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Between the genus and the Γ-genus of an integral quadratic Γ-form

by

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1. Introduction. Let C be a geometrically connected and smooth projective curve defined over the finite field $\mathbb{F}_q$ (q is odd). Let $K = \mathbb{F}_q(C)$ be its function field and let Ω denote the set of all closed points of C. For any point $\mathfrak{p} \in \Omega$ let $v_\mathfrak{p}$ be the induced discrete valuation on K, $\hat{\mathcal{O}}_\mathfrak{p}$ the complete discrete valuation ring with respect to $v_\mathfrak{p}$, and $\hat{K}_\mathfrak{p}$ its fraction field. Any *Hasse set* of K, i.e. a non-empty finite set $S \subset \Omega$, gives rise to an integral domain of K called a *Hasse domain*:

$$\mathcal{O}_S := \{x \in K : v_\mathfrak{p}(x) \geq 0 \ \forall \mathfrak{p} \notin S\}.$$

In what follows, any $\mathcal{O}_S$-scheme is underlined, and omitted in the notation of its generic fiber.

Let Γ be a finite group, faithfully represented by $\rho : \Gamma \hookrightarrow \mathbf{GL}(V)$ where V is a projective $\mathcal{O}_S$-module of rank $n \geq 1$. We briefly write γ instead of $\rho(\gamma)$ when no confusion may occur, and assume $|\Gamma|$ is prime to $\mathrm{char}(K)$. Then Γ acts on $\mathbf{GL}(V)$ on the left by conjugation:

$$\forall \gamma \in \Gamma, A \in \mathbf{GL}(V) : \quad {}^{\gamma}A = \gamma A \gamma^{-1}.$$

Let V be equipped with a degree two homogeneous $\mathcal{O}_S$-form $q : V \to \mathcal{O}_S$, turning (V, q) into an *integral* quadratic $\mathcal{O}_S$-space, represented by a bilinear map $B_q : V \times V \to \mathcal{O}_S$ such that

$$B_q(u, v) = q(u + v) - q(u) - q(v).$$

We assume (V, q) is $\mathcal{O}_S$-*regular*, i.e., the induced homomorphism $V \to V^\vee := \mathrm{Hom}(V, \mathcal{O}_S)$ is an isomorphism. We say (V, q) is *rationally isotropic* (or just *isotropic*) if there exists a non-zero $v \in V$ for which $q(v) = 0$; it is a *Γ-form*

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if $q \circ \gamma = q$ for all $\gamma \in \Gamma$. Two integral forms (V, q) and (V', q') are said to be *R-isomorphic* where R is an extension of $\mathcal{O}_S$ if there exists an R-isomorphism $A : V' \cong V$ such that $q \circ A = q'$; we then call A an *R-isometry.* Two Γ-forms are said to be *Γ-isomorphic* over R if there exists an R-isometry between them which is Γ-equivariant.

The classification of quadratic forms (without a group action) defined over an integral domain of a function field was initially studied by L. Gerstein [Ger] and J. S. Hsia [Hsia] in the late seventies. Later on, J. Morales [Mor] proved that there are only finitely many Γ-isomorphism classes of Γ-forms defined over $\mathbb{Z}$ of a given discriminant. He also showed that the classical Hasse–Minkowski Theorem, stating that two forms defined over a global field K are K-isomorphic if and only if they are $\hat{K}_v$-isomorphic at any place v, does not hold for Γ-forms with Γ-isomorphisms when $K = \mathbb{Q}$. Recently, E. Bayer-Fluckiger, N. Bhaskhar and R. Parimala [BBP] showed that this principle does hold, however, for Γ-forms when K is a global function field.

The failure of this *local-global principle* in the case of integral forms (even without a group action) is measured by the *genus* of such a form. This term and its generalization to integral Γ-forms are defined as follows:

Definition 1.1. The *genus* $c(q)$ of an $\mathcal{O}_S$-form q is the set of isomorphism classes of $\mathcal{O}_S$-forms that are K- and $\hat{\mathcal{O}}_{\mathfrak{p}}$-isomorphic to q for any prime $\mathfrak{p} \notin S$. The *Γ-genus* $c_\Gamma(q)$ of a Γ-form q defined over $\mathcal{O}_S$ is the set of Γ-isomorphism classes of Γ-forms defined over $\mathcal{O}_S$ that are K- and $\hat{\mathcal{O}}_{\mathfrak{p}}$-isomorphic to q for any prime $\mathfrak{p} \notin S$, and these isomorphisms are Γ-isomorphisms.

We denote by $c^+(q)$ and $c^+_\Gamma(q)$ the genus and the Γ-genus of q respectively with respect to proper ($\det = 1$) isomorphisms only.

This paper was motivated by the following question, posed to me by B. Kunyavskiĭ:

Question 1.2. *Suppose two integral Γ-forms share the same Γ-genus and are $\mathcal{O}_S$-isomorphic* (*the Γ-action is forgotten*). *Are they necessarily also Γ-isomorphic?*

Any integral Γ-form representing a class in $c_\Gamma(q)$ clearly represents a class in $c(q)$ as well. So the map $\psi : c_\Gamma(q) \to c(q)$ is well-defined, and we may rephrase Question 1.2 as follows: Is ψ always an injection? Because if the answer is no, and only then, there exist two integral Γ-forms representing two distinct, though $\mathcal{O}_S$-isomorphic classes in $c_\Gamma(q)$.

