easy to show that

$$\mathrm{dom}(C^{-1}) := \Big\{ (\xi_k) \in c_0 : \sum_{k=1}^{\infty} \xi_k = 0 \Big\},$$

and the set on the right hand side is clearly dense in c_0.

It was shown in [46, pp. 341–344] (see also [50]), that C^{-1} does not generate a C_0-semigroup on c_0. The proof of this result in [46] relies on (5.2). We give below a slightly different proof based on Theorem 3.2.

Theorem 5.1. *The operator C^{-1} does not generate a C_0-semigroup on c_0.*

Proof. Suppose to the contrary that C^{-1} generates a C_0-semigroup $(e^{tC^{-1}})_{t\geq 0}$ on c_0. Then observing that

$$G_n x := \int_0^\infty e^{-s/n}\,\frac{J_1(2\sqrt{s})}{\sqrt{s}}\,e^{sC}x\,ds = \int_0^\infty e^{-(1/n+1)s}\,\frac{J_1(2\sqrt{s})}{\sqrt{s}}\sum_{k=0}^\infty \frac{s^k}{k!}\,S^k x\,ds$$
$$= \sum_{k=0}^\infty\left(\int_0^\infty e^{-(1/n+1)s}s^{k-1/2}J_1(2\sqrt{s})\,ds\right)\frac{1}{k!}\,S^k x, \qquad x\in c_0,$$

and using (3.5) for $t=1$ we obtain

$$M := \sup_{n\in N}\|G_n\| < \infty. \tag{5.3}$$

On the other hand, since

$$\sqrt{t}\int_0^\infty e^{-s}s^{-1/2}J_1(2\sqrt{ts})\,ds = 1-e^{-t}, \quad t\geq 0, \tag{5.4}$$

and

$$\sqrt{t}\int_0^\infty e^{-s}s^{k-1/2}J_1(2\sqrt{ts})\,ds = (k-1)!e^{-t}tL_{k-1}^{(1)}(t), \qquad k\in\mathbb{N}, \quad t\geq 0, \tag{5.5}$$

see e.g. [61, § 5.4], we infer that

$$\int_0^\infty e^{-(1+1/n)s}s^{-1/2}J_1(2\sqrt{s})\,ds = 1-e^{-1/(1+1/n)},$$

and

$$\int_0^\infty e^{-(1+1/n)s}s^{k-1/2}J_1(2\sqrt{s})\,ds = \frac{(k-1)!}{(1+1/n)^{k+1}}\,e^{-1/(1+1/n)}L_{k-1}^{(1)}(1/(1+\epsilon)), \quad k\in\mathbb{N}.$$

So, if

$$g_{n,k} = \frac{L_{k-1}^{(1)}(1/(1+1/n))}{k(1+1/n)^{k+1}}\,, \qquad n,k\in\mathbb{N},$$

then

$$G_n x = \sum_{k=1}^\infty g_{n,k}S^k x, \qquad x\in c_0.$$

Therefore if C^{-1} generates a C_0-semigroup on c_0, then by (5.2) and (5.3),

$$\sum_{k=1}^\infty |g_k| \leq \sup_{n\in N}\sum_{k=1}^\infty |g_{n,k}| \leq M,$$

where $g_k := \lim_{n\to\infty} g_{n,k} = k^{-1}L_{k-1}^{(1)}(1)$. Hence the function

$$\psi(z) := \sum_{k=1}^\infty g_k z^k = z\sum_{k=0}^\infty \frac{L_k^{(1)}(1)}{k+1}\,z^k = 1-e^{-z/(1-z)}$$

must be continuous on the closed unit disk. At the same time, $\psi(z)$ is obviously discontinuous at $z=1$, and we arrive at a contradiction. ∎

6. Shift operators on ℓ^p. Now we extend the result of Komatsu to the Banach spaces $\ell^p = \ell^p(\mathbb{N})$, with $1 \le p < \infty$, $p \neq 2$. Our arguments will follow [37] closely.

Consider the left shift Q on ℓ^p defined by

$$Q(\xi_1, \xi_2, \dots) = (\xi_2, \xi_3, \dots). \tag{6.1}$$

Note that the operator

$$A = -I + Q, \tag{6.2}$$

generates a C_0-semigroup $(e^{tA})_{t\ge 0}$ on ℓ^p such that

$$\|e^{tA}\|_p \le e^{-t}\|e^{tQ}\|_p \le 1, \qquad t \ge 0.$$

Since A has trivial kernel, $A^{-1} \in \mathcal{E}(\ell^p)$, $1 < p < \infty$.

The next result is a counterpart of Komatsu's counterexample for all ℓ_p with p different from 2.

Theorem 6.1. *If $p \in (1,2)\cup(2,\infty)$ and A is given by (6.2), then A^{-1} does not generate a C_0-semigroup on ℓ^p.*

To prove Theorem 6.1, we will need the following simple lemma.

Lemma 6.2. *Let X be a Banach space and let $A \in \mathcal{L}(X)$ generate a bounded semigroup on X. Assume that there exists a sequence of A-invariant subspaces $\{X_n : n \ge 1\}$ of X such that $\bigcup_{n\in\mathbb{N}} X_n$ is dense in X. If $A^{-1} \in \mathcal{E}$ and moreover A^{-1} generates a C_0-semigroup on X, then X_n is invariant with respect to $(e^{tA^{-1}})_{t\ge 0}$ for every $n \in \mathbb{N}$ and*

$$e^{tA^{-1}}\big|_{X_n} = e^{tA_n^{-1}}, \qquad \|e^{tA^{-1}}\| = \sup_{n\in\mathbb{N}} \|e^{tA_n^{-1}}\|, \quad t \ge 0,$$

where $A_n = A\big|_{X_n}$ is the restriction of A to X_n.

For each $n \in \mathbb{N}$ consider a finite-dimensional space ℓ^p_n of n-tuples $\xi = (\xi_k)_{k=1}^n$ with the norm

$$\|\xi\|_{n,p} = \Big(\sum_{k=1}^n |\xi_k|^p\Big)^{1/p}.$$

Let also for every $N \in \mathbb{N}$,

$$\mathcal{V}_N := \{x = (x_n)_{n\in\mathbb{N}} \in \ell^p : x_n = 0 \text{ for } n > N\}. \tag{6.3}$$

Clearly, $\{\mathcal{V}_N\}_{N\ge 1}$ are finite-dimensional subspaces of ℓ_p. Moreover, (6.1) and (6.2) imply that $\mathcal{V}_N$ is A-invariant for each N. Let A_N be the restriction of A to $\mathcal{V}_N$. Then A_N has the following matrix form in the standard basis of $\mathcal{V}_N$:

$$A_N = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix}. \tag{6.4}$$

Observe that $e^{tA}|_{\mathcal{V}_N} = e^{tA_N}$, $(e^{tA_N})_{t\ge 0}$ is exponentially stable, and there exists a bounded inverse A_N^{-1}.

Lemma 6.3. *For every $N \in \mathbb{N}$, the semigroup $(e^{tA_N^{-1}})_{t\ge 0}$ has the next matrix representation on ℓ_N^p :*

$$e^{tA_N^{-1}} = e^{-t}\begin{pmatrix} 1 & -tL_0^{(1)}(t)/1 & -tL_1^{(1)}(t)/2 & \dots & -tL_{N-2}^{(1)}(t)/(N-1) \\ 0 & 1 & -tL_0^{(1)}(t)/1 & \dots & -tL_{N-3}^{(1)}(t)/(N-2) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -tL_0^{(1)}(t)/1 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \tag{6.5}$$

where $L_k^{(1)}$, $k \ge 0$, are the first order Laguerre polynomials given by (3.3).

Proof. We have

$$e^{tA_N} = e^{-t}\sum_{m=0}^{N-1}\frac{t^m Q_N^m}{m!}, \text{ where } \quad Q_N := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{6.6}$$

Using now (3.15), (5.4), and (5.5), we obtain:

$$\begin{aligned} e^{tA_N^{-1}} &= I - \sqrt{t}\int_0^\infty e^{-s}s^{-1/2}J_1(2\sqrt{ts})\Big\{I + \sum_{k=1}^{N-1}\frac{s^k Q_N^k}{k!}\Big\}\,ds \\ &= I - \sqrt{t}\int_0^\infty e^{-s}s^{-1/2}J_1(2\sqrt{ts})\,ds - \sqrt{t}\sum_{k=1}^{n-1}\frac{Q_N^k}{k!}\int_0^\infty e^{-s}s^{k-1/2}J_1(2\sqrt{ts})\,ds \\ &= e^{-t}I - e^{-t}t\sum_{k=1}^{N-1}\frac{L_{k-1}^{(1)}(t)}{k}\,Q_N^k, \quad t \ge 0, \end{aligned} \tag{6.7}$$

which is precisely (6.5). ∎

The main technical difficulties are comprised in the following statement. For its proof see [37, Lemma 3].

Lemma 6.4. *For $n \in \mathbb{N}$ and $t_0 > 0$ consider an operator on ℓ_n^p given by the matrix*

$$\mathcal{F}_n(t_0) :=$$

$$\begin{pmatrix} \frac12\cos(2\sqrt{t_0}+\frac{\pi}{4}) & \frac{2^{1/4}}{3}\cos(2\sqrt{2t_0}+\frac{\pi}{4}) & \dots & \frac{n^{1/4}}{n+1}\cos(2\sqrt{nt_0}+\frac{\pi}{4}) \\ 0 & \frac12\cos(2\sqrt{t_0}+\frac{\pi}{4}) & \dots & \frac{(n-1)^{1/4}}{n}\cos(2\sqrt{(n-1)t_0}+\frac{\pi}{4}) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{2^{1/4}}{3}\cos(2\sqrt{2t_0}+\frac{\pi}{4}) \\ 0 & 0 & \dots & \frac12\cos(2\sqrt{t_0}+\frac{\pi}{4}) \end{pmatrix}. \tag{6.8}$$

Then for each $p \in (1,2)\cup(2,\infty)$ and $t_0 = \pi^2/64$,

$$\|\mathcal{F}_{n_s}(t_0)\|_{n,p} \ge \frac{s^{|1/2-1/p|}}{720}, \qquad n_s = (16s)^2+1,\ s \in \mathbb{N}. \tag{6.9}$$

