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Theta type Jacobi forms

by

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1. Introduction. Let V be a real quadratic space of signature $(2, n)$ where $n \in \mathbb{N}$. We will always assume that $n \geq 3$. The bilinear form of V is denoted by $(\cdot, \cdot)$. The group of all isometries of V is called the *orthogonal group* of V and is given by

$$\mathrm{O}(V) = \{g \in \mathrm{GL}(V) \mid \forall v \in V : (gv, gv) = (v, v)\}.$$

We denote by $\mathrm{SO}(V)$ the subgroup of index two which equals the kernel of the determinant character. We obtain another subgroup $\mathrm{O}(V)^+$ of index two as the kernel of the real spinor norm. The intersection of $\mathrm{SO}(V)$ and $\mathrm{O}(V)^+$ is denoted by $\mathrm{SO}(V)^+$. This is the connected component of the identity in $\mathrm{SO}(V)$ and is well-known to be a semisimple and non-compact Lie group (see e.g. [14]). Its maximal compact subgroup is $K = \mathrm{SO}(2) \times \mathrm{SO}(n)$. One can construct a projective model for the Hermitian symmetric space $\mathrm{SO}(V)^+/K$, namely

$$\mathcal{D} = \{[\mathcal{Z}] \in \mathbb{P}(V \otimes \mathbb{C}) \mid (\mathcal{Z}, \mathcal{Z}) = 0,\ (\mathcal{Z}, \overline{\mathcal{Z}}) > 0\}^+ \xrightarrow{\sim} \mathrm{SO}(V)^+/K.$$

Here the superscript $+$ means that we have chosen one of the two connected components. The associated affine cone is defined as

$$\mathcal{D}^\bullet = \{\mathcal{Z} \in V \otimes \mathbb{C} \mid [\mathcal{Z}] \in \mathcal{D}\}.$$

DEFINITION 1.1. A subset $L \subseteq V$ is called a *lattice* if there exist linearly independent vectors $b_1, \ldots, b_N$ in V such that $L = \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_N$. The quantity N is called the *rank* of L and is denoted by $\mathrm{rank}(L)$. We call L *even* if $(l, l) \in 2\mathbb{Z}$ for all $l \in L$. A lattice L is called *full in* V if $\mathrm{rank}\, L = \dim V$, and *positive definite* if $(l, l) > 0$ for all $0 \neq l \in L$. For any $t \in \mathbb{Z}$ we define the *rescaled lattice* $L(t)$ to be the set L equipped with the bilinear form $t(\cdot, \cdot)$.

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Let L_2 be a full lattice in V such that L_2 is even and contains two hyperbolic planes. In terms of the Gramian of L_2 this means that there exists a basis such that

$$\mathrm{Gram}(L_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -S & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{GL}(L_2) \subseteq \mathrm{GL}(V),$$

where S denotes the Gramian of a positive definite even lattice L. We consider the arithmetic subgroup

$$\mathrm{O}(L_2)^+ = \{g \in \mathrm{O}(V)^+ \mid gL_2 \subseteq L_2\}.$$

For any subgroup $\Gamma \leq \mathrm{O}(L_2)^+$ of finite index we consider the modular variety $\Gamma\backslash\mathcal{D}$. This is a non-compact space. We can add cusps to $\Gamma\backslash\mathcal{D}$ such that

$$\Gamma\backslash\mathcal{D}^* = \Gamma\backslash\mathcal{D} \amalg \coprod_{\Pi} X_\Pi \amalg \coprod_{l} Q_l$$

is compact (see e.g. [3]). Here l runs through the finitely many Γ-orbits of isotropic lines, and Π runs through the finitely many Γ-orbits of isotropic planes in $L_2 \otimes \mathbb{Q}$. Each X_Π corresponds to a modular curve and is called a *one-dimensional cusp* of $\Gamma\backslash\mathcal{D}$, and each Q_l corresponds to a point and is called a *zero-dimensional cusp*. One-dimensional cusps are represented by rational two-dimensional totally isotropic subspaces of V, and zero-dimensional cusps are represented by primitive isotropic vectors $c \in L_2$.

Definition 1.2. Let Γ be a subgroup of $\mathrm{O}(L_2)^+$. A *modular form* of weight $k \in \mathbb{Z}$ and character $\chi : \Gamma \to \mathbb{C}^\times$ with respect to Γ is a holomorphic function $F : \mathcal{D}^\bullet \to \mathbb{C}$ such that

$$F(t\mathcal{Z}) = t^{-k}F(\mathcal{Z}) \qquad \text{for all } t \in \mathbb{C}^\times,$$
$$F(g\mathcal{Z}) = \chi(g)F(\mathcal{Z}) \qquad \text{for all } g \in \Gamma.$$

A modular form is called a *cusp form* if it vanishes at every cusp. The space of modular forms of weight k and character χ for the group Γ will be denoted by $\mathcal{M}_k(\Gamma, \chi)$. For the subspace of cusp forms we will write $\mathcal{S}_k(\Gamma, \chi)$.

Orthogonal type modular forms are important tools to investigate the modular variety $\Gamma\backslash\mathcal{D}$. In this text we focus on the Jacobi group $\Gamma^J(L)$ which is a distinguished parabolic type subgroup of $\mathrm{O}(L_2)^+$ (see Section 2 for the definition). Modular forms with respect to $\Gamma^J(L)$ are called *Jacobi forms* and can be considered as a natural generalization of the classical Jacobi forms appearing in [7].

Definition 1.3. Let L be an even lattice in V. The *dual lattice* is the $\mathbb{Z}$-module

$$L^\vee := \{x \in V \mid \forall l \in L : (x, l) \in \mathbb{Z}\}.$$

Since $L \subseteq L^\vee$, in this case we can define the *discriminant group* as the finite abelian group

$$D(L) := L^\vee / L.$$

The lattice L is called *maximal* if for any even lattice M satisfying $\operatorname{rank}(L) = \operatorname{rank}(M)$ and $L \subseteq M$ we have $L = M$.

The group $\mathrm{O}(L_2)^+$ acts on the discriminant group $D(L_2)$. The kernel of this action is denoted by $\widetilde{\mathrm{O}}(L_2)^+$. The following construction is due to Gritsenko—see [11] for the details.

Theorem 1.4 (Gritsenko '93). *Let φ be a Jacobi form with trivial character and weight k where $k \geq 4$. There exists a linear operator* A-Lift *defined on the space of Jacobi forms of weight k into the space of modular forms of weight k such that*

$$\text{A-Lift}(\varphi) \in \mathcal{M}_k(\widetilde{\mathrm{O}}(L_2)^+, 1).$$

For maximal even lattices this lift maps cusp forms to cusp forms.

In Section 2 we review Jacobi forms in many variables. In Section 3 we use Jacobi theta-series and the weak Jacobi form $\phi_{0,1}$ in two variables of weight 0 and index 1 to present pullback constructions for several series of lattices. The forms obtained in this way are called *theta type Jacobi forms.* In the last section we give a new explanation for the existence of the cusp form of weight 24 for the lattice D_4, which is based on its representation as an automorphic product defined by theta type Jacobi forms.

2. Jacobi forms. Our presentation of the theory is influenced by the classical book on Jacobi forms in one variable by Eichler and Zagier [7]. For the description of Jacobi forms in several variables we follow [11] and [6].

Let V be a real quadratic space of signature $(2, 2+N)$ where $N \in \mathbb{N}$. Let L be a positive definite even lattice of rank N with bilinear form $(\cdot,\cdot)_L = (\cdot,\cdot)$ such that $L(-1) \subseteq V$. By construction, $L_2 \subseteq V$ contains two perpendicular hyperbolic planes $U = \langle e, f\rangle$ and $U_1 = \langle e_1, f_1\rangle$. Let F be the totally isotropic plane spanned by e and e_1. We define the parabolic subgroup of $G = \mathrm{SO}(L_2)^+$ fixing F as

$$N_G(F) = \{g \in \mathrm{SO}(L_2)^+ \mid gF = F\}.$$

The matrix realization of $N_G(F)$ with respect to our standard basis of L_2 is well-known.

