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A differential-geometric analysis of the Bergman representative map

Sungmin Yoo (Pohang)

Abstract. We show that the exponential map of the Bochner connection on the restricted holomorphic tangent bundle of a complex manifold admitting the positive-definite Bergman metric coincides with the inverse of Bergman's representative map. We also present a generalization of Lu Qi Keng's theorem, as an application.

1. Introduction. On a complex manifold equipped with the Bergman kernel and metric, the Bergman representative map, originally named "the representative domain" by Stefan Bergman himself, is an important offshoot of the Bergman kernel form [13, Chapter 4]. It is a special holomorphic map which is in a significant contrast with the exponential map of the Riemannian structure given by the real part of the Bergman metric: the Riemannian exponential map is almost never holomorphic. On the other hand, the representative map gives rise to a holomorphic Kähler normal coordinate system with respect to the Bergman metric. One of its best known features is that all holomorphic Bergman isometries become linear mappings in these representative coordinates. In spite of the difficulty that this map need not be well-defined globally, this feature has been proven to be useful in many important works (see, for instance, [23], [1], [30], [14]). However, it was striking to us that no systematic study of this concept has yet been carried out.

In this paper, we describe the relationship between the Bergman representative map and the Riemannian exponential map of the Bergman metric in terms of differential geometry. In particular, we present a construction of the torsion-free flat holomorphic affine connection on the holomorphic tangent bundle of an open dense subdomain of the given complex manifold, whose affine exponential map is the inverse of the representative map (The-

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orem 4.2). This yields a differential-geometric interpretation of the Bergman representative map.

It is worth mentioning that our connection was discovered, at least partially, by several other authors in the articles preceding this paper, even though the information was scattered around in the papers such as [7] (much earlier than the others; in fact, Bochner constructed "normal" coordinates only, which can develop into the connection), [8], and [3]. It is also studied independently in [10] and [20] in relation to the holomorphic part of the Kähler metric connection (a symplectic geometric interpretation can be found in [22] and [27]). More notably, as the connection for the case of "bounded domains", it was studied in [30] for a version of extension theorem for biholomorphic mappings. We hope that this paper presents these concepts in a unified way.

The paper is organized as follows: First, we briefly review the fundamentals of Bergman geometry including the construction of the representative map. Then we present Bochner's normal coordinate system for real analytic Kähler manifolds and the affine connection. We call it the *Bochner connection.* Then, we restrict ourselves to complex manifolds with the Bergman metric, and study the Bochner connnection. In Sections 5 and 6, we analyze the geodesic behavior of the Bochner connection and the removed varieties. In the last section, as an application, we present a generalization of a theorem by Lu Qi-Keng [23], which says that a bounded domain in $\mathbb{C}^n$ whose Bergman metric is complete and of constant holomorphic sectional curvature is biholomorphic to the unit ball. We have been able to generalize this to the case of bounded domains with a *pole of the Bochner connection*, such as a circular domain or a homogeneous domain.

2. Fundamentals of Bergman geometry

2.1. The Bergman kernel and metric for a bounded domain in $\mathbb{C}^n$. Let Ω be a bounded domain in $\mathbb{C}^n$ and $K(z,\overline{w})$ the Bergman kernel of Ω. Since $K(z,\overline{z})>0$, the Bergman metric

$$g_\Omega(z)=\sum_{j,k=1}^{n} g_{j\overline{k}}(z)dz_j\otimes d\overline{z_k}\quad\text{with}\quad g_{j\overline{k}}(z)=g_{j\overline{k}}(z,\overline{z}):=\frac{\partial^2\log K(z,\overline{z})}{\partial z_j\partial\overline{z_k}}$$

is well-defined. In fact, the following result was proved by Bergman himself [2]:

THEOREM 2.1 (Bergman). *The Bergman metric g_Ω is positive-definite at every $z\in\Omega$.*

REMARK 2.2. Note that g_Ω is a Kähler metric. The transformation formula for the Bergman kernel function (under biholomorphisms) implies that every biholomorphism between bounded domains is an isometry with respect to the Bergman metric.

2.2. The Bergman representative map. Let $p \in \Omega$. Since $K(p,\overline{p}) > 0$, there is a neighborhood of p such that $K(z,\overline{w}) \neq 0$ for all z, w in that neighborhood. Denote by $g^{\overline{k}j}(p)$ the (k,j) entry of the inverse matrix of $(g_{j\overline{k}}(p))$.

Definition 2.3. The *Bergman representative map* at p is defined by

$$\operatorname{rep}_p(z) = (\zeta_1(z), \ldots, \zeta_n(z)),$$

where

$$\zeta_j(z) := g^{\overline{k}j}(p)\left\{ \frac{\partial}{\partial \overline{w}_k}\bigg|_{w=p} \log K(z,\overline{w}) - \frac{\partial}{\partial \overline{w}_k}\bigg|_{w=p} \log K(w,\overline{w}) \right\}.$$

Since $\frac{\partial \zeta_k}{\partial z_l}\big|_{z=p} = \delta_{lk}$, this map defines a holomorphic local coordinate system at p. Another special feature appears in the following theorem by Bergman himself.

Theorem 2.4 (Bergman). *If $f : \Omega \to \tilde{\Omega}$ is a biholomorphic mapping of bounded domains, then* $\operatorname{rep}_{f(p)} \circ f \circ \operatorname{rep}_p^{-1}$ *is $\mathbb{C}$-linear.*

Bergman's original proof of this was via a direct computation using the transformation formula. A differential-geometric proof using the Bochner connection will be presented in Section 4 (see Theorem 4.2 and Remark 4.3). Since the Bergman kernel and metric can be defined for complex manifolds [21], this geometric explanation also applies to complex manifolds.

2.3. The Bergman kernel form on a complex manifold. Let M be an n-dimensional complex manifold and $A^2(M)$ the space of holomorphic n-forms f on M satisfying

$$\Big| \int_M f \wedge \overline{f} \Big| < \infty.$$

Let $\{\phi_0, \phi_1, \phi_2, \ldots\}$ be a complete orthonormal system for the Hilbert space $A^2(M)$ and $\overline{M}$ the complex manifold conjugate to M. Define a holomorphic $2n$-form on $M \times \overline{M}$ by

$$K(z,\overline{w}) = \sum_{j \geq 0} \phi_j(z) \wedge \overline{\phi_j(w)}.$$

This construction is independent of the choice of orthonormal basis. By using the diagonal embedding $\iota : M \hookrightarrow M \times \overline{M}$ defined by $\iota(z) = (z, \overline{z})$, and the natural identification of M with $\iota(M)$, $K(z,\overline{z})$ can be considered as a $2n$-form on M. It is called the *Bergman kernel form* of M.