In particular,

$$\limsup_n \|\mathcal{F}_n(t_0)\|_{n,p} = \infty. \tag{6.10}$$

Proof of Theorem 6.1. By the discussion above and Lemma 6.2 it suffices to show that there is $t_0 > 0$ such that

$$\limsup_n \|e^{t_0 A_n^{-1}}\|_{n,p} = \infty. \tag{6.11}$$

Let $Q_{m,k}$ be a zero matrix of size $m \times k$. In view of

$$L_n^{(1)}(t) = \pi^{-1/2} e^{t/2} t^{-3/4} n^{1/4} \cos(2\sqrt{nt} - 3\pi/4) + O(n^{-1/4}), \qquad n \to \infty,\ t > 0,$$

where the bound for remainder holds uniformly in any $[a,b] \subset (0,\infty)$, and (6.5), we infer that for fixed $t_0 > 0$ and $N > 2$ the operator $e^{t_0 A_N^{-1}}$ can be represented as

$$e^{t_0 A_N^{-1}} = -t_0^{1/4} \frac{e^{-t_0/2}}{\sqrt{\pi}} G_N(t_0) + e^{-t_0}(I + R_N(t_0)), \tag{6.12}$$

where

$$G_N(t_0) = \begin{pmatrix} Q_{N-2,2} & \mathcal{F}_{N-2}(t_0) \\ Q_{2,2} & Q_{2,N-2} \end{pmatrix},$$

$$R_N(t_0) = \begin{pmatrix} 0 & a_1 & a_2 & \dots & a_{N-1} \\ 0 & 0 & a_1 & \dots & a_{N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{6.13}$$

and $\mathcal{F}_N(t_0)$ are defined by (6.8). Moreover, $a_k = \mathrm{O}(k^{-5/4})$ as $k \to \infty$, so that $a = \{a_k\}_{k\in\mathbb{N}} \in \ell^1$. Since

$$(R_N(t_0))_N = 0, \quad (R_N(t_0))_m = \sum_{k=1}^{N-m} a_k x_{m+k}, \qquad m = 1, \dots, N-1,$$

by Young's inequality,

$$\|R_N(t_0)x\|_{N,p} \le \Big(\sum_{k=1}^{N-1} |a_k|\Big) \|x\|_{N,p} \le \|a\|_{\ell_1} \cdot \|x\|_{N,p}.$$

Thus, the norm of $R_N(t_0)$ in ℓ^p_N is bounded by a constant independent of N, so for the proof of (6.11) it is enough to show that

$$\limsup \|G_N(t_0)\|_{\ell_{p,N}} = \infty \quad \text{as } N \to \infty. \tag{6.14}$$

Setting now $t_0 = \pi^2/64$, we finally note that (6.10) and

$$\|G_N(t_0)\|_{l_{N,p}} = \|F_{N-2}(t_0)\|_{l_{N-2,p}}, \qquad N > 2,$$

imply (6.14). ∎

It would be instructive to observe that a statement similar to Theorem 6.1 holds for Komatsu's operator C as well.

COROLLARY 6.5. *Let a bounded linear operator C on ℓ^p be defined by* (5.1)*. If $p \in (1,2) \cup (2,\infty)$, then $C^{-1} \in \mathcal{E}(\ell^p)$ but it does not generate a C_0-semigroup in ℓ^p.*

Proof. Clearly, $(e^{tC})_{t\geq 0}$ is a contraction C_0-semigroup on ℓ_p, $1 \leq p \leq \infty$. Let $p \in (1,2)\cup(2,\infty)$ be fixed. If the operator A on ℓ_q, $1/q = 1-1/p$, is given by (6.1) and (6.2), then $C = A^*$. So, if C^{-1} generates a C_0-semigroup on ℓ^p then A^{-1} is the generator of a C_0-semigroup on ℓ^q (see [29, Theorem 1.4.9]), which contradicts Theorem 6.1. ∎

Remark 6.6. Theorem 6.1 does not hold for $p = 2$. While the estimates in the proof of Theorem 6.1 do not lead to a contradiction, one may note that since $(e^{At})_{t\geq 0}$ is a C_0-semigroup of contractions on ℓ^2, A^{-1} generates a C_0-semigroup of contractions as well.

Let us comment on the cases $p = 1$ and $p = \infty$. First observe that Theorem 6.1 is true also on ℓ^1. If $(e^{tA})_{t\geq 0}$ is considered on ℓ^1 then $e^{tA}=(e^{tC})^*$ for every $t \geq 0$, where $C \in \mathcal{L}(c_0)$ is given by (5.1). The latter property implies that $A = C^*$, and $A^{-1} = (C^{-1})^*$, and the claim follows from Komatsu's result. Theorem 6.1 does not hold for $(e^{tA})_{t\geq 0}$ on ℓ^∞ since A^{-1} is not densely defined in this case.

In turn, the operator C cannot serve as a counterexample in both cases. If $p = 1$ then it is easy to check that the range of C is not dense in ℓ^1 (however, it will possibly be a counterexample after restricting to the range of C, see the example of deLaubenfels mentioned in the introduction) so the algebraic inverse of C has non-dense domain in ℓ^1. If $p = \infty$, then C has a non-trivial kernel in ℓ^∞. Hence C has no algebraic inverse in ℓ^∞.

We conclude this section with an observation on norm estimates for $(e^{tA_N^{-1}})_{t\geq 0}$ on ℓ_N^p illustrating Theorem 6.1. From (6.7) and the inequality

$$|L_k^{(1)}(t)| \leq ce^{t/2}t^{-3/4}(k^{1/4} + t^{5/4}), \qquad t > 0 \tag{6.15}$$

(see [60, Ch 6, § 3]), we infer that

$$\begin{aligned}\left\|e^{tA_N^{-1}}\right\|_{N,1} &\leq e^{-t} + e^{-t}t\sum_{k=1}^{N-1}\frac{|L_{k-1}^{(1)}(t)|}{k}\\ &\leq e^{-t} + ce^{-t/2}t^{1/4}\sum_{k=1}^{N-1}\frac{k^{1/4}+t^{5/4}}{k} \leq CN^{1/4}, \qquad t > 0,\end{aligned}$$

for some $C > 0$. Moreover, since $(e^{tA_N})_{t\geq 0}$ is a contraction semigroup on ℓ_N^2, so is the semigroup $(e^{tA_N^{-1}})_{t\geq 0}$. Then by the Riesz–Thorin interpolation theorem (see e.g. [45, § 1.1]),

$$\left\|e^{tA_N^{-1}}\right\|_{N,p} \leq \left\|e^{tA_N^{-1}}\right\|_{N,2}^{2-2/p}\left\|e^{tA_N^{-1}}\right\|_{N,1}^{2/p-1} \leq c_pN^{1/(2p)-1/4}, \qquad p \in (1,2).$$

Moreover, we have

$$\left\|e^{tA_N^{-1}}\right\|_{N,q} = \left\|e^{tA_N^{-1}}\right\|_{N,p}, \qquad 1/p + 1/q = 1,$$

and therefore

$$\left\|e^{tA_N^{-1}}\right\|_{N,p} \leq c_pN^{|1/4-1/(2p)|}, \qquad p \in (1,2)\cup(2,\infty). \tag{6.16}$$

On the other hand, from (6.9), (6.12) and (6.13) it follows that

$$\limsup_N N^{-|1/4-1/(2p)|}\sup_{t\geq 0}\left\|e^{tA_N^{-1}}\right\|_{N,p} > 0, \qquad p \in (1,2)\cup(2,\infty), \tag{6.17}$$

so that, in view of (6.16), (6.17) is sharp.

7. Sufficient conditions for $A^{-1} \in \mathcal{G}_{\rm b}(X)$. Let A be the generator of a bounded C_0-semigroup in a Banach space X, and let A has a densely defined inverse A^{-1}. In this section we give a sufficient condition on the resolvent $R(\lambda, A)$, $\operatorname{Re}\lambda > 0$, implying that A^{-1} is also a generator of a bounded C_0-semigroup (see [32]).

First, recall the following condition for $A \in \mathcal{E}_-(X)$ to generate a bounded C_0-semigroup on X (that is, to belong to $\mathcal{G}_{\rm b}(X)$), see [30] and/or [59] for more on that. If

$$\sup_{\sigma>0} \sigma \int_{\sigma-i\infty}^{\sigma+i\infty} |\langle R^2(\lambda, A)x, y\rangle|\, |d\lambda| < \infty \tag{7.1}$$

for all $x \in X$ and $y \in X^*$, then $A \in \mathcal{G}_{\rm b}(X)$.

If X is a Hilbert space, then (7.1) is also necessary for A to be a generator of a bounded C_0-semigroup. Indeed, if for a Hilbert space X one has $A \in \mathcal{G}_{\rm b}(X)$, then

$$\sigma \int_{\sigma-i\infty}^{\sigma+i\infty} \|R(\lambda, A)x\|^2\, |d\lambda| = 2\pi\sigma \int_0^\infty e^{-2\sigma t}\|e^{tA}x\|^2\, dt \le \pi M^2\|x\|, \qquad \sigma > 0,$$

by Plancherel's theorem, and a similar inequality holds for A^* in view of $A^* \in \mathcal{G}_{\rm b}(X)$. Thus, observing that $|\langle R^2(\lambda, A)x, y\rangle| \le \|(R(\lambda, A)x\|\|R(\bar{\lambda}, A^*y)\|$ we obtain (7.1).

The next result is similar in spirit to the above result from [30] and [59], and it can in fact be considered as its corollary. It also relies on an elementary version of Carleson's embedding theorem, see e.g. [28, Chapter 2.3].

Theorem 7.1. *Let X be a Banach space, and let $A \in \mathcal{G}_{\rm b}(X)$ be such that $A^{-1} \in \mathcal{E}(X)$. Suppose that for all $x \in X$ and $y \in X^*$ there exist $\sigma_0 > 0$, $r_1 \ge r_2 > 0$, and a finite positive Radon measure μ on $(-\infty, \infty)$ such that*

$$\sup_{\sigma\in(0,\sigma_0)} \sigma \int_{\sigma-i\infty}^{\sigma+i\infty} |\langle R^2(\lambda, A)x, y\rangle|\, |d\lambda| < \infty, \tag{7.2}$$

and

$$|\langle R^2(\lambda, A)x, y\rangle| \le \int_{-\infty}^{\infty} \frac{\mu(dt)}{|it-\lambda|^2}\,, \qquad \operatorname{Re}\lambda > 0,\ |\lambda| \in (0, r_2) \cup (r_1, \infty). \tag{7.3}$$

Then $A^{-1} \in \mathcal{G}_{\rm b}(X)$. If $r_1 = r_2$ or X is a Hilbert space, then (7.2) *can be dropped.*

Proof. Since $R(\lambda, A)$ is bounded on every compact subset of the open right half-plane, we can assume that there are $b > 0$ and $r_0 \in (0, b)$ such that (7.3) holds in

$$\Omega(b; r_0) := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0,\ |\lambda \pm ib| > r_0\}$$

with, possibly, μ replaced by $\mu + c\|x\|\|y\|_*\delta_0$, where $c > 0$ and δ_0 is the delta-measure at $t = 0$. Moreover, in view of $(R(\lambda, A))' = -R^2(\lambda, A)$ and simple bounds for Poisson integrals, (7.3) yields

$$|\langle R(\lambda, A)x, y\rangle| \le \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{\mu(dt)}{|it-\lambda|}\,, \qquad \lambda \in \Omega(b; r_0), \tag{7.4}$$

for all $x \in X$ and $y \in X^*$.

