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New recurrence relations and matrix equations for arithmetic functions generated by Lambert series

by

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1. Introduction

1.1. Overview and motivation. We consider new recurrence relations and matrix equations related to *Lambert series* expansions of the form ([3, §27.7], [1, §17.10])

$$L_f(q) := \sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \sum_{m\geq 1} g_f(m)q^m, \quad |q|<1, \tag{1}$$

for prescribed functions $f : \mathbb{Z}^+ \to \mathbb{C}$, and $g_f : \mathbb{Z}^+ \to \mathbb{C}$ defined by $g_f(m) = \sum_{d|m} f(d)$. Our new results provide *exact* matrix-based formulas for a wide range of classical special arithmetic functions expanded in well-known Lambert series of the form (1). The first form of the exact formulas for $f(n)$ is a matrix factorization of the following form for $n \geq 1$:

$$(f(k))_{1\leq k\leq n} = A_n^{-1} \cdot (B_m(f))_{0\leq m<n}. \tag{2}$$

The $n \times n$ matrices A_n in (2) are always independent of the function $f(n)$. Moreover, the right-hand-side vector with entries $B_m(f)$ is independent of the A_n and depends only on a finite sum determined by f for all $m, n \geq 0$.

The matrix equation (2) is a special case of more general *Lambert series factorization* results of the form

$$\sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{C(q)} \sum_{n\geq 0} \sum_{k=1}^{n} s_{n,k} f(k) \cdot q^n;$$

the invertible, lower triangular matrix A_n in (2) corresponds to the entries $s_{n,k}$, which are also independent of f, for fixed $n \geq 1$. In the cases

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of these more general factorizations presented in this article, the series expansion of the factorization parameter $C(q)$ is always defined through the *q-Pochhammer symbol* as $C(q) \equiv (q;q)_\infty$.

Examples of the new formulas for $\mu(n)$, $\phi(n)$, and $\lambda(n)$ that we are able to obtain through the new matrix factorization equations include

$$\mu(n) = \sum_{k=1}^{n}\Big(\sum_{d|n} p(d-k)\mu(n/d)\Big)\cdot B_{k-1}(\mu),$$
$$\phi(n) = \sum_{k=1}^{n}\Big(\sum_{d|n} p(d-k)\mu(n/d)\Big)\cdot B_{k-1}(\phi),$$
$$\lambda(n) = \sum_{k=1}^{n}\Big(\sum_{d|n} p(d-k)\mu(n/d)\Big)\cdot B_{k-1}(\lambda),$$

where $p(n)$ denotes *Euler's partition function* and where the special vector entries $B_m(f)$ from (2) are defined as follows for each $m \geq 0$ (1):

$$B_m(\mu) := [m=0]_\delta + \sum_{b=\pm 1}\sum_{k=1}^{\lfloor\frac{\sqrt{24m+1}-b}{6}\rfloor} (-1)^k [m+1-k(3k+b)/2 = 1]_\delta,$$
$$B_m(\phi) := m+1-\sum_{b=\pm 1}\sum_{k=1}^{\lfloor\frac{\sqrt{24m+1}-b}{6}\rfloor} (-1)^{k+1}\Big(m+1-\frac{k(3k+b)}{2}\Big),$$
$$B_m(\lambda) := [\sqrt{m+1}\in\mathbb{Z}]_\delta - \sum_{b=\pm 1}\sum_{k=1}^{\lfloor\frac{\sqrt{24m+1}-b}{6}\rfloor} (-1)^{k+1}[\sqrt{m+1-k(3k+b)/2}\in\mathbb{Z}]_\delta.$$

The results we prove also include new recurrence relations for the average order of special arithmetic functions, denoted by $\Sigma_{g_f,x} := \sum_{n\leq x} g_f(n)$, for fixed functions $f(n)$ and the corresponding series coefficients $g_f(n)$ defined as in (1) when $x \geq 1$.

