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Regularity of global attractor for a 3D magnetohydrodynamic-α model

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Abstract. We consider a 3D magnetohydrodynamic-α (MHD-α for short) model with periodic boundary conditions. Under suitable assumptions on the external force, we prove both Sobolev regularity and Gevrey regularity of the compact global attractor for the continuous semigroup associated to this model.

1. Introduction. Let $\Omega = (0,L)^3$, $L > 0$, be a periodic box in $\mathbb{R}^3$. We consider the following 3D MHD-α model introduced by Linshiz and Titi [LT]:

$$\begin{aligned} \partial_t v - \nu\Delta v - u \times (\nabla \times v) - (B\cdot\nabla)B \qquad& \\ + \nabla p + \tfrac{1}{2}\nabla|B|^2 = f \quad& \text{in } \Omega\times(0,\infty), \end{aligned} \tag{1.1}$$

$$\partial_t B - \eta\Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u = 0 \quad \text{in } \Omega\times(0,\infty), \tag{1.2}$$

$$v = u - \alpha^2\Delta u \quad \text{in } \Omega\times(0,\infty), \tag{1.3}$$

$$\nabla\cdot u = \nabla\cdot v = \nabla\cdot B = 0 \quad \text{in } \Omega\times(0,\infty), \tag{1.4}$$

subject to the periodic boundary conditions

$$u(0,t) = u(L,t), \qquad B(0,t) = B(L,t), \qquad t > 0, \tag{1.5}$$

and the initial conditions

$$u(x,0) = u^0(x), \qquad B(x,0) = B^0(x), \qquad x \in \Omega. \tag{1.6}$$

Here $u = u(x,t)$ is the unknown velocity, $B = B(x,t)$ is the unknown magnetic field and $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic vis-

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cosity coefficient, $\eta > 0$ is the constant magnetic diffusivity and α is a length scale parameter. When $\alpha = 0$, we formally recover the 3D classical MHD equations of [ST]. Notice that here we only filter the velocity field but not the magnetic field, which contrasts with the so-called Lagrangian-averaged magnetohydrodynamic-α (LAMHD-α) model (also called hyperbolic MHD equations or MHD-Voigt model) of [H].

The MHD-α model (1.1)–(1.4) involves coupling Maxwell's equations governing the magnetic field and the Navier–Stokes-α equations (sometimes called the viscous Camassa–Holm equations). In recent years, the existence and long-time behavior of solutions to this MHD-α model has attracted the attention of many mathematicians. Linshiz and Titi [LT] proved the existence, uniqueness and regularity of solutions with periodic boundary conditions, while Fan and Ozawa [FO] and Liu [Lj] achieved the same result in the whole space $\mathbb{R}^2$ for both cases ($\nu = 1$, $\eta = 0$) and ($\nu = 0$, $\eta = 1$). More recently, Zhou and Fan [ZF1] established regularity criteria to guarantee the existence of smooth solutions for the higher-dimensional case. For the long-time behavior of solutions, the existence of a finite-dimensional global attractor was proved by Catania [C2] in the case of a three-dimensional periodic box, and the time decay rate in $L^2(\mathbb{R}^3)$ of solutions was proved by Jiang and Fan [JF]. When $B = 0$, the above MHD-α model reduces to the well-known Navier–Stokes-α equations, where the existence and some properties of a finite-dimensional global attractor were studied for both periodic and Dirichlet boundary conditions in [FHT, IT, CPS], and the decay rate of solutions on the whole space was proved by Bjorland and Schonbek [BS]. We also refer the interested reader to [C1, CS, HLT, KY, LZZ, ZF2] for results related to other MHD-α models.

In recent years, the Gevrey regularity of solutions and of global attractors for equations in fluid mechanics have been studied by several authors (see for instance [AT, FT, HLT, KLT, K, KV, LO, Lx, PV, PW, S, ZL, YL, YLH]). The Gevrey class regularity of the attractor reveals that the solutions lying in the attractor are analytic with values in a Gevrey class of analytic functions in space. Kalantarov, Levant and Titi [KLT] proved the Gevrey regularity for the attractor of the 3D Navier–Stokes–Voigt equations. The method of proof in [KLT] is the splitting of the velocity into higher and lower Fourier components. Using this method, Zhao and Li [ZL] recently proved the analyticity of the global attractor for 3D regularized magnetohydrodynamics equations, which are regularizations in both the velocity and the magnetic field. Thanks to a nice abstract result of Friz and Robinson [FR], one can use the real analyticity of solutions on the attractor to show that elements of the attractor can be distinguished by their values at a finite number of points in the domain.

In this paper, we study both the Sobolev regularity and the Gevrey regularity of the global attractor for the 3D MHD-α model (1.1)–(1.4). The existence and fractal dimension estimates of the attractor to this model were investigated before by Catania [C2]. Here the Sobolev regularity is proved by following the general lines of the approach used in [R] for 2D Navier–Stokes equations. And the Gevrey regularity is investigated by using some ideas of [FT, R]. Using the results obtained, we show that the attractor of this 3D MHD-α model can be parametrized for large enough k by k nodal values, for almost every choice of k points in Ω.

We can think about the α models as a numerical regularization of the underlying equation, which makes the nonlinearity milder, and hence the solutions of the modified equation are smoother. There are some ways of regularization of the classical MHD equations. In [H], both the velocity field and the magnetic field are filtered. Filtering the magnetic field $B = (1-\alpha_M^2\Delta)B_s$ is equivalent to introducing hyperdiffusivity for the filtered magnetic field B, due to the term $-\eta\alpha_M^2\Delta^2 B_s$. However, as pointed out in [LT], this seems to be unnecessary from the numerical analysis/well-posedness point of view. This is the reason why we choose the MHD-α model (1.1)–(1.4) (the so-called Navier–Stokes-α-MHD model), where only the velocity field is filtered. By using similar arguments, we can also study the regularity of the global attractor for the Modified Leray-α-MHD model, which was obtained by Catania [C2].

The paper is organized as follows. In Section 2, for the convenience of the reader, we recall the functional setting of the 3D MHD-α equations. In Section 3, we prove the Sobolev regularity of the global attractor of the 3D MHD-α equations. The Gevrey regularity of the attractor is proved in Section 4.

2. Functional setting and preliminaries. Let us denote by $\mathcal{V}$ the set of all vector valued trigonometric polynomials defined in Ω such that $\nabla\cdot u = 0$ and $\int_\Omega u(x)\,dx = 0$. Denote by H and V the closures of $\mathcal{V}$ in $L^2(\Omega)^3$ and in $H^1(\Omega)^3$, respectively. Let $(\cdot,\cdot)$ and $|\cdot|$ be the inner product and the norm in H, and $((\cdot,\cdot)) = (\nabla\cdot,\nabla\cdot)$ and $\|\cdot\| = |\nabla\cdot|$ the inner product and the norm in V.