To prove the theorem we verify that the resolvent of A^{-1} satisfies (7.1) for all $x \in X$ and $y \in X^*$. Since

$$R(\lambda, A^{-1}) = \lambda^{-1} - \lambda^{-2}R(\lambda^{-1}, A), \quad \operatorname{Re}\lambda > 0,$$

we have

$$\sigma\int_{\sigma-i\infty}^{\sigma+i\infty}|\langle R^2(\lambda,A^{-1})x,y\rangle|\,|d\lambda|\le\pi\|x\|\|y\|_*+\sigma J_\sigma,\qquad \sigma>0, \tag{7.5}$$

where

$$J_\sigma=\int_{\sigma-\infty}^{\sigma+i\infty}\left\{2|\lambda|^{-3}|\langle R(\lambda^{-1},A)x,y\rangle|+|\lambda|^{-4}|\langle R^2(\lambda^{-1},A)x,y\rangle|\right\}|d\lambda|.$$

Set

$$\Omega_0(b,r_0)=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}\setminus\Omega(b;r_0)=\left\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0,\ |\lambda\pm ib|\le r_0\right\}.$$

The mapping $\lambda\mapsto 1/\lambda$ transforms the line $\operatorname{Re}\lambda=\sigma$, $\sigma\ne 0$, into the circle

$$C_\sigma=\{z\in\mathbb{C}:|z-\epsilon|=\epsilon\},\qquad \epsilon=\frac{1}{2\sigma}\,.$$

Thus, in view of (7.3) and (7.4), after a change of variables, we can write

$$J_\sigma=\int_{C_\sigma}\left\{2|z|\,|\langle R(z,A)x,y\rangle|+|z|^2|\langle R^2(z,A)x,y\rangle|\right\}|dz|\le J_{0,\sigma}+J_{1,\sigma},$$

where

$$J_{1,\sigma}=\frac{\pi}{2}\int_{-\infty}^{\infty}\left\{\int_{C_\sigma}\left\{\frac{2|z|}{|it-z|}+\frac{|z|^2}{|it-z|^2}\right\}|dz|\right\}\mu(dt),$$

and

$$J_{0,\sigma}=\int_{\gamma_\sigma}\left\{2|z|\,|\langle R(z,A)x,y\rangle|+|z|^2|\langle R^2(z,A)x,y\rangle|\right\}|dz|,\qquad \gamma_\sigma:=C_\sigma\cap\Omega_0(b;r_0).$$

We estimate the integrals $J_{1,\sigma}$ and $J_{1,\sigma}$ separately. From

$$\frac{2|z|}{|it-z|}+\frac{|z|^2}{|it-z|^2}\le 8+\frac{2t^2}{|it-z|^2}$$

and the estimate

$$\int_{C_\sigma}\left\{8+\frac{2t^2}{|it-z|^2}\right\}|dz|\le\frac{10\pi}{\sigma}\,,$$

it follows that

$$\sigma J_{1,\sigma}\le 5\pi^2 m,\qquad m=\int_{-\infty}^{\infty}\mu(dt).$$

Let us now consider $J_{0,\sigma}$ assuming that γ_σ is nonempty, i.e., $\sigma\in(0,\sigma_1)$ for some σ_1 depending on b and r_0. Then $|z|\le b+r_0$ for $z\in\gamma_\sigma$, so there exist $c_1>0$ and $c_2>0$ such that

$$l(\gamma_\sigma)\le c_1,\qquad \operatorname{Re}z\ge c_2\sigma \tag{7.6}$$

for all $z\in\gamma_\sigma$ and all $\sigma\in(0,\sigma_1)$. Since $\|R(\lambda,A)\|\le M/(\operatorname{Re}z)$, $\operatorname{Re}z>0$, we have

$$\int_{\gamma_\sigma}|z|\,|\langle R(z,A)x,y\rangle|\,|dz|\le c_3,\qquad \sigma\in(0,\sigma_1)$$

by (7.6).

Since $R(\lambda,A)$ is bounded on $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\ge\sigma_0,\ |\lambda|\le r_1\}$ and

$$\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\int_{-\infty}^{\infty}\frac{\mu(dt)}{|it-\lambda|^2}\right\}|d\lambda|\le\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\frac{ds}{(s-t)^2+\sigma^2}\right\}\mu(dt)=\frac{\pi}{\sigma}\,m,$$

(7.2) and (7.3) imply (7.1).

In other words, for each $\lambda_0 > 0$ the function $\lambda \to \langle R^2(\lambda+\lambda_0, A)x, y\rangle$, belongs to the Hardy space H^1 over the right half-plane. Therefore, setting $\lambda_0 = c_2\sigma$, where c_2 is given by (7.6), and using Carleson's embedding theorem, we obtain

$$\sigma \int_{\gamma_\sigma} |z|^2 |\langle R^2(z, A)x, y\rangle| \, |dz| \le c_4\sigma \int_{\lambda_0 - i\infty}^{\lambda_0+i\infty} |\langle R^2(\lambda, A)x, y\rangle| \, |d\lambda| \le c_5,$$

where $c_5 > 0$ is independent of $\sigma \in (0, \sigma_1)$. Here we have used the fact that if l is arc-length measure, then the Carleson embedding constant N in the inequality

$$l\big(\gamma_\sigma \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \in (\lambda_0, \lambda_0 + h),\ \operatorname{Im}\lambda \in (t, t+h)\}\big) \le Nh, \qquad h > 0,\ t \in \mathbb{R},$$

admits the estimate $N \le 2\pi$ for all $\sigma \in (0, \sigma_1)$, see [32, Lemma 2].

Thus from (7.5) and the estimates for J_σ, $\sigma \in (0, \sigma_1)$ it follows that A^{-1} satisfies (7.1) for all $x \in X$ and $y \in X^*$, so that $A^{-1} \in \mathcal{G}_{\mathrm{b}}(X)$. ∎

Corollary 7.2. *Let X be a Banach space and let $A \in \mathcal{G}_{\mathrm{b}}(X)$. Assume that*

$$\|R(\lambda, A)\| \le c \sum_{k=1}^{n} \frac{1}{|ia_k - \lambda|}, \qquad \operatorname{Re}\lambda > 0,$$

for some $c > 0$ and $(a_k)_{1\le k\le n} \subset \mathbb{R}$. If $A^{-1} \in \mathcal{E}(X)$, then $A^{-1} \in \mathcal{G}_{\mathrm{b}}(X)$.

Remark that (7.3) holds if there exist $r_1 \ge r_2 > 0$ such that

$$\|R(\lambda, A)\| \le \frac{c}{|\lambda|}, \qquad \operatorname{Re}\lambda > 0, \qquad |\lambda| \in (0, r_2) \cup (r_1, \infty). \tag{7.7}$$

Moreover, (7.7) for $\operatorname{Re}\lambda > 0$ and $|\lambda| \in (r_1, \infty)$ is equivalent to $A \in \mathcal{H}_0$, and the same estimate for $\operatorname{Re}\lambda > 0$ and $|\lambda| \in (0, r_2)$ is equivalent to $A^{-1} \in \mathcal{H}_0$, see [44, Chap 9, § 1.7]. Thus we have the following assertion.

Corollary 7.3. *Let X be a Hilbert space. Let both A and A^{-1} belong to $\mathcal{H}_0(X)$. Then*

$$A \in \mathcal{G}_{\mathrm{b}}(X) \ \Leftrightarrow\ A^{-1} \in \mathcal{G}_{\mathrm{b}}(X).$$

In particular, if A is bounded and boundedly invertible, then $A \in \mathcal{G}_{\mathrm{b}}$ if and only if $A^{-1} \in \mathcal{G}_{\mathrm{b}}$.

Note that, in contrast to the situation where $A \in \mathcal{H}(\theta)(X)$, the assumptions $A \in \mathcal{G} \cap \mathcal{H}_0(X)$ and $A^{-1} \in \mathcal{G}(X)$ do not generally imply $A^{-1} \in \mathcal{H}_0(X)$. Indeed, the Volterra operator $-J$ on a Hilbert space $L_2[0,1]$ defined in Section 4 belongs to $\mathcal{G} \cap \mathcal{H}_0$, but its inverse $A = -J^{-1}$ generates the semigroup given by (4.3) which is even not differentiable for $t \in (0, 1)$.

8. A^{-1}-problem for unbounded C_0-semigroups on Hilbert spaces. As we pointed out in Section 1, A^{-1}-problem is still open for generators of bounded C_0-semigroups on (infinite-dimensional) Hilbert spaces. However, the situation changes if we turn to generators of unbounded C_0-semigroups. In this section we show that there exists a C_0-group e^{tA} on a Hilbert space H of arbitrarily slow growth at infinity, such that $A^{-1} \in \mathcal{E}(H)$ but A^{-1} does not generate a C_0-semigroup on H. The construction of such a group is taken from [37].

Denote by $\|\cdot\|_2$ the standard norm in the 2-dimensional Euclidean space $\mathbb{C}^2$. Assume that $\mu > 0$, $\gamma \geq 1$, and let

$$A_{\mu,\gamma} := \mu S_\gamma^{-1} J S_\gamma = \mu \begin{pmatrix} 0 & \gamma \\ -\gamma^{-1} & 0 \end{pmatrix}, \quad S_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{8.1}$$

so that $J^{-1} = J^* = -J$, $\|J\|_2 = 1$, $\|S_\gamma^{-1}\|_2 = 1$, $\|S_\gamma\|_2 = \gamma$. Then straightforward calculations yield the following analogue of Euler's formula:

$$\begin{aligned} e^{tA_{\mu,\gamma}} = S_\gamma^{-1}\Big(\sum_{m=0}^{\infty} \frac{\mu^m J^m}{m!}\, t^m\Big) S_\gamma &= S_\gamma^{-1}(\cos(\mu t) I + \sin(\mu t) J) S_\gamma \\ &= \cos(\mu t) I + \sin(\mu t) S_\gamma^{-1} J S_\gamma, \qquad t \in \mathbb{R}, \end{aligned} \tag{8.2}$$

so

$$e^{tA_{\mu,\gamma}} = \begin{pmatrix} \cos(\mu t) & \gamma \sin(\mu t) \\ -\gamma^{-1} \sin(\mu t) & \cos(\mu t) \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Note that

$$\big\|e^{tA_{\mu,\gamma}}\big\|_2 = \big\|e^{-tA_{\mu,\gamma}}\big\|_2, \qquad t \in \mathbb{R}.$$

From (8.2) we conclude that for $t \in \mathbb{R}$

$$\big\|e^{tA_{\mu,\gamma}}\big\|_2 \leq \|S_\gamma^{-1}\|_2 \; \|\cos(\mu t) I + \sin(\mu t) J\|_2 \; \|S_\gamma\|_2 = \|S_\gamma\|_2 = \gamma, \tag{8.3}$$

and

$$\|e^{tA_{\mu,\gamma}}\|_2 \leq 1 + |\sin(\mu t)| \; \|S_\gamma^{-1} J S_\gamma\|_2 = 1 + |\sin(\mu t)|\, \gamma. \tag{8.4}$$

Moreover, if $|\cos(\mu t)| = 1$, then

$$\big\|e^{sA_{\mu,\gamma}}\big\|_2 = \gamma, \quad s = \frac{\pi}{2\mu}\,. \tag{8.5}$$

Now, since $J^{-1} = -J$, we have

$$A_{\mu,\gamma}^{-1} = -\mu^{-1} S_\gamma^{-1} J S_\gamma = -\mu^{-2} A_{\mu,\gamma},$$

and therefore

$$e^{tA_{\mu,\gamma}^{-1}} = e^{\tau A_{\mu,\gamma}}, \; \tau = -\mu^{-2} t, \qquad t \in \mathbb{R}. \tag{8.6}$$

Let $\{\gamma_n\}_{n \in \mathbb{N}}$ be an unbounded and non-decreasing positive sequence such that

$$\gamma_1 \geq 1, \; \frac{\gamma_n}{n} \leq \frac{\gamma_m}{m} \quad \text{for all } n, m \in \mathbb{N}, \; n \geq m, \tag{8.7}$$

so that the sequence $\{n^{-1}\gamma_n\}_{n \in \mathbb{N}}$ is non-increasing and $1 \leq \gamma_n \leq n$, $n \in \mathbb{N}$. Consider the Hilbert space $H = \bigoplus_{n=1}^{\infty} \mathbb{C}^2$ with the norm

$$\|x\| = \Big\{\sum_{n=1}^{\infty} \|x_n\|_2^2\Big\}^{1/2}, \qquad x = \{x_n\} \in H. \tag{8.8}$$

For a fixed sequence $\{\gamma_n\}_{n \geq 1}$ as above, we now construct a C_0-group $(e^{tA})_{t \in \mathbb{R}}$ on X such that $\|e^{-tA}\| = \|e^{tA}\|$, $t \in \mathbb{R}$,

$$\gamma_m \leq \sup_{t \in [0,m]} \|e^{tA}\| \leq 3\gamma_m, \qquad m \in \mathbb{N}, \tag{8.9}$$

but A^{-1} is not a generator of any C_0-semigroup on X.