1.2. Lambert series generating functions. There are many well-known Lambert series for special arithmetic functions of the form (1). Examples include the following series where $\mu(n)$ denotes the *Möbius function*, $\phi(n)$ denotes *Euler's totient function*, $\sigma_\alpha(n)$ denotes the generalized *sum of divisors function*, and $\lambda(n)$ denotes *Liouville's function* [3, §27.7]:

(1) We use *Iverson's convention* $[\mathtt{cond}=\mathtt{True}]_\delta \equiv \delta_{\mathtt{cond},\mathtt{True}}$ according to whether the condition `cond` is true or false where $\delta_{n,k}$ denotes Kronecker's delta function.

$$(3)\qquad \begin{aligned} \sum_{n\geq 1}\frac{\mu(n)q^n}{1-q^n} &= q, & f, g_f) &:= (\mu(n), [n=1]_\delta),\\ \sum_{n\geq 1}\frac{\phi(n)q^n}{1-q^n} &= \frac{q}{(1-q)^2}, & (f, g_f) &:= (\phi(n), n),\\ \sum_{n\geq 1}\frac{n^\alpha q^n}{1-q^n} &= \sum_{m\geq 1}\sigma_\alpha(n)q^n, & (f, g_f) &:= (n^\alpha, \sigma_\alpha(n)),\\ \sum_{n\geq 1}\frac{\lambda(n)q^n}{1-q^n} &= \sum_{m\geq 1}q^{m^2}, & (f, g_f) &:= (\lambda(n), [n \text{ is a positive square}]_\delta). \end{aligned}$$

1.3. New results. We have two interesting cases of (1) to consider:

I. $f(n)$ is our arithmetic function of interest that we wish to study—in the first, second, and fourth equations in (3); and
II. $g_f(n)$ is the interesting arithmetic function we wish to study, in the third equation of (3).

1.3.1. *Case I.* In the first case, for each $n \geq 1$ and $i \leq n$ we are able to form the matrix solutions for $f(n)$ given in Theorem 1.3, which are expanded in terms of the sequences in the next definition.

Definition 1.1. For integers $n \geq 0$, the sequences $a_f(n)$ and $a_{n,i}$ are defined to be (cf. Remark 1.2 below) (2)

$$a_f(n) = \sum_{i=1}^{n} f(i)\underbrace{\Bigg(\sum_{\substack{(k,s):\,(s+1)i+k(3k\pm 1)/2=n\\ k,s\geq 0}}(-1)^k\Bigg)}_{:=a_{n,i}} + [n=0]_\delta,$$

$$a_{n,i} = \sum_{b=\pm 1}\sum_{s=0}^{\lfloor n/i\rfloor -1}(-1)^{\lfloor\frac{\sqrt{24(n-(s+1)i)+1}-b}{6}\rfloor}\cdot\left[\frac{\sqrt{24(n-(s+1)i)+1}-b}{6}\in\mathbb{Z}\right]_\delta.$$

For $n \geq 1$, we define the $n \times n$ matrices A_n and A_n^{-1} in terms of these sequences as follows:

$$(4)\qquad A_n := (a_{i,j})_{1\leq i,j\leq n}, \qquad A_n^{-1} = (a_{i,j}^{(-1)})_{1\leq i,j\leq n}.$$

The matrices A_n and A_n^{-1} are independent of the choice of the function f for all n and are each invertible, lower triangular square matrices with ones on their diagonals. The independence of these matrices on the choice of f in

(2) The floored terms $\lfloor(\sqrt{24n+1}-b)/6\rfloor$ for $b = \pm 1$ in this and in subsequent formulas in the article correspond to solving for the upper bounds on k in the inequalities (cf. footnote 4) $0 \leq k(3k+b)/2 \leq n \Leftrightarrow 0 \leq k \leq (\sqrt{24n+1}-b)/6$.

the expansions of (1) leads to the Lambert series matrix factorization result phrased by Theorem 1.3 below.

