

## A note on maximal plurifinely plurisubharmonic functions

MOHAMED EL KADIRI (Salé)

**Abstract.** We improve some known results on plurifinely locally maximal plurifinely plurisubharmonic functions. We prove in particular that any locally bounded plurifinely locally maximal plurifinely plurisubharmonic function is maximal, and that any finite plurisubharmonic function on an open subset of  $\mathbb{C}^n$  is maximal if and only if it is plurifinely locally maximal.

**1. Introduction.** The plurifine topology  $\mathcal{F}$  on a Euclidean open set  $\Omega \subset \mathbb{C}^n$  is the smallest topology that makes all plurisubharmonic functions on  $\Omega$  continuous. This construction is completely analogous to the better known fine topology in classical potential theory of H. Cartan. Good references for the latter are [AG, D]. The topology  $\mathcal{F}$  was introduced in [F5], and studied e.g. by Bedford and Taylor [BT], and by El Marzguioui and Wiegerinck [EMW1, EMW2], who proved in particular that this topology is locally connected. Notions relating to the topology  $\mathcal{F}$  are provided with the prefix  $\mathcal{F}$ , e.g. an  $\mathcal{F}$ -domain is an  $\mathcal{F}$ -open set that is connected in  $\mathcal{F}$ .

Just as one introduces finely subharmonic functions on fine domains in  $\mathbb{R}^n$  (see Fuglede's book [F1]), one can introduce plurifinely plurisubharmonic functions on  $\mathcal{F}$ -domains in  $\mathbb{C}^n$ . These functions are called  $\mathcal{F}$ -*plurisubharmonic* (or  $\mathcal{F}$ -psh). In case  $n = 1$ , we recover the finely subharmonic functions on fine domains in  $\mathbb{R}^2$ .

The definition of  $\mathcal{F}$ -plurisubharmonic functions on an  $\mathcal{F}$ -open set of  $\mathbb{C}^n$  was first given in [EK] and [EMW2], where some properties of these functions were studied. The  $\mathcal{F}$ -continuity of  $\mathcal{F}$ -plurisubharmonic functions was established in [EMW3].

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In [EFW], the most important properties of  $\mathcal{F}$ -plurisubharmonic functions were obtained. That paper included a convergence theorem, and the characterization of  $\mathcal{F}$ -plurisubharmonic functions as  $\mathcal{F}$ -locally bounded finely subharmonic functions with the property that they remain finely subharmonic under composition with  $\mathbb{C}$ -isomorphisms of  $\mathbb{C}^n$ . Hence the most important properties of plurisubharmonic functions on Euclidean open sets of  $\mathbb{C}^n$  were extended to  $\mathcal{F}$ -plurisubharmonic functions on  $\mathcal{F}$ -open sets of  $\mathbb{C}^n$ .

In [EKW], the authors obtained a local approximation of  $\mathcal{F}$ -plurisubharmonic functions by plurisubharmonic functions outside a pluripolar set. They also defined the Monge–Ampère measure for finite  $\mathcal{F}$ -plurisubharmonic functions on an  $\mathcal{F}$ -domain  $U$ . This construction was based on the fact (established in [EMW3]) that such a function can be  $\mathcal{F}$ -locally represented as a difference of two bounded plurisubharmonic functions defined on some Euclidean open set, and on the quasi-Lindelöf property of the plurifine topology. The local approximation property allowed them to prove that this Monge–Ampère measure is a positive Borel measure on  $U$  which is  $\mathcal{F}$ -locally finite and does not charge pluripolar sets. It is  $\sigma$ -finite by the quasi-Lindelöf property of the plurifine topology.

In the theory of plurisubharmonic functions on a Euclidean domain, the so-called maximal functions are the analog of harmonic functions in classical potential theory. They play an important role in the resolution of the Dirichlet problem for the Monge–Ampère operator. El Kadiri and Smit [EKS] introduced the notion of  $\mathcal{F}$ -maximal  $\mathcal{F}$ -plurisubharmonic functions, extending the notion of maximal plurisubharmonic functions on a Euclidean domain to an  $\mathcal{F}$ -domain, and they studied some basic properties of these functions. They also studied the connection between  $\mathcal{F}$ -maximality of functions and the Monge–Ampère operator. In particular, they proved that a finite  $\mathcal{F}$ -plurisubharmonic function  $u$  on an  $\mathcal{F}$ -domain  $U$  satisfies  $(dd^c u)^n = 0$  if and only if  $u$  is  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal outside some pluripolar set.