The *integral Heisenberg group* $H(L)$ of the lattice L is generated by the elements

$$[x, y : r] := \begin{pmatrix} 1 & 0 & -y^{\mathrm{tr}}S & -\frac{1}{2}(x,y) - r & -\frac{1}{2}(y,y) \\ 0 & 1 & -x^{\mathrm{tr}}S & -\frac{1}{2}(x,x) & -\frac{1}{2}(x,y) + r \\ 0 & 0 & I_N & x & y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x, y \in L$ and $r \in \mathbb{Z}/2$ are such that $r + \frac{1}{2}(x, y) \in \mathbb{Z}$. The group law of $H(L)$ is

$$[x, y : r] \cdot [x', y' : r'] = \big[x + x', y + y', r + r' + \tfrac{1}{2}[(x, y') - (x', y)]\big].$$

We define an embedding of $\mathrm{SL}(2, \mathbb{Z})$ into $\mathrm{SO}(L_2)^+$ as

$$\{A\} := \begin{pmatrix} A^* & 0 & 0 \\ 0 & I_N & 0 \\ 0 & 0 & A \end{pmatrix} \text{ for } A \in \mathrm{SL}(2, \mathbb{Z}),\ A^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (A^{-1})^{\mathrm{tr}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The group $\mathrm{SL}(2, \mathbb{Z})$ acts on the Heisenberg group by conjugation:

$$A.[x, y : r] = \{A\}[x, y : r]\{A\}^{-1} = [dx - cy, ay - bx : r].$$

Note that $\frac{1}{2}(dx - cy, ay - bx) = \frac{1}{2}(x, y) \bmod \mathbb{Z}$ since $A \in \mathrm{SL}(2, \mathbb{Z})$.

The *integral Jacobi group* $\Gamma^J(L)$ is defined as the subgroup of $N_G(F)$ which acts trivially on the sublattice L. This group is isomorphic to the semidirect product

$$\mathrm{SL}(2, \mathbb{Z}) \ltimes H(L).$$

Let χ be a finite character of the Jacobi group $\Gamma^J(L)$. The structure of the Jacobi group implies

$$\chi = v_{\mathrm{SL}(2,\mathbb{Z})} \cdot \nu_{H(L)}$$

where $v_{\mathrm{SL}(2,\mathbb{Z})}$ is a finite character of $\mathrm{SL}(2, \mathbb{Z})$ and $\nu_{H(L)}$ is a finite character of $H(L)$. The character $\nu_{H(L)}$ satisfies

$$\forall A \in \mathrm{SL}(2, \mathbb{Z})\ \forall [x, y : r] \in H(L) : \quad \nu_{H(L)}(A.[x, y : r]) = \nu_{H(L)}([x, y : r]).$$

We recall the definition of *Dedekind's eta function* $\eta : \mathbb{H} \to \mathbb{C}$ where $\mathbb{H}$ denotes the upper half-plane in $\mathbb{C}$. This function is defined by the product expansion

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad q = e^{2\pi i \tau}.$$

The transformation law for the generators of $\mathrm{SL}(2, \mathbb{Z})$ is

$$\eta(\tau + 1) = e^{\pi i/12} \eta(\tau), \qquad \eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau),$$

where we have chosen the branch of the square root $z^{1/2}$ which is positive if $z > 0$. There exists a multiplier system $v_\eta : \mathbb{H} \to \mathbb{C}^*$ of order 24 whose square is a finite character satisfying

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = v_\eta(A)\eta(\tau) \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

For more details on the function $\eta(\tau)$ we refer to [1, Section 3]. We define $s(L), n(L) \in \mathbb{N}$ as the generators of the ideal in $\mathbb{Z}$ generated by (x, y) and (x, x), respectively, where $x, y \in L$. These quantities are called the *scale* and the *norm* of the lattice L. Obviously $s(L) \mid n(L)$. For a proof of the next proposition see [6] and [17].

Proposition 2.1.

(a) *The group of finite characters of* $\mathrm{SL}(2,\mathbb{Z})$ *is a cyclic group of order* 12 *and is generated by* v_η^2.

(b) *Let* $\nu_{H(L)} : H(L) \to \mathbb{C}^*$ *be a character of finite order such that*

$$\nu_{H(L)}(A[x, y : r]) = \nu_{H(L)}([x, y : r])$$

for all $[x, y : r] \in H(L)$ *and* $A \in \mathrm{SL}(2,\mathbb{Z})$. *Then*

$$\nu_{H(L)}([x, y : r]) = e^{\pi i t((x,x)+(y,y)-(x,y)+2r)}$$

with some $t \in \mathbb{Q}$ *such that* $t \cdot s(L) \in \mathbb{Z}$.

We extend $(\cdot,\cdot)$ to $L \otimes \mathbb{C}$ by $\mathbb{C}$-linearity. Let $k \in \mathbb{Z}$ and $t \in \mathbb{Q}$. We define an action of the Jacobi group on the space of holomorphic functions on $\mathbb{H} \times (L \otimes \mathbb{C})$ by means of the generators of $\Gamma^J(L)$:

$$(2.1)\qquad \begin{aligned} (\varphi|_{k,t}A)(\tau,\mathfrak{z}) &:= (c\tau+d)^{-k} e^{-\pi i t \frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}} \cdot \varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right), \\ (\varphi|_{k,t}[x, y : r])(\tau,\mathfrak{z}) &:= e^{2\pi i t(\frac{1}{2}(x,x)\tau + (x,\mathfrak{z}) + \frac{1}{2}(x,y) + r)} \cdot \varphi(\tau, \mathfrak{z} + x\tau + y), \end{aligned}$$

where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, $[x, y : r] \in H(L)$ and $(\tau,\mathfrak{z}) \in \mathbb{H} \times (L \otimes \mathbb{C})$. The two assignments jointly define an action of the Jacobi group on the space of holomorphic functions. Let φ be a holomorphic function satisfying

$$\forall g \in \Gamma^J(L) : \quad \varphi|_{k,t}g = \chi(g)\varphi$$

where χ is a finite character for the Jacobi group. Then φ is periodic in τ and $\mathfrak{z}$ since

$$\varphi(\tau+1,\mathfrak{z}) = e^{2\pi i D/24}\varphi(\tau,\mathfrak{z}), \qquad \varphi(\tau,\mathfrak{z}+2y) = \nu(0, 2y : 0)\varphi(\tau,\mathfrak{z}) = \varphi(\tau,\mathfrak{z})$$

for any $y \in L$ and some $D \in \mathbb{N}$. Thus φ has a Fourier expansion

$$(2.2)\qquad \varphi(\tau,\mathfrak{z}) = \sum_{\substack{n\in\mathbb{Q},\, n\equiv D/24 \bmod \mathbb{Z} \\ l\in\frac{1}{2}L^\vee}} f(n,l)e^{2\pi i(n\tau+(l,\mathfrak{z}))}.$$

We now introduce the notion of a Jacobi form.

Definition 2.2. Let $k \in \mathbb{Z}$ and $t \in \mathbb{Q}_{\geq 0}$. A holomorphic function

$$\varphi : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$$

is called a *weak Jacobi form* of weight k and index t with character χ if

$$\forall g \in \Gamma^J(L) : \quad \varphi|_{k,t} g = \chi(g)\varphi$$

and φ has a Fourier expansion as in (2.2) where additionally

$$f(n,l) \neq 0 \;\Rightarrow\; n \geq 0.$$

We call φ a *holomorphic Jacobi form* if the Fourier expansion ranges over all n, l with $2nt - (l,l) \geq 0$, and a *Jacobi cusp form* if it ranges over all n, l satisfying $2nt - (l,l) > 0$. We denote by $J^{(\text{weak})}_{k,L;t}(\chi)$ the vector space of weak Jacobi forms, and the corresponding spaces of holomorphic forms and cusp forms are denoted by $J_{k,L;t}(\chi)$ and $J^{(\text{cusp})}_{k,L;t}(\chi)$, respectively. If the character is trivial we also write $J_{k,L;t} = J_{k,L;t}(1)$ for short.

Remark 2.3. (a) The action can be extended to $k \in \frac{1}{2}\mathbb{Z}$ and $\chi|_{\mathrm{SL}(2,\mathbb{Z})}$ being a multiplier system for $\mathrm{SL}(2,\mathbb{Z})$. Here we have to replace $\mathrm{SL}(2,\mathbb{Z})$ by the metaplectic cover $\mathrm{Mp}(2,\mathbb{Z})$ (see e.g. [4]). In this situation we also use the notation $J^{(*)}_{k,L;t}(\chi)$ for the spaces of Jacobi forms.

(b) A Jacobi form is called *singular* if it has weight $N/2$.