The motivation for defining the sequences $a_f(n)$ and $a_{n,i}$ in Definition 1.1 is to provide a compact notation for expressing the left-hand-side terms $A_n[f(1) \ \cdots \ f(n)]^T$ of the matrix equation corresponding to the non-homogeneous recurrence relations for $g_f(n)$ given in Theorems 1.3 and 1.4 (see also Lemma 2.1 below). Particular examples of the sequences $a_f(n)$ include the special case in (5) below for the generalized sum-of-divisors functions $f(n) := \sigma_\alpha(n) = \sum_{d|n} d^\alpha$ for fixed $\alpha \in \mathbb{C}$. The expansion provides a listing of the terms $a_f(n)$ in the previous definition corresponding to the Lambert series pair $(f(n), g_f(n)) := (n^\alpha, \sigma_\alpha(n))$ in (1). This series is intended for comparison with the closely related matrix factorization result for this special case given by Theorem 1.3.

$$\begin{aligned}(5)\quad \sum_{m\geq 0} a_{n^\alpha}(m)q^m &= 1 + q + 2^\alpha q^2 + (-1 - 2^\alpha + 3^\alpha)q^3 + (-1 - 3^\alpha + 4^\alpha)q^4 \\ &\quad + (-1 - 2^\alpha - 3^\alpha - 4^\alpha + 5^\alpha)q^5 + (3^\alpha - 4^\alpha - 5^\alpha + 6^\alpha)q^6 \\ &\quad + (-3^\alpha - 5^\alpha - 6^\alpha + 7^\alpha)q^7 + \cdots .\end{aligned}$$

The first few cases of the matrices $A_n \in \mathbb{Z}_{n\times n}$ and their inverses $A_n^{-1} \in \mathbb{N}_{n\times n}$ are shown in Table 1. In general, we see that for all $n \geq 2$, we have

$$A_{n+1}^{-1} = \left[\begin{array}{c|c} A_n^{-1} & \mathbf{0} \\ \hline r_{n+1,n}, \ldots, r_{n+1,1} & 1 \end{array}\right],$$

where the first several special cases of the sequences $\{r_{n+1,n}, \ldots, r_{n+1,1}\}$ are given in Table 2. The statement of the next theorem employs these sequences and matrix forms. The proof of Theorem 1.3 is given in Section 2.

Remark 1.2. Since the first submission of the manuscript the entries $a_{n,i}$ and $a_{n,i}^{(-1)}$ in (4) have been determined in closed form through joint work by Merca and Schmidt on Lambert series factorizations in 2017. In particular, $a_{n,i}$ corresponds to [3]

$$s_o(n,i) - s_e(n,i) = [q^n]\frac{q^i}{1-q^i}(q;q)_\infty,$$

where $s_o(n,k)$ and $s_e(n,k)$ are respectively the number of k's in all partitions of n into an odd and an even number of distinct parts. Similarly, we can

[3] The bracket notation for coefficient extraction of a (formal) power series is defined by $[q^n]F(q) := f_n$ when $F(q) := \sum_n f_n q^n$ represents the ordinary generating function of the sequence $\langle f_n \rangle_{n\geq 0}$. This notation is also employed in Section 2.

Table 1. The first few special cases of the matrices A_n and A_n^{-1}

n	A_n	A_n^{-1}
1	$[1]$	$[1]$
2	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
3	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
4	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix}$
5	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 4 & 3 & 2 & 1 & 1 \end{bmatrix}$

Table 2. The bottom row sequences in the matrices A_n^{-1}

n	$\{r_{n,n-1}, r_{n,n-2}, \ldots, r_{n,1}\}$
2	$\{1\}$
3	$\{1, 1\}$
4	$\{2, 1, 1\}$
5	$\{4, 3, 2, 1\}$
6	$\{5, 3, 2, 2, 1\}$
7	$\{10, 7, 5, 3, 2, 1\}$
8	$\{12, 9, 6, 4, 3, 2, 1\}$
9	$\{20, 14, 10, 7, 5, 3, 2, 1\}$
10	$\{25, 18, 13, 10, 6, 5, 3, 2, 1\}$
11	$\{41, 30, 22, 15, 11, 7, 5, 3, 2, 1\}$
12	$\{47, 36, 26, 19, 14, 10, 7, 5, 3, 2, 1\}$

derive an exact formula for the inverse matrix entries as

$$a_{n,i}^{(-1)} = \sum_{d|n} p(d-i)\mu(n/d),$$

where $p(n)$ denotes *Euler's partition function* which is generated by the reciprocal of the q-Pochhammer symbol as $p(n) = [q^n]\,(q;q)_\infty^{-1}$ for all $n \geq 0$.