Let

$$\mu_n = \frac{\pi}{2n}, \qquad n \in \mathbb{N}, \tag{8.10}$$

and let $A \in \mathcal{L}(H)$ be defined by

$$A := \operatorname{diag}(B_n), \quad B_n = A_{\gamma_n,\mu_n}, \tag{8.11}$$

so that $A\{x_n\}=\{B_n x_n\}$, $\{x_n\}\in H$. Then e^{tA} is a "diagonal" operator for each $t\in\mathbb{R}$, so

$$\|e^{tA}\| = \sup_{n\geq 1} \|e^{tB_n}\|_2, \qquad t \in \mathbb{R}. \tag{8.12}$$

By (8.3),

$$\|e^{tB_n}\|_2 \leq \gamma_n, \quad n \in \mathbb{N},\ t \in \mathbb{R}. \tag{8.13}$$

Using now (8.4) and (8.7) for $n \geq m \geq t \geq 0$ we obtain

$$\|e^{tB_n}\|_2 \leq 1 + |\sin(\pi t/(2n))|\,\gamma_n \leq 1 + \frac{\pi m}{2n}\,\gamma_n \leq 1 + \left(\frac{\pi\gamma_m}{2\gamma_n}\right)\gamma_n \leq 3\gamma_m. \tag{8.14}$$

Combining (8.14) with (8.12) and (8.13) and using the monotonicity of $\{\gamma_n\}_{n\geq 1}$, we infer that $\|e^{tA}\| \leq 3\gamma_m$, $|t| \leq m$. On the other hand, $\|e^{mB_m}\|_2 = \gamma_m$ (see (8.5)) and therefore $\|e^{mA}\| \geq \gamma_m$. This yields (8.9).

By (8.11) and (8.1) the inverse A^{-1} has the following form:

$$A^{-1} = \operatorname{diag}(B_n^{-1}) = \operatorname{diag}(-\mu_n^{-2} B_n), \tag{8.15}$$

and (see (8.6)) $e^{tB_n^{-1}} = e^{-t\mu_n^{-1}B_n}$. Setting $t_0 = \pi^2/4$, we note that $\|e^{t_0 B_{2m+1}^{-1}}\|_2 = \gamma_{2m+1} \to \infty$ as $m \to \infty$, hence $\sup_{n\in\mathbb{N}} \|e^{t_0 B_n^{-1}}\|_2 = \infty$. The latter property implies that A^{-1} does not generate a C_0-semigroup on H.

Theorem 8.1. *Let $f : [0,\infty) \to (0,\infty)$ be a continuous function increasing to infinity, such that*

$$f(0) = 1, \quad f(t) \leq e^t, \qquad t \geq 0.$$

Then there exists a generator A of C_0-group $(e^{tA})_{t\in\mathbb{R}}$ on H satisfying

$$\|e^{-tA}\| = \|e^{tA}\| \leq cf(t),\ t > 0, \qquad \|e^{tA}\| \to \infty,\ t \to \infty,$$

such that A^{-1} is well-defined but does not generate a C_0-semigroup on H.

Proof. Let a function f satisfying the assumptions be fixed. Define

$$u(t) := g(\ln(1 + \ln t)),\ t \geq 1, \qquad g(t) := \frac{1}{t}\int_0^t f(s)\,ds,\ t > 0, \qquad g(0) = f(0) = 1.$$

By the monotonicity of f,

$$1 \leq g(t) \leq f(t),\ t \geq 0, \qquad u(t) \leq f(\ln(1 + \ln t)) \leq f(t),\ t \geq 1.$$

Moreover, $g(t) \geq f(t/2)/2 \to \infty$ as $t \to \infty$, so that $u(t) \to \infty$ as $t \to \infty$. Observe that

$$\frac{d}{dt}\left(\frac{u(t)}{t}\right) = \left\{e^{-\tau}g'(\tau) - g(\tau)\right\}\frac{1}{t^2}, \qquad \tau = \ln(1 + \ln t),\ t > 1.$$

Since $f(t) \leq e^t$, $t \geq 0$, we have

$$g'(\tau) \leq f(\tau) \leq e^\tau \leq e^\tau g(\tau), \qquad \tau > 0,$$

so that $(t^{-1}u(t))' \leq 0$, $t > 1$.

Let $\gamma_n = u(n)$, $n \in \mathbb{N}$. Then $\{\gamma_n\}_{n\in\mathbb{N}}$ satisfies (8.7), and if A is given by (8.11) then C_0-group $(e^{tA})_{t\in\mathbb{R}}$ satisfies (8.9). Thus, for each $t = n+\alpha$, $n \in \mathbb{N}$, $\alpha \in [0,1)$,

$$\|e^{tA}\| \le \|e^{\alpha A}\|\|e^{nA}\| \le 3c_0\gamma_n, \qquad c_0 = \sup_{t\in[0,1]} \|e^{tA}\| \le 3\gamma_1 = 3.$$

Therefore,

$$\|e^{tA}\| \le 3 \le 3f(t),\ t \in [0,1], \qquad \|e^{tA}\| \le 9\gamma_n \le 9u(t) \le 9f(t),\ t \ge 1,$$

and

$$\gamma_n = \|e^{nA}\| = \|e^{-\alpha A}e^{\alpha A}e^{nA}\| \le \|e^{-\alpha A}\|\|e^{tA}\| \le 3\|e^{tA}\|,$$

so that

$$\|e^{tA}\| \ge u(n)/3 \to \infty \quad \text{as } t = n+\alpha \to \infty.$$

It remains to note that $\|e^{-tA}\| = \|e^{tA}\|$, $t > 0$. ∎

9. Power bounded Cayley transforms. In this section we will study the power boundedness of Cayley transforms of semigroup generators, and relate these studies to the A^{-1}-problem discussed in the preceding sections.

Let X be a Banach space, and let $A \in \mathcal{G}_{\mathrm{b}}(X)$. It is crucial that one can express the powers of $V(A)$ in terms $(e^{At})_{t\ge 0}$:

$$V^n = I - \int_0^\infty e^{-t/2} L_{n-1}^{(1)}(t) e^{(t/2)A}\, dt, \qquad n \in \mathbb{N}, \tag{9.1}$$

where $L_n^{(1)}$ are the first order Laguerre polynomials. The formula above is fundamental in the study of asymptotic behavior of $(V^n)_{n\ge 0}$.

Recall that (9.1) was obtained in [51] for isometric $(e^{At})_{t\ge 0}$. As noted in [10], (9.1) remains true for $A \in \mathcal{G}_{\mathrm{b}}(X)$. Indeed, this fact follows directly from the identity

$$V^n = (1 + 2(A-1)^{-1})^n = \sum_{k=0}^n \frac{(-1)^k n! 2^k}{k!(n-k)!} R^k(1,A), \qquad n \in \mathbb{N},$$

(3.2) with $\lambda = 1$ and (3.3).

Using (9.1), the estimate (see [3])

$$\int_0^\infty e^{-t/2} |L_n^{(1)}(t)|\, dt \le cn^{1/2}, \qquad n \in \mathbb{N}, \tag{9.2}$$

and (6.15), it is easy to obtain a growth bound for powers of V given in the next result.

LEMMA 9.1. *If $A \in \mathcal{G}_{\mathrm{b}}(X)$, then*

$$\|V^n(A)\| \le 1 + cMn^{1/2}, \qquad n \in \mathbb{N}. \tag{9.3}$$

If moreover $A \in \mathcal{G}_{\exp}(X)$, then

$$\|V^n(A)\| \le 1 + c_\omega M n^{1/4}, \qquad n \in \mathbb{N}. \tag{9.4}$$

Note that (9.3) is a partial case of the classical result due to Brenner and Thomée on powers of rational functions, see [9, Theorem 1] for more details.

The estimates (9.3) and (9.4) are the best possible in a sense that the growth rates $n^{1/2}$ and $n^{1/4}$ cannot in general be improved. To justify this claim we first follow an argument similar to the one in [14]. Let $X = L_1(\mathbb{R})$ and let $(\mathcal{D}f)(s) = f'(s)$, with $\mathrm{dom}(\mathcal{D}) = W_1^1(\mathbb{R})$. Note that $\mathcal{D}$ is the generator of a unitary left shift C_0-group

$$(e^{t\mathcal{D}} f)(s) = f(s+t), \qquad s, t \in \mathbb{R},$$

hence by (9.1),

$$(V^n(\mathcal{D})f)(s) = f(s) - \int_0^\infty e^{-t/2} L_{n-1}^{(1)}(t) f(s+t/2)\, dt, \qquad s \in \mathbb{R},\ n \in \mathbb{N}.$$

From this, using the inequality (see [3])

$$\int_0^\infty e^{-t/2} |L_n^{(1)}(t)|\, dt \geq c_1 n^{1/2}, \qquad n \in \mathbb{N},$$

for some $c_1 > 0$ and an abstract result on operator norms from [43, Theorem XI.1.4], it follows that

$$1 + \|V^n(\mathcal{D})\|_{L_1(\mathbb{R})} \geq c_1 n^{1/2}, \qquad n \in \mathbb{N},$$

therefore (9.3) is, in general, sharp.

Let now $X = L_1[0,1]$, and let an operator A on X be defined by (4.2) with $p = 1$. If $A_0 := 2A - I$, then A_0 is the generator of a nilpotent C_0-semigroup on X, and in particular, $A_0 \in \mathcal{G}_{\exp}(X)$. Observe that

$$V(A_0) = A(A-I)^{-1} = (I+J)^{-1},$$

where J is the classical Volterra operator given by (4.1). Then (4.5) for $p = 1$ shows that (9.4) is optimal.

The above examples motivate the problem of providing conditions on A to ensure that $V(A)$ is power bounded. The following statement from [35] gives such conditions in terms of the resolvent of A.