(c) The notion of a Jacobi form is compatible with Definition 1.2. To see this we note that we have an affine model for the homogeneous domain $\mathcal{D}$ given by

$$\mathcal{H}(L_2) = \left\{ \begin{pmatrix} \omega \\ \mathfrak{z} \\ \tau \end{pmatrix} \in \mathbb{C} \times (L \otimes \mathbb{C}) \times \mathbb{C} \;\middle|\; \begin{matrix} \omega_i, \tau_i > 0, \\ 2\omega_i\tau_i - (\mathfrak{z}_i, \mathfrak{z}_i) > 0 \end{matrix} \right\} \tag{2.3}$$

where we have used the abbreviations

$$\omega_i := \mathrm{Im}(\omega), \qquad \tau_i := \mathrm{Im}(\tau), \qquad \mathfrak{z}_i := \mathrm{Im}(\mathfrak{z}).$$

Let $\varphi \in J_{k,L;t}(\chi)$ where $k \in \mathbb{Z}$. We define a holomorphic function on $\mathcal{H}(L_2)$ by

$$\widetilde{\varphi}(\tau, \mathfrak{z}) = \varphi(\tau, \mathfrak{z}) e^{2\pi i t\omega}.$$

Since $\mathcal{D}$ and $\mathcal{H}(L_2)$ are biholomorphically equivalent, we can interpret $\widetilde{\varphi}$ as an element of $\mathcal{M}_k(\Gamma^J(L), \chi)$.

(d) Suppose φ is a non-vanishing Jacobi form of index t and weight k with character $\chi = v_\eta^D \cdot \nu$. Consider the extended Jacobi form $\widetilde{\varphi}(\tau, \mathfrak{z}) = \varphi(\tau, \mathfrak{z}) e^{2\pi i t\omega}$. By (b) we know that $\widetilde{\varphi}$ can be interpreted as a modular form. If we transform this function by $g = [0, 0 : 1] \in H(L)$, an investigation of the modular behaviour on the affine domain $\mathcal{H}(L_2)$ yields $e^{2\pi i t} = \nu([0,0:1])$ and Proposition 2.1 implies

$$\nu([x,y:r]) = e^{\pi i t((x,x)+(y,y)-(x,y)+2r)}.$$

(e) The Jacobi forms for a fixed lattice L and with trivial character form a bigraded ring. The transformation behaviour of elements belonging to $J_{k_1,L;t_1} J_{k_1,L;t_2}$ is that of a Jacobi form of weight k_1+k_2 and index t_1+t_2. If both factors are holomorphic then the product is also a holomorphic Jacobi form by the identity

$$k_1 + k_2 - \frac{(x+y,x+y)}{2(t_1+t_2)} = \left(k_1 - \frac{(x,x)}{2t_1}\right) + \left(k_2 - \frac{(y,y)}{2t_2}\right) + \frac{(t_1 y - t_2 x, t_1 y - t_2 x)}{2t_1 t_2 (t_1+t_2)},$$

which holds for all $k_1, k_2 \in \mathbb{Q}_{\geq 0}$, $t_1, t_2 > 0$ and $x, y \in L \otimes \mathbb{R}$.

We finish this section with several examples of Jacobi forms for the root lattice $A_1 \cong (\mathbb{Z}, 2x^2)$ which correspond to Jacobi forms in the sense of [7]. We denote by $(\cdot,\cdot)_N$ the standard scalar product in $\mathbb{R}^N$. The standard basis of $\mathbb{R}^N$ is denoted by $\varepsilon_1, \ldots, \varepsilon_N$. For $\tau \in \mathbb{H}$ we set $q := \exp(2\pi i \tau)$.

EXAMPLE 2.4. (a) Dedekind's eta function is a Jacobi form of weight $1/2$ and index 0 for every positive definite even lattice L, thus $\eta \in J^{(\mathrm{cusp})}_{1/2,L;0}(v_\eta)$ where v_η is not a character but a multiplier system for $\mathrm{SL}(2,\mathbb{Z})$. For $m \in \mathbb{Z}$ one defines

$$\left(\frac{-4}{m}\right) := \begin{cases} 1, & m \equiv 1 \bmod 4, \\ -1, & m \equiv -1 \bmod 4, \\ 0, & m \equiv 0 \bmod 2. \end{cases}.$$

We mention the following infinite sum expansion for $\eta^3(\tau)$:

$$\eta^3(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z},\, n \equiv 1 \bmod 2} \left(\frac{-4}{n}\right) n q^{n^2/8}, \tag{2.4}$$

which follows from the Jacobi triple identity.

(b) Eichler and Zagier [7] constructed the Jacobi–Eisenstein series $e_{k,1} \in J_{k,A_1;1}$. One defines a Jacobi cusp form $\varphi_{12,A_1} \in J_{12,A_1;1}$ as

$$\varphi_{12,A_1}(\tau,z) := \tfrac{1}{144}\big(e_4^2(\tau) e_{4,1}(\tau,z) - e_6(\tau) e_{6,1}(\tau,z)\big)$$

where

$$e_k(\tau) = 1 - \frac{2k}{B_k} \cdot \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

denotes the classical Eisenstein series of weight k for the group $\mathrm{SL}(2,\mathbb{Z})$ (see e.g. [15, p. 161]).

(c) We define

$$\phi_{0,1}(\tau,z) := \frac{\varphi_{12,A_1}(\tau,z)}{\eta^{24}(\tau)} = \frac{e_4^2(\tau)e_{4,1}(\tau,z) - e_6(\tau)e_{6,1}(\tau,z)}{\frac{1}{12}(e_4^3(\tau) - e_6^2(\tau))}.$$

In accordance with the classical theory of elliptic modular forms we write

$$\Delta_{12}(\tau) := \eta^{24}(\tau)$$

for the first cusp form for $\mathrm{SL}(2,\mathbb{Z})$. One has $\phi_{0,1} \in J_{0,A_1;1}^{(\mathrm{weak})}(1)$ since

$$\phi_{0,1}(\tau,z) = \sum_{\substack{n,s\in\mathbb{Z},\, n\geq 0\\ 4n-s^2\geq -1}} q^n r^s = (r + 10 + r^{-1}) + q(\dots)$$

with the convention

$$r = \exp(2\pi i z).$$

Moreover the function defined by $(\tau,z) \mapsto \eta^6(\tau)\phi_{0,1}(\tau,z)$ is a holomorphic Jacobi form of weight 3 and index 1 for the lattice A_1, with Fourier expansion

$$\eta^6(\tau)\phi_{0,1}(\tau,z) = q^{1/4}(r + 10 + r^{-1}) + q^{5/4}(\cdots).$$

(d) The Jacobi theta-series of characteristic $(1/2, 1/2)$ is given by

$$\begin{aligned}\vartheta(\tau,z) &= \sum_{n\in\mathbb{Z}} \left(\frac{-4}{n}\right) q^{n^2/8} r^{n/2}\\ &= \sum_{n\in\mathbb{Z},\, n\equiv 1 \bmod 2} (-1)^{(n-1)/2} \exp\left(\frac{\pi i n^2 \tau}{4} + \pi i n z\right)\end{aligned}$$

where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$ and $r = e^{2\pi i z}$, $z \in \mathbb{C}$. This function was originally discovered by Carl Gustav Jacob Jacobi. Gritsenko and Nikulin [13] reinterpreted this function as a modular form of half-integral weight and index. The same proof as in [15, Section I.6] can be used to show that ϑ defines a holomorphic function on $\mathbb{H} \times \mathbb{C}$. The transformation behaviour under the action of $H(A_1)$ is

$$\vartheta(\tau, z + x\tau + y) = (-1)^{x+y} \exp(-\pi i(x^2\tau + 2xz))\vartheta(\tau,z).$$

Moreover,

$$\begin{aligned}\vartheta(\tau+1,z) &= \exp(\pi i/4)\vartheta(\tau,z),\\ \vartheta(-1/\tau, z/\tau) &= \exp(-3\pi i/4)\tau^{1/2}\exp(\pi i z^2/\tau)\vartheta(\tau,z),\end{aligned}$$

by the theta transformation formula. This shows that ϑ transforms like a Jacobi form of weight $1/2$ and index $1/2$. For the extended Jacobi form

$\widetilde{\vartheta}(\tau,z,\omega)=\vartheta(\tau,z)e^{\pi i\omega}$ we obtain

$$\begin{aligned}\widetilde{\vartheta}(\tau,z,\omega)|_{1/2}[x,y:r] = &\ e^{\pi i(x+y)}\exp(-\pi i(x^2\tau+2xz))\vartheta(\tau,z)\\ &\times\exp(\pi i(\omega+2xz+x^2\tau+xy+r))\\ = &\ e^{\pi i(x+y+xy+r)}\,\widetilde{\vartheta}(\tau,z,\omega)\\ =:&\ \nu_{H(A_1)}([x,y:r])\widetilde{\vartheta}(\tau,z,\omega).\end{aligned}$$