Theorem 1.3 (Matrix factorization equations for $f(n)$). *Let $f(n)$ be as in* (1)*. Then for all $n \geq 1$, we have the matrix factorization equation*

$$\begin{bmatrix} f(1) \\ \vdots \\ f(n) \end{bmatrix} = \tag{6}$$

$$A_n^{-1} \underbrace{\left(g_f(m+1) - \sum_{b=\pm 1} \sum_{k=1}^{\lfloor \frac{\sqrt{24m+1}-b}{6} \rfloor} (-1)^{k+1} g_f(m+1-k(3k+b)/2) \right)_{0 \leq m < n}}_{:=(B_{g_f,m})}.$$

1.3.2. *Case II.* In the second case, we have recurrence relations for $g_f(n)$ in (1), of the form stated in Theorem 1.4 below. Moreover, if we define $\Sigma_{g_f,x} := \sum_{n \leq x} g_f(n)$ to be the average order of $g_f(n)$, then we can also prove easily by induction that $\Sigma_{g_f,x}$ itself also satisfies the related form of the recurrence relation given in Corollary 1.5.

THEOREM 1.4 (Recurrence relations for $g_f(n)$). *For all $n \geq 1$,*

$$g_f(n+1) = \sum_{b=\pm 1} \Bigg(\sum_{k=1}^{\lfloor \frac{\sqrt{24n+1}-b}{6} \rfloor} (-1)^{k+1} g_f\big(n+1-k(3k+b)/2\big) \Bigg) + a_f(n+1).$$

COROLLARY 1.5 (Recurrence relations for $\Sigma_{g_f,x}$). *For all $n \geq 1$,*

$$\Sigma_{g_f,n+1} = \sum_{b=\pm 1} \Bigg(\sum_{k=1}^{\lfloor \frac{\sqrt{24n+1}-b}{6} \rfloor + 1} (-1)^{k+1} \Sigma_{g_f,n+1-k(3k+b)/2} \Bigg) + \sum_{k=1}^{n} a_f(k+1). \tag{7}$$

Theorem 1.4 and Corollary 1.5 are proved in Section 2.

1.3.3. *Algorithms for computing $f(n)$ and $g_f(n)$ in polynomial time.* Since the determinant of an $(n-1) \times (n-1)$ matrix can be computed in $O(n^3)$ time if $g_f(n)$ can be computed in constant time, our theorem yields an $O(n^4)$ time algorithm to compute any function $f(n)$ in (1). If we instead use Gaussian elimination with back substitution, we can compute $f(n)$ in $O(n^3)$ time. Similarly, if $g_f(n)$ can be computed in $O(h_{g_f}(n))$ time, then we can compute any function $f(n)$ in (1) in $O(h_{g_f}(n)\sqrt{n} + n^4)/O(h_{g_f}(n)\sqrt{n} + n^3)$ time.

However, each of the special arithmetic functions on the left-hand side of (3) can be computed more efficiently using sieves and other prime factorization algorithms. Despite more efficient known methods for computing the classical arithmetic functions involved in the expansions of (3), this observation is still useful since it implies that there are now known polynomial-time algorithms for computing the pairs $(f(n), g_f(n))$ in *any* Lambert series expansion provided a polynomial-time method of computation for either one of the functions in the corresponding pair is available.

1.4. Organization of the article. The proofs of Theorems 1.3 and 1.4 and of Corollary 1.5 are given in Section 2. In Section 3, we provide several concrete applications of these new results to the classical arithmetic functions $\sigma_1(n)$, $\phi(n)$, $\mu(n)$, and $\lambda(n)$. In the concluding remarks of Section 4, we suggest an analogous approach to a generalized class of Lambert series expansions $L_f(\alpha, \beta; a, b, c, d; q)$ as a future avenue of research.

2. Proofs of the theorems. In order to obtain recurrence relations between the sequences in the definition of (1), we first observe that for all $m \geq 0$ we have the next series expansions of the partial sums of the Lambert series $L_f(q)$, where $(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ denotes the *q-Pochhammer symbol* [3, §17.2], and where the functions $\mathrm{poly}_{i,m}(f;q)$ denote polynomials in q with coefficients depending on f for $i = 1, 2$ and whose degree is linear m.