Assume that this form is non-zero at any point of M. In a local coordinate system $(U, (z_1, \dots, z_n))$, the Bergman kernel form can be written as

$$K(z, \overline{z}) = K_U^*(z, \overline{z}) dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z_1} \wedge \cdots \wedge d\overline{z_n},$$

where $K_U^*(z, \overline{z})$ is a well-defined function on U. Set

$$ds_M^2 := \sum_{j,k=1}^{n} g_{j\overline{k}}(z) dz_j \otimes d\overline{z_k} = \sum_{j,k=1}^{n} \frac{\partial^2 \log K_U^*(z, \overline{z})}{\partial z_j \partial \overline{z_k}} dz_j \otimes d\overline{z_k}.$$

This is independent of the choice of local coordinate system. When the matrix $G(z) := (g_{j\overline{k}}(z))$ is positive-definite for each $z \in M$, ds_M^2 is called the *Bergman metric* of M.

2.4. Bergman representative coordinates. From now on, suppose that M is a complex manifold which possesses the Bergman metric. (In fact, many non-compact complete Kähler manifold with negative curvature admit the Bergman metric [15, Theorem H].) In a local coordinate system $(U \times \overline{V}, (z_1, \dots, z_n, \overline{w_1}, \dots, \overline{w_n}))$ for $M \times \overline{M}$,

$$K(z, \overline{w}) = K_{U \times \overline{V}}^*(z, \overline{w}) dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{w_1} \wedge \cdots \wedge d\overline{w_n},$$

where $K_{U \times \overline{V}}^*(z, \overline{w})$ is a well-defined function on $U \times \overline{V}$. Given $\overline{p} \in \overline{V}$, define the following holomorphic coordinate system centered at p ([9], [13], [12]).

Definition 2.5. The *Bergman representative coordinate system* at p is defined by

$$\mathrm{rep}_p(z) = (\zeta_1(z), \dots, \zeta_n(z)),$$

where

$$\zeta_j(z) := g^{\overline{k}j}(p) \left\{ \frac{\partial}{\partial \overline{w_k}}\bigg|_{w=p} \log K_{U \times \overline{V}}^*(z, \overline{w}) - \frac{\partial}{\partial \overline{w_k}}\bigg|_{w=p} \log K_{V \times \overline{V}}^*(w, \overline{w}) \right\}.$$

Remark 2.6. The above construction is independent of the choice of the coordinate system $(U, (z_1, \dots, z_n))$ for M, but it depends on the choice of a local coordinate system $(V, (w_1, \dots, w_n))$. Note that rep_p extends to an almost global function, well-defined on the whole of M minus the analytic variety $Z_0^p := \{z \in M : K(z, \overline{p}) = 0\}$ (this set is well-defined because of the transformation formula (4.1)).

3. Bochner's normal coordinates and connection. For real analytic Kähler manifolds, Bochner constructed the Kähler normal coordinate system, a version of the representative coordinate system, from the Kähler potential [7]. This normal coordinate system is strongly related to the exponential map of the Kähler metric [10]. We feel that this relation can be better explained via the language of vector bundles and connections [20]. Therefore, we reorganize this information, scattered in the literature.

3.1. Bochner's normal coordinates. Suppose that M is a Kähler manifold with real analytic Kähler metric g. In [7], a Kähler normal coordinate system is defined as follows:

Proposition 3.1 (Bochner's normal coordinates). *Given $p \in M$, there exist holomorphic coordinates $(\zeta_1, \ldots, \zeta_n)$, unique up to unitary linear transformations, satisfying*

(i) $\zeta(p) = 0$,
(ii) $g_{j\overline{k}}(p) = \delta_{jk}$,
(iii) $dg_{j\overline{k}}(p) = 0$,
(iv) $\dfrac{\partial^I g_{j\overline{k}}}{\partial\zeta_1^{i_1}\cdots\partial\zeta_n^{i_n}}(p) = 0$ *for all* $I \geq 1$ *and* $i_1 + \cdots + i_n = I$.

In [3], Bochner's coordinate system was rediscovered in the context of mathematical physics. There, the Bochner coordinates were called the *canonical coordinates.*

Proposition 3.2 (Bershadsky, Cecotti, Ooguri and Vafa [3]). *Bochner's normal coordinates $(\zeta_1, \ldots, \zeta_n)$ can be expressed in terms of the Kähler potential $\psi(z, \overline{z})$:*

$$\zeta_j(z) = \sqrt{g}^{\overline{k}j}(p)\left\{ \frac{\partial}{\partial\overline{w_k}}\bigg|_{w=p} \psi(z, \overline{w}) - \frac{\partial}{\partial\overline{w_k}}\bigg|_{w=p} \psi(w, \overline{w}) \right\},$$

where $\sqrt{g}^{\overline{k}j}(p)$ is defined as follows: Since $G(p) := (g_{j\overline{k}}(p))$ is a positive-definite Hermitian matrix, there exists a matrix A such that $G(p) = A\overline{A^t}$. Denote by $\sqrt{g}^{\overline{k}j}(p)$ the (k, j) entry of the inverse matrix of A.

Corollary 3.3. *Bochner's normal coordinate system for a manifold with the Bergman metric is the same as the Bergman representative coordinate system of the Kähler potential* $\log K(z, \overline{z})$ *up to the normalization factor $\sqrt{g}^{\overline{k}j}(p)$.*

We explain how to separate the holomorphic part from the Riemannian exponential map and show that it coincides with the inverse map of the Bochner normal coordinate system. Our exposition follows those of [10], [11], and [20].

3.2. The holomorphic exponential map. Let M be a real-analytic Kähler manifold. The construction of the holomorphic exponential map from the Riemannian exponential map $\exp_p : T_pM \to M$ consists of two steps: (1) complexification, (2) restriction.

Step 1: *Complexification* (*Polarization*). Note that the real-analytic manifold M of real dimension n can be embedded as a totally real submanifold of a complex manifold $\mathbb{C}M$ of complex dimension n.

Theorem 3.4 (Whitney–Bruhat [31]). *Every real analytic manifold M can be embedded as a totally real submanifold of a complex manifold. This embedding is unique in the sense that, if $\iota_1 : M \hookrightarrow \mathbb{C}M_1$ and $\iota_2 : M \hookrightarrow \mathbb{C}M_2$ are such embeddings* (*in spaces of the same dimension*), *then there exist neighborhoods U_1 and U_2 of M in $\mathbb{C}M_1$ and $\mathbb{C}M_2$ respectively and a biholomorphism $f : U_1 \to U_2$ such that $\iota_2 = f \circ \iota_1$.*

Since M is a Kähler manifold (and of course a complex manifold) in our setting, we have two embeddings, the complexification $T_pM \hookrightarrow T_p^{\mathbb{C}}M$ and the diagonal embedding $\iota : M \hookrightarrow M \times \overline{M}$ as the embedding in Theorem 3.4. Then, we apply the following lemma to the exponential map $\exp_p : T_pM \to M$.