From the infinite sum expansion we get

$$\left.\frac{\partial\vartheta(\tau,z)}{\partial z}\right|_{z=0}=\pi i\sum_{n\in\mathbb{Z},\,n\equiv 1\bmod 2}\left(\frac{-4}{n}\right)nq^{n^2/8}=2\pi i\eta(\tau)^3$$

by (2.4). This shows that ϑ transforms like a Jacobi form with mutiplier system $v_\eta^3\times\nu_{H(A_1)}$. Moreover Jacobi's triple identity yields

$$\vartheta(\tau,z)=-q^{1/8}r^{-1/2}\prod_{n\geq 1}(1-q^{n-1}r)(1-q^nr^{-1})(1-q^n).$$

Since each summand in the infinite sum defining ϑ is of the shape $e^{2\pi i\tau n^2/8}e^{2\pi i\cdot 2(z,m/4)_1}$, we see that ϑ is a holomorphic Jacobi form. This function has the properties

$$\begin{aligned}\vartheta(\tau,-z)&=-\vartheta(\tau,z),\\ \vartheta(\tau,z+x\tau+y)&=(-1)^{x+y}\exp(-\pi i(x^2\tau+2xz))\vartheta(\tau,z),\end{aligned}$$

for all $x,y\in\mathbb{Z}$. Hence the divisor of ϑ contains $\{z=x\tau+y\mid x,y\in\mathbb{Z}\}$. Finally, a classical argument from the theory of complex functions yields

$$\operatorname{div}(\vartheta)=\{x\tau+y\,|\,x,y\in\mathbb{Z}\}$$

(see for example [5, Section 5.6]).

3. Theta type Jacobi forms for several root lattices. In this section we give several pullback constructions for Jacobi forms. The constructions are motivated by [10] where Grotsenko constructed towers of reflective modular forms by means of Jacobi forms. All lattices appearing in this section are even. The next lemma can be found in [6, Proposition 3.1].

LEMMA 3.1. *Let $L\leq M$ be a sublattice of M such that* $\operatorname{rank}L<\operatorname{rank}M$, *and let $\varphi\in J_{k,M,t}(\chi)$ be a Jacobi form of weight k and index t for the character χ. Consider the decomposition $\mathfrak{z}_M=\mathfrak{z}_L\oplus\mathfrak{z}_{L^\perp}\in M\otimes\mathbb{C}=(L\oplus(L)^\perp_M)\otimes\mathbb{C}$. Define the pullback of φ to L as the function $\varphi\lfloor_L$ on $\mathbb{H}\times(L\otimes\mathbb{C})$ given by*

$$\varphi\lfloor_L(\tau,\mathfrak{z}_L):=\varphi(\tau,\mathfrak{z}_L\oplus 0).$$

Then $\varphi\lfloor_L\in J_{k,L,t}(\chi|_L)$ and the pullback maps cusp forms to cusp forms.

DEFINITION 3.2. Let $L\leq M$ be a sublattice of M such that $\operatorname{rank}L<\operatorname{rank}M$, and let $\varphi\in J^{(\text{weak})}_{k,M,t}$ be a Jacobi form of weight k and index t. Let

$\psi \in J^{(\mathrm{weak})}_{k,L;t}$. We say that ψ is a *pullback* of φ if there exists some $\alpha \in \mathbb{C}^\times$ such that $\psi = \alpha \cdot \varphi\lfloor_L$. In this case we write $\varphi \to \psi$. We set $\varphi\lfloor_L := \varphi$ if $\operatorname{rank} L = \operatorname{rank} M$.

Let $L \subseteq \mathbb{R}^m$ be a positive definite even lattice. We define

$$\vartheta_L : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}, \qquad \vartheta_L(\tau, \mathfrak{z}) := \prod_{j=1}^{m} \vartheta(\tau, (\mathfrak{z}, \varepsilon_j)).$$

We now introduce Jacobi forms of theta type.

Definition 3.3. Let $L \subseteq \mathbb{R}^m$ be a positive definite even lattice and $\varphi \in J_{k,L;t}$. We say that φ is of *theta type* if there exists a sublattice $L' \subseteq L$, $\alpha \in \mathbb{C}^\times$ and $a, b \in \mathbb{Z}_{\geq 0}$ such that

$$\varphi\lfloor_{L'}(\tau, \mathfrak{z}') = \alpha \cdot \eta(\tau)^a\, \vartheta_{L'}(\tau, \mathfrak{z}')^b.$$

Proposition 3.4. *Let $L \subseteq \mathbb{R}^m$ be a positive definite even lattice. Assume that its bilinear form is $(\cdot,\cdot) := (\cdot,\cdot)_m$.*

(a) *The function ϑ_L transforms like a Jacobi form of index 1 and weight $m/2$ for the lattice L with a character having trivial Heisenberg part. The Fourier expansion of this function is*

$$\vartheta_L(\tau, \mathfrak{z}) = \sum_{\substack{n \in \mathbb{Q}_{\geq 0},\, n \equiv \frac{D}{24} \bmod \mathbb{Z} \\ l \in \frac{1}{2}\mathbb{Z}^m \\ 2n - (l,l) = 0}} f(n,l) e^{2\pi i (n\tau + (l, \mathfrak{z}))}$$

for some $D \in \mathbb{N}$.

(b) *Let $L \leq M$ and suppose $J \subseteq \{1, \ldots, m\}$ is such that $M \subseteq \langle \varepsilon_j \mid j \in J \rangle_{\mathbb{R}}$. Assume that there exists a $\kappa \in J$ such that*

$$L = M \cap \langle \varepsilon_j \mid j \in J,\, j \neq \kappa \rangle_{\mathbb{R}}.$$

Define a function ${}_{L(2)}\prod^{M(2)}(\vartheta_L^2, \phi_{0,1}) : \mathbb{H} \times (M(2) \otimes \mathbb{C}) \to \mathbb{C}$ by

$${}_{L(2)}\prod^{M(2)}(\vartheta_L^2, \phi_{0,1})(\tau, \mathfrak{z}) := \phi_{0,1}(\tau, (\mathfrak{z}, \varepsilon_\kappa)) \cdot \vartheta_L^2(\tau, \mathfrak{z}).$$

Then ${}_{L(2)}\prod^{M(2)}(\vartheta_L^2, \phi_{0,1})$ transforms like a Jacobi form of index 2 for the lattice M with a character having trivial Heisenberg part, and satisfies ${}_{L(2)}\prod^{M(2)}(\vartheta_L^2, \phi_{0,1}) \to \vartheta_L^2$.

Proof. (a) For any $\mathfrak{z} \in L \otimes \mathbb{C}$ we have the identity

$$\mathfrak{z} = \sum_{j=1}^{m} (\mathfrak{z}, \varepsilon_j)\varepsilon_j.$$

Hence $(\mathfrak{z},\mathfrak{z}) = \sum_{j=1}^{m} (\mathfrak{z},\varepsilon_j)^2$ and for any $x \in L$ we obtain

$$\sum_{j=1}^{m} (x,\varepsilon_j) = \sum_{j=1}^{m} (x,\varepsilon_j)^2 = (x,x) = 0 \bmod 2\mathbb{Z}.$$

From Example 2.4(d) we deduce that ϑ_L has the transformation behaviour and character claimed above. The same example shows that each summand of the Fourier expansion is of the shape

$$\alpha \prod_{j\in J} \left(q^{n_j^2/8} \exp(\pi i n_j(\mathfrak{z},\varepsilon_j))\right)$$

where $J \subseteq \{1,\dots,m\}$, $n_j \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}$. Note that $\sharp J \geq \operatorname{rank}(L)$. Now define

$$v := \frac{1}{2}\sum_{j\in J} n_j\varepsilon_j \in \frac{1}{2}\mathbb{Z}^m$$

to write the above summand as

$$\alpha \exp(2\pi i(\mathfrak{z},v)) \cdot \prod_{j\in J} q^{n_j^2/8}.$$

This completes the proof of (a).