After showing in Section 2 that $[H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{SO}}^\Gamma_V)]$ $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{O}}^\Gamma_V)$ [properly] classifies the integral Γ-forms that are locally Γ-isomorphic to (V, q), where $[\underline{\mathbf{SO}}^\Gamma_V]$ $\underline{\mathbf{O}}^\Gamma_V$ stands for the [special] orthogonal group of (V, q), we compare in Section 3 the genus and the Γ-genus of (V, q), and give a necessary and sufficient condition for the positive answer to Question 1.2. In order to provide a counter-example, we refer in Section 4 more concretely to the case where $(\underline{\mathbf{O}}_V^{\Gamma})^0$ is the special orthogonal group of another isotropic form (V', q'). Based on a result established in [Bit1] and [Bit2], stating that if q is isotropic of rank ≥ 3, then $c(q) \cong \mathrm{Pic}(\mathcal{O}_S)/2$, we show that if q' is isotropic of rank 2 then it may possess twisted forms which are only *stably isomorphic*, i.e. isomorphic over (any) regular non-trivial extension of V'. So our suggested obstruction to Question 1.2 arises from the failure of the Witt cancellation theorem over $\mathcal{O}_S$.

2. Classification via flat cohomology. The following general framework that appears in [CF, §2.2.4] allows us to extend some known facts about the classification of integral forms via flat cohomology to integral Γ-forms.

PROPOSITION 2.1. *Let R be a scheme and X_0 be an R-form, i.e. an object of a fibered category of schemes defined over R. Let $\mathbf{Aut}_{X_0}$ be its R-group of automorphisms. Let $\mathfrak{Forms}(X_0)$ be the category of R-forms that are locally isomorphic to X_0 in some topology, and let $\mathfrak{Tors}(\mathrm{Aut}_{X_0})$ be the category of Aut_{X_0}-torsors in that topology. The functor*

$$\varphi : \mathfrak{Forms}(X_0) \to \mathfrak{Tors}(\mathbf{Aut}_{X_0}), \qquad X \mapsto \mathbf{Iso}_{X_0,X},$$

is an equivalence of fibered categories.

We first implement this proposition on split torsion R-groups. For a non-negative integer m we consider the R-group $\underline{\mu}_m := \mathrm{Spec}\, R[t]/(t^m - 1)$. The pointed set $H^1_{\mathrm{fl}}(R, \underline{\mu}_m)$ classifies $\underline{\mu}_m$-*torsors*, i.e. R-groups that are locally isomorphic to $\underline{\mu}_m$ in the flat topology. We briefly introduce another description of these elements, as can be found for example in [AG, §5.1].

An m-degree *R-Kummer pair* is a couple $\Lambda = (L, h)$ consisting of a rank 1 projective R-module L and an isomorphism $h : R \cong L^{\otimes m}$. It gives rise to a $\underline{\mu}_m$-torsor E_Λ assigning to any extension R'/R the group

$$E_\Lambda(R') = \{\varphi \in L^\vee \otimes R' : \varphi^{\otimes m} = h\}$$

where $L^\vee := \mathrm{Hom}(R, L)$.

In particular, for $m = 2$, we set X_0 to be the quadratic R-algebra $R \oplus R$ with the standard involution $(r_1, r_2) \mapsto (r_1, -r_2)$, thus $\mathbf{Aut}_{X_0} = \underline{\mu}_2$. Let L be a fractional ideal of order 2 in $\mathrm{Pic}(R)$. Then any 2-degree Kummer pair $\Lambda = (L, h)$ gives rise to an R-algebra $X = R \oplus L$ with multiplication defined by $(0, l_1) \cdot (0, l_2) = (h^{-1}(l_1 \otimes l_2), 0)$, viewed as an R-form, whence according to Proposition 2.1, Λ corresponds to the $\underline{\mu}_2$-torsor

$$E_\Lambda = \mathbf{Iso}(R \oplus R, R \oplus L),$$

in which the isomorphism induced by h is $\varphi : (r_1, r_2) \mapsto (r_1, l)$ where l is such that $l \otimes l = h(r_2)$, i.e., $\varphi \otimes \varphi = h$.

Let $\underline{\mathbf{O}}_V$ be the *orthogonal group* of (V,q), i.e. the $\mathcal{O}_S$-group assigning to any $\mathcal{O}_S$-algebra R the group of self-isometries of q defined over R:

$$\underline{\mathbf{O}}_V(R) = \{A \in \mathbf{GL}(V \otimes R) : q \circ A = q\}.$$

The pointed set $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{O}}_V)$ classifies the integral quadratic forms of rank n [Knu, IV.5.3.1], which are all locally isomorphic to (V,q) in the flat topology. We may generalize this to Γ-forms as follows: Suppose (V,q) is a Γ-form. Then if we restrict φ in Proposition 2.1 to the $\mathcal{O}_S$-group of Γ-automorphisms $\underline{\mathbf{O}}_V^\Gamma$ for the flat topology, the corresponding forms are the integral Γ-forms that are locally Γ-isomorphic to (V,q) in the flat topology. Modulo $\mathcal{O}_S$-isomorphisms we get $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{O}}_V^\Gamma)$.