The problem of whether an  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal  $\mathcal{F}$ -plurisubharmonic function is  $\mathcal{F}$ -maximal was settled in [EKS]. In [HV1] and [HV2], it is proved that any continuous or bounded  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal  $\mathcal{F}$ -plurisubharmonic function is  $\mathcal{F}$ -maximal.

In the present note we shall prove that, more generally, any locally bounded  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal  $\mathcal{F}$ -plurisubharmonic function is  $\mathcal{F}$ -maximal. As a consequence, a finite  $\mathcal{F}$ -plurisubharmonic function  $u$  on an  $\mathcal{F}$ -open subset of  $\mathbb{C}^n$  is  $\mathcal{F}$ -locally maximal if and only if  $(dd^c u)^n = 0$ , and any finite  $\mathcal{F}$ -locally maximal function on a Euclidean open subset of  $\mathbb{C}^n$  is maximal, improving some results in [EKS]. The last result cannot be improved, as shown by [EKS, Example 4.19].

**2. Preliminaries.** Some elements of (resp. plurifine) pluripotential theory that will be used can be found in [BT, W]. The *plurifine topology*  $\mathcal{F}$  on a Euclidean open set of  $\mathbb{C}^n$  is the smallest topology that makes all plurisubharmonic functions on it continuous.

For this article let  $n$  be an integer  $\geq 1$ . If  $A \subset \mathbb{C}^n$ , we denote the closure of  $A$  in the Euclidean, respectively plurifine topologies by  $\bar{A}$  and  $\bar{A}^{\mathcal{F}}$ . The Euclidean boundary and the  $\mathcal{F}$ -boundary of  $A$  are denoted by  $\partial A$  and  $\partial_{\mathcal{F}} A$  respectively. For a function  $f : A \rightarrow \bar{\mathbb{R}}$ , we denote by  $\limsup_{x \in A, x \rightarrow y} f(x)$  and  $\mathcal{F}\text{-lim sup}_{x \in A, x \rightarrow y} f(x)$  the lim sup with respect to the Euclidean topology of  $\mathbb{C}^n$  and the plurifine topology of  $\mathbb{C}^n$  respectively, and likewise for other limits.

DEFINITION 2.1. Let  $U$  be an  $\mathcal{F}$ -open set in  $\mathbb{C}^n$ . A function  $u : U \rightarrow [-\infty, +\infty)$  is said to be  $\mathcal{F}$ -*plurisubharmonic* if it is  $\mathcal{F}$ -upper semicontinuous, and for every complex line  $l$  in  $\mathbb{C}^n$ , the restriction of  $u$  to any  $\mathcal{F}$ -component of the finely open subset  $l \cap U$  of  $l$  is either finely subharmonic or  $= -\infty$ . The set of all  $\mathcal{F}$ -plurisubharmonic functions in  $U$  is denoted by  $\mathcal{F}\text{-PSH}(U)$ .

DEFINITION 2.2. Let  $U$  be an  $\mathcal{F}$ -open set in  $\mathbb{C}^n$  and let  $\text{QB}(U)$  be the trace of  $\text{QB}(\mathbb{C}^n)$  on  $U$ , where  $\text{QB}(\mathbb{C}^n)$  denotes the  $\sigma$ -algebra on  $\mathbb{C}^n$  generated by the Borel sets and the pluripolar subsets of  $\mathbb{C}^n$ . Assume that  $u \in \mathcal{F}\text{-PSH}(U)$  is finite. By the quasi-Lindelöf property of the plurifine topology and [EKW, Theorem 2.17], there exist a pluripolar set  $E \subset U$ , a sequence  $\{O_j\}$  of  $\mathcal{F}$ -open subsets of  $U$  and bounded plurisubharmonic functions  $f_j, g_j$  defined on Euclidean open neighborhoods of  $\bar{O}_j$  such that  $U = E \cup \bigcup_{j=1}^{\infty} O_j$  and  $u = f_j - g_j$  on  $O_j$ . We define  $O_0 := \emptyset$  and

$$(2.1) \quad \int_A (dd^c u)^n := \sum_{j=1}^{\infty} \int_{A \cap (O_j \setminus \bigcup_{k=0}^{j-1} O_k)} (dd^c (f_j - g_j))^n, \quad A \in \text{QB}(U).$$

By [EKW, Theorem 3.6], the measure defined by (2.1) is independent of  $E, \{O_j\}, \{f_j\}$  and  $\{g_j\}$ . It is called the *complex Monge–Ampère measure* for the  $\mathcal{F}$ -plurisubharmonic function  $u$ .