THEOREM 9.2. *Let $A \in \mathcal{G}_{\mathrm{b}}$. Suppose that for all $x \in X$ and $y \in X^*$ there exist $\sigma_0 > 0$, $r_0 \geq 0$ and a finite positive Radon measure μ on* R *such that*

$$\sup_{\sigma \in (0,\sigma_0)} \sigma \int_{\sigma - i\infty}^{\sigma + i\infty} |\langle R^2(\lambda, A)x, y\rangle|\, |d\lambda| < \infty, \tag{9.5}$$

$$|\langle R^2(\lambda, A)x, y\rangle| \leq \int_{-\infty}^{\infty} \frac{\mu(dt)}{|it - \lambda|^2}, \qquad \mathrm{Re}\,\lambda > 0,\ |\lambda| \in (r_0, \infty). \tag{9.6}$$

Then

$$\sup_{n \in \mathbb{N}} \|V^n(A)\| < \infty. \tag{9.7}$$

If $r_0 = 0$ or X is a Hilbert space, then (9.5) *can be omitted.*

The proof of Theorem 9.2 is similar to that of Theorem 7.1. One verifies that the resolvent of $V(A)$ satisfies

$$\sup_{r>1} (r-1) \int_{|\lambda| = r} |\langle R^2(\lambda, V(A))x, y\rangle|\, |d\lambda| < \infty, \tag{9.8}$$

for all $x \in X$, $y \in X^*$. By [15, Lemma 2.1] this implies (9.7).

Note that (9.6) holds for $A \in \mathcal{E}_-(X)$ such that

$$\|R(\lambda, A)\| \leq c/|\lambda|^{-1}, \qquad \mathrm{Re}\,\lambda > 0,\ |\lambda| > r_0.$$

If $A \in \mathcal{E}(X)$ then the latter estimate is equivalent to fact that A is the generator of a (sectorially bounded if $r_0 = 0$) holomorphic C_0-semigroup, see [44, Chap. 9, § 1.4]. Thus Theorem 9.2 implies the following assertion, originally obtained in [31], [39], and in [4] in the particular case of bounded generator.

Corollary 9.3. *Let H be a Hilbert space and let $A \in \mathcal{G}_{\rm b}(H)$ be the generator of a holomorphic C_0-semigroup. Then the Cayley transform $V(A)$ is power-bounded. In particular, $V(A)$ is power-bounded if $A \in \mathcal{L}(H)$.*

Corollary 9.3 was apparently the first positive statement on power boundedness of Cayley transforms for generators of not necessarily contractive semigroups. Unfortunately, if A is not bounded, then the situation in the setting of Hilbert spaces becomes much more complicated, and only partial results are available so far.

10. Growth of powers for Cayley's transforms in Hilbert space. In this section we will show that for generators of bounded C_0-semigroups on Hilbert spaces the powers of their Cayley transforms grow at most logarithmically. The exposition is based on the results from [31].

We start from the next auxiliary assertion of independent interest (see [31, Lemma 1]).

Lemma 10.1. *Let $g_j : [0,1) \to \infty$, $j = 1, 2$, be continuous functions. Let T be a bounded linear operator on a Hilbert space H satisfying*

$$\sup_{r\in(0,1)} g_1(r) \sum_{n=0}^{\infty} \|T^n x\|^2 r^{2n} \le M^2(T)\|x\|^2,$$

$$\sup_{r\in(0,1)} g_2(r) \sum_{n=0}^{\infty} \|T^{*n} x\|^2 r^{2n} \le M^2(T^*)\|x\|^2,$$

for all $x \in H$, and some $M(T), M(T^*) > 0$. Then

$$ng\left(\frac{n}{n+1}\right)\|T^n\| \le eM(T)M(T^*), \qquad n \in \mathbb{N}, \tag{10.1}$$

where $g(r) := \sqrt{g_1(r)g_2(r)}$, $r \in [0, 1)$.

Proof. For all $x, y \in H$ and $r \in (0, 1)$,

$$nr^n \langle T^n x, y\rangle = \sum_{k=1}^{n} \langle T^n x, y\rangle r^n = \sum_{k=1}^{n} \langle T^{n-k} x, T^{*k} y\rangle r^{n-k} r^k,$$

hence

$$\begin{aligned} nr^n |\langle T^n x, y\rangle| &\le \sum_{k=1}^{n} \|r^{n-k} T^{n-k} x\|\, \|r^k T^{*k} y\| \\ &\le \Big(\sum_{k=1}^{n} \|T^k x\|^2 r^{2k}\Big)^{1/2} \Big(\sum_{k=1}^{n} \|T^{*k} y\|^2 r^{2k}\Big)^{1/2} \le \frac{M(T)M(T^*)}{\sqrt{g_1(r)g_2(r)}}\, \|x\|\, \|y\|. \end{aligned}$$

Setting $r = n/(n+1)$, we have

$$e^{-1} ng\left(\frac{n}{n+1}\right)|\langle T^n x, y\rangle| \le M(T)M(T^*)\|x\|\, \|y\|.$$

Since $x, y \in H$ are arbitrary, (10.1) follows. ∎

For $g_j(r) = 1-r$, $j = 1, 2$, Lemma 10.1 yields the estimate for powers of T first proved in [11, Proposition 2.1] and [39, Theorem 8.3].

COROLLARY 10.2. *Let T be a bounded linear operator on a Hilbert space H with the spectral radius $r(T) \le 1$. Assume that for all $x \in X$ there exist $M(T), M(T^*) > 0$ such that*

$$\sup_{r\in(0,1)} (1-r)\sum_{n=0}^{\infty} \|T^n x\|^2 r^{2n} \le M(T)\|x\|^2,$$
$$\sup_{r\in(0,1)} (1-r)\sum_{n=0} \|T^{*n} x\|^2 r^{2n} \le M(T^*)\|x\|^2.$$

Then

$$\|T^n\| \le eM(T)M(T^*), \qquad n \in \mathbb{N}.$$

We will also need a kind of Parseval identity for powers of Cayley transform.

PROPOSITION 10.3. *Let $A \in \mathcal{G}_{\rm b}(H)$. Then for every $x \in H$,*

$$(1-r^2)\sum_{n=1}^{\infty} \|V^n x - x\|^2 r^{2n} = 4r^2\epsilon^2 \int_0^\infty \int_0^\infty \langle e^{As}x, e^{At}x\rangle e^{-\epsilon(1+r^2)(s+t)} u(s,t;r)\, ds\, dt, \qquad r \in (0,1), \tag{10.2}$$

where

$$u(s,t;r) := r\,\frac{(s+t)}{\sqrt{st}}\, I_1(4r\epsilon\sqrt{st}) - (1+r^2) I_0(4r\epsilon\sqrt{st}), \qquad \epsilon = \frac{1}{1-r^2}\,, \tag{10.3}$$

and $I_k(\cdot)$ is the modified Bessel function.

Proof. By virtue of (9.1), for $r \in (0,1)$, we have

$$\sum_{n=1}^{\infty} \|V^n(A)x - x\|^2 r^{2n} = 4r^2 \sum_{n=0}^{\infty} r^{2n} \left\| \int_0^\infty e^{-t} L_n^{(1)}(2t) e^{tA}x\, dt \right\|^2$$
$$= 4r^2 \sum_{n=0}^{\infty} r^{2n} \int_0^\infty \int_0^\infty e^{-(s+t)} L_n^{(1)}(2s) L_n^{(1)}(2t) \langle e^{sA}x, e^{tA}x\rangle\, ds\, dt. \tag{10.4}$$

Then, by Fubini's theorem, using (9.2), the identity

$$\sum_{n=0}^{\infty} L_n^{(1)}(t) L_n^{(1)}(s) r^{2n} = \epsilon^3 s^{-r^2\epsilon(s+t)} \left[-r\,\frac{(s+t)}{\sqrt{st}}\, I_1(2r\epsilon\sqrt{st}) + (1+r^2) I_0(2r\epsilon\sqrt{st}) \right]$$

and (10.4), we obtain (10.2). ■

Now we are ready to obtain a logarithmic bound for powers of Cayley transforms.

THEOREM 10.4. *Let $A \in \mathcal{G}_{\rm b}(H)$. Then*

$$\sup_{n\in\mathbb{N}} \frac{\|V^n(A)\|}{\log(n+1)} < \infty.$$

Proof. Taking into account Lemma 10.1 and Proposition 10.3 it suffices to establish the estimate

$$(1-r^2)Q(r) \le a + b\log(1-r), \qquad r \in (0,1),$$

where

$$Q(r) := \epsilon^2 \int_0^\infty \int_0^\infty e^{-\epsilon(1+r^2)(s+t)} |u(s,t;r)|\, ds\, dt, \qquad \epsilon = \frac{1}{1-r^2}, \qquad r \in (0,1).$$

Next note that if u is defined by (10.3) then

$$u(s,t;r) = (1-r^2) I_1(4r\epsilon\sqrt{st}) + (1+r^2)\big[I_0(4r\epsilon\sqrt{st}) - I_1(4r\epsilon\sqrt{st})\big] - r\, \frac{(\sqrt{s}-\sqrt{t})^2}{\sqrt{st}}\, I_1(4r\epsilon\sqrt{st}).$$

Since the modified Bessel functions I_0 and I_1 satisfy ([26])

$$0 \le I_1(t) \le c\frac{e^t}{\sqrt{t}}, \qquad 0 \le I_0(t) - I_1(t) \le c\frac{e^t}{t\sqrt{t}}, \quad t > 0,$$

for some $c > 0$, we have

$$|u(s,t;r)| \le c_1 \frac{e^{4r\epsilon\sqrt{st}}}{\epsilon^{1/2}(st)^{1/4}} \left[(1-r)^2 + \frac{1}{\epsilon(st)^{1/2}} + \frac{(\sqrt{s}-\sqrt{t})^2}{(st)^{1/2}}\right],$$

for every $r \in [1/2, 1)$. Hence

$$Q(r) \le c_1 \epsilon^{3/2} \int_0^\infty \int_0^s \frac{e^{-\epsilon(1+r^2)(s+t)} e^{-4r\epsilon\sqrt{st}}}{(st)^{1/4}} \left[(1-r)^2 + \frac{1}{\epsilon(st)^{1/2}} + \frac{(\sqrt{s}-\sqrt{t})^2}{(st)^{1/2}}\right] dt\, ds,$$

so letting $t = s\tau^2$, $\tau \in (0,1)$, we obtain

$$\begin{aligned}
Q(r) &\le 2c_1 \epsilon^{3/2} \int_0^\infty \int_0^1 e^{-\epsilon[(1+r^2)(1+\tau^2)-4r\tau]s} (s\tau)^{1/2} \left[(1-r)^2 + \frac{1}{\epsilon s\tau} + \frac{(1-\tau)^2}{\tau}\right] d\tau\, ds \\
&= 2c_1 \epsilon^{3/2} \int_0^1 \int_0^\infty e^{-\epsilon[(1+r^2)(1+\tau^2)-4r\tau]s} (s\tau)^{1/2} \left[(1-r)^2 + \frac{1}{\epsilon s\tau} + \frac{(1-\tau)^2}{\tau}\right] ds\, d\tau \\
&\le c_2 \int_0^1 \tau^{1/2} \left[(1-r)^2 + \frac{(1-\tau)^2}{\tau}\right] \frac{d\tau}{[(1+r^2)(1+\tau^2)-4r\tau]^{3/2}} \\
&\quad + c_2 \int_0^1 \frac{d\tau}{\tau^{1/2}[(1+r^2)(1+\tau^2)-4r\tau]^{1/2}} \\
&\le c_3 + c_4 \int_{1/2}^1 \frac{d\tau}{[(1-\tau)^2+(1-r)^2]^{1/2}} \\
&= c_3 + c_4 \int_0^{1/(2(1-r))} \frac{ds}{(s^2+1)^{1/2}} \le a + b\,|\log(1-r)|, \qquad r \in (0,1). \quad \blacksquare
\end{aligned}$$

It was shown in [4], [31] and [39] that if $A \in \mathcal{G}_b(H)$ then $V(A)$ is power bounded if *both* A and A^{-1} generate bounded C_0-semigroups. The question whether one can omit the assumption on A^{-1} remains still open.