(b) The statement follows from the identity

$$_{L(2)}\prod^{M(2)}(\vartheta_L^2,\phi_{0,1})(\tau,\mathfrak{z}) = \phi_{0,1}(\tau,(\mathfrak{z},\varepsilon_\kappa)) \cdot \prod_{j\in J\setminus\{\kappa\}} \vartheta^2(\tau,(\mathfrak{z},\varepsilon_j))$$

and a reasoning analogous to that in part (a). The statement on the pullback follows from $\phi_{0,1}(\tau,0) = 12$ and $(\mathfrak{z}_L,\varepsilon_\kappa) = 0$ if $\mathfrak{z}_L \in L\otimes\mathbb{C}$. ∎

We now restrict ourselves to some special types of (root) lattices. We first note that for any $t,k \in \mathbb{N}$ the space of classical Jacobi forms of weight k and index t as defined in [7] coincides with $J_{k,A_1(t);1} = J_{k,A_1;t^2}$. For the next lemma we consider the lattice NA_1 which can be realized as $\mathbb{Z}^N$ equipped with the bilinear form $2(\cdot,\cdot)_N$. Note that every $\mathfrak{z} \in NA_1\otimes\mathbb{C}$ can be written in the form

$$\mathfrak{z} = \sum_{j=1}^{N} z_j\varepsilon_j, \qquad z_j \in \mathbb{C}. \tag{3.1}$$

LEMMA 3.5. *Let $N,t,k \in \mathbb{N}$ and $\varphi \in J^{(\mathrm{weak})}_{k,NA_1,t^2}$. Then the function*

$$\varphi\otimes\phi_{0,1} : \mathbb{H}\times((N+1)A_1\otimes\mathbb{C}) \to \mathbb{C}$$

defined by

$$\varphi\otimes\phi_{0,1}(\tau,\mathfrak{z}_{(N+1)A_1}) := \varphi(\tau,\mathfrak{z}_{NA_1})\phi_{0,1}(\tau,tz_{N+1})$$

is an element of $J^{(\mathrm{weak})}_{k,(N+1)A_1;t^2}$. Moreover, $\varphi\otimes\phi_{0,1} \to \varphi$.

Proof. Let $\tau \in \mathbb{H}$, $\mathfrak{z} \in ((N+1)A_1 \otimes \mathbb{C})$ and $x, y \in (N+1)A_1$. We consider the decomposition

$$\mathfrak{z} = \mathfrak{z}_{NA_1} \oplus z_{N+1}\varepsilon_{N+1}$$

and decompose x, y in the same manner. For every $A \in \mathrm{SL}(2,\mathbb{Z})$ and $x, y \in \mathbb{Z}$ one has

$$\begin{aligned}\phi_{0,1}\left(\frac{a\tau+b}{c\tau+d}, \frac{tz_{N+1}}{c\tau+d}\right) &= (c\tau+d)^k \exp\left(2t^2\pi i\frac{z_{n+1}^2}{c\tau+d}\right)\phi_{0,1}(\tau, tz_{N+1}),\\ \phi_{0,1}(\tau, tz_{N+1} + tx_{N+1}\tau + ty_{N+1}) &\\ &= \exp(-2\pi it^2(x_{N+1}^2 + 2x_{N+1}z_{N+1}))\phi_{0,1}(\tau, tz_{N+1}),\end{aligned}$$

which shows that $\varphi \otimes \phi_{0,1}$ has the correct transformation behaviour. Since $NA_1(t^2) \subseteq \mathbb{R}^N$, we conclude that $\varphi \otimes \phi_{0,1}$ has a Fourier expansion of the shape

$$\varphi \otimes \phi_{0,1}(\tau, \mathfrak{z}) = \sum_{\substack{n\in\mathbb{N}\\ l\in\frac{1}{2t^2}\mathbb{Z}^N}} f(n,l)e^{2\pi i(n\tau+(l,\mathfrak{z}))}$$

where $(\cdot,\cdot) := 2t^2(\cdot,\cdot)_N$. ∎

Following [10] and [6] we can construct Jacobi forms for the special case $N \le 4$ and $t = 1$.

PROPOSITION 3.6. *Let $N \in \{1, \ldots, 4\}$ and consider the coordinates* (3.1).

(a) *We have $\vartheta_{NA_1} \in J_{N/2,NA_1;1/2}(v_\eta^{3N} \times \nu_{H(NA_1)})$ where*

$$\nu_{H(NA_1)}([x, y : r]) = (-1)^{r+\sum_{j=1}^N x_j+y_j+x_jy_j} \quad \textit{for all } [x, y : r] \in H(NA_1).$$

(b) *Define $\varrho_{6-N,NA_1}, \psi_{12-2N,NA_1} : \mathbb{H} \times (NA_1 \otimes \mathbb{C}) \to \mathbb{C}$ by*

$$\begin{aligned}\varrho_{6-N,NA_1}(\tau, \mathfrak{z}_{NA_1}) &:= \eta(\tau)^{12-3N}\vartheta_{NA_1}(\tau, \mathfrak{z}_{NA_1}),\\ \psi_{12-2N,NA_1} &:= \varrho_{6-N,NA_1}^2.\end{aligned}$$

Then we have $\varrho_{6-N,NA_1} \in J_{6-N,NA_1;1/2}(v_\eta^{12} \times \nu_{H(NA_1)})$ and $\psi_{12-2N,NA_1} \in J_{12-2N,NA_1;1}$.

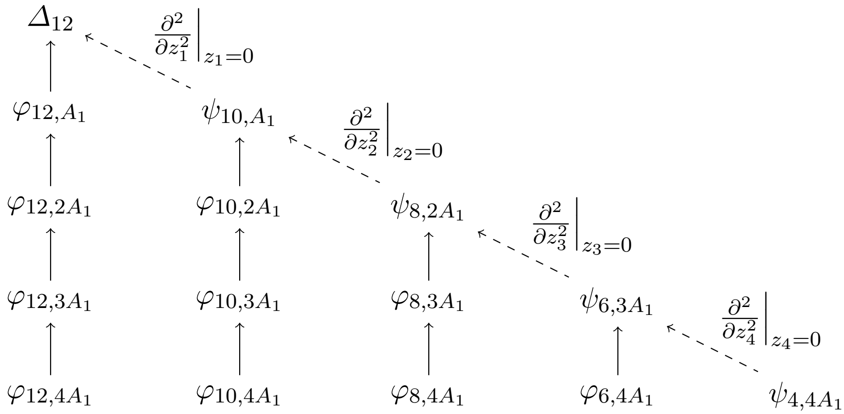
Proof. Since by our construction the ambient vector space of NA_1 is $\mathbb{R}^N$, we can derive the holomorphicity of ϑ_{NA_1} by the Fourier expansion given in Proposition 3.4(a). The Heisenberg part of these functions is a binary character since the bilinear form of NA_1 is $2(\cdot,\cdot)_N$ instead of $(\cdot,\cdot)_N$. The last fact is also responsible for the appearance of a half-integral index in this case. The other statements are direct consquences of Proposition 3.4 and Example 2.4(a), (d). ∎

The previous considerations show that there exists an operator

$$\otimes\ \phi_{0,1} : J_{k,NA_1;1}^{(\mathrm{weak})} \to J_{k,(N+1)A_1;1}^{(\mathrm{weak})}, \quad \varphi \mapsto \varphi \otimes \phi_{0,1}, \tag{3.2}$$

which extends to the space of Jacobi forms of index zero, which is isomorphic to the space of elliptic modular forms $\mathcal{M}_k(\mathrm{SL}(2,\mathbb{Z}))$ of weight k by the same construction. This enables us to build a tower of theta type Jacobi forms for $4A_1$.

PROPOSITION 3.7. *Let $N = 1, \ldots, 4$. We have the following diagram of Jacobi forms:*



where $\varphi_{k,NA_1} \in J_{k,NA_1;1}$. Except for those in the last row, all forms are cusp forms. The functions at the tail of each arrow are defined by application of $\otimes\,\phi_{0,1}$ to the function at the arrowhead. Here the quasi-pullbacks on the diagonal are taken with respect to the coordinates (3.1).

Proof. The pullback structure of the diagram follows from Propositions 3.6 and 3.4. If we rewrite the Fourier expansion of $\phi_{0,1}(\tau, z)$ given in Example 2.4(c) according to Definition 2.2, we find that the hyperbolic norm of all indices belonging to non-vanishing Fourier coefficients is bounded from below by $-1/2$. Note that multiplication with $\eta(\tau)^6$ adds $1/2$ to this bound. If we take into account that NA_1 consists of N perpendicular copies of A_1, we find that this proves the assertion on the holomorphicity and on cusp forms. Part (d) of the same example yields the quasi-pullback structure of the diagonal. ∎

We consider another series of root lattices. For $N \geq 3$ the root system D_N is described by

$$(\langle \varepsilon_2 + \varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2 \ldots, \varepsilon_N - \varepsilon_{N-1} \rangle_{\mathbb{Z}}, (\cdot,\cdot)_N)$$

and D_2, D_1 can be considered as $(\varepsilon_2 + \varepsilon_1, \varepsilon_2 - \varepsilon_1, (\cdot,\cdot)_2)$ and $(2\varepsilon_1, (\cdot,\cdot)_1)$, respectively. The first of the following two constructions also appeared in [10] but in different coordinates.