THEOREM 2.3 ([EKW, Theorem 4.6]). *Let  $U$  be an  $\mathcal{F}$ -open set in  $\mathbb{C}^n$  and let  $u \in \mathcal{F}\text{-PSH}(U)$  be finite. Then  $(dd^c u)^n$  is a nonnegative  $\mathcal{F}$ -locally finite measure on  $\text{QB}(U)$  (cf. [EKW, Remark 4.7]).*

**3. Maximal and  $\mathcal{F}$ -locally maximal  $\mathcal{F}$ -plurisubharmonic functions.** In analogy with maximal plurisubharmonic functions, which play a role in pluripotential theory comparable to that of harmonic functions in classical potential theory, we will introduce  $\mathcal{F}$ -maximal  $\mathcal{F}$ -plurisubharmonic functions. These relate similarly to finely harmonic functions and constitute

the plurifine analog of maximal plurisubharmonic functions on Euclidean open sets.

DEFINITION 3.1. Let  $U \subset \mathbb{C}^n$  be an  $\mathcal{F}$ -open set and let  $u \in \mathcal{F}\text{-PSH}(U)$ . We say that  $u$  is  $\mathcal{F}$ -maximal if for every bounded  $\mathcal{F}$ -open set  $G$  of  $\mathbb{C}^n$  such that  $\overline{G} \subset U$ , and for every function  $v \in \mathcal{F}\text{-PSH}(G)$  that is bounded from above on  $G$  and extends  $\mathcal{F}$ -upper semicontinuously to  $\overline{G}^{\mathcal{F}}$ , the following holds:  $v \leq u$  on  $\partial_{\mathcal{F}}G \Rightarrow v \leq u$  on  $G$ . We denote by  $\mathcal{F}\text{-MPSH}(U)$  the set of  $\mathcal{F}$ -maximal  $\mathcal{F}$ -plurisubharmonic functions on  $U$ .

THEOREM 3.2 ([EKS, Theorem 4.8]). *Let  $U$  be an  $\mathcal{F}$ -open set in  $\mathbb{C}^n$  and let  $u$  be a finite  $\mathcal{F}$ -maximal  $\mathcal{F}$ -plurisubharmonic function in  $U$ . Then  $(dd^c u)^n = 0$  in  $\text{QB}(U)$ .*

DEFINITION 3.3. An  $\mathcal{F}$ -psh function  $u$  in an  $\mathcal{F}$ -open set  $U$  of  $\mathbb{C}^n$  is said to be  $\mathcal{F}$ -locally maximal if each point of  $U$  has an  $\mathcal{F}$ -open neighborhood  $V$  such that the restriction of  $u$  to  $V$  is  $\mathcal{F}$ -maximal.

It is clear that an  $\mathcal{F}$ -maximal  $\mathcal{F}$ -psh in an  $\mathcal{F}$ -domain  $U$  of  $\mathbb{C}^n$  is  $\mathcal{F}$ -locally maximal in  $U$ . The question of whether the converse is true was raised in [EKS]. A partial answer is given by

PROPOSITION 3.4 ([HV1, Proposition 3.2]). *Let  $U$  be an  $\mathcal{F}$ -open set in  $\mathbb{C}^n$  and  $u \in \mathcal{F}\text{-PSH}(U)$ . If  $u$  is a bounded  $\mathcal{F}$ -locally maximal function on  $U$ , then  $u$  is  $\mathcal{F}$ -maximal on  $U$ .*

The following result is more general than the main theorems of [HV1] and [HV2]:

THEOREM 3.5. *Let  $U$  be an  $\mathcal{F}$ -open set in  $\mathbb{C}^n$  and  $u \in \mathcal{F}\text{-PSH}(U)$ . If  $u$  is locally bounded (relative to the Euclidean topology)  $\mathcal{F}$ -locally maximal function on  $U$ , then  $u$  is  $\mathcal{F}$ -maximal on  $U$ .*