Let P be the Helmholtz–Leray orthogonal projection in $L^2(\Omega)^3$ onto H. Following the notation for the MHD-α equations, we set

$$\mathcal{B}(u,v) = P(u\cdot\nabla)v \quad \text{and} \quad \widetilde{\mathcal{B}}(u,v) := -P(u\times(\nabla\times v)), \qquad \forall u,v\in V.$$

Using the identity

$$(b\cdot\nabla)a + \sum_{j=1}^{3} a_j\nabla b_j = -b\times(\nabla\times a) + \nabla(a\cdot b),$$

one can easily show that

$$\langle \widetilde{\mathcal{B}}(u,v), w\rangle_{V',V} = \langle \mathcal{B}(u,v), w\rangle_{V',V} - \langle \mathcal{B}(w,v), u\rangle_{V',V}. \tag{2.1}$$

For a Banach space X, we write $\langle \cdot,\cdot\rangle_{X',X}$ for the pairing between X and its dual space X'.

We denote by $A = -P\Delta$ the Stokes operator with domain $D(A) = H^2(\Omega)^3 \cap V$. Notice that for periodic boundary conditions, $A = -\Delta|_{D(A)}$ is a self-adjoint positive operator with compact inverse. Hence there exists a complete orthogonal set $\{w_j\}_{j=1}^{\infty} \subset H$ of eigenfunctions such that $Aw_j = \lambda_j w_j$ with $(2\pi/L)^2 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \sim j^{2/3}L^{-2} \leq \cdots$.

For any $s \in \mathbb{R}$, we can define the Hilbert space $V_s := D(A^{s/2})$ with the inner product and norm

$$(u,v)_s = \sum_{j\in\mathbb{Z}^3} u_j \cdot v_j |j|^{2s}, \qquad \|u\|_s^2 = (u,u)_s,$$

for all $u, v \in V_s$, where u_j, v_j are the Fourier coefficients of u and v, respectively. One can see that $H = V_0$ and $V = V_1$.

We have the following Poincaré type inequalities for any $m, j \in \mathbb{N}_+$:

$$\|u\|^2 \geq \lambda_1 |u|^2 \quad \text{for all } u \in V, \tag{2.2}$$

$$\|u\|_m^2 \geq \lambda_1^j \|u\|_{m-j}^2 \quad \text{for all } u \in V_m. \tag{2.3}$$

From the definition of $\mathcal{B}$ we see that

$$\langle \mathcal{B}(u,v), w\rangle_{V',V} = -\langle \mathcal{B}(u,w), v\rangle_{V',V} \quad \text{for all } u,v,w \in V, \tag{2.4}$$

and in particular

$$\langle \mathcal{B}(u,v), v\rangle_{V',V} = 0 \quad \text{for all } u,v \in V. \tag{2.5}$$

Since (2.1), we also have

$$\langle \widetilde{\mathcal{B}}(u,v), u\rangle_{V',V} = 0 \quad \text{for all } u,v \in V. \tag{2.6}$$

We apply the projection P to (1.1)–(1.6) to obtain the equivalent system of equations

$$\frac{dv}{dt} + \nu A v + \widetilde{\mathcal{B}}(u,v) - \mathcal{B}(B,B) = Pf, \tag{2.7}$$

$$v = u + \alpha^2 A u, \tag{2.8}$$

$$\frac{dB}{dt} + \eta A B + \mathcal{B}(u,B) - \mathcal{B}(B,u) = 0, \tag{2.9}$$

with the initial conditions

$$u(0) = u^0, \qquad B(0) = B^0. \tag{2.10}$$

The existence and uniqueness of solutions to problem (1.1)–(1.6) was proved in [LT]. Moreover, the following result on the existence of a finite-dimensional global attractor for this system was proved by Catania [C2].

THEOREM 2.1. *For any* $(u^0, B^0) \in V \times H$ *and* $f \in H$, *there is a (unique) compact global attractor* $\mathcal{A} \subset V \times H$ *of solutions* (u, B) *to* (1.1)–(1.6). *Moreover, we have an upper bound for the Hausdorff dimension* $d_H(\mathcal{A})$ *and the fractal dimension* $d_F(\mathcal{A})$ *of* $\mathcal{A}$: *there is a positive constant* C *such that*

$$d_H(\mathcal{A}) \le d_F(\mathcal{A}) \le CG^{6/5}\left(\frac{L}{\alpha}\right)^3 \left[\left(\frac{L}{\alpha}\right)^{3/5} + G^{6/5}\left(\frac{L}{\alpha}\right)^{9/5} + G^{3/10}\right],$$

where $G = L^{3/2}|f|/\mu^2$ *with* $\mu = \min\{\nu, \eta\}$.

The purpose of this paper is to study the regularity of the global attractor $\mathcal{A}$. First, we show that $\mathcal{A}$ is bounded in $V_{M+1} \times V_M$, for $M \in \mathbb{N}^*$, when the force f belongs to V_{M-1}. Then we prove that if f is real analytic then the functions in $\mathcal{A}$ are all real analytic, in a uniform way.

3. Sobolev regularity of the global attractor. We first prove some estimates on the nonlinear term $\mathcal{B}$.

LEMMA 3.1. *For any integer* $m \ge 1$, *there exist positive constants* c_i, $i = 1, \dots, 8$, *such that:*

$$|(\mathcal{B}(u,v), A^m w)| \le c_1 \lambda_1^{-(m-1)/2} \|u\|_m \|v\|_m^{1/2} \|v\|_{m+1}^{1/2} \|w\|_{m+1} \quad \forall u \in V_m,\ v \in V_{m+1},\ w \in V_{m+1}; \tag{3.1}$$

$$|(\mathcal{B}(A^m u,v), w)| \le c_2 \lambda_1^{-(m-1)/2} \|u\|_{m+1} \|v\|_m^{1/2} \|v\|_{m+1}^{1/2} \|w\|_m \quad \forall u \in V_{m+1},\ v \in V_{m+1},\ w \in V_m; \tag{3.2}$$

$$|(\mathcal{B}(u,v), A^m w)| \le c_3 \lambda^{-(m-1)/2} \|u\|_m \|v\|_m \|w\|_{m+1}^{1/2} \|w\|_{m+2}^{1/2} \quad \forall u \in V_m,\ v \in V_m,\ w \in V_{m+2}; \tag{3.3}$$

$$|(\mathcal{B}(A^m u,v), w)| \le c_4 \lambda^{-(m-1)/2} \|u\|_{m+1}^{1/2} \|u\|_{m+2}^{1/2} \|v\|_m \|w\|_m \quad \forall u \in V_{m+2},\ v \in V_m,\ w \in V_m; \tag{3.4}$$

$$|(\mathcal{B}(u,v), A^m w)| \le c_5 \lambda^{-(m-1)/2} \|u\|_{m-1} \|v\|_m^{1/2} \|v\|_{m+1}^{1/2} \|w\|_{m+2} \quad \forall u \in V_{m-1},\ v \in V_{m+1},\ w \in V_{m+2}; \tag{3.5}$$

