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Approximation properties of doubly overconvergent power series

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Abstract. We investigate some approximation properties of doubly universal Taylor series defined on a simply connected domain Ω. In particular, we study the approximation properties inside Ω and the case of doubly overconvergent power series smooth on the boundary.

1. Introduction. Let $\Omega \subset \mathbb{C}$ be a simply connected domain. We denote by $H(\Omega)$ the space of all holomorphic functions in Ω, endowed with the topology of uniform convergence on compacta. Without loss of generality we may assume that $0 \in \Omega$. For a function $f \in H(\Omega)$, we denote by

$$S_n(f)(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k, \qquad n = 1, 2, \ldots,$$

the partial sums of the Taylor expansion of f around 0. We consider zero as the center of expansion in order to simplify our notation. However, the results presented in this paper may be generalized to any point $\zeta \in \Omega$.

For a compact set $K \subset \mathbb{C}$, we write K^o for the interior of K. We consider the space $C(K)$ of all functions continuous on K and

$$A(K) = \{g \in C(K) : g|_{K^o} \text{ is holomorphic}\}$$

endowed with the topology of uniform convergence on K.

Our aim is to investigate whether some well known properties of universal Taylor series are true in the case of doubly universal Taylor series. Let us recall the definition of universal Taylor series [19], [18].

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Definition 1.1. A function $f \in H(\Omega)$ belongs to the class $U(\Omega, 0)$ if the set $\{S_n(f) : n \in \mathbb{N}\}$ is dense in $A(K)$ for every compact set $K \subseteq \mathbb{C} \setminus \Omega$ with connected complement.

In 1996, V. Nestoridis [19] proved that if $\Omega \subset \mathbb{C}$ is a simply connected domain, then $U(\Omega, 0)$ is a dense G_δ subset of $H(\Omega)$. Note that functions in $U(\Omega, 0)$ have overconvergent power series outside Ω. Actually if $f \in U(\Omega, 0)$ then f has overconvergent power series inside Ω as well. This fact is derived by an elegant and deep result first presented in [8] for bounded domains Ω (see also [17], [14]).

Our goal is to show that the functions in the class of doubly universal Taylor series also have overconvergent power series inside Ω. The notion of doubly universal Taylor series was recently introduced by G. Costakis and N. Tsirivas [7] for Ω being the open unit disk and it is connected with the phenomenon of disjoint universality [2], [3]. In [4], the notion was generalized to any simply connected domain Ω. Let us give the definition that appears in [4].

Definition 1.2. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive integers. A function $f \in H(\Omega)$ belongs to the class $U(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$ if the set $\{(S_n(f), S_{\lambda_n}(f)) : n \in \mathbb{N}\}$ is dense in $A(K_1) \times A(K_2)$ for any compact sets $K_1, K_2 \subseteq \mathbb{C} \setminus \Omega$ with connected complements.

The class $U(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$ is non-empty if and only if $\limsup_{n\in\mathbb{N}} \lambda_n/n = \infty$. If it is non-empty, it is G_δ and dense in $H(\Omega)$ [4], [7].

In Section 2 we will prove that every doubly universal Taylor series realizes approximation inside Ω.

In Section 3 we investigate a problem for doubly universal Taylor series motivated by an article of Ch. Kariofillis, Ch. Konstadilaki and V. Nestoridis [10] on universal Taylor series. More precisely, universal (and consequently doubly universal) Taylor series cannot be smooth on the boundary of Ω (see [13]). In an effort to obtain a result towards this direction, we consider (as in [10]) functions with universal approximation properties outside $\overline{\Omega}$ (in the sense of Luh [11] and Chui and Parnes [5]). Hence, we study the concept of double universality on the space of holomorphic functions in Ω whose derivatives extend continuously to the boundary of Ω.

2. Double universality in $H(\Omega)$. The main result of this section is that if $f \in U(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$ then f has overconvergent power series in Ω. Let us give a definition.

Definition 2.1. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive integers, and Ω be a simply connected domain with $0 \in \Omega$. A function $f \in H(\Omega)$ belongs to the class $U_{\text{ins}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$ if for any compact sets $K_1, K_2 \subseteq \mathbb{C} \setminus \Omega$ with connected complements and for every pair $(g_1, g_2) \in$

$A(K_1) \times A(K_2)$, there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of positive integers such that

$$\|S_{\mu_n}(f) - g_1\|_{K_1} \xrightarrow{n\to\infty} 0, \qquad \|S_{\lambda_{\mu_n}}(f) - g_2\|_{K_2} \xrightarrow{n\to\infty} 0,$$

and for every compact set $\Gamma \subset \Omega$,

$$\|S_{\mu_n}(f) - f\|_{\Gamma} \xrightarrow{n\to\infty} 0, \qquad \|S_{\lambda_{\mu_n}}(f) - f\|_{\Gamma} \xrightarrow{n\to\infty} 0.$$

Clearly $U_{\mathrm{ins}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0) \subseteq U(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$. We will show that the two classes actually coincide. The following result proved by J. Müller and A. Yavrian [16] is a crucial part of the proof. For the notion of thinness we refer to [20].