LEMMA 2.1 (Partial sums of the Lambert series $L_f(q)$). *For a fixed pair of functions $(f(n), g_f(n))$ in (1) and for all integers $m \geq 0$ we have*

$$g_f(m+1) = [q^m]\left(\frac{1}{q} \times \sum_{n=1}^{m+1} \frac{f(n)q^n}{1-q^n}\right) \tag{8a}$$

$$= [q^m]\left(\frac{\frac{1}{q} \cdot (q;q)_{m+1}\left[\frac{f(1)\cdot q}{1-q} + \frac{f(2)\cdot q^2}{1-q^2} + \cdots + \frac{f(n)\cdot q^{m+1}}{1-q^{m+1}}\right]}{(1-q)(1-q^2)\cdots(1-q^{m+1})}\right) \tag{8b}$$

(8c)
$$= [q^m]\left(\frac{\sum_{1\leq i\leq m+1} a_f(i)q^i + q^{m+2}\cdot \mathrm{poly}_{1,m}(f;q)}{1 + \sum_{b=\pm 1}\sum_{k=1}^{\lfloor\frac{\sqrt{24m+25}+1}{6}\rfloor}(-1)^k q^{k(3k+b)/2} + q^{m+2}\cdot \mathrm{poly}_{2,m}(f;q)}\right).$$

Proof. To justify (8a), we observe that for all integers $m \geq 1$ and $1 \leq i \leq m$, we have

$$[q^i]\left(L_f(q) - \sum_{n>m} \frac{f(n)q^n}{1-q^n}\right) = 0,$$

i.e., the mth partial sums of $L_f(q)$ generate the coefficients $[q^k]L_f(q)$ for each k in the range $1 \leq k \leq m$. This is easy enough to see by considering the numerator multiples q^n of the geometric series $(1-q^n)^{-1}$ in the individual Lambert series terms from (1). The result in (8b) follows immediately from (8a) by combining the terms in the first partial sum, and implies the third result in (8c) in two key ways.

First, the form of the denominator terms in (8c) follows from *Euler's pentagonal number theorem* [1, §19.9, Thm. 353], which states that

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = 1 + \sum_{n\geq 1}(-1)^n(q^{k(3k-1)/2} + q^{k(3k+1)/2})$$
$$= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots .$$

In particular, this shows that

$$[q^i](1-q)(1-q^2)\cdots(1-q^n) = \begin{cases} (-1)^k & \text{if } i = k(3k\pm 1)/2, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \leq n$ by a contradiction argument. Since $1-q^i$ is a factor of $(q;q)_n$ for all $1 \leq i \leq n$, we see that both the numerator and the denominator of (8b) are polynomials in q, each of degree greater than $m+1$. This implies the correctness of the denominator in (8c).

Secondly, since

$$\frac{1}{1-q^i} = \sum_{s\geq 0} q^{si}$$

for each finite $i \geq 1$, by the definition of $a_f(n)$ in Definition 1.1 we see that the first $m+1$ terms of the numerator expansion in (8c) are correct. Since this numerator is polynomial in q, we conclude that (8c) holds as well. ■

Proof of Theorem 1.4. We use (8c). If we let $\mathrm{Num}_m(q)$ and $\mathrm{Denom}_m(q)$ denote the numerator and denominator polynomials in (8c), respectively, we see that $\deg_q\{\mathrm{Num}_m(q)\} < \deg_q\{\mathrm{Denom}_m(q)\}$. Next, for any sequence $\langle f_n\rangle_{n\geq 0}$ generated by a rational generating function of the form

$$\sum_{n\geq 0} f_n q^n = \frac{a_0 + a_1 q + a_2 q^2 + \cdots + a_{k-1}q^{k-1}}{1 - b_1 q - b_2 q^2 - \cdots - b_k q^k}$$

for some fixed $k \geq 1$, we can prove that f_n satisfies at most a k-order finite difference equation with constant coefficients of the form [2, §2.3]

$$f_n = \sum_{i=1}^{\min(k,n)} b_i f_{n-i} + a_n[0 \leq n < k]_\delta .$$

Then since we define $f(n) = 0$ for all $n < 1$ in (1), since the mth partial sums of $L_f(q)$ generate $g_f(i)$ for all $1 \leq i \leq m$ by the lemma, and since $g_f(i) = 0$ for all $i < 1$, we see that (8c) implies our result. ■