$$|(\mathcal{B}(A^m u,v), w)| \le c_6 \lambda^{-(m-1)/2} \|u\|_{m+2} \|v\|_m^{1/2} \|v\|_{m+1}^{1/2} \|w\|_{m-1} \quad \forall u \in V_{m+2},\ v \in V_{m+1},\ w \in V_{m-1}; \tag{3.6}$$

$$|(\mathcal{B}(u,v), A^m w)| \le c_7 \lambda_1^{-(m-1)/2} \|u\|_m^{1/2} \|u\|_{m+1}^{1/2} \|v\|_m \|w\|_{m+1} \quad \forall u \in V_{m+1},\ v \in V_m,\ w \in V_{m+1}; \tag{3.7}$$

$$|(\mathcal{B}(A^m u,v), w)| \le c_8 \lambda_1^{-(m-1)/2} \|u\|_{m+1} \|v\|_m \|w\|_m^{1/2} \|w\|_{m+1}^{1/2} \quad \forall u \in V_{m+1},\ v \in V_m,\ w \in V_{m+1}. \tag{3.8}$$

Proof. To prove (3.1), using the Hölder inequality, the inequality $\|\varphi\|_{L^3(\Omega)} \le c\|\varphi\|^{1/2}_{L^2(\Omega)}\|\nabla\varphi\|^{1/2}_{L^2(\Omega)}$ in dimension three, the embedding $H^1(\Omega) \subset L^6(\Omega)$ and the Poincaré inequality (2.3), we obtain

$$\begin{aligned}(\mathcal{B}(u,v), A^m w) &= \int_\Omega (u\cdot\nabla)v\cdot A^m w\,dx = \int_\Omega \nabla^{m-1}((u\cdot\nabla)v)\cdot\nabla^{m+1}w\,dx\\
&\le \int_\Omega \sum_{j=0}^{m-1} |\nabla^j u|\,|\nabla^{m-j}v|\,|\nabla^{m+1}w|\,dx\\
&\le \sum_{j=0}^{m-1} \|\nabla^j u\|_{L^6(\Omega)}\|\nabla^{m-j}v\|_{L^3(\Omega)}\|\nabla^{m+1}w\|_{L^2(\Omega)}\\
&\le c_1\sum_{j=0}^{m-1} \|\nabla^{j+1}u\|_{L^2(\Omega)}\|\nabla^{m-j}v\|^{1/2}_{L^2(\Omega)}\|\nabla^{m+1-j}v\|^{1/2}_{L^2(\Omega)}\|\nabla^{m+1}w\|_{L^2(\Omega)}\\
&\le c_1\lambda_1^{-(m-1)/2}\|u\|_m\|v\|_m^{1/2}\|v\|_{m+1}^{1/2}\|w\|_{m+1}.\end{aligned}$$

For (3.2), as in the above proof, we have

$$\begin{aligned}(\mathcal{B}(A^m u,v), w) &= \int_\Omega (A^m u\cdot\nabla)v\cdot w\,dx = \int_\Omega \nabla^{m+1}u\cdot\nabla^{m-1}(\nabla v\cdot w)\,dx\\
&\le \int_\Omega \sum_{j=0}^{m-1} |\nabla^{m+1}u|\,|\nabla^{j+1}v|\,|\nabla^{m-1-j}w|\,dx\\
&\le \sum_{j=0}^{m-1} \|\nabla^{m+1}u\|_{L^2(\Omega)}\|\nabla^{j+1}v\|_{L^3(\Omega)}\|\nabla^{m-1-j}w\|_{L^6(\Omega)}\\
&\le c_2\sum_{j=0}^{m-1} \|\nabla^{m+1}u\|_{L^2(\Omega)}\|\nabla^{j+1}v\|^{1/2}_{L^2(\Omega)}\|\nabla^{j+2}v\|^{1/2}_{L^2(\Omega)}\|\nabla^{m-j}w\|_{L^2(\Omega)}\\
&\le c_2\lambda_1^{-(m-1)/2}\|u\|_{m+1}\|v\|_m^{1/2}\|v\|_{m+1}^{1/2}\|w\|_m.\end{aligned}$$

In a similar way, one can get (3.3)–(3.8). ■

THEOREM 3.2. *Let $M \in \mathbb{N}^*$. If $f \in V_{M-1}$ then the attractor $\mathcal{A}$ of solutions (u,B) to* (1.1)–(1.6) *is bounded in $V_{M+1} \times V_M$.*

Proof. Step 1. We first prove that when $(u^0, B^0) \in V_{m+1} \times V_m$ and $f \in V_{m-1}$ for any $m \ge 0$, then the corresponding solution (u,B) belongs to $L^\infty(0,T;V_{m+1}) \times L^\infty(0,T;V_m)$ for all $T > 0$. Moreover, there exists a positive constant ρ_m depending on $\|f\|_{m-1}$, $\|u^0\|_{m+1}$ and $\|B^0\|_m$ such that for all $t \in [0,T]$,

$$\|u(t)\|_m^2 + \alpha^2\|u(t)\|_{m+1}^2 + \|B(t)\|_m^2 + \int_0^t (\|u(s)\|_{m+1}^2 + \alpha^2\|u(s)\|_{m+2}^2 + \|B(s)\|_{m+1}^2)\,ds \le \rho_m. \tag{3.9}$$

We will prove (3.9) by induction on $m \in \mathbb{N}$.

For $m = 0$, multiplying (2.7) by u and (2.9) by B, then integrating over Ω and using (2.4)–(2.6), we obtain

$$\frac{1}{2}\frac{d}{dt}(|u|^2 + \alpha^2\|u\|^2 + |B|^2) + \nu(\|u\|^2 + \alpha^2|Au|^2) + \eta\|B\|^2 = (f,u).$$

By the Cauchy inequality, we have

$$|(f,u)| \le \begin{cases} \dfrac{1}{2\nu}|A^{-1/2}f|^2 + \dfrac{\nu}{2}\|u\|^2, \\ \dfrac{1}{2\nu\alpha^2}|A^{-1}f|^2 + \dfrac{\nu\alpha^2}{2}|Au|^2. \end{cases}$$

Therefore,

$$\frac{d}{dt}(|u|^2 + \alpha^2\|u\|^2 + |B|^2) + \nu(\|u\|^2 + \alpha^2|Au|^2) + \eta\|B\|^2 \le K_{-1},$$

where

$$K_{-1} = \min\left\{\frac{1}{\nu}\|f\|_{V_{-1}}^2, \frac{1}{\nu\alpha^2}\|f\|_{V_{-2}}^2\right\}.$$

Hence, from the Poincaré type inequalities (2.2)–(2.3), we deduce that

$$\frac{d}{dt}(|u|^2 + \alpha^2\|u\|^2 + |B|^2) + \mu\lambda_1(|u|^2 + \alpha^2\|u\|^2 + |B|^2) \le K_{-1}, \tag{3.10}$$

where $\mu = \min\{\nu,\eta\}$. Integrating (3.10) from 0 to t, we get (3.9) with $m = 0$.