THEOREM 2.2 (Müller–Yavrian). *Let Γ be a compact and connected subset of $\mathbb{C}$, but not a singleton. Let $E \subset \mathbb{C}$ be a closed set that is non-thin at ∞. Also suppose $(P_n)_{n\in\mathbb{N}}$ is a sequence of polynomials with* $\deg(P_n) \le d_n$ *for some increasing sequence $(d_n)_{n\in\mathbb{N}}$ of positive integers and with the following properties:*

(a) *there exists a function $f : \Gamma \to \mathbb{C}$ with*

$$\limsup_{n\to\infty} \|f - P_n\|_{\Gamma}^{1/d_n} < 1,$$

(b) *for all $z \in E$,*

$$\limsup_{n\to\infty} |P_n(z)|^{1/d_n} \le 1.$$

Then:

(i) *if the sequence $(d_{n+1}/d_n)_{n\in\mathbb{N}}$ is bounded, then f extends to an entire function and for every compact set $K \subset \mathbb{C}$ we have*

$$\limsup_{n\to\infty} \|f - P_n\|^{1/d_n} < 1,$$

(ii) *if f is analytic on Γ, then f extends to a holomorphic function having a simply connected domain of existence $G_f \subset \mathbb{C}$ (G_f denotes the unique largest domain to which f extends as a holomorphic function; observe that G_f exists in this case) and for every compact set $K \subset G_f$ we have*

$$\limsup_{n\to\infty} \|f - P_n\|^{1/d_n} < 1.$$

THEOREM 2.3. *The classes $U(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$ and $U_{\mathrm{ins}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$ coincide.*

Proof. If $\limsup_{n\in\mathbb{N}} \lambda_n/n < \infty$, then $U(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0) = \emptyset$ (see [4]) and since $U_{\mathrm{ins}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0) \subseteq U(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$, there is nothing to prove.

So assume that $\limsup_{n\in\mathbb{N}} \lambda_n/n = \infty$. Let $f \in U(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$. It suffices to prove that $f \in U_{\mathrm{ins}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$. Let $K_1, K_2 \subset \Omega^c$ be compact sets with connected complements and $(g_1, g_2) \in A(K_1) \times A(K_2)$. Choose $R > 0$ such that $K_1 \cup K_2 \subset D(0, R)$. Let ω be the union of the bounded components (if any) of $(\Omega \cup D(0,R))^c$. It follows easily that $\tilde{\Omega} := \Omega \cup D(0,R) \cup \omega$ is a simply connected domain. Applying [21, Lemma 2.2] to $\tilde{\Omega}$ (see also [7]), we can fix an increasing sequence $(E_k)_{k\in\mathbb{N}}$ of compact sets with the following properties: $E_k \subset \tilde{\Omega}^c$, E_k^c is connected for every $k \in \mathbb{N}$ and $E = \bigcup_{k\in\mathbb{N}} E_k$ is closed and non-thin at ∞.

For every $k \in \mathbb{N}$, the sets $K_1 \cup E_k$ and $K_2 \cup E_k$ are compact with connected complements. Since $f \in U(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$, there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that

$$\|S_{n_k}(f) - g_1\|_{K_1} < 1/k, \quad \|S_{\lambda_{n_k}}(f) - g_2\|_{K_2} < 1/k,$$
$$\|S_{n_k}(f) - 0\|_{E_k} < 1/k, \quad \|S_{\lambda_{n_k}}(f) - 0\|_{E_k} < 1/k.$$

Let $z \in E$. Then there exists $k_0 \in \mathbb{N}$ such that $z \in E_k$ for all $k \geq k_0$. For $k \geq k_0$, $|S_{n_k}(f)(z)| \leq \|S_{n_k}(f)\|_{E_k} \leq 1$ and $|S_{\lambda_{n_k}}(f)(z)| \leq \|S_{\lambda_{n_k}}(f)\|_{E_k} \leq 1$. Consequently,

(1a) $$\limsup_{k\to\infty} |S_{n_k}(f)(z)|^{1/n_k} \leq 1,$$

(2a) $$\limsup_{k\to\infty} |S_{\lambda_{n_k}}(f)(z)|^{1/\lambda_{n_k}} \leq 1.$$

Now let ρ be the radius of convergence of the Taylor series of f around 0. Let $\Delta = \overline{D(0,r)}$ for $0 < r < \rho$. Then

(1b) $$\limsup_{k\to\infty} \|S_{n_k}(f) - f\|_{\Delta}^{1/n_k} < 1,$$

(2b) $$\limsup_{k\to\infty} \|S_{\lambda_{n_k}}(f) - f\|_{\Delta}^{1/\lambda_{n_k}} < 1.$$