Proof of Theorem 1.3. The theorem is a consequence of Definition 1.1 applied to Theorem 1.4. Specifically, by rearranging terms in the result from the previous theorem, we see that

$$A_n \begin{bmatrix} f(1) \\ f(2) \\ \vdots \\ f(n) \end{bmatrix} = \begin{bmatrix} B_{g_f,0} \\ B_{g_f,1} \\ \vdots \\ B_{g_f,n-1} \end{bmatrix}, \tag{9}$$

where (6) defines the sequence of $B_{g_f,m}$. Then by the definition of $a_{n,i}$ given in Definition 1.1 (cf. Remark 1.2), it is easy to see that A_n is lower triangular with ones on its diagonal, and so is invertible for all $n \geq 1$. Thus by applying A_n^{-1} to both sides of (9), we have proved (6). ∎

Proof of Corollary 1.5. By direct computation, the statement is true for $n = 1$. For some $j \geq 1$, suppose that (7) is true for $n = j$. Then

$$\begin{aligned}
\widetilde{\Sigma}_{g_f,j+1} &= \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24j+25}-b}{6} \right\rfloor} (-1)^k [\Sigma_{g_f,j+1-k(3k+b)/2} + g_f(j+2-k(3k+b)/2)] \\
&\quad + \sum_{k=1}^{j+1} a_f(k+1) \\
&= \Sigma_{g_f,j+1} + \sum_{b=\pm 1} \sum_{k=1}^{\left\lfloor \frac{\sqrt{24j+25}-b}{6} \right\rfloor} g_f(j+2-k(3k+b)/2) + a_f(j+2)
\end{aligned}$$

(by induction hypothesis)

$$= \Sigma_{g_f,j+1} + g_f(j+2) = \Sigma_{g_f,j+2}.$$

The second to last equality follows from Theorem 1.4 and the fact that $\lfloor(\sqrt{24n+25}-b)/6\rfloor \geq \lfloor(\sqrt{24n+1}-b)/6\rfloor$ and $g_f(i) = 0$ for all $i < 1$. ∎

3. Examples of the new results

3.1. The generalized sum-of-divisors functions. For any $n, x \geq 0$, we have the following recurrence relations following from Theorem 1.4 and Corollary 1.5 ([4]):

([4]) Here, we make use of a natural conjecture from (5) that $a_n(m) = (-1)^k(m+1)$ $[m = k(3k \pm 1)/2]_\delta$, where the *pentagonal numbers*, $\omega_{k,b} := k(3k+b)/2$, between 1 and $n+1$ are given by the following sets for $b = \pm 1$ (see footnote 2):

$$\{\omega_{1,b}, \omega_{2,b}, \ldots, \omega_{\lfloor(\sqrt{24n+1}-b)/6\rfloor,b}\}.$$

$$\begin{aligned}\sigma_1(n+1) &= \sum_{b=\pm 1} \sum_{k=1}^{\lfloor \frac{\sqrt{24m+1}-b}{6} \rfloor} (-1)^{k+1} \sigma_1(n+1-k(3k+b)/2) \\ &\quad + (-1)^k (n+1)[n+1 = k(3k\pm 1)/2]_\delta, \\ \Sigma_{n,x+1} &= \sum_{b=\pm 1} \Bigg(\sum_{k=1}^{\lfloor \frac{\sqrt{24x+1}-b}{6} \rfloor + 1} (-1)^{k+1} \Sigma_{n,x+1-k(3k+b)/2} \\ &\qquad + \sum_{k=1}^{\lfloor \frac{\sqrt{24x+25}-b}{6} \rfloor} (-1)^{k+1} \frac{k(3k+b)}{2} \Bigg).\end{aligned}$$