As $2 \in \mathcal{O}_S^\times$ and (V,q) is $\mathcal{O}_S$-regular, $\underline{\mathbf{O}}_V$ is smooth and its connected component, the *special orthogonal group* of (V,q), is $\underline{\mathbf{SO}}_V = \ker[\underline{\mathbf{O}}_V \xrightarrow{\det} \underline{\mu}_2]$, containing only the proper isometries [Con, Thm. 1.7, Cor. 2.5]. The push-forward homomorphism $\det_* : H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{O}}_V) \to H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mu}_2)$ induced by flat cohomology assigns to any quadratic form (V', q') (taken up to an $\mathcal{O}_S$-isomorphism) the class of its *discriminant module* $D(q') = D(V', q') = (\bigwedge^n V', \det(q'))$ [Knu, IV, 4.6, 5.3.1]. Let $\underline{\mathbf{SO}}_V^\Gamma$ be its Γ-invariant subgroup. Our assumption that $|\Gamma| \in \mathcal{O}_S^\times$ guarantees that $\underline{\mathbf{SO}}_V^\Gamma$ remains smooth [CGP2, Proposition A.8.10(2)]. Any representative (V', q') of a class in $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V^\Gamma)$ represents a class in $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{O}}_V^\Gamma)$, though $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V^\Gamma)$ may not embed in $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{O}}_V^\Gamma)$ (notice that these pointed sets do not have to be groups). More precisely, any class in $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V^\Gamma)$ is represented by a triple (V', q', θ') where θ' is the trivialization of $D(q')$, i.e. an isomorphism $\theta' : D(q') \otimes S \cong D(q)$ where S is some 2-degree flat extension of $\mathcal{O}_S$. Any proper Γ-isometry $A : V' \cong V : q \circ A = q'$ induces an isomorphism $D(A) : D(q') \otimes S \cong D(q)$. The question is whether these additional data $D(q')$ and $D(A)$ (when q' is a Γ-form) are also Γ-equivariant.

As mentioned before, in order to answer this question we may consider the $\mathcal{O}_S$-module $D(q')$, of rank 1 and isomorphic to $\mathcal{O}_S$ over some at most 2-degree étale extension, as a 2-degree Kummer pair $(L = D(q'), h)$, giving rise to the $\underline{\mu}_2$-torsor $E_{q'}$ for which $E_{q'}(\mathcal{O}_S) = \{\varphi \in L^\vee : \varphi \otimes \varphi = h\}$. Since q' is a Γ-form, it admits the following commutative diagram:

$$\begin{array}{ccc} \Gamma & \xhookrightarrow{\ \rho\ } & \underline{\mathbf{O}}_{V'}(\mathcal{O}_S) \\ & \searrow & \big\downarrow {\scriptstyle \det} \\ & & \underline{\mu}_2(\mathcal{O}_S) \end{array}$$

from which we see that Γ acts on $E_{q'}(\mathcal{O}_S)$ through its determinant in $\underline{\mu}_2(\mathcal{O}_S)$. But $\underline{\mu}_2(\mathcal{O}_S)$ is the automorphism group of $\mathcal{O}_S \oplus \mathcal{O}_S$ with respect to its standard involution $\tau = (\mathrm{id}, -\mathrm{id})$, and $E_{q'}(\mathcal{O}_S)$ is stable (not pointwise fixed) under τ, as $-l \otimes -l = l \otimes l$. Furthermore, the correspondence $A \rightsquigarrow D(A)$ is functorial, thus if we consider any $\gamma \in \Gamma$ as an isometry, we get

$$\gamma A \gamma^{-1} = A \ \Leftrightarrow \ D(\gamma) D(A) D(\gamma)^{-1} = D(A),$$

i.e. $D(A)$ is Γ-invariant as well. So Proposition 2.1 also applies to the proper classification.

Corollary 2.2. *Given an integral Γ-form base-point (V,q), the pointed set $H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{O}}^{\Gamma}_V)$ $[H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{\mathbf{SO}}^{\Gamma}_V)]$ [properly] classifies the integral Γ-forms that are locally Γ-isomorphic to (V,q) in the flat topology.*

3. The [proper] genus and [proper] Γ-genus. Consider the ring of S-integral adèles $\mathbb{A}_S := \prod_{\mathfrak{p} \in S} \hat{K}_{\mathfrak{p}} \times \prod_{\mathfrak{p} \notin S} \hat{\mathcal{O}}_{\mathfrak{p}}$, a subring of the adèles $\mathbb{A}$. The *S-class set* of an affine and flat $\mathcal{O}_S$-group $\underline{G}$ is the set of double cosets

$$\mathrm{Cl}_S(\underline{G}) := \underline{G}(\mathbb{A}_S) \backslash \underline{G}(\mathbb{A}) / G(K)$$

(where for any prime $\mathfrak{p}$ the geometric fiber $\underline{G}_{\mathfrak{p}}$ of $\underline{G}$ is taken). According to Nisnevich [Nis, Theorem I.3.5], $\underline{G}$ admits the exact sequence of pointed sets

$$1 \to \mathrm{Cl}_S(\underline{G}) \to H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}) \to H^1(K, G) \times \prod_{\mathfrak{p} \notin S} H^1_{\mathrm{fl}}(\hat{\mathcal{O}}_{\mathfrak{p}}, \underline{G}_{\mathfrak{p}}) \tag{3.1}$$

in which the left exactness reflects the fact that $\mathrm{Cl}_S(\underline{G})$ is the *genus* of $\underline{G}$, i.e. the set of (classes of) $\underline{G}$-torsors that are generically and locally at $\mathfrak{p} \notin S$ isomorphic to $\underline{G}$. If $\underline{G}$ has the property

$$\forall \mathfrak{p} \notin S: \quad H^1_{\mathrm{fl}}(\hat{\mathcal{O}}_{\mathfrak{p}}, \underline{G}_{\mathfrak{p}}) \hookrightarrow H^1_{\mathrm{fl}}(\hat{K}_{\mathfrak{p}}, G_{\mathfrak{p}}), \tag{3.2}$$

then due to [Nis, Corollary 3.6] the Nisnevich sequence (3.1) simplifies to

$$1 \to \mathrm{Cl}_S(\underline{G}) \to H^1_{\mathrm{fl}}(\mathcal{O}_S, \underline{G}) \to H^1(K, G), \tag{3.3}$$

which indicates that any two $\underline{G}$-torsors belong to the same genus if and only if they are K-isomorphic.