Proposition 3.8.

(a) *We have* $\vartheta_{D_N} \in J_{N/2,D_N,1}(v_\eta^{3N})$ *and for* $N = 1, \ldots, 8$ *there are Jacobi forms* $\psi_{12-N,D_N} \in J_{12-N,D_N;1}$ *given by*

$$\psi_{12-N,D_N}(\tau, \mathfrak{z}_{D_N}) = \eta(\tau)^{24-3N} \vartheta_{D_N}(\tau, \mathfrak{z}_{D_N}).$$

For $N < 8$ *these functions define Jacobi cusp forms. These forms are obtained by iterative quasi-pullbacks of* ϑ_{D_8}*, namely*

$$\left.\frac{\partial \psi_{12-N,D_N}}{\partial z_N}\right|_{z_N=0} = 2\pi i \psi_{12-N+1,D_{N-1}}.$$

(b) *We have the following diagram of holomorphic Jacobi forms where all forms except those in the first column are cusp forms of weight indicated by the index:*

$$\begin{array}{cccccccc}
\psi_{4,D_8} & \psi^2_{5,D_7} & \psi^2_{6,D_6} & \psi^2_{7,D_5} & \psi^2_{8,D_4} & \psi^2_{9,D_3} & \psi^2_{10,D_2} & \psi^2_{11,D_1} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \\
\phi_{10,D_8(2)} & \phi_{12,D_7(2)} & \phi_{14,D_6(2)} & \phi_{16,D_5(2)} & \phi_{18,D_4(2)} & \phi_{20,D_3(2)} & \phi_{22,D_2(2)} & \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & \\
\phi_{12,D_8(2)} & \phi_{14,D_7(2)} & \phi_{16,D_6(2)} & \phi_{18,D_5(2)} & \phi_{20,D_4(2)} & \phi_{22,D_3(2)} & & \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & & & \\
\phi_{14,D_8(2)} & \phi_{16,D_7(2)} & \phi_{18,D_6(2)} & \phi_{20,D_5(2)} & \phi_{22,D_4(2)} & & & \\
\nearrow & \nearrow & \nearrow & \nearrow & & & & \\
\phi_{16,D_8(2)} & \phi_{18,D_7(2)} & \phi_{20,D_6(2)} & \phi_{22,D_5(2)} & & & & \\
\nearrow & \nearrow & \nearrow & & & & & \\
\phi_{18,D_8(2)} & \phi_{20,D_7(2)} & \phi_{22,D_6(2)} & & & & & \\
\nearrow & \nearrow & & & & & & \\
\phi_{20,D_8(2)} & \phi_{22,D_7(2)} & & & & & & \\
\nearrow & & & & & & & \\
\phi_{22,D_8(2)} & & & & & & &
\end{array}$$

where the functions at the tail of each arrow are defined by application of ${}_{D_{N-1}(2)}\prod^{D_N(2)}(\cdot, \phi_{0,1})$ *to the function at the arrowhead. More precisely, we have*

$$\phi_{k,D_N(2)} \in J_{k,D_N(2),1}$$

and $\phi_{k,D_N(2)}$ is a cusp form if and only if $N < 8$. All the functions except those in the first row have the property

$$ {}_{D_{N-1}(2)}\prod^{D_N(2)}(\phi_{k,D_{N-1}(2)},\phi_{0,1}) \to \phi_{k,D_{N-1}(2)} \text{ where } N \geq 2,$$

with the convention $\phi_{26-2N,D_{N-1}(2)} := \psi^2_{13-N,D_{N-1}}$.

The arithmetic lifting A-Lift(ψ_{11,D_1}) is the cusp form of weight 11 for the Siegel paramodular group of level 2.

Proof of Proposition 3.8. (a) We first note that our model for D_N is compatible with the requirements of Proposition 3.4. Hence the statements on the modular behaviour and the identification of the character follow directly, as also does the property under quasi-pullbacks. It remains to investigate the holomorphicity. Since $D_N \subseteq \mathbb{R}^N$ is a full lattice, we conclude from Proposition 3.4(a) that ϑ_{D_N} is indeed holomorphic at infinity. Moreover for $N < 8$ the forms ψ_{12-N,D_N} are cuspidal because they equal the product of a holomorphic Jacobi form of singular weight and a cusp form of index 0.

(b) From (a) and the iterative construction we can easily derive that $\phi_{k,D_N(2)}$ is a holomorphic Jacobi form whose Fourier coefficients $f(n,l)$ are parametrized by all pairs $(n,l) \in \mathbb{N} \times D_N(2)^\vee$ such that $2n - (l,l) \geq (8-N)/2$. Now Proposition 3.4 completes the proof. ∎

We finish this section by considering root lattices of type A_N. Here we use the model

$$(\langle \varepsilon_2 - \varepsilon_1, \ldots, \varepsilon_{N+1} - \varepsilon_N \rangle_{\mathbb{Z}}, (\cdot,\cdot)_{N+1})$$

for A_N where $N \geq 1$. Note that in this case $A_N \subseteq \mathbb{R}^{N+1}$. Hence A_N is not a full lattice according to our realization.

Proposition 3.9.

(a) *Let $N \in \{2,\ldots,7\}$. Define*

$$\vartheta^{(1)}_{A_N} : \mathbb{H} \times (A_N \otimes \mathbb{C}) \to \mathbb{C}, \qquad \vartheta^{(1)}_{A_N}(\tau,\mathfrak{z}) := \eta(\tau)^{-1}\vartheta_{A_N}(\tau,\mathfrak{z}).$$

Then $\vartheta^{(1)}_{A_N} \in J_{N/2,A_N;1}(v_\eta^{3N+2})$ and there are Jacobi forms $\psi_{11-N,A_N} \in J_{11-N,A_N;1}$ given by

$$\psi_{11-N,A_N}(\tau,\mathfrak{z}_{A_N}) = \eta(\tau)^{21-3N}\vartheta_{A_N}(\tau,\mathfrak{z}_{A_N}).$$

For $N < 7$ these functions define Jacobi cusp forms. These forms are obtained by iterative quasi-pullbacks of ψ_{A_7}, namely

$$\left.\frac{\partial \psi_{11-N,A_N}}{\partial z_N}\right|_{z_N=0} = 2\pi i \psi_{11-N+1,A_{N-1}}.$$

(b) *We have the following diagram of holomorphic Jacobi forms where all forms except those in the first column are cusp forms of weight indicated by the index.*

$$\begin{array}{cccccc}
\psi_{4,A_7} & \psi^2_{5,A_6} & \psi^2_{6,A_5} & \psi^2_{7,A_4} & \psi^2_{8,A_3} & \psi^2_{9,A_2} \\
\nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \\
\phi_{10,A_7(2)} & \phi_{12,A_6(2)} & \phi_{14,A_5(2)} & \phi_{16,A_4(2)} & \phi_{18,A_3(2)} & \\
\nearrow & \nearrow & \nearrow & \nearrow & & \\
\phi_{12,A_7(2)} & \phi_{14,A_6(2)} & \phi_{16,A_5(2)} & \phi_{18,A_4(2)} & & \\
\nearrow & \nearrow & \nearrow & & & \\
\phi_{14,A_7(2)} & \phi_{16,A_6(2)} & \phi_{18,A_5(2)} & & & \\
\nearrow & \nearrow & & & & \\
\phi_{16,A_7(2)} & \phi_{18,A_6(2)} & & & & \\
\nearrow & & & & & \\
\phi_{18,A_7(2)} & & & & &
\end{array}$$

where the functions at the tail of each arrow are defined by application of ${}_{A_{N-1}(2)}\prod^{A_N(2)}(\cdot, \phi_{0,1})$ *to the function at the arrowhead. More precisely, we have*

$$\phi_{k,A_N(2)} \in J_{k,A_N(2),1}$$

and $\phi_{k,A_N(2)}$ *is a cusp form if and only if* $N < 7$. *All the functions except those in the first row have the property*

$${}_{A_{N-1}(2)}\coprod^{A_N(2)}(\phi_{k,A_{N-1}(2)}, \phi_{0,1}) \to \phi_{k,A_{N-1}(2)} \quad \textit{where } N \geq 3,$$