*Proof.* Suppose that  $u$  is a locally bounded  $\mathcal{F}$ -locally maximal function in  $U$ . By the Lindelöf property there is an increasing sequence  $(V_j)$  of Euclidean open sets such that  $U \subset \bigcup_j V_j$  and  $u$  is bounded on each  $\mathcal{F}$ -open set  $V_j \cap U$ . Since  $u$  is  $\mathcal{F}$ -locally maximal and bounded on  $V_j \cap U$  for any  $j$ , it follows from Proposition 3.4 that  $u$  is  $\mathcal{F}$ -maximal on each  $V_j \cap U$ . Now let  $G$  be a bounded  $\mathcal{F}$ -open subset of  $\mathbb{C}^n$  such that  $\overline{G} \subset U$ . Then there exists  $j \in \mathbb{N}$  such that  $\overline{G} \subset V_j \cap U$ . Let  $v$  be an  $\mathcal{F}$ -upper semicontinuous function on  $\overline{G}^{\mathcal{F}}$  that is bounded from above and such that  $v \in \mathcal{F}\text{-PSH}(G)$  and  $v \leq u$  on  $\partial_{\mathcal{F}}G$ . Then  $v \leq u$  on  $G$  because  $u$  is  $\mathcal{F}$ -maximal on  $U \cap V_j$ . It follows that  $u$  is  $\mathcal{F}$ -maximal. ■

As a corollary we obtain the following result of Hong and Viet [HV1, Main Theorem]:

COROLLARY 3.6. *Let  $U$  be an  $\mathcal{F}$ -open set in  $\mathbb{C}^n$  and  $u \in \mathcal{F}\text{-PSH}(U)$ . If  $u$  is a continuous  $\mathcal{F}$ -locally maximal function on  $U$ , then  $u$  is  $\mathcal{F}$ -maximal on  $U$ .*

The following corollaries follow easily from Theorem 3.5 and [EKS, Proposition 2.5, Theorems 4.1 and 4.14]:

COROLLARY 3.7. *Let  $u$  be a locally bounded  $\mathcal{F}$ -psh function on an  $\mathcal{F}$ -open subset  $U$  of  $\mathbb{C}^n$ . Then  $u$  is  $\mathcal{F}$ -maximal if and only if  $(dd^c u)^n = 0$ .*

COROLLARY 3.8. *Let  $u$  be a continuous  $\mathcal{F}$ -psh function on an  $\mathcal{F}$ -open subset  $U$  of  $\mathbb{C}^n$ . Then  $u$  is  $\mathcal{F}$ -maximal if and only if  $(dd^c u)^n = 0$ .*

**4. The case  $n = 1$ .** For more details on finely superharmonic functions on a finely open subset of  $\mathbb{C}^n \cong \mathbb{R}^{2n}$  we refer the reader to the monograph of Fuglede [F1]. We recall here the definition of the sweeping in a fine domain  $U$  of  $\mathbb{C}$ . Let  $U$  be a fine domain of  $\mathbb{C}$ ,  $f : U \rightarrow [0, +\infty]$  and  $A \subset U$ . Then we define

$$R_f^A := \inf \{s \in \mathcal{S}^+(U) \cup \{+\infty\} : s \geq f \text{ on } A\}$$

and let  $\widehat{R}_f^A$  be the finely lower semicontinuous regularized function of  $R_f^A$  on  $U$ , that is,  $\widehat{R}_f^A(z) = \mathcal{F}\text{-}\liminf_{y \rightarrow z} R_f^A(z)$  for every  $z \in U$ , where  $\mathcal{S}^+(U)$  denotes the cone of nonnegative finely superharmonic functions on  $U$ .

A function  $s \in \mathcal{S}^+(U)$  is called *invariant* if there exists a sequence  $(V_j)$  of finely open subsets of  $U$  such that  $\bigcup_j V_j = U$ ,  $\overline{V_j} \subset U$  and  $\widehat{R}_u^{U \setminus V_j} = u$  for every  $j$ .