Now, assume that (3.9) holds for some $m \ge 0$. Taking the inner product of (2.7) by $A^{m+1}u$, (2.9) by $A^{m+1}B$, and then integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{m+1}^2 + \alpha^2\|u\|_{m+2}^2) + \nu(\|u\|_{m+2}^2 + \alpha^2\|u\|_{m+3}^2) = -(\widetilde{\mathcal{B}}(u,v), A^{m+1}u) + (\mathcal{B}(B,B), A^{m+1}u) + (f, A^{m+1}u)$$

and

$$\frac{1}{2}\frac{d}{dt}\|B\|_{m+1}^2 + \eta\|B\|_{m+2}^2 = -(\mathcal{B}(u,B), A^{m+1}B) + (\mathcal{B}(B,u)A^{m+1}B).$$

Summing up these two equalities gives

$$\tag{3.11} \begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u\|_{m+1}^2 + \alpha^2\|u\|_{m+2}^2 + \|B\|_{m+1}^2) \\ &\quad + \nu(\|u\|_{m+2}^2 + \alpha^2\|u\|_{m+3}^2) + \eta\|B\|_{m+2}^2 \\ &= -(\widetilde{\mathcal{B}}(u,v), A^{m+1}u) + (\mathcal{B}(B,B), A^{m+1}u) \\ &\quad - (\mathcal{B}(u,B), A^{m+1}B) + (\mathcal{B}(B,u)A^{m+1}B) + (f, A^{m+1}u). \end{aligned}$$

We first use the Cauchy inequality to get

$$|(f, A^{m+1}u)| \leq \begin{cases} \|f\|_m\|u\|_{m+2}, \\ \|f\|_{m-1}\|u\|_{m+3} \end{cases} \leq \begin{cases} \dfrac{2\|f\|_m^2}{\nu} + \dfrac{\nu}{8}\|u\|_{m+2}^2, \\ \dfrac{2\|f\|_{m-1}^2}{\nu\alpha^2} + \dfrac{\nu\alpha^2}{8}\|u\|_{m+3}^2. \end{cases}$$

So

$$\tag{3.12} |(f, A^{m+1}u)| \leq \frac{\nu}{8}(\|u\|_{m+2}^2 + \alpha^2\|u\|_{m+3}^2) + K_m,$$

where

$$K_m = \min\left\{\frac{2\|f\|_m^2}{\nu}, \frac{2\|f\|_{m-1}^2}{\nu\alpha^2}\right\}.$$

Now, by using (2.1) with $v = u + \alpha^2 Au$ and by (2.5) we get

$$\tag{3.13} \begin{aligned} (\widetilde{\mathcal{B}}(u,v), A^{m+1}u) &= (\mathcal{B}(u,u), A^{m+1}u) + \alpha^2(\mathcal{B}(u,Au), A^{m+1}u) \\ &\quad - \alpha^2(\mathcal{B}(A^{m+1}u, Au), u). \end{aligned}$$

From (3.1) and the Young inequality, we deduce that

$$\tag{3.14} \begin{aligned} (\mathcal{B}(u,u), A^{m+1}u) &\leq c_1\lambda_1^{-m/2}\|u\|_{m+1}^{3/2}\|u\|_{m+2}^{3/2} \\ &\leq \frac{54c_1^4}{\nu^3\lambda_1^{2m}}\|u\|_{m+1}^6 + \frac{\nu}{8}\|u\|_{m+2}^2. \end{aligned}$$

Using (3.3), (3.4) and the Young inequality, we have

$$\tag{3.15} \begin{aligned} &\alpha^2(\mathcal{B}(u,Au), A^{m+1}u) - \alpha^2(\mathcal{B}(A^{m+1}u, Au), u) \\ &\quad \leq (c_3 + c_4)\alpha^2\lambda_1^{-m/2}\|u\|_{m+1}\|u\|_{m+2}^{1/2}\|u\|_{m+3}^{3/2} \\ &\quad \leq \frac{54(c_3+c_4)^4\alpha^2}{\nu^3\lambda_1^{2m}}\|u\|_{m+1}^4\|u\|_{m+2}^2 + \frac{\nu\alpha^2}{8}\|u\|_{m+3}^2. \end{aligned}$$

Combining (3.13)–(3.15), we get

$$(3.16)\qquad (\widetilde{\mathcal{B}}(u,v), A^{m+1}u) \le \frac{54(c_1^4+(c_3+c_4)^4)}{\nu^3\lambda_1^{2m}}(\|u\|_{m+1}^6+\alpha^2\|u\|_{m+1}^4\|u\|_{m+2}^2) + \frac{\nu}{8}(\|u\|_{m+2}^2+\alpha^2\|u\|_{m+3}^2).$$

Now, (3.5) and the Cauchy inequality yield

$$(3.17)\qquad |(\mathcal{B}(B,B), A^{m+1}u)| \le c_5\lambda_1^{-m/2}\|B\|_m\|B\|_{m+1}^{1/2}\|B\|_{m+2}^{1/2}\|u\|_{m+3} \le \frac{c_5^4}{\eta\nu^2\alpha^4\lambda_1^{2m}}\|B\|_m^4\|B\|_{m+1}^2+\frac{\eta}{4}\|B\|_{m+2}^2+\frac{\nu\alpha^2}{4}\|u\|_{m+3}^2.$$

Using (3.1) and (3.4) and the Young inequality, we have

$$(3.18)\qquad -(\mathcal{B}(u,B), A^{m+1}B)+(\mathcal{B}(B,u), A^{m+1}B) \le (c_1+c_4)\lambda_1^{-m/2}\|u\|_{m+1}\|B\|_{m+1}^{1/2}\|B\|_{m+2}^{3/2} \le \frac{27(c_1+c_4)^4}{4\nu^3\lambda_1^{2m}}\|u\|_{m+1}^4\|B\|_{m+1}^2+\frac{\eta}{4}\|B\|_{m+2}^2.$$

Substituting (3.12) and (3.16)–(3.18) into (3.11), we deduce that

$$(3.19)\qquad \frac{d}{dt}(\|u\|_{m+1}^2+\alpha^2\|u\|_{m+2}^2+\|B\|_{m+1}^2)+\mu(\|u\|_{m+2}^2+\alpha^2\|u\|_{m+3}^2+\|B\|_{m+2}^2) \le \frac{\ell}{\nu^2\mu\lambda_1^{2m}}\Big(\|u\|_{m+1}^6+\alpha^2\|u\|_{m+1}^4\|u\|_{m+2}^2+\|u\|_{m+1}^4\|B\|_{m+1}^2 + \frac{1}{\alpha^4}\|B\|_m^4\|B\|_{m+1}^2\Big)+K_m,$$

where

$$\ell = 54\big(c_1^4+(c_3+c_4)^4+c_5^4+(c_1+c_4)^4/8\big).$$

Using the inductive assumption that (3.9) holds for m, we have

$$\|u\|_{m+1}^6+\alpha^2\|u\|_{m+1}^4\|u\|_{m+2}^2+\|u\|_{m+1}^4\|B\|_{m+1}^2+\frac{1}{\alpha^4}\|B\|_m^4\|B\|_{m+1}^2 \le \|u\|_{m+1}^4(\|u\|_{m+1}^2+\alpha^2\|u\|_{m+2}^2)+\frac{1}{\alpha^4}\|B\|_m^4\|B\|_{m+1}^2 \le \frac{2\rho_m^2}{\alpha^4}(\|u\|_{m+1}^2+\alpha^2\|u\|_{m+2}^2+\|B\|_{m+1}^2).$$