with the convention $\phi_{24-2N,A_{N-1}(2)} := \psi^2_{12-N,A_{N-1}}$

Proof. (a) We start with $\vartheta^{(1)}_{A_N}$. As before, the transformation behaviour under modular substitutions and the shape of the character follow directly from Proposition 3.4. The only thing which remains to be shown is the holomorphicity. In this case the proof is less obvious than in the previous cases, since the dimension of the ambient vector space of A_N is strictly greater than N. We start with the Fourier expansion of $\vartheta_{A_2}(\tau, \mathfrak{z}_{A_2})$. Each summand is of the shape

$$q^{(n_1^2+n_2^2+n_3^2)/8} \exp\bigl(\pi i(-n_1 z_1 + n_2 z_2 + n_3(z_1 - z_2))\bigr)$$

where $n_1 \equiv n_2 \equiv n_3 \equiv 1 \bmod 2$. Since $A_2^\vee \subseteq \frac{1}{6}A_2$, this can be rewritten as

$$q^{(n_1^2+n_2^2+n_3^2)/8}\exp(2\pi i(v,\mathfrak{z}_{A_2}))$$

where

$$v := \tfrac{1}{6}[n_1(2\varepsilon_1-\varepsilon_2-\varepsilon_3)+n_2(2\varepsilon_3-\varepsilon_1-\varepsilon_2)+n_3(2\varepsilon_2-\varepsilon_1-\varepsilon_3)].$$

The hyperbolic norm of the last expression equals

$$\begin{aligned}
&2\frac{n_1^2+n_2^2+n_3^2}{8}-(v,v)\\
&\qquad=\frac{n_1^2+n_2^2+n_3^2}{4}-\frac{n_1^2+n_2^2+n_3^2}{6}+\frac{n_1n_2+n_1n_3+n_2n_3}{6}\\
&\qquad=\frac{(n_1+n_2+n_3)^2}{12}\geq\frac{1}{12}
\end{aligned}$$

since n_1, n_2, n_3 are odd. This shows that $\vartheta_{A_2}^{(1)}$ is indeed a holomorphic Jacobi form since

$$\eta(\tau)^{-1}=q^{-1/24}(1+q(\dots)).$$

Now let $N>2$. We decompose $\mathfrak{z}\in A_N\otimes\mathbb{C}$ as

$$\mathfrak{z}=\mathfrak{z}_{A_{N-1}}\oplus\mathfrak{z}_K,\qquad K=(A_{N-1})_{A_N}^\perp.$$

With respect to this basis, $\vartheta_{A_N}^{(1)}$ decomposes as

$$\vartheta_{A_N}^{(1)}(\tau,\mathfrak{z})=\vartheta_{A_{N-1}}^{(1)}(\tau,\mathfrak{z}_{A_{N-1}})\cdot\vartheta_K(\tau,\mathfrak{z}_K)$$

where K is realized as a lattice in $\mathbb{R}$. Now induction and the holomorphicity of ϑ_K show that $\vartheta_{A_N}^{(1)}$ is a holomorphic Jacobi form. The statements on ψ_{11-N,A_N} are obvious.

(b) The proof is completely analogous to the proof of Proposition 3.8(b). ∎

The length of the towers can be arbitrarily increased if we multiply them by powers of Δ_{12} to preserve the holomorphicity.

4. Additional symmetries for the lattice D_4. In this section we use theta type Jacobi forms to explain the existence of a particular modular form for the lattice D_4. For any $r\in L_2$ we define the reflection in the hyperplane $r^\perp$ as

$$\sigma_r: V\to V,\qquad \sigma_r(v)=v-2\frac{(r,v)}{(r,r)}r.$$

Let $r\in L_2\otimes\mathbb{Q}$ be such that $(r,r)<0$. The rational quadratic divisor with respect to r is given as

$$\mathcal{D}_r(L_2):=\{[\mathcal{Z}]\in\mathcal{D}(L_2)\mid(\mathcal{Z},r)=0\}.$$

In particular

$$[\mathcal{Z}] \in \mathcal{D}_r(L_2) \Leftrightarrow \sigma_r([\mathcal{Z}]) = [\mathcal{Z}].$$

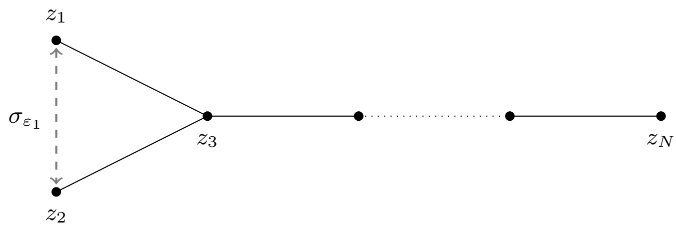
Note that $\sigma_r \in \mathrm{O}(L_2 \otimes \mathbb{Q})^+ \Leftrightarrow (r,r) < 0$.

We now state a variant of Borcherds multiplicative lifting which can be found in [9].

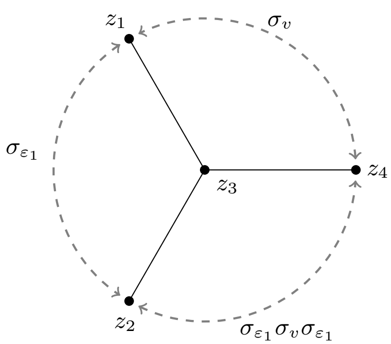
Theorem 4.1 (Borcherds, Gritsenko). *Let φ be a weakly holomorphic Jacobi form with Fourier coefficients $f(n,l)$. Assume that $f(n,l) \in \mathbb{Z}$ if $2n - (l,l) \leq 0$. Then there exists a modular form F_φ of weight $\frac{1}{2}f(0,0)$ with respect to $\widetilde{\mathrm{O}}(L_2)^+$ satisfying*

$$\mathrm{div}(F_\varphi) = \sum_{\substack{h \in L_2^\vee \text{ primitive} \\ (h,h)<0}} \alpha_h \mathcal{D}_h(L_2) \quad \text{where } \alpha_h = \sum_{\substack{d>0 \\ (h,h)=2n-(l,l) \\ h \equiv l \bmod L_2}} f(d^2 n, dl).$$

For the details of the construction of F_φ we refer to [9] and [2]. The root lattices of type D_N have been defined in the last section. Using the same coordinates as before, let $\mathfrak{z} \in D_N \otimes \mathbb{C}$. In general the only symmetries of the Coxeter diagram of D_N are given by the permutation of $z_1 := (\mathfrak{z}, \varepsilon_1)$ and $z_2 := (\mathfrak{z}, \varepsilon_2)$:



If $N = 4$ the diagram has additional symmetries:



$$v = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4}{2} \in D_4^\vee$$

In this case we can define two additional singular Jacobi forms

$$\vartheta_{D_4}^{(2)} := \vartheta_{D_4}|_{2,1}\sigma_{\varepsilon_1}\sigma_v\sigma_{\varepsilon_1}, \qquad \vartheta_{D_4}^{(3)} := \vartheta_{D_4}|_{2,1}\sigma_v.$$

If we use our standard coordinates, we obtain

$$\vartheta_{D_4}(\tau,\mathfrak{z}_{D_4}) = \vartheta(\tau,z_1-z_2)\vartheta(\tau,z_1+z_2-z_3)\vartheta(\tau,z_3-z_4)\vartheta(\tau,z_4),$$
$$\vartheta^{(2)}_{D_4}(\tau,\mathfrak{z}_{D_4}) = \vartheta(\tau,z_2)\vartheta(\tau,z_3-z_2)\vartheta(\tau,z_1-z_3+z_4)\vartheta(\tau,z_1-z_4),$$
$$\vartheta^{(3)}_{D_4}(\tau,\mathfrak{z}_{D_4}) = \vartheta(\tau,z_1)\vartheta(\tau,z_1-z_3)\vartheta(\tau,z_4-z_2)\vartheta(\tau,-z_2+z_3-z_4).$$

For any $0 \neq l \in L_2$ we define $\operatorname{div}(l) \in \mathbb{N}$ to be the generator of the ideal $((l,h) \mid h \in L_2)_{\mathbb{Z}}$ and we set $l^* := l/\operatorname{div}(l)$. A vector $l \in L_2$ is called *primitive* if

$$\forall k \in \mathbb{Z}, h \in L_2 : \quad (l = kh \Rightarrow k = \pm 1).$$

The following proposition is useful if one wants to determine the orbits of the divisor of a multiplicative lifting under the orthogonal group. A proof can be found in [12].