Remark 3.1. Since $\operatorname{Spec} \mathcal{O}_S$ is normal, i.e. integrally closed locally everywhere (by the smoothness of C), any finite étale covering of $\mathcal{O}_S$ arises from its normalization in some separable unramified extension of K [Len, Theorem 6.13]. Consequently, if $\underline{G}$ is a finite $\mathcal{O}_S$-group, then $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ is embedded in $H^1(K, \underline{G})$.

Lemma 3.2. *Let $\varphi : \underline{G} \to \underline{G}'$ be a monomorphism of smooth affine $\mathcal{O}_S$-groups and let $\widetilde{\underline{Q}}$ be the sheafification of $\underline{Q} :=$ coker(φ). Then:*

- *The map $\underline{G}'(\mathcal{O}_S) \to \widetilde{\underline{Q}}(\mathcal{O}_S)$ is surjective iff* $\ker[Cl_S(\underline{G}) \xrightarrow{\psi} Cl_S(\underline{G}')] = 1$.
- *If, moreover, $\underline{G}$ is locally of finite presentation, $\underline{G}$ and $\underline{G}'$ have property (3.2), $\underline{Q}$ is a finite $\mathcal{O}_S$-group and $G'(K) \to Q(K)$ is surjective, then ψ is surjective.*

Proof. As a pointed set, $\mathrm{Cl}_S(\underline{G})$ is bijective to the first Nisnevich cohomology set $H^1_{\mathrm{Nis}}(\mathcal{O}_S, \underline{G})$ [Nis, I. Theorem 2.8], classifying $\underline{G}$-torsors for the Nisnevich topology. But Nisnevich covers are étale, so $\mathrm{Cl}_S(\underline{G})$ is a subset of $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$. The monomorphism φ does not have to be a closed immersion, hence $\underline{Q}$ may not be representable, and so we may not be able to apply flat cohomology to the resulting short exact sequence of $\mathcal{O}_S$-schemes. Restricting, however, to the small site of flat extensions of $\mathcal{O}_S$, we have $\widetilde{\underline{G}}(R) \subseteq \widetilde{\underline{G}'}(R)$ for any such extension $R/\mathcal{O}_S$, where $\widetilde{\underline{G}}$ and $\widetilde{\underline{G}'}$ stand for the sheafifications of $\underline{G}$ and $\underline{G}'$, respectively. Then flat cohomology applied to the exact sequence of flat sheaves

$$1 \to \widetilde{\underline{G}} \xrightarrow{i} \widetilde{\underline{G}'} \to \widetilde{\underline{Q}} \to 1 \tag{3.4}$$

yields a long exact sequence in which $H^1_{\mathrm{fl}}(\mathcal{O}_S, \widetilde{*}) = H^1_{\mathrm{fl}}(\mathcal{O}_S, *) = H^1_{\text{ét}}(\mathcal{O}_S, *)$ for both smooth groups $* = \underline{G}'$ and $\underline{G}$, whence $\widetilde{\underline{G}'}(\mathcal{O}_S) = \underline{G}'(\mathcal{O}_S) \to \widetilde{\underline{Q}}(\mathcal{O}_S)$ is surjective if and only if $\ker[H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \xrightarrow{\psi} H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}')] = 1$, which is equivalent (by restriction to $\mathrm{Cl}_S(\underline{G})$) to $\ker[\mathrm{Cl}_S(\underline{G}) \xrightarrow{\psi} \mathrm{Cl}_S(\underline{G}')] = 1$, since any twisted form in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ can be $\mathcal{O}_S$-isomorphic to $\underline{G}$ only if it belongs to $\mathrm{Cl}_S(\underline{G})$.

Now suppose $\underline{G}$ is locally of finite presentation and $\underline{Q}$ is representable as a finite $\mathcal{O}_S$-group. Then $\underline{Q}$ is smooth as well [SGA3, $\mathrm{VI_B}$, Proposition 9.2 xii], and so given furthermore that $\underline{G}$ and $\underline{G}'$ have property (3.2) and $G'(K) \to Q(K)$ is surjective, étale cohomology applied to the sequence (3.4) over $\mathcal{O}_S$ and over K extends the exactness of the sequence (3.3) to the commutative diagram

$$\begin{array}{ccccccc}
 & & \mathrm{Cl}_S(\underline{G}) & \xrightarrow{\ \psi\ } & \mathrm{Cl}_S(\underline{G}') & & \\
 & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i'} & & \\
1 & \longrightarrow & H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) & \xrightarrow{\ \psi\ } & H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}') & \xrightarrow{\ d\ } & H^1_{\text{ét}}(\mathcal{O}_S, \underline{Q}) \\
 & & \big\downarrow{\scriptstyle m} & & \big\downarrow{\scriptstyle m'} & & \big\downarrow{\scriptstyle m''} \\
1 & \longrightarrow & H^1(K, G) & \xhookrightarrow{\ h\ } & H^1(K, G') & \xrightarrow{\ d'\ } & H^1(K, Q)
\end{array}$$