Hence (3.19) becomes

$$\frac{d}{dt}(\|u\|_{m+1}^2+\alpha^2\|u\|_{m+2}^2+\|B\|_{m+1}^2)+\mu(\|u\|_{m+2}^2+\alpha^2\|u\|_{m+3}^2+\|B\|_{m+2}^2) \leq \frac{2\ell\rho_m^2}{\alpha^4\nu^2\mu\lambda_1^{2m}}(\|u\|_{m+1}^2+\alpha^2\|u\|_{m+2}^2+\|B\|_{m+1}^2)+K_m.$$

Integrating this inequality from 0 to t, we get (3.9) with m replaced by $m+1$.

Step 2. We now suppose as an inductive hypothesis that for any (u^0,B^0) in $\mathcal{A}$,

$$\|u^0\|_m^2+\alpha^2\|u^0\|_{m+1}^2+\|B^0\|_m^2 \leq I_m, \tag{3.20}$$

$$\int_0^1(\|u(t)\|_{m+1}^2+\alpha^2\|u(t)\|_{m+2}^2+\|B(t)\|_{m+1}^2)\,dt \leq R_{m+1}. \tag{3.21}$$

Since the attractor is bounded in $V_2\times V_1$, (3.20) holds for $m=1$. Moreover, using (3.9) with $m=1$, we deduce (3.21) for $m=1$.

We now show that while $m\leq M$, (3.20) and (3.21) hold with m replaced by $m+1$.

For any $(u,B)\in\mathcal{A}$, since $\mathcal{A}$ is invariant we have $(u,B)=S(1)(u^0,B^0)$ for some $(u^0,B^0)\in\mathcal{A}$, where $\{S(t)\}_{t\geq 0}$ is the semigroup associated to the system (1.1)–(1.4). It follows from (3.21) that there exists $t_0\in[0,1]$ such that

$$\|u(t_0)\|_{m+1}^2+\alpha^2\|u(t_0)\|_{m+2}^2+\|B(t_0)\|_{m+1}^2 \leq R_{m+1}. \tag{3.22}$$

We now consider the solution starting at (u^0,B^0) and note that $(u,B)=S(1-t_0)(u^0,B^0) =: (u(1-t_0),B(1-t_0))$. As in (3.19), we have

$$\begin{aligned}\frac{d}{dt}&(\|u\|_{m+1}^2+\alpha^2\|u\|_{m+2}^2+\|B\|_{m+1}^2)+\mu(\|u\|_{m+2}^2+\alpha^2\|u\|_{m+3}^2+\|B\|_{m+2}^2)\\ &\leq \frac{\ell}{\nu^2\mu\lambda_1^{2m}}\Big(\|u\|_{m+1}^6+\alpha^2\|u\|_{m+1}^4\|u\|_{m+2}^2\\ &\qquad+\|u\|_{m+1}^4\|B\|_{m+1}^2+\frac{1}{\alpha^4}\|B\|_m^4\|B\|_{m+1}^2\Big)+K_m.\end{aligned} \tag{3.23}$$

Since $(u^0,B^0)\in V_{m+1}\times V_m$ and $f\in V_{m-1}$, from (3.9) we have

$$\|u\|_{m+1}^4\leq\frac{\rho_m^2}{\alpha^4} \quad\text{and}\quad \|B\|_m^4\leq\rho_m^2.$$

So

$$\begin{aligned}
&\|u\|_{m+1}^6 + \alpha^2\|u\|_{m+1}^4\|u\|_{m+2}^2 + \|u\|_{m+1}^4\|B\|_{m+1}^2 + \frac{1}{\alpha^4}\|B\|_m^4\|B\|_{m+1}^2 \\
&\quad \le \|u\|_{m+1}^4(\|u\|_{m+1}^2 + \alpha^2\|u\|_{m+2}^2 + \|B\|_{m+1}^2) + \frac{1}{\alpha^4}\|B\|_m^4\|B\|_{m+1}^2 \\
&\quad \le \frac{2\rho_m^2}{\alpha^4}(\|u\|_{m+1}^2 + \alpha^2\|u\|_{m+2}^2 + \|B\|_{m+1}^2).
\end{aligned}$$

Thus, (3.23) becomes

$$\begin{aligned}
(3.24)\qquad &\frac{d}{dt}(\|u\|_{m+1}^2 + \alpha^2\|u\|_{m+2}^2 + \|B\|_{m+1}^2) \\
&\qquad + \mu(\|u\|_{m+2}^2 + \alpha^2\|u\|_{m+3}^2 + \|B\|_{m+2}^2) \\
&\qquad\qquad \le \frac{2\ell\rho_m^2}{\alpha^4\nu^2\mu\lambda_1^{2m}}(\|u\|_{m+1}^2 + \alpha^2\|u\|_{m+2}^2 + \|B\|_{m+1}^2) + K_m.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{m+1}^2 + \alpha^2\|u\|_{m+2}^2 + \|B\|_{m+1}^2) \\
&\qquad \le \frac{2\ell\rho_m^2}{\alpha^4\nu^2\mu\lambda_1^{2m}}(\|u\|_{m+1}^2 + \alpha^2\|u\|_{m+2}^2 + \|B\|_{m+1}^2) + K_m.
\end{aligned}$$

So, using the Gronwall inequality, we get

$$\begin{aligned}
(3.25)\qquad &\|u\|_{m+1}^2 + \alpha^2\|u\|_{m+2}^2 + \|B\|_{m+1}^2 \\
&\quad = \|u(1-t_0)\|_{m+1}^2 + \alpha^2\|u(1-t_0)\|_{m+2}^2 + \|B(1-t_0)\|_{m+1}^2 \\
&\quad \le \Bigg(\|u(t_0)\|_{m+1}^2 + \alpha^2\|u(t_0)\|_{m+2}^2 + \|B(t_0)\|_{m+1}^2 \\
&\qquad + \frac{\alpha^4\nu^2\mu\lambda_1^{2m}}{2\ell\rho_m^2}K_m\Bigg)\exp\left(\frac{2\ell\rho_m^2}{\alpha^4\nu^2\mu\lambda_1^{2m}}|1-2t_0|\right).
\end{aligned}$$