Notice that the previous two equations imply exact closed-form formulas for $\sigma_1(m)$ and $\Sigma_{n,m}$ at each $m \geq 1$, and similarly for fixed m and all $1 \leq n \leq m$. Moreover, we conjecture that the zeros of the polynomials $\widetilde{Q}_n(q) := q^n \cdot Q_n(1/q)$, or alternatively the reciprocal zeros of the polynomials

$$Q_n(q) := 1 - \sum_{b=\pm 1} \sum_{k=1}^{\lfloor \frac{\sqrt{24n+1}-b}{6} \rfloor} (-1)^k q^{k(3k+b)/2}, \tag{10}$$

have maximum magnitude of a little over 1 (depending on n). This suggests that it may be possible to use these results to obtain better error bounds on the known average order sums [3, §27.11]

$$\begin{aligned}\sum_{n\leq x} \sigma_1(n) &= \frac{\pi^2}{12} x^2 + O(x \log x), \\ \sum_{n \leq x} \sigma_\alpha(n) &= \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + O(x^\beta), \qquad \alpha > 0,\, \alpha \neq 1,\, \beta = \max(1, \alpha),\end{aligned}$$

using the explicit sequence formulas in (5).

3.2. Euler's totient function. We provide a computation of (6) to demonstrate the use of our results:

$$\begin{bmatrix} \phi(1) \\ \phi(2) \\ \phi(3) \\ \phi(4) \\ \phi(5) \\ \phi(6) \\ \phi(7) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 & 1 & 0 & 0 \\ 5 & 3 & 2 & 2 & 1 & 1 & 0 \\ 10 & 7 & 5 & 3 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 4 \\ 2 \\ 6 \end{bmatrix}.$$

In this case, we can solve for the right-hand-side vector $(B_{g_f,n})$ explicitly. More precisely, when $(f, g_f) := (\phi(n), n)$ from (3), we see by straightforward summation that

$$\begin{aligned}B_{n,m} &= m+1\\ &\quad - \tfrac{1}{8}\big(8 - 5\cdot(-1)^{u_1} - 4(-2+(-1)^{u_1}+(-1)^{u_2})m\\ &\quad + 2(-1)^{u_1}u_1(3u_1+2) + (-1)^{u_2}(6u_2^2+8u_2-3)\big),\end{aligned}$$

where $u_1 \equiv u_1(m) := \lfloor(\sqrt{24m+1}+1)/6\rfloor$ and $u_2 \equiv u_2(m) := \lfloor(\sqrt{24m+1} - 1)/6\rfloor$. The first terms of the sequence $\{B_{n,m}\}_{m\geq 0}$ corresponding to the Lambert series for Euler's phi function are given by

$$\{B_{n,m}\}_{m\geq 0} = \{1,1,0,-1,-2,-2,-2,-1,0,1,2,3,3,3,3,2,1,0,-1,-2,-3,\ldots\}.$$

3.3. The Möbius function. We similarly compute (6):

$$\begin{bmatrix}\mu(1)\\ \mu(2)\\ \mu(3)\\ \mu(4)\\ \mu(5)\\ \mu(6)\\ \mu(7)\end{bmatrix} = \begin{bmatrix}1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&1&1&0&0&0&0\\ 2&1&1&1&0&0&0\\ 4&3&2&1&1&0&0\\ 5&3&2&2&1&1&0\\ 10&7&5&3&2&1&1\end{bmatrix}\begin{bmatrix}1\\ -1\\ -1\\ 0\\ 0\\ 1\\ 0\end{bmatrix} = \begin{bmatrix}1\\ -1\\ -1\\ 0\\ -1\\ 1\\ -1\end{bmatrix}.$$

In this case the vector $(B_{g_f,m})$ is given by the formula

$$B_{[n=1]_\delta,m} = [m=0]_\delta + \sum_{b=\pm 1}\sum_{k=1}^{\lfloor\frac{\sqrt{24m+1}-b}{6}\rfloor}(-1)^k[m+1-k(3k+b)/2=1]_\delta.$$

The first terms of the sequence $\{B_{g_f,m}\}_{m\geq 0}$ for the Lambert series of the Möbius function are given by

$$\{B_{[n=1]_\delta,m}\}_{m\geq 0} = \{1,-1,-1,0,0,1,0,0,1,0,0,0,0,-1,0,0,-1,0,0,0,0,\ldots\}.$$