Lemma 3.5. *Let M and N be totally real submanifolds of complex manifolds $\mathbb{C}M$ and $\mathbb{C}N$, and $f : M \to N$ a real-analytic diffeomorphism. Then there are neighborhoods U and V of M and N, and a unique holomorphic map $f^{\mathbb{C}} : U \to V$ extending f.*

Proof. See [29, proof of Lemma 1.2.1]. ∎

Denote by $\exp_p^{\mathbb{C}}$ the unique holomorphic extension of $\exp_p$.

Step 2: *Restriction.* Use the decomposition $T_p^{\mathbb{C}}M \cong T'_pM \oplus T''_pM$ where T'_pM is the holomorphic tangent space and T''_pM is the anti-holomorphic tangent space of the given real-analytic Kähler manifold M. Then we restrict the complexified map $\exp_p^{\mathbb{C}}$ to T'_pM.

Definition 3.6. The restriction map $\exp_p^{\mathbb{C}}|_{T'_pM}(\zeta) := \exp_p^{\mathbb{C}}(\zeta, 0)$ is called the *holomorphic exponential map*.

We remark that the above definition is the same as the following definition which first appeared in [10].

Definition 3.7. Take the power series expansion of the exponential map of the Kähler metric $\exp_p : T'_pM \oplus T''_pM \to M$ and the decomposition

$$\exp_p(\zeta, \overline{\zeta}) = f(\zeta) + g(\zeta, \overline{\zeta})$$

on some neighborhood of 0, where f is holomorphic in ζ and g is the sum of all monomials in the power series expansion of $\exp_p(\zeta, \overline{\zeta})$ which are not holomorphic in ζ. Then the *holomorphic part* of the exponential map at p is defined to be

$$\mathrm{exph}_p(\zeta) := f(\zeta).$$

3.3. The Bochner connection. We present the construction of the holomorphic affine connection whose affine exponential map is the *holomorphic exponential map* exph_p. We also show that exph_p is the same as the inverse to the Bochner normal coordinate system, using the affine geodesic equations of the connection.

THEOREM 3.8 (Kapranov [20]). *There exists a holomorphic affine connection $\nabla^{\mathbb{C}}$ on $T'(M \times \overline{M})$, defined over a neighborhood of $\iota(M)$, whose affine exponential map is $\exp_p^{\mathbb{C}}$. The restriction of $\nabla^{\mathbb{C}}$ to T'_pM is also a holomorphic affine connection defined over a neighborhood of p in M. The affine exponential map of $\nabla^{\mathbb{C}}|_{T'_pM}$ is* exph_p.

Proof. Let ∇ be the Kähler connection defined by the Christoffel symbols $\Gamma^j_{kl}(z,\overline{z}) = \frac{\partial g_{k\overline{m}}(z,\overline{z})}{\partial z_l} g^{\overline{m}j}(z,\overline{z})$. Denote by $\nabla^{\mathbb{C}}$ the analytic continuation (complexification) of ∇. Then $\nabla^{\mathbb{C}}$ is an affine connection defined by the coefficients of the connection 1-form:

$$\Gamma^j_{kl}(z,\overline{w}) = \frac{\partial g_{k\overline{m}}(z,\overline{w})}{\partial z_l} g^{\overline{m}j}(z,\overline{w}),$$

where $(z,\overline{w})$ are holomorphic coordinates for $M \times \overline{M}$. Moreover, its affine exponential map is the same as $\exp_p^{\mathbb{C}}$, since the complexification of $\exp_p$ is unique.

To prove the second statement, take the decomposition

$$T'_{(p,\overline{p})}(M \times \overline{M}) = T'_pM \oplus T'_{\overline{p}}\overline{M},$$

and restrict $\nabla^{\mathbb{C}}$ to T'_pM. This yields a holomorphic affine connection on T'_pM, defined only in some neighborhood of p in M. The affine exponential map of this connection is the holomorphic exponential map exph_p. ∎

From now on, we denote $\nabla^{\mathbb{C}}|_{T'_pM}$ by ∇^p, and call it the *Bochner connection* at p. The following lemma gives the affine geodesic equations for the holomorphic exponential map exph_p.

LEMMA 3.9. *The geodesic of the Bochner connection ∇^p emanating from p in the initial direction $\zeta \in T'_pM$ satisfies the following system of second order ODE:*

$$\begin{cases} \frac{d^2 z_j(t)}{dt^2} + \Gamma^j_{kl}(z(t),\overline{p}) \frac{dz_k(t)}{dt}\frac{dz_l(t)}{dt} = 0, \\ z(0) = p, \quad \frac{dz_j}{dt}(0) = \zeta_j. \end{cases} \tag{3.1}$$

Proof. The curve $\exp_p^{\mathbb{C}}(\zeta t, \overline{\xi} t)$ constructed by using the affine exponential map of $\nabla^{\mathbb{C}}$ satisfies

$$\begin{cases} \frac{d^2 z_j(t)}{dt^2} + \Gamma^j_{kl}(z(t),\overline{w}(t)) \frac{dz_k(t)}{dt}\frac{dz_l(t)}{dt} = 0, \\ z(0) = p, \quad \frac{dz_j}{dt}(0) = \zeta_j, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \frac{d^2 \overline{w_j}(t)}{dt^2} + \Gamma^{\overline{j}}_{\overline{k}\overline{l}}(z(t),\overline{w}(t)) \frac{d\overline{w_k}(t)}{dt}\frac{d\overline{w_l}(t)}{dt} = 0, \\ \overline{w}(0) = \overline{p}, \quad \frac{d\overline{w_j}}{dt}(0) = \overline{\xi_j}, \end{cases} \tag{3.3}$$

where $(\zeta,\overline{\xi}) \in T'_pM \oplus T''_pM = \mathbb{C}T_pM$. It suffices to let $\xi \equiv 0$, since the solution of (3.3) becomes the constant map $(w_1,\dots,w_n) \equiv (p_1,\dots,p_n)$. ∎

Using the above lemma, we prove the follwing proposition, which appeared first in [20].