in which m'' is injective as $\underline{Q}$ is finite, by Remark 3.1. We then get the surjectivity of ψ:

$$\begin{aligned}
x' \in \mathrm{Cl}_S(\underline{G}') \ &\Rightarrow\ m''(d(i'(x'))) = d'(m'(i'(x')) = 0) = 0 \ \Rightarrow\ d(i'(x')) = 0 \\
&\Rightarrow\ \exists y \in H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) : \psi(y) = i'(x') \ \Rightarrow\ m'(\psi(y)) = h(m(y)) = 0 \\
&\Rightarrow\ m(y) = 0 \ \Rightarrow\ \exists x \in \mathrm{Cl}_S(\underline{G}) : \psi(i(x) = y) = i'(x') \ \Rightarrow\ \psi(x) = x'. \ \blacksquare
\end{aligned}$$

We return now to our integral Γ-form (V, q) for which $\mathrm{Cl}_S(\underline{\mathbf{O}}_V^\Gamma) = c_\Gamma(q)$ and $\mathrm{Cl}_S((\underline{\mathbf{O}}_V^\Gamma)^0) = c^+_\Gamma(q)$. Since $\underline{\mathbf{O}}_V^\Gamma$ and $(\underline{\mathbf{O}}_V^\Gamma)^0$ are smooth, their flat cohomology sets over $\operatorname{Spec}\mathcal{O}_S$ coincide with the étale ones [SGA4, VIII Corollaire 2.3]. According to Lemma 3.2, if $\underline{\mathbf{O}}_V(\mathcal{O}_S) \to (\underline{\mathbf{O}}_V/\underline{\mathbf{O}}_V^{\Gamma})(\mathcal{O}_S)$ is surjective then $\ker[c_{\Gamma}(q) \xrightarrow{\psi} c(q)] = 1$, which means that for any $[q'] \in c_{\Gamma}(q)$, if q' is not Γ-isomorphic to q then neither is it $\mathcal{O}_S$-isomorphic to it. This still does not imply that $c_{\Gamma}(q)$ injects into $c(q)$ since both are not necessarily groups, so there may be two classes of forms other than $[q]$, which are distinct in $c_{\Gamma}(q)$, yet $\mathcal{O}_S$-isomorphic. We summarize the above in

Proposition 3.3. *Question 1.2 is answered in the affirmative if and only if $\underline{\mathbf{O}}_{V'}(\mathcal{O}_S) \to (\underline{\mathbf{O}}_{V'}/\underline{\mathbf{O}}_{V'}^{\Gamma})(\mathcal{O}_S)$ is surjective for any $[(V', q')] \in c(q)$.*

Lemma 3.4. *The group scheme $(\underline{\mathbf{O}}_V^{\Gamma})^0$ is reductive.*

Proof. The group scheme $\underline{\mathbf{SO}}_V$ is reductive as it is smooth, affine and all its fibers are reductive. As mentioned before, its affine subgroup $\underline{\mathbf{SO}}_V^{\Gamma}$ is smooth as well, so we may refer to its neutral component $(\underline{\mathbf{SO}}_V^{\Gamma})^0$ as defined in [SGA3, VIB, Théorème 3.10]. The reduction $(\overline{\mathbf{SO}}_V)_{\mathfrak{p}}$ defined over the residue field $k_{\mathfrak{p}}$ at any prime $\mathfrak{p}$ is reductive, hence according to [CGP2, Proposition A.8.12] its subgroup $((\overline{\mathbf{SO}}_V)_{\mathfrak{p}}^{\Gamma})^0 = (\overline{\mathbf{SO}}_V^{\Gamma})_{\mathfrak{p}}^0$ remains reductive, so that $(\underline{\mathbf{SO}}_V^{\Gamma})^0$ is reductive. The latter being a smooth, open and connected subgroup of $\underline{\mathbf{O}}_V^{\Gamma}$, it coincides with $(\underline{\mathbf{O}}_V^{\Gamma})^0$, thus it is reductive. ∎

Remark 3.5. The group $(\underline{\mathbf{O}}_V^{\Gamma})^0$ has property (3.2) by Lang's Theorem (recall that all residue fields are finite). By reductivity over $\operatorname{Spec}\mathcal{O}_S$ (Lemma 3.4), this property holds for $\underline{\mathbf{O}}_V^{\Gamma}$ as well if $\underline{\mathbf{O}}_V^{\Gamma}/(\underline{\mathbf{O}}_V^{\Gamma})^0$ is representable as a finite $\mathcal{O}_S$-group [CGP1, proof of Proposition 3.14].