3.4. Liouville's lambda function. Again, we compute (6):

$$\begin{bmatrix}\lambda(1)\\ \lambda(2)\\ \lambda(3)\\ \lambda(4)\\ \lambda(5)\\ \lambda(6)\\ \lambda(7)\end{bmatrix} = \begin{bmatrix}1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&1&1&0&0&0&0\\ 2&1&1&1&0&0&0\\ 4&3&2&1&1&0&0\\ 5&3&2&2&1&1&0\\ 10&7&5&3&2&1&1\end{bmatrix}\begin{bmatrix}1\\ -1\\ -1\\ 1\\ -1\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}1\\ -1\\ -1\\ 1\\ -1\\ 1\\ -1\end{bmatrix}.$$

In this case the vector $(B_{g_f,m})$ is given by

$$B_{[n \text{ is a positive square}]_\delta,m} = [\sqrt{m+1} \in \mathbb{Z}]_\delta - \sum_{b=\pm 1} \sum_{k=1}^{\lfloor \frac{\sqrt{24m+1}-b}{6} \rfloor} (-1)^{k+1} [\sqrt{m+1-k(3k+b)/2} \in \mathbb{Z}]_\delta.$$

The first few terms of the sequence $\{B_{g_f,m}\}_{m\geq 0}$ for the Lambert series corresponding to Liouville's function are given by

$$\{B_{[n\,=\,k^2]_\delta,m}\}_{m\geq 0} = \{1,-1,-1,1,-1,0,0,1,2,-1,0,0,-1,1,0,0,-1,-1,-1,0,\ldots\}.$$

4. Conclusions. The examples from (3) for $\sigma_1(n)$, $\phi(n)$, $\mu(n)$, and $\lambda(n)$ given in Section 3 are easily extended to Lambert series of other special functions, such as those for the logarithmic derivatives of the *Jacobi theta functions* $\vartheta_i(z,q)$, cited in [3, §20.5(ii)]. There are also well-known Lambert series expansions involving von Mangoldt's function $\Lambda(n)$, $|\mu(n)|$, the number of distinct primes dividing n, or $\omega(n)$, and Jordan's totient function $J_t(n)$, which provide still other applications of our results. To the best of our knowledge, these results, and certainly the interpretations of their proofs, are new in the literature.

We can generalize these results to obtain analogous matrix equations and new recurrence relations for the sequences $\widetilde{f}(n)$ and $\widetilde{g}(n)$ in the expansions of generalized Lambert series of the form

$$L_{\widetilde{f}}(\alpha,\beta;a,b,c,d;q) := \sum_{n\geq 1} \frac{\widetilde{f}(n)\alpha^n q^{d(an+b)}}{1-\beta q^{c(an+b)}} = \sum_{m\geq 1} \widetilde{g}(m) q^m$$

for some $\alpha,\beta,a,b,c,d \in \mathbb{C}$ with $\alpha,\beta,a,c,d \neq 0$ such that $\max(|\alpha q^{da}|,|\beta q^{ca}|) < 1$. In particular, if we know the forms of the coefficients of the power series expansions of the infinite q-Pochhammer products $(\beta q^{cb}; q^{ca})_\infty$ with respect to q (which may or may not be obvious depending on the application), then we can extend the proof of Lemma 2.1 to obtain analogous results for these generalized Lambert series expansions.

One immediate application of these generalized results is a Lambert series generating function for the *sum-of-squares function* $r_2(n)$, given by [1, §17.10, Thm. 311] [3, §27.13(iv)]

$$\sum_{n\geq 1} \frac{4\cdot(-1)^{n+1} q^{2n+1}}{1-q^{2n+1}} = \sum_{m\geq 1} r_2(m) q^m, \qquad (\widetilde{f},\widetilde{g}) := ((-1)^{n+1}, r_2(n)).$$

However, we point out that the coefficients of the power series expansion of the q-Pochhammer symbol $(q;q^2)_\infty$ in q do not appear to have a known closed-form formula, but instead only the related q-series expansions proved in the references. Nonetheless, the study of the analogous results to those proved in this article corresponding to these generalized cases is an interesting new direction for more careful future study.

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