PROPOSITION 3.10. *The inverse to the holomorphic exponential map at p of the real analytic Kähler metric is the Bochner normal coordinate system at p, up to unitary linear transformations.*

Proof. Let φ be the inverse to the Bochner normal coordinate system and $\tilde{\gamma}(t)$ the curve in M given by $\tilde{\gamma}(t) = \varphi(vt)$ where $v \in \mathbb{C}^n \cong T'_pM$. It is enough to show that $\tilde{\gamma}(t) = (z_1(t), \ldots, z_n(t))$ satisfies (3.1). By the definition of the normal coordinates, we obtain $\tilde{\gamma}(0) = \varphi(0) = p$ and

$$\frac{\partial \zeta_k}{\partial z_l} = g^{\bar{j}k}(p) g_{l\bar{j}}(z, \overline{p}), \qquad \frac{\partial z_k}{\partial \zeta_r} = g_{r\overline{\lambda}}(p) g^{\overline{\lambda}k}(z, \overline{p}). \tag{3.4}$$

Since $\frac{\partial \zeta_k}{\partial z_l}\big|_{z=p} = \delta_{lk}$, we have $\tilde{\gamma}'(0) = \big(\frac{dz_1}{dt}(0), \ldots, \frac{dz_n}{dt}(0)\big) = \big(\frac{d\zeta_1}{dt}(0), \ldots, \frac{d\zeta_n}{dt}(0)\big) = v$. Then the holomorphicity of the Bochner normal coordinates implies that

$$\begin{aligned}
&\frac{d^2 z_j(t)}{dt^2} + \Gamma^j_{kl}(\tilde{\gamma}(t), \overline{p}) \frac{dz_k(t)}{dt} \frac{dz_l(t)}{dt} \\
&\qquad = \frac{\partial^2 z_j}{\partial \zeta_r \partial \zeta_s} \frac{d\zeta_r}{dt} \frac{d\zeta_s}{dt} + \Gamma^j_{kl}(\tilde{\gamma}(t), \overline{p}) \frac{\partial z_k}{\partial \zeta_r} \frac{d\zeta_r}{dt} \frac{\partial z_l}{\partial \zeta_s} \frac{d\zeta_s}{dt} \\
&\qquad = \left\{ \frac{\partial^2 z_j}{\partial \zeta_r \partial \zeta_s} + \Gamma^j_{kl}(\tilde{\gamma}(t), \overline{p}) \frac{\partial z_k}{\partial \zeta_r} \frac{\partial z_l}{\partial \zeta_s} \right\} v_r v_s.
\end{aligned}$$

Thus it suffices to show that the following analytic differential equations hold:

$$\frac{\partial^2 z_j}{\partial \zeta_r \partial \zeta_s} + \Gamma^j_{kl}(\tilde{\gamma}(t), \overline{p}) \frac{\partial z_k}{\partial \zeta_r} \frac{\partial z_l}{\partial \zeta_s} = 0. \tag{3.5}$$

The matrix equation $d(A \cdot A^{-1}) = 0$ and (3.4) yield

$$\begin{aligned}
\frac{\partial^2 z_j}{\partial \zeta_r \partial \zeta_s} &= g_{r\overline{\lambda}}(p) \frac{\partial g^{\overline{\lambda}j}(z, \overline{p})}{\partial z_l} \frac{\partial z_l}{\partial \zeta_s} \\
&= -g_{r\overline{\lambda}}(p) g^{\overline{\lambda}k}(z, \overline{p}) \frac{\partial g_{k\overline{m}}(z, \overline{p})}{\partial z_l} g^{\overline{m}j}(z, \overline{p}) \frac{\partial z_l}{\partial \zeta_s} \\
&= -\frac{\partial z_k}{\partial \zeta_r} \frac{\partial g_{k\overline{m}}(z, \overline{p})}{\partial z_l} g^{\overline{m}j}(z, \overline{p}) \frac{\partial z_l}{\partial \zeta_s}.
\end{aligned}$$

Therefore, we arrive at (3.5). ■

REMARK 3.11. For a bounded domain with the Bergman metric, it is known that the above analytic equations hold for the Bergman representative map [24]. This is, of course, closely analogous to the analysis of flows of vector fields in the context of Riemannian geometry.

4. The Bochner connection on a manifold with the Bergman metric. Let M be a complex manifold with the Bergman metric, and p

a point in M. Since the Bergman metric is a real-analytic Kähler metric, the Bochner connection can be constructed in an open neighborhood of p as in Section 3.3. On the other hand, we show that the Bochner connection actually extends to the whole manifold minus possibly an analytic variety.

4.1. The extended Bochner connection. Suppose that M is a complex manifold which possesses the Bergman metric. Recall that in a local coordinate system $(U\times\overline{V},(z_1,\dots,z_n,\overline{w_1},\dots,\overline{w_n}))$ for $M\times\overline{M}$, the Bergman kernel form is

$$K(z,\overline{w}) = K^*_{U\times\overline{V}}(z,\overline{w})dz_1\wedge\cdots\wedge dz_n\wedge d\overline{w_1}\wedge\cdots\wedge d\overline{w_n},$$

where $K^*_{U\times\overline{V}}(z,\overline{w})$ is a well-defined function on $U\times\overline{V}$. Define a tensor on $M\times\overline{M}$ by

$$G(z,\overline{w}) := \sum_{j,k=1}^{n} g_{j\overline{k}}(z,\overline{w})dz_j\otimes d\overline{w_k} = \sum_{j,k=1}^{n}\frac{\partial^2\log K^*_{U\times V}(z,\overline{w})}{\partial z_j\partial\overline{w_k}}dz_j\otimes d\overline{w_k}.$$

Let $(\widetilde{U}\times\overline{\widetilde{V}},(\widetilde{z}_1,\dots,\widetilde{z}_n,\overline{\widetilde{w}_1},\dots,\overline{\widetilde{w}_n}))$ be another coordinate system. Then, in $(U\times\overline{V})\cap(\widetilde{U}\times\overline{\widetilde{V}})$, the transformation formula

$$K^*_{U\times\overline{V}}(z,\overline{w}) = K^*_{\widetilde{U}\times\overline{\widetilde{V}}}(\widetilde{z},\overline{\widetilde{w}})\det J_U^{\widetilde{U}}(z)\overline{\det J_V^{\widetilde{V}}(w)} \tag{4.1}$$

holds, where $J_U^{\widetilde{U}}(z) = \left(\frac{\partial\widetilde{z}_k}{\partial z_j}\right)_{n\times n}$ and $J_V^{\widetilde{V}}(w) = \left(\frac{\partial\widetilde{w}_k}{\partial w_j}\right)_{n\times n}$. In terms of matrices,

$$G_{U\times\overline{V}}(z,\overline{w}) = J_U^{\widetilde{U}}(z)\cdot\widetilde{G}_{\widetilde{U}\times\overline{\widetilde{V}}}(\widetilde{z},\overline{\widetilde{w}})\cdot\overline{J_V^{\widetilde{V}}(w)}^{t} \tag{4.2}$$

where

$$G_{U\times\overline{V}}(z,\overline{w}) = \left[\frac{\partial^2\log K^*_{U\times\overline{V}}(z,\overline{w})}{\partial z_j\partial\overline{w_k}}\right]_{n\times n},$$
$$\widetilde{G}_{\widetilde{U}\times\overline{\widetilde{V}}}(\widetilde{z},\overline{\widetilde{w}}) = \left[\frac{\partial^2\log K^*_{\widetilde{U}\times\overline{\widetilde{V}}}(\widetilde{z},\overline{\widetilde{w}})}{\partial\widetilde{z}_j\partial\overline{\widetilde{w}_k}}\right]_{n\times n}.$$

Given a point $\overline{p}\in\overline{M}$, define the analytic varieties

$$Z_0^p := \{z\in M : K(z,\overline{p})=0\},$$
$$Z_1^p := \{z\in M - Z_0^p : \det(G(z,\overline{p}))=0\}.$$

LEMMA 4.1. *If $f: M\to\widetilde{M}$ is a biholomorphism with $q=f(p)$, then*

(1) $f(Z_0^p) = \widetilde{Z}_0^q$,
(2) $f(Z_1^p) = \widetilde{Z}_1^q$,
(3) $f(M^p) = \widetilde{M}^q$,

where $M^p := M-(Z_0^p\cup Z_1^p)$ and $\widetilde{M}^q := \widetilde{M}-(\widetilde{Z}_0^q\cup\widetilde{Z}_1^q)$.