In [EKS, Proposition 2.10] we have proved that for any invariant function  $u$  on a finely open subset of  $\mathbb{C}$ , the function  $-u$  is  $\mathcal{F}$ -locally maximal. Conversely, we have the following:

THEOREM 4.1. *Let  $U$  be a regular finely open subset of  $\mathbb{C}$  and  $u$  a non-negative finely superharmonic function on  $U$ . If  $-u$  is maximal then  $u$  is invariant.*

*Proof.* We may suppose that  $U$  is a fine domain of  $\mathbb{C}$ . Suppose that  $-u$  maximal. By the Riesz decomposition of  $u$ , there exist a fine potential  $p$  on  $U$  and an invariant function  $h$  on  $U$  such that  $u = p + h$ . By [F3, Theorem 4.4] and [F4, Corollaire, p. 132], there exists an increasing sequence  $(V_j)$  of finely open subsets of  $U$  such that  $\bigcup_j V_j = U$ ,  $\overline{V_j} \subset V_{j+1}$  for each  $j$ ,  $\inf_j \widehat{R}_p^{U \setminus V_j} = 0$  q.e. (that is, outside a polar subset of  $U$ ) and  $R_h^{U \setminus V_j} = h$  for any  $j$ . Let  $j > 0$  be an integer and  $s$  a finely superharmonic and nonnegative function on  $U$  such that  $s \geq u$  on  $U \setminus V_j$ . Since  $-u$  is maximal, we have  $u \leq s$  on  $V_j$ . We then deduce that  $u \leq R_u^{U \setminus V_j}$ , and hence  $u = R_u^{U \setminus V_j} = \widehat{R}_u^{U \setminus V_j}$ . It follows that

$p = \widehat{R}_p^{U \setminus V_j}$  for every  $j$ ; by passing to the infimum over  $j$ , we obtain  $p = 0$ , so that  $u = h$  is invariant. ■

**COROLLARY 4.2.** *Let  $u$  be a finely subharmonic function on a finely open subset  $U$  of  $\mathbb{C}$ , bounded from above by some constant  $c$ . Then  $u$  is maximal if and only if  $c - u$  is invariant.*

*Proof.* Apply Theorem 4.1 to the function  $c - u$ . ■

**REMARK 4.3.** 1. It follows from [F2, Théorème 2.3] that any finely harmonic function on  $U$  is  $\mathcal{F}$ -maximal. Conversely, any finite nonnegative  $\mathcal{F}$ -maximal function on  $U$  is finely harmonic because any finite invariant function is finely harmonic.

2. [EKS, Example 4.19] shows that the converse of [EKS, Proposition 2.10] is not true. However, we do not know if, in general, the converse of Theorem 4.1 is true.

**5.  $\mathcal{F}$ -local maximality and  $\mathcal{F}$ -maximality for finite  $\mathcal{F}$ -plurisubharmonic functions.** In [EKS, Example 4.19], it is proved that an  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal  $\mathcal{F}$ -psh function need not be  $\mathcal{F}$ -maximal. In the present section we study the  $\mathcal{F}$ -local maximality and  $\mathcal{F}$ -maximality for finite  $\mathcal{F}$ -psh functions.

Let us recall the following result:

**THEOREM 5.1** ([EKS, Theorem 4.15]). *Let  $U$  be an  $\mathcal{F}$ -open subset of  $\mathbb{C}^n$  and let  $u \in \mathcal{F}\text{-PSH}(U)$  be finite. Then  $(dd^c u)^n = 0$  if and only if  $u$  is  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal on the complement of an  $\mathcal{F}$ -closed pluripolar subset of  $U$ .*

The following result improves Theorem 5.1:

**THEOREM 5.2.** *Let  $u$  be a finite  $\mathcal{F}$ -psh function on an  $\mathcal{F}$ -open subset  $U$  of  $\mathbb{C}^n$ . Then  $u$  is  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal if and only if  $(dd^c u)^n = 0$ .*

*Proof.* The “only if” part follows from Theorem 5.1. Let us prove the “if” part. Suppose that  $(dd^c u)^n = 0$ . For any integer  $j > 0$  set  $V_j = \{-j < u < j\}$ . Then  $V_j$  is  $\mathcal{F}$ -open,  $u$  is bounded on  $V_j$ , and therefore  $u$  is maximal on  $V_j$  by Corollary 3.7. Since  $U = \bigcup_j V_j$ , it follows that  $u$  is  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal on  $U$ . ■

It is proved in [EKS, Theorem 4.22] that in a Euclidean open subset of  $\mathbb{C}^n$ , any finite plurisubharmonic function which is locally maximal is necessarily maximal. This leads to the following question:

**QUESTION 5.3.** Is any  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal finite  $\mathcal{F}$ -plurisubharmonic function on an  $\mathcal{F}$ -open subset of  $\mathbb{C}^n$   $\mathcal{F}$ -maximal?