PROPOSITION 4.2 (Eichler criterion). *If $r,s \in L_2$ are primitive, $(s,s) = (r,r)$ and $r^* \equiv s^* \bmod L_2$ then there exists a $g \in \widetilde{\mathrm{SO}}(L_2)^+$ such that $g(r) = s$.*

The next theorem can be found in [10].

THEOREM 4.3 (Gritsenko '10). *Let $N \in \{1,\dots,8\}$. There exists a modular form $G^{D_N}_{12-N} \in \mathcal{M}_{12-N}(\Gamma,\chi)$ where*

$$\Gamma = \begin{cases} \mathrm{O}(L_2(D_N))^+ & \text{if } N \neq 4, \\ \langle \widetilde{\mathrm{O}}(L_2(D_4))^+, \sigma_{\varepsilon_1}\rangle & \text{if } N = 4. \end{cases}$$

In the last case $\Gamma \leq \mathrm{O}(L_2(D_4))^+$ is a subgroup of index three and the character is defined as

$$\chi : \Gamma \to \Gamma/\widetilde{\mathrm{O}}(L_2(D_N))^+ \xrightarrow{\text{sign}} \{\pm 1\}.$$

The divisor of $G^{D_N}_{12-N}$ is

$$\sum_{\substack{\pm r \in L_2(D_N)\\ (r,r)=-4\\ \operatorname{div}(r)=2}} \mathcal{D}_r(L_2(D_N)) = \widetilde{\mathrm{O}}(L_2(D_N))^+.\mathcal{D}_{\varepsilon_1}(L_2(D_N))$$

if $N \neq 4$, and for D_4 it is the orbit

$$\widetilde{\mathrm{O}}(L_2(D_N))^+.\mathcal{D}_{\varepsilon_1}(L_2(D_N)).$$

If $N \neq 8$ then $G^{D_N}_{12-N}$ is a cusp form.

The proof uses Theorem 4.1. The case $N = 4$ plays a particular role in this tower because the maximal modular group and the divisor of $G^{D_4}_8$ are smaller.

In the next theorem we use theta type Jacobi forms to construct a modular form of weight 24 which can be considered as the correct replacement for $G_8^{D_4}$.

Theorem 4.4. *There is a cusp form $\Delta_{24}^{D_4} \in \mathcal{S}_{24}(\mathrm{O}(L_2(D_4))^+, \chi)$ with a binary character*

$$\chi : \Gamma \to \Gamma/\widetilde{\mathrm{O}}(L_2(D_4))^+ \xrightarrow{\text{sign}} \{\pm 1\} \quad \textit{where} \quad \Gamma = \mathrm{O}(L_2(D_4))^+.$$

Its divisor is equal to

$$\sum_{\substack{\pm r \in L_2(D_4) \\ (r,r)=-4 \\ \mathrm{div}(r)=2}} \mathcal{D}_r(L_2(D_4)).$$

This function first appeared in [8] and later in [16] where Krieg determined the graded ring of modular forms for the Hurwitz order.

Proof of Theorem 4.4. For the proof we will first verify that

$$\text{A-Lift}(\eta(\tau)^{12}\vartheta_{D_4}),\ \text{A-Lift}(\eta(\tau)^{12}\vartheta_{D_4}^{(2)}),\ \text{A-Lift}(\eta(\tau)^{12}\vartheta_{D_4}^{(3)})$$

are multiplicative liftings as in Theorem 4.1. The discriminant group of D_4 has the four classes

$$D_4,\ \varepsilon_1 + D_4,\ v + D_4,\ v + \varepsilon_1 + D_4$$

where v was defined in the diagram above. The group $\mathrm{O}(L_2(D_4))^+$ is generated by $\widetilde{\mathrm{O}}(L_2(D_4))^+$ and the natural embeddings of σ_{ε_1} and σ_v into $\mathrm{O}(L_2(D_4))^+$.

We start by investigating $\psi_{8,D_4}(\tau,\mathfrak{z}) = \eta(\tau)^{12}\vartheta_{D_4}(\tau,\mathfrak{z})$. From the Fourier–Jacobi criterion for automorphic products [9, Corollary 3.3] we deduce that

$$\varphi_{0,D_4} = -\frac{2^{-1}\psi_{8,D_4}|\mathrm{T}_-(2)}{\psi_{8,D_4}}$$

if $\text{A-Lift}(\psi_{8,D_4}) = F_{\varphi_{0,D_4}}$ where $\mathrm{T}_-(2)$ is the Hecke operator defined in [11]. A direct computation yields

$$\varphi_{0,D_4}(\tau,\mathfrak{z}) = 16 + s_1 + s_2 + s_3 + s_4 + s_1^{-1} + s_2^{-1} + s_3^{-1} + s_4^{-1} + q(\dots)$$

where

$$s_j = \exp(2\pi i(\mathfrak{z}, \varepsilon_j)).$$

Hence φ_{0,D_4} is a weak Jacobi form which satisfies the assumptions of Theorem 4.1. If $L = D_N$, the hyperbolic norm in the Fourier exansion of a weak Jacobi form is bounded from below by the quantity

$$-\max_{l \in L^\vee/L}\Big(\min_{v \in l+L}(v,v)\Big) = -N/4.$$

Now assume there is a pair $(n,l) \in \mathbb{N} \times L^\vee$ such that $2n - (l,l) < 0$. Then

$$2n - (l,l) \in \{-N/4, -1\}.$$

Since the lattice $D_N^\vee$ represents all such values under $-(\cdot,\cdot)_{D_N}$, we know that all the information about the divisor is completely contained in the q^0 part of φ_{0,D_4}, because the Fourier coefficients of a Jacobi form depend only on the hyperbolic norm and the class of l. Now Theorem 4.1 and Proposition 4.2 imply that the divisor of $F_{\varphi_{0,D_4}}$ is exactly

$$\widetilde{\mathrm{O}}(L_2(D_4))^+.\mathcal{D}_{\varepsilon_1}(L_2(D_4)).$$

By construction this divisor is contained in the divisor of $\text{A-Lift}(\eta(\tau)^{12}\vartheta_{D_4})$. By Koecher's principle we can divide the last function by $F_{\varphi_{0,D_4}}$ and conclude that the last two functions coincide up to a multiple in $\mathbb{C}^\times$, and a second application of the Fourier–Jacobi criterion for automorphic products shows that this constant is actually 1, in accordance with Theorem 4.3. So far we have just repeated the proof of Theorem 4.3 for $N=4$. Now the same construction can be applied to $\text{A-Lift}(\eta(\tau)^{12}\vartheta_{D_4}^{(2)})$ and $\text{A-Lift}(\eta(\tau)^{12}\vartheta_{D_4}^{(3)})$, which yields again modular forms of weight 8 where the divisor is determined by the orbits of v and ε_1+v, respectively. We consider the product

$$\psi(\tau,\mathfrak{z}) := \eta(\tau)^{36}\vartheta_{D_4}(\tau,\mathfrak{z})\vartheta_{D_4}^{(2)}(\tau,\mathfrak{z})\vartheta_{D_4}^{(3)}(\tau,\mathfrak{z}).$$

Then ψ has the representation

$$\psi(\tau,\mathfrak{z}) = \eta(\tau)^{36}\cdot\prod_{j=1}^{4}\vartheta(\tau,(\mathfrak{z},\varepsilon_j))\vartheta(\tau,(\mathfrak{z},\sigma_v(\varepsilon_j)))\vartheta(\tau,(\mathfrak{z},\sigma_{\varepsilon_1}\sigma_v\sigma_{\varepsilon_1}(\varepsilon_j))).$$

From this representation and the above representation in coordinates we deduce

$$\psi|_{24,3}\sigma_{\varepsilon_1} = -\psi, \qquad \psi|_{24,3}\sigma_v = -\psi.$$

Since these two elements generate the group of the graph automorphisms of the Coxeter diagram of D_4 which is isomorphic to the symmetric group $\mathcal{S}_3$ on three letters, we see that ψ is $\mathcal{S}_3$-modular with respect to the sign character. If we define $\Delta_{24}^{D_4}$ as the product

$$\text{A-Lift}(\eta(\tau)^{12}\vartheta_{D_4})\,\text{A-Lift}(\eta(\tau)^{12}\vartheta_{D_4}^{(2)})\,\text{A-Lift}(\eta(\tau)^{12}\vartheta_{D_4}^{(3)}),$$

the definition of A-Lift yields a cusp form whose divisor is the composition of the divisors of the factors, which is $\mathrm{O}(L_2(D_4))^+$-modular with respect to the binary character induced by the sign character. ∎

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