Proof. The transformation formulae (4.1) and (4.2) prove that these sets are well-defined and invariant under biholomorphisms. ∎

Let $T'M^p$ be the holomorphic tangent bundle over M^p.

THEOREM 4.2. *There exists a holomorphic affine connection ∇^p on $T'M^p$ satisfying:*

(1) *∇^p is locally flat, i.e. the torsion and curvature of ∇^p are zero.*
(2) *(3.1) are the affine geodesic equations for ∇^p.*
(3) *$f_*(\nabla^p_X Y) = \nabla^q_{\widetilde{X}} \widetilde{Y}$ for all $X, Y \in T'M^p$ where $\widetilde{X} = f_*(X)$, $\widetilde{Y} = f_*(Y)$ and f is as in the preceding lemma.*

Proof. Define the connection 1-forms as follows: Note that $G := G_{U\times\overline{V}}(z,\overline{p})$ is an invertible holomorphic $(n\times n)$-matrix on $U \cap M^p$ so that G^{-1} is well-defined on $U \cap M^p$. Define the $(n\times n)$-matrix ω of holomorphic 1-forms by $\omega := \partial G \cdot G^{-1}$, locally defined on $U \cap M^p$. In other words,

$$\omega_i^j(z) = \Gamma_{ik}^j(z,\overline{p})dz_k = \frac{\partial g_{i\overline{m}}(z,\overline{p})}{\partial z_k} g^{\overline{m}j}(z,\overline{p})dz_k.$$

Since $\partial G = \omega\cdot G$, the transformation formula (4.2) yields the *transformation rule for connection* 1*-forms*:

$$\omega \cdot J = \partial J + J \cdot \widetilde{\omega}. \tag{4.3}$$

To show (1), observe that ∇^p is torsion-free, because $\frac{\partial}{\partial z_k} g_{i\overline{j}} = \frac{\partial}{\partial z_i} g_{k\overline{j}}$. Moreover, its curvature form $\Omega := d\omega - \omega \wedge \omega$ is also zero, because $G := (g_{j\overline{k}}(z,\overline{p}))$ is holomorphic. More precisely,

$$\begin{aligned} d(\partial G \cdot G^{-1}) - (\partial G \cdot G^{-1}) \wedge (\partial G \cdot G^{-1}) \\ &= \partial(\partial G \cdot G^{-1}) + \partial G \wedge \partial G^{-1} \cdot G \cdot G^{-1} \\ &= -\partial G \wedge \partial G^{-1} + \partial G \wedge \partial G^{-1} = 0. \end{aligned}$$

Now, (2) follows from the construction, and (3) follows by (4.3). ∎

REMARK 4.3. The last statement in Theorem 4.2 implies the $\mathbb{C}$-linearity of the Bergman representative coordinates: Since geodesics are straight lines in this coordinate system, $\mathrm{rep}_{f(p)} \circ f \circ \mathrm{rep}_p^{-1}$ maps straight lines to straight lines. Thus it is $\mathbb{R}$-linear. Since the representative map is holomorphic, it is $\mathbb{C}$-linear. This is the geometric proof of Theorem 2.4, promised in Section 2.2.

It is possible to find a formula for the inverse of the affine exponential map of ∇^p not only at p but also at an arbitrary point $q \in M^p$. The proof is the same as that of Theorem 3.10.

PROPOSITION 4.4. *Denote by* exph_q *the affine exponential map of ∇^p at q. Let $(\zeta_1, \ldots, \zeta_n)$ be the coordinate system for the holomorphic tangent* *space* $T'_q M^p$ *at* q. *Then, in the local coordinate neighborhood* $(U, (z_1, \ldots, z_n))$ *containing* q,

$$\mathrm{exph}_q^{-1}(z) = (\zeta_1(z), \ldots, \zeta_n(z)),$$

where

$$\zeta_j(z) = \sqrt{g}^{\overline{k}j}(q, \overline{p}) \left\{ \frac{\partial}{\partial \overline{w_k}}\bigg|_{w=p} \log K^*_{U\times\overline{V}}(z, \overline{w}) - \frac{\partial}{\partial \overline{w_k}}\bigg|_{w=p} \log K^*_{U\times\overline{V}}(q, \overline{w}) \right\}.$$

Remark 4.5. The above proposition implies that ${\mathrm{exph}_q}^{-1}$ is a linear transform of ${\mathrm{exph}_p}^{-1}$. Therefore M^p can be covered by copies (by linear maps) of the representative coordinates. This can be explained by the concept of affine structures of [25], to be introduced in the next subsection.

Remark 4.6. For a real-analytic Kähler manifold, it is known that such an affine structure exists on a local neighborhood of a given point [23].

4.2. Affine structure of M^p. The proof of Theorem 3.10 also implies

Proposition 4.7. *Let* U *be a local neighborhood of* p *and* V *a local neighborhood of* 0 *such that* ${\mathrm{exph}_p}^{-1} : U \to V$ *is biholomorphic. Take any straight line* l *in* V (*not necessarily through* p). *Then* $\mathrm{exph}_p(l)$ *is a geodesic of* ∇^p.

This proposition follows immediately from the affine structures as recalled below:

Definition 4.8. Let X be a complex manifold of dimension n and $\mathcal{M} = \{U_i, \phi_i\}_{i\in I}$ the maximal atlas. A subset $\mathcal{A} = \{U_j, \phi_j\}_{j\in J}$, $J \subset I$, of $\mathcal{M}$ is called an *affine atlas* of X if all transition maps are complex affine transformation of $\mathbb{C}^n$. We say that each maximal affine atlas defines a complex *affine structure* of X.

Theorem 4.9 (Gunning [16], Matsushima [25]). *There is a one-to-one correspondence between the set of all complex affine structures on a complex manifold* X *and the set of all locally flat holomorphic affine connections on* X.

Remark 4.10. For any $x, y \in M^p$, $\mathrm{exph}_y \circ \mathrm{exph}_x^{-1}$ is an affine transformation of $\mathbb{C}^n$. Thus M^p has a complex affine structure and the Bochner connection ∇^p is the corresponding locally flat holomorphic affine connection.