From the sequence (3.3) and Remark 3.5 we then get:

Corollary 3.6. *We have*

$$c_{\Gamma}^{+}(q) \cong \ker[H^1_{\text{ét}}(\mathcal{O}_S, (\underline{\mathbf{O}}_V^{\Gamma})^0) \to H^1(K, (\mathbf{O}_V^{\Gamma})^0)].$$

If $\underline{\mathbf{O}}_V^{\Gamma}/(\underline{\mathbf{O}}_V^{\Gamma})^0$ is a finite $\mathcal{O}_S$-group then

$$c_{\Gamma}(q) \cong \ker[H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_V^{\Gamma}) \to H^1(K, \mathbf{O}_V^{\Gamma})].$$

Corollary 3.7. *If (V, q) of rank ≥ 3 is regular and isotropic then $c^{+}(q) = c(q)$.*

Proof. Any representative (V', q') of a class in $c(q)$ is $\mathcal{O}_S$-regular, thus $\underline{\mathbf{O}}_{V'}/\underline{\mathbf{SO}}_{V'} \cong \underline{\mu}_2$ ([Con, Thm. 1.7]; notice that $\underline{\mathbb{Z}/2} := \operatorname{Spec}\mathbb{Z}[t]/(t(t-1))$ is isomorphic to $\underline{\mu}_2$ as $2 \in \mathcal{O}_S^{\times}$). Moreover, being K-isomorphic to q (by Corollary 3.6), q' is isotropic as well, therefore as $\operatorname{rank}(V) \geq 3$, the map $\underline{\mathbf{O}}_{V'}(\mathcal{O}_S) \xrightarrow{\det} \underline{\mu}_2(\mathcal{O}_S) = \mu_2(K) = \{\pm 1\}$ is surjective [Bit2, Lemma 4.3]. So setting $\underline{G} = \underline{\mathbf{SO}}_{V'}$ and $\underline{G'} = \underline{\mathbf{O}}_{V'}$ in Lemma 3.2, we find that $\ker[c^{+}(q') \xrightarrow{\psi}$

$c(q') = c(q)] = 1$ and ψ is surjective. As mentioned before, this holds for any $[q'] \in c(q)$, which amounts to ψ being the identity. ∎

Remark 3.8. When $|S| = 1$, i.e. $S = \{\infty\}$ where ∞ is an arbitrary closed point of K, Lemma 3.7 is automatic for any regular q of rank ≥ 3, since in that case (V, q) must be isotropic [Bit1, proof of Prop. 4.4].

Lemma 3.9. *Suppose that* $\underline{\mathbf{O}}_V^\Gamma \cong (\underline{\mathbf{O}}_V^\Gamma)^0 \rtimes \underline{Q}$ *where* $\underline{Q}$ *is a finite* $\mathcal{O}_S$*-group. Then* $\ker[c_\Gamma^+(q) \xrightarrow{\psi} c_\Gamma(q)] = 1$ *and* ψ *is surjective.*

Proof. As $\underline{Q}$ embeds as a semidirect factor in $\underline{\mathbf{O}}_V^\Gamma$, $\underline{\mathbf{O}}_V^\Gamma(\mathcal{O}_S)$ surjects onto $\underline{Q}(\mathcal{O}_S)$, and $\mathbf{O}_V^\Gamma(K)$ onto $Q(K)$. Given furthermore that $\underline{Q}$ is a finite $\mathcal{O}_S$-group, both $(\underline{\mathbf{O}}_V^\Gamma)^0$ and $\underline{\mathbf{O}}_V^\Gamma$ have property (3.2) (see Remark 3.5), so all conditions in Lemma 3.2 are satisfied, and the assertion follows. ∎

Lemma 3.10. *If* $\operatorname{rank}(V) = 2$ *then* $c_\Gamma(q) \subseteq c(q)$, *i.e. Question* 1.2 *is then answered positively.*

Proof. If $\operatorname{rank}(V) = 2$ then $\underline{\mathbf{SO}}_V$ is a one-dimensional torus. We first claim that $\underline{\mathbf{O}}_V^\Gamma$ cannot be $\underline{\mathbf{SO}}_V$. Indeed, since q is a Γ-form, Γ embeds by definition in $\underline{\mathbf{O}}_V(\mathcal{O}_S)$. As $(\underline{\mathbf{O}}_V/\underline{\mathbf{SO}}_V)(\mathcal{O}_S) = \underline{\mu}_2(\mathcal{O}_S)$ does not commute with $\underline{\mathbf{O}}_V(\mathcal{O}_S)$, Γ cannot have a non-trivial image in it and still stabilize $\underline{\mathbf{SO}}_V$, i.e. it must embed in $\underline{\mathbf{SO}}_V(\mathcal{O}_S)$ only. But then, being finite, the Γ-image must be the group $\{\pm I_2\}$, which does not kill $\underline{\mu}_2(\mathcal{O}_S)$. So either $\underline{\mathbf{O}}_V^\Gamma = \underline{\mathbf{O}}_V$ for which the assertion is trivial, or $\underline{\mathbf{O}}_V^\Gamma$ is a finite group for which $c_\Gamma(q) \overset{(3.6)}{\cong} \ker[H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{O}}_V^\Gamma) \to H^1(K, \mathbf{O}_V^\Gamma)]$ is trivial according to Remark 3.1. ∎

As mentioned in the proof of Lemma 3.10, the non-equality $c_\Gamma(q) \subsetneq c(q)$ for $\operatorname{rank}(V) = 2$ may occur only when $\underline{\mathbf{O}}_V^\Gamma$ is finite. For example, suppose $\underline{\mathbf{O}}_V^\Gamma = \underline{\mu}_m$. Then étale cohomology applied to the related Kummer exact sequence of smooth $\mathcal{O}_S$-groups

$$1 \to \underline{\mu}_m \to \underline{\mathbb{G}}_{\mathrm{m}} \to \underline{\mathbb{G}}_{\mathrm{m}} \to 1$$

yields the exactness of

$$1 \to \mathcal{O}_S^\times/(\mathcal{O}_S^\times)^m \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_m) \to {}_m\mathrm{Pic}(\mathcal{O}_S) \to 1 \tag{3.5}$$

where the right non-trivial term stands for the m-torsion part of $\mathrm{Pic}(\mathcal{O}_S)$. According to Remark 3.1, since $\operatorname{Spec} \mathcal{O}_S$ is normal, both $\mathcal{O}_S^\times/(\mathcal{O}_S^\times)^m$ and the preimage of ${}_m\mathrm{Pic}(\mathcal{O}_S)$ in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_m)$ are embedded in $K^\times/(K^\times)^m \cong H^1(K, \mu_m)$. The following example demonstrates this embedding, which yields the above non-equality.