The answer is yes if  $n = 1$ . Indeed, a finite  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal function on an  $\mathcal{F}$ -open subset  $U$  of  $\mathbb{C}$  is finely locally finely harmonic on  $U$ , and hence finely harmonic on  $U$  by [F1, 8.6, p. 70], that is,  $\mathcal{F}$ -maximal on  $U$ .

The following theorem improves [EKS, Theorem 4.22]:

**THEOREM 5.4.** *Let  $u$  be a finite psh function on an open subset  $U$  of  $\mathbb{C}^n$ . Then  $u$  is maximal if and only if  $u$  is  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal.*

*Proof.* We only have to prove the “if” part. Suppose that  $u$  is  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal. Then  $u$  is  $\mathcal{F}$ -maximal on each  $\mathcal{F}$ -open set  $V_j = \{-j < u\}$ ,  $j \in \mathbb{N}^*$ . Hence  $u$  is maximal on each  $V_j$  by Theorem 3.5, because it is locally bounded there ( $u_j$  being upper semicontinuous on  $U$  and hence on  $V_j$ ). It follows that  $(dd^c u)^n = 0$  on  $V_j$  for any  $j$ , so that  $(dd^c u)^n = 0$  on  $U$ . For each  $j$ , let  $u_j = \max\{u, -j\}$ . Then, as in [EKS, proof of Proposition 4.22],  $\lim (dd^c u_j)^n = 0$  weakly on  $U$ . It follows from [B, Theorem 4.4] that  $u$  is maximal. ■

**REMARK 5.5.** We cannot drop the hypothesis that  $u$  is finite in the above theorem: see [EKS, Example 4.19].

**COROLLARY 5.6.** *Let  $u$  be a finite plurisubharmonic function on an open subset of  $\mathbb{C}^n$ . Then  $u$  is maximal if and only if  $(dd^c u)^n = 0$ .*

*Proof.* The “only if” part follows from [EKS, Theorem 4.8]. If  $(dd^c u)^n = 0$ , then  $u$  is  $\mathcal{F}$ -locally maximal by Theorem 5.2, hence it is maximal by Theorem 5.4. ■

In [HV2] it is proved that if  $u$  is a bounded  $\mathcal{F}$ -psh function on an  $\mathcal{F}$ -open subset  $U$  of  $\mathbb{C}^n$ , then for any integer  $m \geq 1$  the function  $u \circ \pi$  is  $\mathcal{F}$ -maximal on  $U \times \mathbb{C}^m$  where  $\pi$  is the canonical projection from  $\mathbb{C}^{n+m}$  into  $\mathbb{C}^n$ . Here we generalize this result and give a simple and direct proof of it.

**PROPOSITION 5.7.** *Let  $u$  be a finite  $\mathcal{F}$ -psh function on an  $\mathcal{F}$ -open subset  $U$  of  $\mathbb{C}^n$ . Then for any integer  $m \geq 1$  the function  $u \circ \pi$  is  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal on  $U \times \mathbb{C}^m$ , where  $\pi : \mathbb{C}^{n+m} \rightarrow \mathbb{C}^n$  is the canonical projection.*

*Proof.* For any  $z \in U$ , by [EMW3, Theorem 3.1] there exist an  $\mathcal{F}$ -open neighborhood  $V_z$  of  $z$  and a Euclidean open set  $O_z$  such that  $z \in O_z$  and  $V_z \subset U$ , and bounded plurisubharmonic functions  $u_1$  and  $u_2$  on  $O_z$  such that  $u = u_1 - u_2$  on  $O_z \cap V_z$ . The functions  $u_1 \circ \pi$  and  $u_2 \circ \pi$  are maximal plurisubharmonic functions on  $O_z \times \mathbb{C}^m$  by [HH, Proposition 3.5], and we have  $u \circ \pi = u_1 \circ \pi - u_2 \circ \pi$  on  $(O_z \times \mathbb{C}^m) \cap (V_z \times \mathbb{C}^m)$  (this last set being an  $\mathcal{F}$ -open subset of  $\mathbb{C}^{n+m}$ ). Let  $G$  be a bounded  $\mathcal{F}$ -open subset of  $U \times \mathbb{C}^m$  such that  $G \subset \overline{G} \subset (O_z \times \mathbb{C}^m) \cap (V_z \times \mathbb{C}^m)$  and let  $v$  be an  $\mathcal{F}$ -psh function on  $G$  that can be extended upper semicontinuously to  $\overline{G}^{\mathcal{F}}$  with  $v \leq u \circ \pi$  on  $\partial_{\mathcal{F}} G$ . Then  $v + u_2 \circ \pi \leq u_1 \circ \pi$  on  $\partial_{\mathcal{F}} G$ , and hence  $v + u_2 \circ \pi \leq u_1 \circ \pi$  on  $G$  since