5. Geodesics of the Bochner connection ∇^p

5.1. Incompleteness of ∇^p. The behavior of geodesics of ∇^p played an important role in the proof of the following theorem, which generalizes Fefferman's extension theorem.

THEOREM 5.1 (Webster [30]). *Let $f : \Omega \to \widetilde{\Omega}$ be a biholomorphism between bounded domains with smooth boundaries. Suppose that their Bergman kernels are smooth up to the boundaries. Then f extends smoothly to a dense open subset of $\partial\Omega$.*

REMARK 5.2. In the proof, Webster used incompleteness of ∇^p, that is, geodesics of ∇^p extend through the boundary points.

On the other hand, one might expect to find a suitable Kähler metric, compatible with ∇^p. But this is impossible because the exponential map of a Kähler metric is holomorphic if and only if the metric is flat (the Euclidean metric). However, using the image of geodesics under $\mathrm{rep}_p = \mathrm{exph}_p^{-1}$, it is possible to define a distance between two points in a connected manifold M^p.

DEFINITION 5.3 (Intrinsic distance). Let M be a connected manifold with the Bergman metric and $\mathcal{A} = \{U_i, \phi_i\}_{i\in I}$ the affine structure of M^p, given by the Bochner normal coordinate system. If $x, y \in U_i$ for some U_i, then define $\delta^p(x, y)$ to be the Euclidean norm of the vector

$$\left(\dots, \frac{\partial}{\partial \overline{w_k}}\bigg|_{w=p} \log K(x, \overline{w}) - \frac{\partial}{\partial \overline{w_k}}\bigg|_{w=p} \log K(y, \overline{w}), \dots\right).$$

In general, if x, y are arbitrary points in M^p, then define the *intrinsic distance* by

$$\mathrm{d}^p(x, y) := \inf \sum_{j=1}^{N} \delta^p(p_{j-1}, p_j),$$

where the infimum is taken over all possible partitions with $x = p_0, \dots, p_N = y$.

This is well-defined since there always exists a broken geodesic between two points in the connected affine manifold (M^p, ∇^p).

REMARK 5.4. For the symmetry $\mathrm{d}^p(x, y) = \mathrm{d}^p(y, x)$, we do not use the normalization factor of the Bochner normal coordinate system. Although d^p is not a biholomorphic invariant, its finiteness between two points is a biholomorphic invariant.

The following theorem shows the relation between the intrinsic distance d^p and the analytic variety Z_0^p.

THEOREM 5.5. *If $q \in Z_0^p$, then there are no geodesics toward q such that the intrinsic distance is finite.*

Proof. Suppose that there exists a geodesic toward q such that the intrinsic distance is finite. Then the Bochner normal coordinate system is well-defined at q. Note that $\frac{\partial}{\partial \overline{w_k}} K(q, \overline{w})/K(q, \overline{w})$ is an anti-holomorphic function in w for each k. Fix an index k; then in the w_k-section, this is a function of one variable. Since $K(q, \overline{p}) = 0$, it has a simple pole at p so that the value is infinite. This implies that δ^p is infinite, which contradicts the finiteness of the intrinsic distance. ■

5.2. Examples. Let Ω be a bounded domain in $\mathbb{C}^n$. Denote by ∇ the Levi-Civita connection of the Bergman metric of Ω. Given a point $p \in \Omega$, denote the Bochner connection by ∇^p. In this section, we show the difference between the connections ∇ and ∇^p by comparing their geodesics.

5.2.1. *Unit ball.* Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$. Then the Riemannian exponential map of $\mathbb{B}^n$ at 0 is

$$\exp_0(\zeta) = \frac{\tanh(|\zeta|)}{|\zeta|} \cdot \zeta = \left(\frac{\tanh(|\zeta|)}{|\zeta|}\zeta_1, \dots, \frac{\tanh(|\zeta|)}{|\zeta|}\zeta_n\right),$$

where $\zeta = (\zeta_1, \dots, \zeta_n)$ is the standard complex coordinate for $\mathbb{C}^n \cong T_0\mathbb{B}^n$. On the other hand, the representative map at 0 is

$$\mathrm{rep}_0(z) = \sqrt{n+1} \cdot z = (\sqrt{n+1}\, z_1, \dots, \sqrt{n+1}\, z_n).$$

Therefore, in the unit ball, geodesics of ∇ and ∇^0 at 0 move along the same direction but with different speed.

5.2.2. *Unit polydisk.* Let $\mathbb{D}^n$ be the unit polydisk in $\mathbb{C}^n$. Then the Riemannian exponential map of $\mathbb{D}^n$ at 0 is

$$\exp_0(\zeta) = \left(\frac{\tanh(|\zeta_1|)}{|\zeta_1|}\zeta_1, \dots, \frac{\tanh(|\zeta_n|)}{|\zeta_n|}\zeta_n\right),$$

where ζ is the standard complex coordinate for $\mathbb{C}^n \cong T_0D^n$.

On the other hand, the representative map at 0 is

$$\mathrm{rep}_0(z) = \sqrt{2} \cdot z = (\sqrt{2}\, z_1, \dots, \sqrt{2}\, z_n).$$

Unlike the unit ball, this shows that the geodesics of ∇ and ∇^0 at 0 move along different directions with different speed.

5.2.3. *Skwarczyński's annulus.* Let $A := \{z \in \mathbb{C} : 0 < r < |z| < 1\}$. Then the Bergman kernel of A is

$$K(z, \overline{w}) = \frac{\wp(\log z\overline{w}) + \eta_1/\omega_1}{\pi z\overline{w}},$$

where $\wp$ is the Weierstrass elliptic function with half-periods $\omega_1 = \log(1/r)$ and $\omega_2 = \pi i$, and η_1 is the increment of the Weierstrass zeta function ζ with respect to ω_1.

The geodesics of the Levi-Civita connection ∇ were already studied in [17]. To study the geodesics of the Bochner connection ∇^p, we need the following:

Zeros of $K(z,\overline{p})$. Define $h(\lambda) := \wp(\log\lambda) + \eta_1/\omega_1$ on the set $\widetilde{A} := \{\lambda \in \mathbb{C} : r^2 < |\lambda| < 1\}$. Then $K(z,\overline{w}) = h(\lambda)/(\pi\lambda)$, where $\lambda = z\overline{w}$. In [28], Skwarczyński proved that:

- $h(\lambda)$ is real for all $\lambda \in \mathbb{R}$.
- For $r < e^{-2}$, there exists a point $\lambda \in \widetilde{A}$ such that $h(\lambda) = 0$.

Later, Błocki improved the above result as follows [4, Theorem 3.4]:

- $h(-1) = h(-r^2) < 0$ and $h(-r) > 0$.
- For $r < 1$, there exist only two solutions λ_1, λ_2 of the equation $h(\lambda) = 0$ in $\widetilde{A} = \{\lambda \in \mathbb{C} : r^2 < |\lambda| < 1\}$, where $\lambda_2 \in (-1,-r)$ and $\lambda_1 \in (-r,-r^2)$.