Example 3.11. Consider the elliptic curve $C = \{Y^2Z = X^3 + XZ^2\}$ defined over $\mathbb{F}_{11}$. Removing the closed point $\infty = (0 : 1 : 0)$ results in an affine curve with coordinate ring

$$C^{\mathrm{af}} = \{y^2 = x^3 + x\} \quad \text{and} \quad \mathcal{O}_S = \mathbb{F}_{11}[C^{\mathrm{af}}].$$

Let $V = \mathcal{O}_S^2$ be generated by the standard basis over $\mathcal{O}_S$ endowed with the form q represented by $B_q = 1_2$. Then

$$\underline{\mathbf{SO}}_V = \underline{\mathbf{SO}}_2 = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x^2 + y^2 = 1 \right\}$$

is a one-dimensional $\mathcal{O}_S$-torus, and as -1 is not a square, it is isomorphic to the non-split norm torus $\underline{N} := R^{(1)}_{\mathcal{O}_S(i)/\mathcal{O}_S}(\underline{\mathbb{G}}_{\mathrm{m}})$, fitting into the exact sequence of $\mathcal{O}_S$-tori:

$$1 \to \underline{N} \to \underline{R} := R_{\mathcal{O}_S(i)/\mathcal{O}_S}(\underline{\mathbb{G}}_{\mathrm{m}}) \xrightarrow{\det} \underline{\mathbb{G}}_{\mathrm{m}} \to 1.$$

Then since $\underline{R}(\mathcal{O}_S) \xrightarrow{\det} \mathcal{O}_S^\times = \mathbb{F}_{11}^\times$ is surjective ($x^2 + y^2$ gets any value in $\mathbb{F}_{11}^\times$), étale cohomology together with Shapiro's Lemma gives rise to the exact sequence

$$1 \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{N}) \to \mathrm{Pic}(\mathcal{O}_S(i)) \to \mathrm{Pic}(\mathcal{O}_S)$$

from which we see that $c^+(q) = \mathrm{Cl}_S(\underline{\mathbf{SO}}_V) = H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SO}}_V \cong \underline{N})$ [Bit1, Prop. 4.2] is far from being trivial (say, by the Hasse–Weil bound: $|\mathrm{Pic}(\mathcal{O}_S)| = |C^{\mathrm{af}}(\mathbb{F}_{11})| < 11 + 1 + 2\sqrt{11} < 19$, while $|\mathrm{Pic}(\mathcal{O}_S(i))| = |C^{\mathrm{af}}(\mathbb{F}_{11}(i))| > 121 + 1 - 2\sqrt{121} = 100$, see Example 4.2). As $\underline{\mathbf{O}}_V/\underline{\mathbf{SO}}_V \cong \underline{\mu}_2$ and $\underline{\mathbf{O}}_V(\mathcal{O}_S) \xrightarrow{\det} \underline{\mu}_2(\mathcal{O}_S)$ is surjective by $\mathrm{diag}(1,-1) \mapsto -1$, setting $\underline{G} = \underline{\mathbf{SO}}_V$ and $\underline{G}' = \underline{\mathbf{O}}_V$ in Lemma 3.2 we see that $c(q) = c^+(q)$, thus is not trivial either.

Now let $\Gamma = S_3 = \langle \tau, \sigma \rangle$ be represented in V by

$$\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma \mapsto \begin{pmatrix} 5 & -8 \\ 8 & 5 \end{pmatrix}.$$

One can easily check that q is a Γ-form and

$$\underline{\mathbf{SO}}_V^\Gamma = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x^2 = 1 \right\} \cong \underline{\mu}_2.$$

For $L = \langle x, y \rangle \in \mathrm{Pic}(\mathcal{O}_S)$ one has

$$L \otimes L = \langle x^2, xy, y^2 \rangle = \langle x^2, xy, x^3 + x \rangle \subseteq \langle x \rangle.$$

But $x = y^2 - x^3 \in L \otimes L$, thus $L \otimes L = \langle x \rangle$. So the 2-degree Kummer pair (L, h) gives rise to the $\underline{\mu}_2$-torsor $\mathcal{O}_S \oplus L$ isomorphic to $\mathcal{O}_S^2$ over $\mathcal{O}_S[1/\sqrt{x}]$ which is not contained in K. The same happens for the other $\underline{\mu}_2$-torsors, i.e., $c_\Gamma(q) \overset{(3.6)}{\cong} \ker[H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2) \to H^1(K, \mu_2)] = 1 \subsetneq c(q)$.