11. Growth of $e^{tA^{-1}}$ in terms of growth of powers of $V(A)$. In this section, based on the paper [36], we show that if $A \in \mathcal{G}_{\exp}(X)$ then growth bounds for $V^n(A)$, yield similar growth bounds for $\left(e^{tA^{-1}}\right)_{t\ge 0}$.

THEOREM 11.1. *Let X be a Banach space and assume that for any $A \in \mathcal{G}_{\exp}(X)$ there exists $M_A > 0$ such that*

$$\|V^n(A)\| \le M_A g(n), \qquad n \in \mathbb{N}, \tag{11.1}$$

where g is a strictly positive non-decreasing function on $[0,\infty)$. Then the semigroup $(e^{tA^{-1}})_{t\ge 0}$ satisfies a similar estimate. More precisely, if there exist $M \ge 1$ and $\omega > 0$ such that $\|e^{tA}\| \le Me^{-\omega t}$, $t \ge 0$, then

$$\|e^{tA^{-1}}\| \le \tilde{M}_A g\left(\frac{2et}{\omega}\right), \qquad t \ge 0, \tag{11.2}$$

for some $\tilde{M}_A > 0$.

Proof. By Lemma 9.1, we may assume without loss of generality that $g(n) \le c_0(1+\sqrt[4]{n})$. It is easy to see that $A_0 = 2\omega^{-1}A + I \in \mathcal{G}_{\exp}(X)$. Furthermore, $V(A_0) = I + \omega A^{-1}$, and

$$\|e^{t\omega A^{-1}}\| = e^{-t}\|e^{tV(A_0)}\|, \qquad t \ge 0. \tag{11.3}$$

Since $A_0 \in \mathcal{G}_{\exp}(X)$, $V(A_0)$ satisfies (11.1) for some $M_{A_0} > 0$. From this and Stirling's formula [1, § 1.4], it follows that

$$\begin{aligned}\|e^{tV(A_0)}\| &\le \sum_{n=0}^{\infty} \frac{t^n\|V^n(A_0)\|}{n!} \le M_{A_0}\sum_{n=0}^{\infty}\frac{g(n)t^n}{n!}\\ &\le c_1\Bigg[\sum_{n\ge 2et}\left(\frac{et}{n}\right)^n \frac{g(n)}{\sqrt{n}} + \sum_{n\le 2et}\frac{g(n)t^n}{n!}\Bigg] \le c_1\Bigg[\sum_{n=1}^{\infty}\frac{g(n)}{2^n\sqrt{n}} + g(2et)\sum_{n=0}^{\infty}\frac{t^n}{n!}\Bigg]\\ &= c_2 + c_1 g(2et)e^t, \qquad t \ge 1, \end{aligned} \tag{11.4}$$

where we used the estimate $g(n) \le c_0(1+\sqrt[4]{n})$, $n \in \mathbb{N}$. Combining (11.4) and (11.3) we obtain (11.2). ∎

Theorem 11.1 has several several important corollaries. We start with the one relating power boundedness of $V(A)$ and boundedness of $(e^{tA^{-1}})_{t\ge 0}$.

COROLLARY 11.2. *Suppose there exist a Banach space X and $A \in \mathcal{G}_{\exp}(X)$ such that $A^{-1} \notin \mathcal{G}_{\mathrm{b}}(X)$. Then there exists $A_0 \in \mathcal{G}_{\exp}(X)$ such that $V(A_0)$ is not power bounded.*

Proof. Assume that for every $A_0 \in \mathcal{G}_{\exp}(X)$ its Cayley transform $V(A_0)$ is power bounded. Then, Theorem 11.1 with $g(n) \equiv 1$ in (11.1) implies that $A^{-1} \in \mathcal{G}_{\mathrm{b}}(X)$, whenever $A \in \mathcal{G}_{\exp}(X)$, a contradiction with e.g. Zwart's counterexample mentioned in the introduction. ∎

If X is finite-dimensional, then the function g in Theorem 11.1 can always be chosen to be constant. However, this constant may depend on the dimension of X, see e.g. (35) in [37]. Theorem 11.1 shows that on finite-dimensional spaces the best estimates for $\sup_{t\ge 0}\|e^{tA^{-1}}\|$ and $\sup_{n\in\mathbb{N}}\|V^n(A)\|$ are asymptotically the same when $\dim X \to \infty$.

Using Theorem 11.1, we can provide new proofs for known estimates of $\|e^{tA^{-1}}\|$. The first result follows directly from Theorem 11.1 and Lemma 9.1. It has been originally obtained in [64] by a different argument.

Corollary 11.3. *Let X be a Banach space. If $A \in \mathcal{G}_{\exp}(X)$, then there exists $M_0 > 0$ such that*

$$\|e^{tA^{-1}}\| \le 1 + M_0 t^{1/4}, \qquad t \ge 0.$$

If X is a Hilbert space then Theorem 10.4 implies that one can put $g(n) = \ln(n+1)$ in (11.1). This observation together with Theorem 11.1 yields the following estimate proved in [65].

Corollary 11.4. *Let H be a Hilbert space. If $A \in \mathcal{G}_{\exp}(H)$ then there exists $M_0 > 0$ such that*

$$\|e^{tA^{-1}}\| \le M_0 \ln(t+2), \qquad t \ge 0.$$

The next result, obtained in [36, Theorem 3.5], can be seen as a converse to Theorem 11.1.

Theorem 11.5. *Let X be a Banach space. Assume that for every $A \in \mathcal{G}_{\exp}(X)$ there exists $M_A > 0$ and a positive non-decreasing function g (not depending on A) such that*

$$\left\|e^{tA^{-1}}\right\| \le \tilde{M}_A g(t), \qquad t \ge 0.$$

Then for every $A \in \mathcal{G}_{\exp}(X)$ and every $\alpha > 1$ there exists $M_{\alpha,A} > 0$ such that

$$\|V^n(A)\| \le M_{\alpha,A} g(\alpha n), \qquad n \in \mathbb{N}.$$

If A generates a bounded C_0-semigroup on a Hilbert space H, then it is still not known whether its Cayley transform $V(A)$ is always power bounded (whenever $V(A)$ is well-defined). However, one can prove that $V(A)$ is a power bounded, if in addition A^{-1} exists and generate a bounded C_0-semigroup too, i.e., when both A and A^{-1} are in $\mathcal{G}_{\mathrm{b}}(H)$. Below we will give a new proof of this result relying on the Lyapunov equations technique worked out in details in [36]. To this aim, we will need several facts on infinite-dimensional Lyapunov equations. Their proofs can be found in [17, Exercise 4.29] or [23, Section II.6].

Lemma 11.6. *Let Q be a linear bounded operator on a Hilbert space H. If there exists a positive solution $P \in \mathcal{L}(H)$ of the Lyapunov equation*

$$Q^* P Q - P = -I, \tag{11.5}$$

then $\sum_{n=0}^{\infty} \|Q^n x\|^2 < \infty$ for all $x \in X$. Conversely, if $\sum_{n=0}^{\infty} \|Q^n x\|^2$ is finite for every $x \in X$, then there exists a unique solution of (11.5). Furthermore, this solution is positive, and

$$\langle Px, x\rangle = \sum_{n=0}^{\infty} \|Q^n x\|^2, \qquad x \in H. \tag{11.6}$$

Assume that A and A^{-1} belong to $\mathcal{E}(H)$, $\lambda, \lambda^{-1} \in \rho(A)$, and $\lambda \ge 1$. It will be useful to consider the following Lyapunov equations:

$$(\lambda^2 - 1) R^*(\lambda, A) P_1 R(\lambda, A) - P_1 = -I, \tag{11.7}$$

$$(\lambda^2 - 1) R^*(\lambda, A^{-1}) P_2 R(\lambda, A^{-1}) - P_2 = -I. \tag{11.8}$$

with unknown operators P_1 and P_2.

If we assume that $1 \in \rho(A)$, it will also be convenient to consider the additional Lyapunov equation

$$\left(\frac{\lambda-1}{\lambda+1}\right)V^*(A)P_V V(A) - P_V = -I, \tag{11.9}$$

defined by means of $V(A)$.

Each of the Lyapunov equations above can be rewritten using the inner product $\langle\cdot,\cdot\rangle$ in H. For instance, (11.7) then takes the form:

$$(\lambda^2-1)\big\langle P_1 R(\lambda,A)x_1, R(\lambda,A)x_2\big\rangle - \langle P_1 x_1, x_2\rangle = -\langle x_1, x_2\rangle, \qquad x_1,x_2 \in H.$$

Choosing $z_k = R(\lambda,A)x_k \in \operatorname{dom}(A)$, $k=1,2$, we obtain

$$(\lambda^2-1)\langle P_1 z_1, z_2\rangle - \big\langle P_1(\lambda I - A)z_1, (\lambda I - A)z_2\big\rangle = -\big\langle(\lambda I - A)z_1, (\lambda I - A)z_2\big\rangle,$$

and, after simple algebraic manipulations,

$$\lambda\langle P_1 z_1, Az_2\rangle + \lambda\langle P_1 Az_1, z_2\rangle - \langle P_1 Az_1, Az_2\rangle - \langle P_1 z_1, z_2\rangle = -\langle(\lambda I - A)z_1, (\lambda I - A)z_2\rangle.$$

To simplify the notation, given $A \in \mathcal{E}(H)$ and $P \in \mathcal{L}(H)$, define a bilinear form $B_\lambda[A,P](\cdot,\cdot)$ on $\operatorname{dom}(A)$ by

$$\begin{aligned} B_\lambda[A,P](z_1,z_2) &:= \lambda\langle PAz_1, z_2\rangle + \lambda\langle Pz_1, Az_2\rangle \\ &\quad - \langle PAz_1, Az_2\rangle - \langle Pz_1, z_2\rangle, \qquad z_1,z_2 \in \operatorname{dom}(A). \end{aligned} \tag{11.10}$$

Then (11.7) can be rewritten as

$$B_\lambda[A,P_1](z_1,z_2) = -\big\langle(\lambda I - A)z_1, (\lambda I - A)z_2\big\rangle, \tag{11.11}$$

for all $z_1,z_2 \in \operatorname{dom}(A)$. Similarly, (11.8) is equivalent to

$$B_\lambda[A,P_2](z_1,z_2) = -\big\langle(I - \lambda A)z_1, (I - \lambda A)z_2\big\rangle, \qquad z_1,z_2 \in \operatorname{dom}(A), \tag{11.12}$$

and (11.9) can be recast as

$$B_\lambda[A,P_V](z_1,z_2) = -\frac{(\lambda+1)}{2}\,\langle(A-I)z_1, (A-I)z_2\rangle, \qquad z_1,z_2 \in \operatorname{dom}(A). \tag{11.13}$$

Lemma 11.7. *Let $A, A^{-1} \in \mathcal{E}(H)$ and let $\lambda \in (1,\infty)$ be such that $\lambda, \lambda^{-1} \in \rho(A)$. Moreover, assume $1 \in \rho(A)$. Then*