Fix $p \in A$. The symmetry of the annulus allows one to assume that $p \in (r,1)$ on the real line. Let λ_1^p and λ_2^p be the solutions of the equation $h(z\overline{p}) = 0$ satisfying $\lambda_1 = \lambda_1^p\overline{p}$ and $\lambda_2 = \lambda_2^p\overline{p}$. Since $\lambda_2^p \in (-1/p,-r/p)$ and $\lambda_1^p \in (-r/p,-r^2/p)$, the number of elements of the zero set $\{z \in A : K(z,\overline{p}) = 0\}$ depends on the location of p (for details, see [19, Corollary 2.5]). For example, if p is close enough to 1 (or r), then there exists only one solution of $K(z,\overline{p}) = 0$ in A, located in $(-1,-r)$ (or $(-r,-r^2)$).

Geodesics of the Bochner connection ∇^p. Recall that the exponential map of the Bochner connection ∇^p is the inverse of rep_p. A simple computation shows that

$$\mathrm{rep}_p(z) = C_1 \cdot \frac{\wp'(\log z\overline{p})}{\wp(\log z\overline{p}) + \eta_1/\omega_1} + C_2,$$

where C_1 and C_2 are constants, and $\wp'$ is the derivative of the Weierstrass elliptic function $\wp$. Then rep_p is also an elliptic function that shares the same periods with $\wp$ and has three simple poles. Since the image of any elliptic function contains $\mathbb{C}$, the pre-image of each straight line consists of three curves (counting multiplicity) in the annulus $\{z \in \mathbb{C} : r^2 < |z\overline{p}| < 1\}$.

Note that the geodesics of ∇^p are the images of straight lines under the holomorphic exponential map $\mathrm{exph}_p = \mathrm{rep}_p^{-1}$. Therefore, it is enough to study the pre-images of straight lines under the elliptic function $f(\lambda) := \frac{\wp'(\lambda)}{\wp(\lambda)+c}$, where the lattice of periods is $\Lambda = \{\mathbb{Z}(2\omega_1) + \mathbb{Z}(2\omega_2)\}$. Since $\omega_1 \in \mathbb{R}$ and $\omega_2 \in i\mathbb{R}$, the function $\wp$ and its derivative $\wp'$ have rectangular lattice so that $\wp(z) = \overline{\wp(\overline{z})}$ and $\wp'(z) = \overline{\wp'(\overline{z})}$. This implies that for all $t \in \mathbb{R}$,

- $\wp(t2\omega_j) = \overline{\wp(\overline{t2\omega_j})} = \overline{\wp(t2\omega_j)}$ and $\wp(\omega_i + t2\omega_j) = \overline{\wp(\omega_i + t2\omega_j)}$,
- $\wp'(t2\omega_j) = \pm\overline{\wp'(t2\omega_j)}$ and $\wp'(\omega_i + t2\omega_j) = \pm\overline{\wp'(\omega_i + t2\omega_j)}$.

Since $c = \eta_1/\omega_1 \in \mathbb{R}$, we see that

- $f(\mathbb{R}) \subset \mathbb{R}$ and $f(i\mathbb{R}) \subset i\mathbb{R}$.
- $f(\mathbb{R} + \omega_2) \subset \mathbb{R}$ and $f(\omega_1 + i\mathbb{R}) \subset i\mathbb{R}$.

Therefore, $f^{-1}(\mathbb{C} - (\mathbb{R} \cup i\mathbb{R}))$ consists of four open subrectangles which are divided by ω_1 and ω_2 in the fundamental region $\{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 2\omega_1, 0 \le \mathrm{Im}(z) \le \mathrm{Im}(2\omega_2)\}$. This provides an approximate, but useful, information on the location of geodesics.

6. On the variety $Z_0^p \cup Z_1^p$. Let Ω be a bounded domain in $\mathbb{C}^n$ and $K(z, \overline{w})$ the Bergman kernel of Ω. Fix $p \in \Omega$. Then the Bochner connection ∇^p can be constructed over $\Omega^p = \Omega - (Z_0^p \cup Z_1^p)$. Whether Z_0^p is empty is related to the well-known Lu Qi-Keng conjecture; a bounded domain Ω is called a *Lu Qi-Keng domain* if Z_0^p is empty for each $p \in \Omega$.

REMARK 6.1. It was anticipated in the 1960's that every bounded domain should be a Lu Qi-Keng domain, called the *Lu Qi-Keng conjecture*. However, many counterexamples have been discovered ([28], [5, 6], and others), and in contrast, many domains are Lu Qi-Keng.

On the other hand, note that

$$\det[G(z, \overline{p})] = \frac{\det\big[K(z, \overline{p}) \frac{\partial^2}{\partial z_i \partial \overline{w_j}}\big|_{w=p} K(z, \overline{w}) - \frac{\partial}{\partial z_i} K(z, \overline{p}) \frac{\partial}{\partial \overline{w_j}}\big|_{w=p} K(z, \overline{w})\big]}{K(z, \overline{p})^{2n}}.$$

Set

$$\widehat{Z}_1^p := \left\{ z \in \Omega : \det\left[K(z, \overline{p}) \frac{\partial^2 K(z, \overline{w})}{\partial z_i \partial \overline{w_j}}\bigg|_{w=p} - \frac{\partial K(z, \overline{p})}{\partial z_i} \frac{\partial K(z, \overline{w})}{\partial \overline{w_j}}\bigg|_{w=p} \right] = 0 \right\}.$$

It is known that if $n > 1$, then $Z_0^p \subset \widehat{Z}_1^p$, and the statement is false if $n = 1$ [12, Lemma 2.1]. In particular, $Z_0^p \not\subseteq \widehat{Z}_1^p$ for the annulus in the complex plane. Therefore, it is natural to ask whether a domain of higher dimension satisfying $Z_0^p \subsetneq \widehat{Z}_1^p$ exists.

THEOREM 6.2. *There exists a smooth bounded strongly pseudoconvex domain Ω satisfying $Z_0^p \subsetneq \widehat{Z}_1^p$ for some $p \in \Omega$.*

Proof. We first prove that when r is small enough, the product domain $A_r \times D$ has a point p such that $Z_0^p \subsetneq \widehat{Z}_1^p$, where $A_r := \{z \in \mathbb{C} : 0 < r < |z| < 1\}$ is the annulus and D is the unit disk in $\mathbb{C}$.