Using (3.22) we infer from (3.25) that

$$(3.26)\qquad \|u\|_{m+1}^2 + \alpha^2\|u\|_{m+2}^2 + \|B\|_{m+1}^2 \le I_{m+1}$$

with

$$I_{m+1} = \left(R_{m+1} + \frac{\alpha^4\nu^2\mu\lambda_1^{2m}}{2\ell\rho_m^2}K_m\right)\exp\left(\frac{2\ell\rho_m^2}{\alpha^4\nu^2\mu\lambda_1^{2m}}\right).$$

Returning to (3.24) using (3.26), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_m^2 + \alpha^2\|u\|_{m+1}^2 + \|B\|_m^2) + \mu(\|u\|_{m+1}^2 + \alpha^2\|u\|_{m+2}^2 + \|B\|_{m+1}^2) \\
&\qquad \le \frac{2\ell\rho_m^2}{\alpha^4\nu^2\mu\lambda_1^{2m}}I_{m+1} + K_m.
\end{aligned}$$

Integrating this inequality from 0 to 1, starting at $u(0) = u$, $B(0) = B$, we deduce that

$$\begin{aligned}\|u(1)\|_{m+1}^2 + \alpha^2\|u(1)\|_{m+2}^2 + \|B(1)\|_{m+1}^2 \\ + \mu\int_0^1 (\|u(t)\|_{m+2}^2 + \alpha^2\|u(t)\|_{m+3}^2 + \|B(t)\|_{m+2}^2)\,dt \\ \le \left(1 + \frac{2\ell\rho_m^2}{\alpha^4\nu^2\mu\lambda_1^{2m}}\right) I_{m+1} + K_m =: R_{m+2}. \ \blacksquare\end{aligned}$$

4. Gevrey regularity of the global attractor. We will use the following Gevrey class:

$$G_\tau^r := D(A^{r/2}e^{\tau A^{1/2}}) = \Big\{u \in H \;\Big|\; |A^{r/2}e^{\tau A^{1/2}}u| = \sum_{j\in\mathbb{Z}^3} |u_j|^2|j|^{2r}e^{2\tau|j|} < \infty\Big\}.$$

The space G_τ^r has the inner product

$$(u, v)_{r,\tau} = (A^{r/2}e^{\tau A^{1/2}}u, A^{r/2}e^{\tau A^{1/2}}v) = \sum_{j\in\mathbb{Z}^3} u_j \cdot v_j|j|^{2r}e^{2\tau|j|},$$

and the associated norm

$$\|u\|_{r,\tau} = (u, u)_{r,\tau}^{1/2},$$

for $u, v \in G_\tau^r$. As noticed in [LO], the space $C^\omega(\Omega)$ of real analytic functions has the following representation:

$$C^\omega(\Omega) = \bigcup_{\tau>0} G_\tau^r$$

for any $r \ge 0$.

We have the following estimates for the nonlinear terms $\mathcal{B}$ and $\widetilde{\mathcal{B}}$.

Lemma 4.1 ([FT, YLH]). *For any $\tau > 0$, we have*

$$|(\mathcal{B}(u, v), w)_{1,\tau}| \le \kappa_1\|u\|_{1,\tau}^{1/2}\|u\|_{2,\tau}^{1/2}\|v\|_{1,\tau}\|w\|_{2,\tau} \quad \forall u \in G_\tau^2,\ v \in G_\tau^1,\ w \in G_\tau^2, \tag{4.1}$$

and

$$|(\widetilde{\mathcal{B}}(u, v), w)_{1,\tau}| \le \kappa_2\|u\|_{1,\tau}^{1/2}\|u\|_{2,\tau}^{1/2}\|v\|_{1,\tau}\|w\|_{2,\tau} \quad \forall u \in G_\tau^2,\ v \in G_\tau^1,\ w \in G_\tau^2, \tag{4.2}$$

for some positive constants κ_1, κ_2.

Theorem 4.2. *Suppose that $f \in G_{\tau_0}^0$ for some $\tau_0 > 0$. Then there exist $\sigma > 0$ and a constant R_G that depends only on $\|f\|_{0,\tau_0}$ such that*

$$\|u\|_{1,\sigma}^2 + \alpha^2\|u\|_{2,\sigma}^2 + \|B\|_{1,\sigma}^2 \le R_G, \quad \forall (u, B) \in \mathcal{A}.$$

Proof. We argue in two steps.

Step 1. We first show that if $f \in G^0_{\tau_0}$ for some $\tau_0 > 0$, then the solution $(u(t), B(t))$ of (2.7)–(2.10) satisfies $(u(t), B(t)) \in G^2_{\phi(t)} \times G^1_{\phi(t)}$ for all $0 \leq t \leq T(\|f\|_{0,\tau_0}, \|u^0\|_2, \|B^0\|)$, where $\phi(t) = \min\{t, \tau_0\}$, and moreover

$$\|u(t)\|^2_{1,\phi(t)} + \alpha^2\|u(t)\|^2_{2,\phi(t)} + \|B(t)\|^2_{1,\phi(t)} \leq 1 + 2(\|u^0\|^2 + \alpha^2\|u^0\|_2^2 + \|B^0\|^2). \tag{4.3}$$

To do this, taking the inner product in $G^1_{\phi(t)}$ of (2.7) with u and of (2.9) with B, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u\|^2_{1,\phi(t)} + \alpha^2\|u\|^2_{2,\phi(t)}) + \nu(\|u\|^2_{2,\phi(t)} + \alpha^2\|u\|^2_{3,\phi(t)}) \\ &\quad = \dot{\phi}(Ae^{\phi(t)A^{1/2}}u, A^{1/2}e^{\phi(t)A^{1/2}}u) + \dot{\phi}\alpha^2(A^{3/2}e^{\phi(t)A^{1/2}}u, Ae^{\phi(t)A^{1/2}}u) \\ &\qquad + (e^{\phi(t)A^{1/2}}f, Ae^{\phi(t)A^{1/2}}u) \\ &\qquad - (\widetilde{\mathcal{B}}(u, u + \alpha^2 Au), u)_{1,\phi(t)} + (\mathcal{B}(B,B), u)_{1,\phi(t)} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|B\|^2_{1,\phi(t)} + \eta\|B\|^2_{2,\phi(t)} &= \dot{\phi}(Ae^{\phi(t)A^{1/2}}B, A^{1/2}e^{\phi(t)A^{1/2}}B) \\ &\quad - (\mathcal{B}(u,B), B)_{1,\phi(t)} + (\mathcal{B}(B,u), B)_{1,\phi(t)}, \end{aligned} \tag{4.5}$$

where $\dot{\phi} = d\phi/dt$. Here we have used the following results:

$$\begin{aligned} &\left(A^{1/2}e^{\phi(t)A^{1/2}}\frac{dv}{dt}, A^{1/2}e^{\phi(t)A^{1/2}}u\right) \\ &\quad = \left(\frac{d}{dt}\big(A^{1/2}e^{\phi(t)A^{1/2}}(u + \alpha^2 Au)\big), A^{1/2}e^{\phi(t)A^{1/2}}u\right) \\ &\qquad - \dot{\phi}\big(Ae^{\phi(t)A^{1/2}}(u + \alpha^2 Au), A^{1/2}e^{\phi(t)A^{1/2}}u\big) \\ &\quad = \frac{1}{2}\frac{d}{dt}(\|u\|^2_{1,\phi(t)} + \alpha^2\|u\|^2_{2,\phi(t)}) - \dot{\phi}(Ae^{\phi(t)A^{1/2}}u, A^{1/2}e^{\phi(t)A^{1/2}}u) \\ &\qquad - \dot{\phi}\alpha^2(A^{(r+2)/2}e^{\phi(t)A^{1/2}}u, Ae^{\phi(t)A^{1/2}}u) \end{aligned}$$

and

$$\begin{aligned} &\left(A^{1/2}e^{\phi(t)A^{1/2}}\frac{dB}{dt}, A^{1/2}e^{\phi(t)A^{1/2}}B\right) \\ &\quad = \left(\frac{d}{dt}(A^{1/2}e^{\phi(t)A^{1/2}}B), A^{1/2}e^{\phi(t)A^{1/2}}B\right) - \dot{\phi}(Ae^{\phi(t)A^{1/2}}B, A^{1/2}e^{\phi(t)A^{1/2}}B) \\ &\quad = \frac{1}{2}\frac{d}{dt}\|B\|^2_{1,\phi(t)} - \dot{\phi}(Ae^{\phi(t)A^{1/2}}B, A^{1/2}e^{\phi(t)A^{1/2}}B). \end{aligned}$$

Using the fact that $|\dot{\phi}| \leq 1$ and the Cauchy inequality, we get

$$\begin{aligned} \text{(4.6)} \quad & \dot{\phi}(Ae^{\phi(t)A^{1/2}}u, A^{1/2}e^{\phi(t)A^{1/2}}u) \\ & \quad + \dot{\phi}\alpha^2(A^{(r+2)/2}e^{\phi(t)A^{1/2}}u, Ae^{\phi(t)A^{1/2}}u) \\ & \leq \|u\|_{2,\phi(t)}\|u\|_{1,\phi(t)} + \|u\|_{3,\phi(t)}\|u\|_{2,\phi(t)} \\ & \leq \frac{\nu}{8}(\|u\|^2_{2,\phi(t)} + \alpha^2\|u\|^2_{3,\phi(t)}) + \frac{2}{\nu}(\|u\|^2_{1,\phi(t)} + \alpha^2\|u\|^2_{2,\phi(t)}) \end{aligned}$$

and

$$\begin{aligned} \text{(4.7)} \quad \dot{\phi}(Ae^{\phi(t)A^{1/2}}B, A^{1/2}e^{\phi(t)A^{1/2}}B) & \leq \|B\|_{2,\phi(t)}\|B\|_{1,\phi(t)} \\ & \leq \frac{\eta}{4}\|B\|^2_{2,\phi(t)} + \frac{1}{\eta}\|B\|^2_{1,\phi(t)}. \end{aligned}$$

By the Cauchy inequality,

$$\begin{aligned} \text{(4.8)} \quad (e^{\phi(t)A^{1/2}}f, Ae^{\phi(t)A^{1/2}}u) & \leq \|f\|_{0,\phi(t)}\|u\|_{2,\phi(t)} \\ & \leq \frac{\nu}{8}\|u\|^2_{2,\phi(t)} + \frac{2}{\nu}\|f\|^2_{0,\phi(t)}. \end{aligned}$$

Using (4.2) and the Young inequality, we have

$$\begin{aligned} \text{(4.9)} \quad & -(\widetilde{\mathcal{B}}(u, u + \alpha^2 Au), u)_{1,\phi(t)} \\ & \leq |(\widetilde{\mathcal{B}}(u,u),u)_{1,\phi(t)}| + \alpha^2|(\widetilde{\mathcal{B}}(u,Au),u)_{1,\phi(t)}| \\ & \leq \kappa_2(\|u\|^{3/2}_{1,\phi(t)}\|u\|^{3/2}_{2,\phi(t)} + \alpha^2\|u\|^{1/2}_{1,\phi(t)}\|u\|^{3/2}_{2,\phi(t)}\|u\|_{3,\phi(t)}) \\ & \leq \kappa_2\left(\|u\|^{3/2}_{1,\phi(t)}\|u\|^{3/2}_{2,\phi(t)} + \frac{\alpha^2}{\lambda_1^{1/2}}\|u\|^{3/2}_{2,\phi(t)}\|u\|^{3/2}_{3,\phi(t)}\right) \\ & \leq \frac{\nu}{8}(\|u\|^2_{2,\phi(t)} + \alpha^2\|u\|^2_{3,\phi(t)}) + \frac{54\kappa_2^4}{\nu^3}\left(\|u\|^6_{1,\phi(t)} + \frac{\alpha^2}{\lambda_1^2}\|u\|^6_{2,\phi(t)}\right). \end{aligned}$$

Here we have used the Poincaré type inequality

$$\|u\|^{1/2}_{1,\phi(t)} \leq \frac{1}{\lambda_1^{1/4}}\|u\|^{1/2}_{2,\phi(t)} \leq \frac{1}{\lambda_1^{1/2}}\|u\|^{1/2}_{3,\phi(t)}.$$

Using (4.1) and the Cauchy inequality, we obtain

$$\begin{aligned} \text{(4.10)} \quad |(\mathcal{B}(B,B),u)_{1,\phi(t)}| & \leq \kappa_1\|B\|^{3/2}_{1,\phi(t)}\|B\|^{1/2}_{2,\phi(t)}\|u\|_{2,\phi(t)} \\ & \leq \frac{8\kappa_1^2}{\eta\nu^2}\|B\|^6_{1,\phi(t)} + \frac{\eta}{8}\|B\|^2_{2,\phi(t)} + \frac{\nu}{8}\|u\|^2_{2,\phi(t)}, \end{aligned}$$

$$\begin{aligned} \text{(4.11)} \quad |(\mathcal{B}(u,B),B)_{1,\phi(t)}| & \leq \kappa_1\|u\|^{1/2}_{1,\phi(t)}\|u\|^{1/2}_{2,\phi(t)}\|B\|_{1,\phi(t)}\|B\|_{2,\phi(t)} \\ & \leq \frac{8\kappa_1^4}{\eta^2\nu}\|u\|^2_{1,\phi(t)}\|B\|^4_{1,\phi(t)} + \frac{\eta}{8}\|B\|^2_{2,\phi(t)} + \frac{\nu}{8}\|u\|^2_{2,\phi(t)}, \end{aligned}$$

and

$$|(\mathcal{B}(B,u),B)_{1,\phi(t)}| \le \kappa_1 \|B\|_{1,\phi(t)}^{1/2} \|B\|_{2,\phi(t)}^{3/2} \|u\|_{1,\phi(t)} \le \frac{54\kappa_1^4}{\eta^3} \|u\|_{1,\phi(t)}^4 \|B\|_{1,\phi(t)}^2 + \frac{\eta}{8} \|B\|_{2,\phi(t)}^2. \tag{4.12}$$