$u_1 \circ \pi$  is  $\mathcal{F}$ -maximal on  $O_z \times \mathbb{C}^m$  (and hence on  $(O_z \times \mathbb{C}^m) \cap (V_z \times \mathbb{C}^m)$ ). It follows that  $v \leq u \circ \pi$  on  $G$ . Hence  $u \circ \pi$  is  $\mathcal{F}$ -locally maximal on  $U \times \mathbb{C}^m$ . ■

**COROLLARY 5.8.** *Let  $u$  be a locally bounded  $\mathcal{F}$ -psh function on an  $\mathcal{F}$ -open subset  $U$  of  $\mathbb{C}^n$ . Then for any integer  $m \geq 1$  the function  $u \circ \pi$  is  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal on  $U \times \mathbb{C}^m$ , where  $\pi : \mathbb{C}^{n+m} \rightarrow \mathbb{C}^n$  is the canonical projection.*

*Proof.* The corollary is an immediate consequence of Proposition 5.7 and Theorem 3.5. ■

**LEMMA 5.9.** *Let  $(u_j)$  be a decreasing sequence of maximal psh functions on an  $\mathcal{F}$ -open  $U$  of  $\mathbb{C}^n$ . Then  $u = \inf u_j$  is maximal on  $U$ .*

*Proof.* Let  $G$  be a bounded  $\mathcal{F}$ -open subset of  $\mathbb{C}^n$  such that  $\overline{G} \subset U$ , and let  $v$  be an  $\mathcal{F}$ -upper semicontinuous function on  $\overline{G}^{\mathcal{F}}$  that is bounded from above such that  $v \in \mathcal{F}\text{-PSH}(G)$  and  $v \leq u$  on  $\partial_{\mathcal{F}}G$ . Then for any  $j$  we have  $v \leq u_j$  on  $G$  because  $u_j$  is  $\mathcal{F}$ -maximal on  $U$ . It follows that  $v \leq u$  on  $U$ . Hence  $u$  is  $\mathcal{F}$ -maximal on  $U$ . ■

**COROLLARY 5.10.** *Let  $u$  be a locally upper bounded  $\mathcal{F}$ -psh function on an  $\mathcal{F}$ -open subset  $U$  of  $\mathbb{C}^n$ . Then for any integer  $m \geq 1$  the function  $u \circ \pi$  is  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal on  $U \times \mathbb{C}^m$  where  $\pi : \mathbb{C}^{n+m} \rightarrow \mathbb{C}^n$  is the canonical projection.*

*Proof.* For  $j \in \mathbb{N}$ , the function  $u_j = \max\{u, -j\}$  is locally bounded on  $U$ , hence  $u_j \circ \pi$  is maximal in  $U$  by Corollary 5.8 and Theorem 3.5. It follows from Lemma 5.9 that  $u \circ \pi = \inf_j (u_j \circ \pi)$  is  $\mathcal{F}$ -maximal in  $U$ . ■

**COROLLARY 5.11.** *Let  $v$  be an invariant function on a finely open subset  $U$  of  $\mathbb{C}$ . Then for any integer  $m \geq 1$  the function  $u = -v \circ \pi$  is  $\mathcal{F}$ -locally  $\mathcal{F}$ -maximal on  $U \times \mathbb{C}^m$ , where  $\pi : \mathbb{C}^{1+m} \rightarrow \mathbb{C}^n$  is the canonical projection.*

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Mohamed El Kadiri  
Résidence Hasnae B1, Appt 14, rue Chorafa  
Salé-Tabriquet, Morocco  
E-mail: elkadiri@fsr.ac.ma