4. An explicit obstruction. The criterion exhibited in Proposition 3.3 for Question 1.2 to be answered in the affirmative, namely that $\underline{\mathbf{O}}_{V'}(\mathcal{O}_S) \to (\underline{\mathbf{O}}_{V'}/\underline{\mathbf{O}}_{V'}^\Gamma)(\mathcal{O}_S)$ is surjective for any $[(V', q')] \in c(q)$, is somewhat vague. We would like to refer to the case in which $(\underline{\mathbf{O}}_V^\Gamma)^0$ is the special orthogonal group of another isotropic Γ-form. It is shown in [Bit1, Proposition 4.4] for $|S| = 1$

and more generally in [Bit2, Theorem 4.6] for any finite S that if $\mathrm{rank}(V) \geq 3$ (q is isotropic), then $c(q) \cong \mathrm{Pic}(\mathcal{O}_S)/2$. For $\mathrm{rank}(V) = 2$, however, this genus might be larger. This means that there may be two integral forms of rank 2 that are only *stably isomorphic*, i.e. become isomorphic after being extended by any non-trivial regular common extension. This failure of the Witt cancellation theorem over $\mathcal{O}_S$ invokes a case in which Question 1.2 is answered negatively, namely when $\mathrm{rank}(V) \geq 3$ and $(\underline{\mathbf{O}}_V^\Gamma)^0$ is the special orthogonal group of another integral Γ-form (V', q') of rank 2, whose genus decreases over V.

PROPOSITION 4.1. *Let (V, q) be a regular Γ-form of rank ≥ 3 such that $(\underline{\mathbf{O}}_V^\Gamma)^0$ is the special orthogonal group of an isotropic form of rank 2, and suppose this group is a semidirect factor in $\underline{\mathbf{O}}_V^\Gamma$, while the quotient is a finite $\mathcal{O}_S$-group. If $-1 \in (\mathbb{F}_q^\times)^2$ and $\exp(\mathrm{Pic}(\mathcal{O}_S)) > 2$ then $c_\Gamma(q)$ does not inject into $c(q)$, i.e., Question 1.2 is then answered negatively.*

Proof. Given that $\mathrm{rank}(V') = 2$ and $-1 \in (\mathbb{F}_q^\times)^2$, $\underline{\mathbf{O}}_{V'}^0 \cong \underline{\mathbb{G}}_{\mathrm{m}}$, one has

$$c_\Gamma^+(q) = c^+(q') \overset{(3.6)}{\cong} \ker[H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_{\mathrm{m}}) \to H^1(K, \mathbb{G}_{\mathrm{m}})].$$

By Shapiro's Lemma and Hilbert's 90 Theorem this kernel is isomorphic to $\mathrm{Pic}(\mathcal{O}_S)$. Since $(\underline{\mathbf{O}}_V^\Gamma)^0$ is a normal semidirect factor in $\underline{\mathbf{O}}_V^\Gamma$ and the quotient is a finite $\mathcal{O}_S$-group, by Lemma 3.9 we have $\ker[c_\Gamma^+(q) \xrightarrow{\psi} c_\Gamma(q)] = 1$, so $\ker[c_\Gamma(q) \to c(q)]$ cannot vanish, because if it did, the composition, being a morphism of abelian groups

$$c_\Gamma^+(q) \cong \mathrm{Pic}(\mathcal{O}_S) \to c(q) \cong \mathrm{Pic}(\mathcal{O}_S)/2,$$

would be injective, which is impossible whenever $\exp(\mathrm{Pic}(\mathcal{O}_S)) > 2$. ∎

EXAMPLE 4.2. Let C be an elliptic $\mathbb{F}_q$-curve such that $-1 \in (\mathbb{F}_q^\times)^2$ and $\exp(C(\mathbb{F}_q)) > 2$. Let ∞ be an $\mathbb{F}_q$-rational point. Then $\mathrm{Pic}(\mathcal{O}_S) \cong C(\mathbb{F}_q)$ [Bit1, Example 4.8]. Let (V, q) be the quadratic space generated by the standard basis and represented by $B_q = 1_n$ for $n \geq 4$. Then as mentioned before, $c(q) \cong \mathrm{Pic}(\mathcal{O}_S)/2 \cong C(\mathbb{F}_q)/2$. Let the permutations in $\Gamma = S_{n-2}$ be canonically represented by monomial matrices in the lower right $(n-2) \times (n-2)$ block of $\underline{\mathbf{O}}_V \subset \underline{\mathbf{GL}}(V)$:

$$\Gamma \hookrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S_{n-2} \end{pmatrix}$$

turning q into a Γ-form. Now if n is even then $\underline{\mathbf{SO}}_V^\Gamma \cong \underline{\mathbb{G}}_{\mathrm{m}} \times \{\pm I_{n-2}\}$. Otherwise, if n is odd then $\underline{\mathbf{SO}}_V^\Gamma$ is a semidirect product of $\underline{\mathbb{G}}_{\mathrm{m}} \times \{I_{n-2}\} \cong \underline{\mathbb{G}}_{\mathrm{m}}$ and $\mathrm{diag}(1, -1) \times \{-I_{n-2}\} \cong \underline{\mu}_2$. In both cases, $(\underline{\mathbf{O}}_V^\Gamma)^0 = (\underline{\mathbf{SO}}_V^\Gamma)^0 \cong \underline{\mathbb{G}}_{\mathrm{m}}$ is a normal semidirect factor in $\underline{\mathbf{O}}_V^\Gamma$ and the quotient is a finite $\mathcal{O}_S$-group, therefore according to Proposition 4.1, $c_\Gamma(q)$ cannot inject into $c(q)$.

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