Summing up (4.4)–(4.12), we deduce that

$$\begin{aligned} &\frac{d}{dt}(\|u\|_{1,\phi(t)}^2 + \alpha^2\|u\|_{2,\phi(t)}^2 + \|B\|_{1,\phi(t)}^2) \\ &\quad + \frac{3\nu}{4}(\|u\|_{2,\phi(t)}^2 + \alpha^2\|u\|_{3,\phi(t)}^2) + \frac{3\eta}{4}\|B\|_{2,\phi(t)}^2 \\ &\le \frac{4}{\nu}(\|u\|_{1,\phi(t)}^2 + \alpha^2\|u\|_{2,\phi(t)}^2) + \frac{2}{\eta}\|B\|_{1,\phi(t)}^2 \\ &\quad + \frac{4}{\nu}\|f\|_{0,\phi(t)}^2 + \frac{108\kappa_2^4}{\nu^3}(\|u\|_{1,\phi(t)}^6 + \frac{\alpha^2}{\lambda_1^2}\|u\|_{2,\phi(t)}^6) \\ &\quad + \frac{16\kappa_1^4}{\eta\nu^2}\|B\|_{1,\phi(t)}^6 + \frac{16\kappa_1^4}{\eta^2\nu}\|u\|_{1,\phi(t)}^2\|B\|_{1,\phi(t)}^4 + \frac{108\kappa_1^4}{\eta^3}\|u\|_{1,\phi(t)}^4\|B\|_{1,\phi(t)}^2. \end{aligned} \tag{4.13}$$

Set

$$y(t) = 1 + \|u\|_{1,\phi(t)}^2 + \alpha^2\|u\|_{2,\phi(t)}^2 + \|B\|_{1,\phi(t)}^2,$$

$$M = \frac{4}{\nu}\|f\|_{0,\tau_0}^2 + \frac{12}{\nu} + \frac{6}{\eta} + \frac{108\kappa_2^4}{\nu^3} + \frac{108\kappa_2^4}{\nu^3\alpha^4\lambda_1^2} + \frac{16\kappa_1^4}{\eta\nu^2} + \frac{48\kappa_1^4}{\eta^2\nu} + \frac{324\kappa_1^4}{\eta^3}.$$

Then from (4.13) we have

$$\frac{dy}{dt} \le My^3.$$

Thus,

$$y(t) \le \frac{1}{\sqrt{y(0)^{-2} - 2Mt}},$$

and so $y(t) \le 2y(0)$ for $t \le 3/(8y(0)^2M)$. Since $\phi(0) = 0$, we have

$$y(0) = 1 + \|u^0\|^2 + \alpha^2\|u^0\|_2^2 + \|B^0\|^2.$$

Hence

$$\|u(t)\|_{1,\phi(t)}^2 + \alpha^2\|u(t)\|_{2,\phi(t)}^2 + \|B(t)\|_{1,\phi(t)}^2 \le 1 + 2(\|u^0\|^2 + \alpha^2\|u^0\|_2^2 + \|B^0\|^2)$$

for all $0 \le t \le T(\|f\|_{0,\tau_0}, \|u^0\|, \|u^0\|_2, \|B^0\|)$.

Step 2. By the embedding $G_{\tau_0}^0 \hookrightarrow H$, the conclusion of Theorem 3.2 holds. Take $(u,B) \in \mathcal{A}$. Then $(u,B) = S(T)(u^0,B^0)$ for some $(u^0,B^0) \in \mathcal{A}$ and some $T > 0$. Since $(u^0,B^0) \in \mathcal{A}$, from Theorem 3.2 we get

$$\|u^0\|^2 + \alpha^2\|u^0\|_2^2 + \|B^0\|^2 \le R_1.$$

Hence, $T(\|f\|_{0,\tau_0}, \|u^0\|, \|u^0\|_2, \|B^0\|) \le T(\|f\|_{0,\tau_0}, R_1)$. So if we take $T = T(\|f\|_{0,\tau_0}, R_1)$ from Step 1, we get from (4.3)

$$\begin{aligned}\|u\|^2_{1,\phi(T)} + \alpha\|u\|^2_{2,\phi(T)} + \|B\|^2_{1,\phi(T)} \\ = \|u(T)\|^2_{1,\phi(T)} + \alpha\|u(T)\|^2_{2,\phi(T)} + \|B(T)\|^2_{1,\phi(T)} \le 1 + 2R_1 =: R_G\end{aligned}$$

uniformly over $\mathcal{A}$. ∎

As a particular case of Theorem 4.2, if the external force f is real analytic, then so is the global attractor $\mathcal{A}$. Thanks to the result of Friz and Robinson [FR], we can use this regularity to parametrize the attractor by using a finite number of nodal values.

Let us recall the result of [FR] (see also [FKR, KR]).

THEOREM 4.3 ([FR]). *Let Ω be a periodic domain in $\mathbb{R}^n$. Suppose that $\mathcal{A}$ is a compact connected subset of $\mathbb{L}^2(\Omega) = (L^2(\Omega))^m$ which has finite fractal dimension $d_F(\mathcal{A})$ and is uniformly bounded in G^r_σ for some $r \ge 0$ and $\sigma > 0$. Then, provided that $k > 16d_F(\mathcal{A}) + 1$, the map from $\mathcal{A}$ into $\mathbb{R}^{mk}$ given by*

$$E_{\mathbf{x}}[u] : u \mapsto (u(x_1), \dots, u(x_k))$$

is 1-1 *between $\mathcal{A}$ and its image for almost every choice x of k nodes $x_1, \dots, x_k$ in Ω (with respect to nk-dimensional Lebesgue measure).*

In particular, the values $(u(x_1), \dots, u(x_k))$ provide a parametrization of $\mathcal{A}$ which is continuous from $\mathbb{R}^{mk}$ into $\mathbb{L}^2(\Omega)$.

We now show that if the external force f is real analytic then all assumptions of Theorem 4.3 hold for the global attractor $\mathcal{A}$ of the 3D MHD-α model (1.1)–(1.6).

Indeed, by Theorem 2.1, the attractor $\mathcal{A}$ is a compact connected subset of $V \times H$ with finite fractal dimension $d_F(\mathcal{A})$. Moreover, Theorem 4.2 shows that $\mathcal{A}$ is bounded in $G^2_\sigma \times G^1_\sigma$ for some $\sigma > 0$. Hence $\mathcal{A}$ can be parametrized by k nodes in Ω with $k > 16d_F(\mathcal{A}) + 1$.

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