Let $K((z_1, z_2), (\overline{w_1}, \overline{w_2})) = K_{A_r}(z_1, \overline{w_1}) K_D(z_2, \overline{w_2})$ be the Bergman kernel of $A_r \times D$. Set

$$F_\Omega(z, \overline{w}) := \det\left[K_\Omega(z, \overline{w}) \frac{\partial^2}{\partial z_i \partial \overline{w_j}} K_\Omega(z, \overline{w}) - \frac{\partial}{\partial z_i} K_\Omega(z, \overline{w}) \frac{\partial}{\partial \overline{w_j}} K_\Omega(z, \overline{w}) \right].$$

Then

$$F_{A_r \times D}((z_1, z_2), (\overline{w_1}, \overline{w_2})) = K_{A_r}(z_1, \overline{w_1})^2 K_D(z_2, \overline{w_2})^2 F_{A_r}(z_1, \overline{w_1}) F_D(z_2, \overline{w_2}).$$

Since $K_D(z_2, \overline{w_2})^2 F_D(z_2, \overline{w_2}) \neq 0$ for all $(z_2, \overline{w_2}) \in D \times \overline{D}$, it suffices to show that there exists a point $(z,\overline{p}) \in A_r \times \overline{A_r}$ satisfying $K_{A_r}(z,\overline{p}) \neq 0$ and

$$K_{A_r}(z,\overline{p})^2 F_{A_r}(z,\overline{p}) = K_{A_r}(z,\overline{p})^4 \partial_z \overline{\partial_w}|_{w=p} \log K_{A_r}(z,\overline{w}) = 0.$$

It is known that if r is sufficiently close to 0, and p is on the real axis, such a point $(z,\overline{p})$ exists, where z is near the imaginary axis (see [12, proof of Theorem 1.5]).

Now we modify this example to the case of irreducible strictly pseudoconvex domains as follows: Consider a strictly pseudoconvex exhaustion Ω_j for $A_r \times D$. Note that these are irreducible domains [18]. On the other hand, the Bergman kernel of Ω_j and its derivatives uniformly converge on compacta to those of $A_r \times D$ [26]. By Hurwitz's theorem, Ω_j satisfies $Z_0^p \subsetneq \widehat{Z}_1^p$ when j is large enough. ∎

7. A generalization of the Lu theorem. We present an application of the Bochner connection. Let Ω be a bounded domain in $\mathbb{C}^n$ and M a complex manifold with the positive-definite Bergman metric. Denote their Bergman metric by β_Ω and β_M, respectively. Call the point $p \in \Omega$ a *pole of the Bochner connection* ∇^p whenever $\mathrm{rep}_p : \Omega \to \mathbb{C}^n$ is one-to-one.

THEOREM 7.1. *Suppose that Ω has a pole p of ∇^p. If there is a surjective holomorphic map $f : \Omega \to M$ satisfying $f^*\beta_M = \beta_\Omega$, then f is a biholomorphism.*

This theorem is a generalization of the following well-known result.

THEOREM 7.2 (Lu Qi-Keng [23]). *If Ω is a bounded domain in $\mathbb{C}^n$ whose Bergman metric is complete and has constant holomorphic sectional curvature, then Ω is biholomorphic to the unit ball.*

Proof that Theorem 7.1 ⇒ Theorem 7.2. Although the proof below is included in [13, proof of Theorem 4.2.2], we recall it for convenience. Let c be the (constant) holomorphic sectional curvature of Ω.

If $c > 0$, then Ω would be a complete Riemannian manifold with all sectional curvatures $\geq c/4 > 0$. Thus Myers' theorem in Riemannian geometry implies that Ω is compact, a contradiction.

If $c = 0$, then the covering space is $\mathbb{C}^n$. Therefore the covering map is constant by Liouville's theorem, which is impossible.

Consequently, $c < 0$. In that case, it is known that the universal covering space of Ω is biholomorphic to the unit ball $\mathbb{B}^n$ and the covering map $f : \mathbb{B}^n \to \Omega$ is a Bergman isometry. Therefore f has to be one-to-one by Theorem 7.1, and hence the conclusion follows. ∎

REMARK 7.3. Notice that Theorem 7.1 does not assume the completeness of the Bergman metric β_Ω. Moreover, the bounded domain Ω need not have constant holomorphic sectional curvature. Besides the unit ball, the following domains satisfy the hypothesis of Theorem 7.1:

- Every complete circular domain; the center is a pole.
- Every bounded homogeneous domain; every point is a pole [32].

More generally, every bounded domain which has a point p such that the matrix $G(z,\overline{p})$ is independent of z, satisfies the hypothesis (the point p is a pole). This is called a *representative domain* [24].

With a slight modification of [13, proof of Theorem 4.2.2], we present

Proof of Theorem 7.1. We have only to show that f is one-to-one. Since $f^*\beta_M = \beta_\Omega$ implies that df is non-singular, f is locally invertible. Let V be a neighborhood of p, and U a neighborhood of $q := f(p)$ such that $f|_V : V \to U$ is a biholomorphism. Denote by g_0 the inverse of $f|_V$.

On the other hand, $\nabla_\Omega = f^*\nabla_M$, since $f^*\beta_M = \beta_\Omega$, where ∇ denotes the Bergman metric connection. The uniqueness of the polarization and the holomorphicity of f yield $\nabla^p_\Omega = f^*\nabla^q_M$. This means that f maps geodesics of ∇^p to geodesics of ∇^q, and one sees that $A := \mathrm{rep}_q \circ f|_V \circ \mathrm{rep}_p^{-1}$ is $\mathbb{C}$-linear as in Remark 4.3. Thus $g_0 := f|_V^{-1} = \mathrm{rep}_p^{-1} \circ A \circ \mathrm{rep}_q$, where A is an invertible $\mathbb{C}$-linear map.

Note that $\mathrm{rep}_q \circ f = A^{-1} \circ \mathrm{rep}_p$ on $\Omega - (Z^p \cup f^{-1}(Z^q))$, where $Z^p := Z^p_0 \cup Z^p_1$ and $Z^q := Z^q_0 \cup Z^q_1$. Then the restriction map $\mathrm{rep}_p^{-1}|_{A \circ \mathrm{rep}_q(M - (f(Z^p) \cup Z^q))}$ is a well-defined holomorphic map. Since A is everywhere defined and rep_q extends to a holomorphic mapping of $M - Z^q$, so does g_0. Denote by g the extension of g_0.

Let $X := f^{-1}(Z^q)$. Then $g \circ f : \Omega - X \to \mathbb{C}^n$ is holomorphic and $g \circ f(z) = z$ for every $z \in \Omega - X$. Therefore, for every $\zeta \in M - Z^q$, choose $x \in \Omega$ such that $f(x) = \zeta$. Since $g(\zeta) = g(f(x)) = x$, we have $g(M - Z^q) \subset \Omega$. Note that g is a bounded holomorphic map on the connected manifold $M - Z^q$. By the Riemann extension theorem, g extends to a holomorphic mapping of M into $\mathbb{C}^n$. This shows that g is the left inverse to f, and hence f is one-to-one. ∎

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Sungmin Yoo
Department of Mathematics
Pohang University of Science and Technology
37673 Pohang, Republic of Korea
E-mail: sungmin@postech.ac.kr