1) (11.7) *has a bounded solution P_1 if and only if* (11.8) *has a bounded solution P_2. Furthermore, the solutions are related by the formula*

$$P_2 = (I - \lambda A)^* R(\bar{\lambda}, A^*) P_1 (I - \lambda A) R(\lambda, A). \tag{11.14}$$

2) *If* (11.7) *and* (11.8) *have bounded solutions P_1 and P_2 respectively, then a bounded solution P_V of* (11.9) *is given by*

$$P_V = \frac{1}{2\lambda}\,(P_1 + P_2 + \lambda I - I). \tag{11.15}$$

Proof. To prove 1), let $P_1 \in \mathcal{L}(H)$ be a solution of (11.7) (or (11.11)) and define a bounded operator

$$P_2 := (I - \lambda A^*) R(\bar{\lambda}, A^*) P_1 (I - \lambda A) R(\lambda, A). \tag{11.16}$$

Then, in view of

$$[(I - \lambda A^*) R(\bar{\lambda}, A^*)]^* = (I - \lambda A) R(\lambda, A),$$

and

$$(I-\lambda A)R(\lambda,A)Az = AR(\lambda,A)(I-\lambda A)z, \qquad z\in D(A),$$

we infer that

$$\begin{aligned}B_\lambda[A,Q](z_1,z_2) &= B_\lambda[A,P_1]\langle R(\lambda,A)(I-\lambda A)z_1, R(\lambda,A)(I-\lambda A)z_2\rangle\\ &= -\langle(\lambda I-A)R(\lambda,A)(I-\lambda A)z_1,(\lambda-A)R(\lambda,A)(I-\lambda A)z_2\rangle\\ &= -\langle(I-\lambda A)z_1,(I-\lambda A)z_2\rangle,\end{aligned}$$

for all $z_1,z_2\in\operatorname{dom}(A)$. So, P_2 is a solution of (11.8) and (11.12).

To show 2), define $\tilde Q\in\mathcal{L}(H)$ as

$$\tilde Q := P_1+P_2+(\lambda-1)I,$$

where P_1 and P_2 be solutions of (11.11) and (11.12), respectively. Then

$$\begin{aligned}B_\lambda[A,\tilde Q](z_1,z_2) &= B_\lambda[A,P_1](z_1,z_2)+B_\lambda[A,P_2](z_1,z_2)+(\lambda-1)B_\lambda[A,I](z_1,z_2)\\ &= -\langle(\lambda I-A)z_1,(\lambda I-A)z_2\rangle-\langle(I-\lambda A)z_1,(I-\lambda A)z_2\rangle\\ &\quad +(\lambda-1)\big[\lambda\langle Az_1,z_2\rangle+\lambda\langle z_1,Az_2\rangle-\langle Az_1,Az_2\rangle-\langle z_1,z_2\rangle\big]\\ &= \lambda(\lambda+1)\big[\langle Az_1,z_2\rangle+\langle z_1,Az_2\rangle-\langle Az_1,Az_2\rangle-\langle z_1,z_2\rangle\big]\\ &= -\lambda(\lambda+1)\langle(A-I)z_1,(A-I)z_2\rangle,\end{aligned}$$

for all $z_1,z_2\in\operatorname{dom}(A)$. Hence, by choosing $P_V=(2\lambda)^{-1}\tilde Q$, it follows that

$$B_\lambda[A,P_V](z_1,z_2) = -\frac{(\lambda+1)}{2}\langle(A-I)z_1,(A-I)z_2\rangle,$$

for all $z_1,z_2\in\operatorname{dom}(A)$, so that P_V satisfies (11.13), and thus also (11.9). ∎

Now we will show the Lyapunov equations technique in action. The proof of the first item can found in [31, Theorem 2]. However, we present a new proof below. (Note that there is a typo in the formulation of that result in [31].)

THEOREM 11.8. *Let $A\in\mathcal{E}(H)$ be such that $\sigma(A)\subset\{\lambda\in\mathbb{C}:\operatorname{Re}(\lambda)\le 0\}$. Suppose that the (algebraic) inverse A^{-1} of A exists and $A^{-1}\in\mathcal{E}$.*

1) *If, for $x\in H$ and $\lambda>1$,*

$$G_A(x;\lambda) := \frac{1}{\lambda^2}\sum_{n=0}^{\infty}(\lambda^2-1)^n\big[\|R^n(\lambda,A)x\|^2+\|R^n(\lambda,A^{-1})x\|^2\big], \tag{11.17}$$

then setting $\lambda=(1+r^2)/(1-r^2)$, $r\in(0,1)$, we have

$$G_A(x;\lambda)+\frac{(\lambda-1)}{\lambda^2}\|x\|^2 = \frac{2(1-r^2)}{(1+r^2)}\sum_{n=0}^{\infty}\|V^n(A)x\|^2r^{2n}, \qquad x\in H. \tag{11.18}$$

2) *If $A,A^{-1}\in\mathcal{G}_{\mathrm b}(H)$, then $V(A)$ is power bounded and*

$$\|V^n(A)\|\le\frac{e}{2}\left(\frac14+M^2+M_1^2\right), \tag{11.19}$$

where $M=\sup_{t>0}\|e^{tA}\|$ and $M_1=\sup_{t>0}\|e^{tA^{-1}}\|$.

Proof. Starting with the proof of 1), we note that $G_A(x;\lambda)$ is finite for any $x \in H$, since for $\lambda > 0$ the spectra of $\lambda R(\lambda, A)$ and $\lambda R(\lambda, A^{-1})$ are located outside the closed unit disk. Thus by Lemma 11.6 there exists a unique solution P_1 of the Lyapunov equation (11.7). Furthermore, this solution satisfies

$$\langle P_1 x, x\rangle = \sum_{n=0}^{\infty} (\lambda^2 - 1)^n \|R^n(\lambda, A)x\|^2, \tag{11.20}$$

for all $x \in H$. Similarly, there exists a unique solution $P_2 \in \mathcal{L}(H)$ of (11.8), satisfying

$$\langle P_2 x, x\rangle = \sum_{n=0}^{\infty} (\lambda^2 - 1)^n \|R^n(\lambda, A^{-1})x\|^2, \tag{11.21}$$

for all $x \in H$. By Lemma 11.7, the operator P_V defined by (11.15) is a unique solution of (11.9). Since the operators P_1 and P_2 are positive, and since $\lambda > 1$, we conclude that P_V is positive as well, and so by Lemma 11.6,

$$\langle P_V x, x\rangle = \sum_{n=0}^{\infty} \left(\frac{\lambda - 1}{\lambda + 1}\right)^n \|V^n(A)x\|^2, \qquad x \in H. \tag{11.22}$$

By rewriting (11.15) in the inner product form:

$$\langle (P_1 + P_2 + \lambda I - I)x, x\rangle = 2\lambda \langle P_V x, x\rangle,$$

and substituting (11.20)–(11.22) we find that

$$\sum_{n=0}^{\infty} (\lambda^2 - 1)^n \|R^n(\lambda, A)x\|^2 + \sum_{n=0}^{\infty} (\lambda^2 - 1)^n \|R^n(\lambda, A^{-1})x\|^2$$
$$+ (\lambda - 1)\|x\|^2 = 2\lambda \sum_{n=0}^{\infty} \left(\frac{\lambda - 1}{\lambda + 1}\right)^n \|V^n(A)x\|^2, \qquad x \in H,$$

dividing by λ^2 and substituting $\lambda = (1 + r^2)/(1 - r^2)$ into the right-hand side, we arrive at (11.18).

Let us now prove 2). Since $A, A^{-1} \in \mathcal{G}_{\mathrm{b}}(H)$, from (3.2) it follows that there exist $M \geq 1$ and $M_1 \geq 1$ such that

$$\|R^n(\lambda, A)\| \leq M\lambda^{-n}, \quad \|R^n(\lambda, A^{-1})\| \leq M_1 \lambda^{-n}, \quad \lambda > 0, \ n \in \mathbb{N}. \tag{11.23}$$

Substituting this in (11.17), we deduce that for any $x \in H$,

$$G_A(x, \lambda) \leq \frac{M^2 + M_1^2}{\lambda^2} \sum_{n=0}^{\infty} \left[\frac{\lambda^2 - 1}{\lambda^2}\right]^n \|x\|^2 = (M^2 + M_1^2)\|x\|^2. \tag{11.24}$$

Next, using an elementary calculus, we observe that

$$r \in (0, 1) \Leftrightarrow \lambda = \frac{1 + r^2}{1 - r^2} \in (1, \infty), \qquad r > 0,$$

and

$$\frac{(\lambda - 1)}{\lambda^2} \leq \frac{1}{4}, \quad \lambda > 1, \qquad \frac{1 + r}{1 + r^2} \geq 1, \quad r \in (0, 1).$$

Hence, by employing (11.18) and (11.24), it follows that

$$(1-r)\sum_{n=0}^{\infty}\|V^n(A)x\|^2 r^{2n}\le \frac{M^2+M_1^2+1/4}{2}\,\|x\|^2,\qquad r\in(0,1),\ x\in H.$$

Since (11.23) also holds for A substituted by A^* and A^{-1} substituted by $(A^{-1})^*$, we obtain as above

$$(1-r)\sum_{n=0}^{\infty}\|V^n(A^*)x\|^2 r^{2n}\le \frac{M^2+M_1^2+1/4}{2}\,\|x\|^2,\qquad r\in(0,1),\ x\in H.$$

In view of $V(A^*)=V^*(A)$, Corollary 10.2 yields (11.19). ■

In the previous theorem we have shown that if A and A^{-1} are both the generators of bounded C_0-semigroups on a Hilbert space, then the Cayley transform of A is power bounded. Next we observe that this result is invariant with respect to "shifts" of A^{-1} along the imaginary axis, see [36].

Theorem 11.9. *Let H be a Hilbert space and let $A\in\mathcal{E}_-(H)$. If one of the following conditions holds*:

1) *There exists $s\in\mathbb{R}$ such that $is\in\rho(A)$ and $(A-is)^{-1}\in\mathcal{G}_{\rm b}(H)$.*
2) *There exists a non-zero $s\in\mathbb{R}$ such that $(A-isI)^{-1}$ and $(A+is^{-1}I)^{-1}$ are in $\mathcal{G}_{\rm b}(H)$,*

then $V(A)$ is power bounded.

Remark that condition 2) of Theorem 11.9 is, in general, weaker than condition 1), since in 1) it is assumed that $(A-isI)^{-1}$ is a bounded operator, whereas in 2) this operator is supposed to be merely the generator of a C_0-semigroup.

Now we show that shifts of A^{-1} along the positive real axis do not perturb power boundedness of $V(A)$ too. The following theorem extends the second statement of Theorem 11.8.

Theorem 11.10. *Let H be a Hilbert space and let $A\in\mathcal{G}_{\rm b}(H)$. Assume that there exists the* (*algebraic*) *inverse of A. Then the following statements are equivalent*:

1) *For each $\varepsilon>0$ one has $-R(\varepsilon,A)\in\mathcal{G}_{\rm b}(H)$, and*

$$\sup_{\epsilon>0}\sup_{t\ge 0}\|e^{-tR(\varepsilon,A)}\|<\infty.$$

2) $A^{-1}\in\mathcal{G}_{\rm b}(H)$.
3) *For each $\delta>0$ the Cayley transform $V(\delta A)$ is power bounded and*

$$\sup_{\delta>0}\sup_{n\ge 0}\|V^n(\delta A)\|<\infty. \tag{11.25}$$

Proof. The implication 1) ⇒ 2) follows from Trotter–Kato's approximation theorem, see [25, Theorem III.4.9].

To prove 2) ⇒ 3), observe that if A generates a bounded C_0-semigroup, then so does δA for each $\delta>0$. Furthermore,

$$\sup_{t\ge 0}\|e^{t\delta A}\|=\sup_{t\ge 0}\|e^{tA}\|=M.$$

Similarly, we have

$$\sup_{t\ge 0}\|e^{t(\delta A)^{-1}}\| = \sup_{t\ge 0}\|e^{tA^{-1}}\| = M_1.$$

By 2) of Theorem 11.8, $V(\delta A)$ is power bounded, with a bound for its powers independent of δ.

Passing to the proof of 3) ⇒ 1) and using (11.25), note that

$$\|e^{tV(\delta A)}\| \le Ce^t, \quad t\ge 0, \tag{11.26}$$

for every $\delta > 0$. Moreover, for each $\varepsilon > 0$,

$$e^{-2\varepsilon tR(\varepsilon,A)} = e^{2t(\varepsilon^{-1}A-I)^{-1}} = e^{-t}e^{tV(\varepsilon^{-1}A)}.$$

Choosing $\delta = \varepsilon^{-1}$ and combining this with (11.26), we conclude that $\|e^{-tR(\epsilon,A)}\| \le C$, $t\ge 0$, and 1) follows. ■

The above result shows that if $A \in \mathcal{G}_{\rm b}(H)$ then $A^{-1} \in \mathcal{G}_{\rm b}(H)$ if and only if $(A-\varepsilon I)^{-1} \in \mathcal{G}_{\rm b}$ for all $\varepsilon > 0$ and $(e^{-tR(\varepsilon,A)})_{t\ge 0}$ is bounded by a constant independent of t and ε.

Now we would like to further extend this result by replacing $R(\varepsilon, A)$ with $R(\lambda, A)$, $\operatorname{Re}\lambda \ge 0$. For the proof of this generalization, we invoke (3.15) relating the semigroups generated by $A \in \mathcal{G}_{\rm exp}(H)$ and by $A^{-1} \in \mathcal{L}(H)$. The next auxiliary lemma will also be crucial.

Lemma 11.11. *Let H is a Hilbert space and let $f : [0,\infty) \to H$ be a continuous function decaying exponentially at infinity such that $f(0) = 0$. If*

$$\hat f(\tau) := \sqrt{\tau}\int_0^\infty \frac{J_1(2\sqrt{\tau t})}{\sqrt{t}}\, f(t)\,dt, \quad \tau > 0, \tag{11.27}$$

then the following Plancherel type theorem holds:

$$\int_0^\infty \|\hat f(\tau)\|^2\,\frac{d\tau}{\tau} = \int_0^\infty \|f(t)\|^2\,\frac{dt}{t}\,. \tag{11.28}$$

Proof. For scalar f, the statement follows easily from the identity ([18, § 7.3])

$$\int_0^\infty \left|\mathcal{H}[f](\tau)\right|^2\tau\,d\tau = \int_0^\infty |f(t)|^2 t\,dt,$$

for the (first order) classical Hankel transformation given by

$$\mathcal{H}[f](\tau) := \int_0^\infty tf(t)J_1(\tau t)\,dt, \quad f\in L_2((0,\infty); t\,dt)$$

combined with the relation

$$\hat f(\tau) = \sqrt{\tau}(\mathcal{H}(f_0))(\sqrt{\tau}),$$

where $f_0(t) = \frac{1}{t}f(t^2/4)$. A passage to the vector-valued setting can be justified as for Plancherel's theorem for Fourier transforms, see e.g. [2, Chapter 1]. ■

By a standard procedure, we can extend the transform defined in (11.27) to all functions f for which the right-hand side of (11.28) is finite.

Theorem 11.12. *Let H be a Hilbert space, and let A and A^{-1} belong to $\mathcal{G}_{\rm b}(H)$. Let also*

$$M := \sup_{t\ge 0}\|e^{tA}\|, \qquad M_1 := \sup_{t\ge 0}\big\|e^{tA^{-1}}\big\|.$$

Then for any $\lambda=\varepsilon+i\tau$ with $\varepsilon>0$, one has $-R(A,\varepsilon+i\tau)\in\mathcal{G}_{\rm b}(H)$. Moreover, for all $t\ge 0$,

$$\big\|e^{-tR(\varepsilon+i\tau,A)}\big\| \le 2\bigg[e^2\Big(M^2+M_1^2+\frac14\Big)^2+M^2\ln\Big(1+\frac{\tau^2}{4\varepsilon^2}\Big)\bigg]. \tag{11.29}$$

Proof. From Theorem 11.10 it follows that $-R(A,\varepsilon)\in\mathcal{G}_{\rm b}(H)$ for each $\varepsilon>0$, and moreover

$$\big\|e^{-tR(\varepsilon,A)}\big\|\le \frac{e}{2}\Big(M^2+M_1^2+\frac14\Big),\qquad t\ge 0. \tag{11.30}$$

Next, by (3.15), for any $x\in H$ we have

$$e^{-tR(\varepsilon,A)}x-e^{-tR(\varepsilon+i\tau,A)}x=\sqrt{t}\int_0^\infty \frac{J_1(2\sqrt{ts})}{\sqrt{s}}\,[e^{-i\tau s}-1]e^{-\varepsilon s}e^{sA}x\,ds,\qquad t>0,$$

and thus by Lemma 11.11,

$$\int_0^\infty \big\|e^{-tR(\varepsilon,A)}x-e^{-tR(\varepsilon+i\tau,A)}x\big\|^2\,\frac{dt}{t}=\int_0^\infty \frac{|1-e^{i\tau t}|^2}{t}\,e^{-2\epsilon t}\|e^{tA}x\|^2\,dt. \tag{11.31}$$

In view of

$$\int_0^\infty \frac{|1-e^{i\tau t}|^2}{t}\,e^{-2\varepsilon t}\,dt=4\int_0^\infty\frac{\sin^2(\tau t/2)}{t}\,e^{-2\varepsilon t}\,dt=\ln\Big(1+\frac{\tau^2}{4\varepsilon^2}\Big),$$

we conclude by (11.31) that

$$\int_0^\infty \big\|e^{-tR(\varepsilon,A)}x-e^{-tR(\epsilon+i\tau,A)}x\big\|^2\,\frac{dt}{t}\le M^2\ln\Big(1+\frac{\tau^2}{4\epsilon^2}\Big)\|x\|^2, \tag{11.32}$$

for any $x\in H$.

Therefore, for $t>0$ and $x\in H$,

$$\begin{aligned}
&\frac1t\int_0^t \|e^{-sR(\varepsilon+i\tau,A)}x\|^2\,ds\\
&\quad\le \frac2t\int_0^t \big\|e^{-sR(\varepsilon,A)}x-e^{-sR(\varepsilon+i\tau,A)}x\big\|^2\,ds+\frac2t\int_0^t\big\|e^{-sR(\varepsilon,A)}x\big\|^2\,ds\\
&\quad\le 2\int_0^t \big\|e^{-sR(\varepsilon,A)}-e^{-sR(\varepsilon+i\tau,A)}x\big\|^2\,\frac{ds}{s}+2\|x\|^2\sup_{t\ge0}\|e^{-tR(\varepsilon)}\|^2\\
&\quad\le 2M^2\ln\Big(1+\frac{\tau^2}{4\varepsilon^2}\Big)\|x\|^2+\frac{e^2}{2}\Big(M^2+M_1^2+\frac14\Big)^2\|x\|^2,
\end{aligned}$$

where we have used (11.32) and (11.30). Hence,

$$\frac1t\int_0^t \big\|e^{-sR(\varepsilon+i\tau,A)}x\big\|^2\,ds\le\bigg[\frac{e^2}{2}\Big(M^2+M_1^2+\frac14\Big)^2+2M^2\ln\Big(1+\frac{\tau^2}{4\varepsilon^2}\Big)\bigg]\|x\|^2,$$

for all $x\in H$ and $t>0$. Clearly, a similar estimate holds for the adjoint semigroup $\big(e^{-tR(\varepsilon-i\tau,A^*)}\big)_{t\ge0}$. So, using a counterpart of Proposition 10.1 for C_0-semigroups (see e.g. [11, Proposition 3.1]), we obtain (11.29). ∎

Note that the above theorem holds if we assume that there exists $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$, $\lambda \neq 0$, such that A and $(A-\lambda I)^{-1}$ are in $\mathcal{G}_{\mathrm{b}}(H)$. The proof follows the same lines as above.

So far we have been concentrating mainly on the implication

$$A \in \mathcal{G}_{\mathrm{b}}(H) \Rightarrow \sup_n \|V^n(A)\| < \infty,$$

and showed that it is true under extra assumption $A^{-1} \in \mathcal{G}_{\mathrm{b}}(H)$. The following theorem shows that for exponentially stable semigroups the assumption is in fact necessary.

THEOREM 11.13. *Let A generate an exponentially stable C_0-semigroup on a Hilbert space H and $V(A)$ be power bounded. Then A^{-1} generates a bounded C_0-semigroup on H.*

Proof. Setting $A_1 = A - I$, we note that

$$V(A) = (A+I)(A-I)^{-1} = I + 2A_1^{-1}.$$

Then

$$\|e^{2tA_1^{-1}}\| = e^{-t}\|e^{tV(A)}\| \leq M_d, \qquad M_d := \sup_{n\geq 0}\|V^n(A)\|,$$

hence A_1^{-1} generates a bounded semigroup. Since A generates an exponentially stable C_0-semigroup,

$$\int_0^\infty \|e^{tA}x - e^{tA_1}x\|^2 \frac{dt}{t} = \int_0^\infty (1-e^{-t})^2\|e^{tA}x\|^2 \frac{dt}{t} < \infty.$$

Combining this with (3.15) and Lemma 11.11, we conclude that for every $x \in H$,

$$\int_0^\infty \|e^{tA^{-1}}x - e^{tA_1^{-1}}x\|^2 \frac{dt}{t} < \infty.$$

Since A_1^{-1} generates a bounded semigroup on H, arguing as in the proof of Theorem 11.12 we infer that A^{-1} generates a bounded semigroup as well. ∎

For the discrete analogues of the two preceding theorems we refer to [8].

Finally, using Theorems 11.8 and 11.13, we obtain the following corollary on power boundedness of scaled Cayley transforms.

COROLLARY 11.14. *Let H be a Hilbert space. If $A \in \mathcal{G}_{\mathrm{exp}}(H)$ and $V(A)$ is power bounded, then $V(\alpha A)$ is power bounded for all $\alpha > 0$.*

Proof. Let $\alpha > 0$ be fixed. Theorem 11.13 implies that A^{-1} generates a bounded C_0-semigroup. Therefore, $(\alpha A)^{-1}$ generates a bounded C_0-semigroup as well. Since αA generates a bounded C_0-semigroup too, we infer by Theorem 11.8 that $V(\alpha A)$ is power bounded. ∎

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