From inequalities (1a) and (1b), we see that conditions (a) and (b) of the Müller–Yavrian Theorem 2.2 are fulfilled. Therefore for every compact set $\Gamma \subset \Omega$, we have

$$\limsup_{k\to\infty} \|S_{n_k}(f) - f\|_{\Gamma}^{1/n_k} < 1.$$

The same holds for (2a) and (2b). Hence for every compact set $\Gamma \subset \Omega$,

$$\limsup_{k\to\infty} \|S_{\lambda_{n_k}}(f) - f\|_{\Gamma}^{1/\lambda_{n_k}} < 1.$$

As a result, the sequence $(n_k)_{k\in\mathbb{N}}$ satisfies

$$\|S_{n_k}(f) - g_1\|_{K_1} \to 0, \quad \|S_{\lambda_{n_k}}(f) - g_2\|_{K_2} \to 0,$$

and for every compact set $\Gamma \subset \Omega$,

$$\|S_{n_k}(f) - f\|_{\Gamma} \to 0, \quad \|S_{\lambda_{n_k}}(f) - f\|_{\Gamma} \to 0.$$

Therefore $f \in U_{\text{ins}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$. ∎

Next, we will prove that the class $U(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$ is *densely lineable*, i.e. it contains a dense vector subspace with zero removed. Arguing as in [1] (see also [15]), we start by defining the following auxiliary class of functions.

DEFINITION 2.4. Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$ be strictly increasing sequences of positive integers, and Ω be a simply connected domain with $0 \in \Omega$. A function $f \in H(\Omega)$ belongs to the class $W(\Omega, (\lambda_n)_{n\in\mathbb{N}}, (\alpha_n)_{n\in\mathbb{N}}, 0)$ if for any compact sets $K_1, K_2 \subseteq \mathbb{C} \setminus \Omega$ with connected complements and for every pair $(g_1, g_2) \in A(K_1) \times A(K_2)$, there exists a strictly increasing subsequence $(\mu_n)_{n\in\mathbb{N}}$ of $(\alpha_n)_{n\in\mathbb{N}}$ such that

$$\|S_{\mu_n}(f) - g_1\|_{K_1} \to 0, \qquad \|S_{\lambda_{\mu_n}}(f) - g_2\|_{K_2} \to 0.$$

Following the method used in [4, Theorems 2.2 and 3.2] we obtain the following proposition.

PROPOSITION 2.5. *The class* $W(\Omega, (\lambda_n)_{n\in\mathbb{N}}, (\alpha_n)_{n\in\mathbb{N}}, 0)$ *is dense and* G_δ *in* $H(\Omega)$ *if and only if* $\limsup_{n\in\mathbb{N}} \lambda_{\alpha_n}/\alpha_n = \infty$.

THEOREM 2.6. *Let* Ω *be a simply connected domain with* $0 \in \Omega$. *Let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a strictly increasing sequence of positive integers such that* $\limsup_{n\in\mathbb{N}} \lambda_n/n = \infty$. *Then* $U(\Omega, (\lambda_n), 0) \cup \{0\}$ *contains a dense vector subspace.*

Proof. The result follows easily if we combine the previous proposition with the theory presented in [1]. ∎

3. Double universality in $X^\infty(\Omega)$. Let Ω be a simply connected domain such that $0 \in \Omega$ and the complement of $\overline{\Omega}$ in the extended complex plane $\mathbb{C}_\infty$ is connected. Consider an exhausting sequence $(\Gamma_k)_{k\in\mathbb{N}}$ of compact subsets of $\overline{\Omega}$ with $\mathbb{C} \setminus \Gamma_k$ connected for every $k \in \mathbb{N}$. We denote by $A^\infty(\Omega)$ the space of all holomorphic functions in Ω whose derivatives extend continuously to $\overline{\Omega}$, endowed with the topology defined by the seminorms $\sup_{z\in\Gamma_k} |f^{(\ell)}(z)|$ for all $k, \ell \in \mathbb{N}$. We also denote by $X^\infty(\Omega)$ the closure of the polynomials in $A^\infty(\Omega)$. Both $A^\infty(\Omega)$ and $X^\infty(\Omega)$ are Fréchet spaces, and therefore Baire's Theorem is at our disposal. The results of this section concern the following class of functions.

DEFINITION 3.1. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive integers and Ω be a simply connected domain with $0 \in \Omega$. A function $f \in A^\infty(\Omega)$ belongs to the class $U_{\mathrm{sm}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$ if for any compact sets $K_1, K_2 \subseteq \mathbb{C} \setminus \overline{\Omega}$ with connected complements and for any polynomials g_1, g_2, there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of positive integers such that

$$\|S_{\mu_n}(f) - g_1\|_{K_1} \xrightarrow{n\to\infty} 0, \qquad \|S_{\lambda_{\mu_n}}(f) - g_2\|_{K_2} \xrightarrow{n\to\infty} 0,$$
$$\|S_{\mu_n}^{(\ell)}(f) - f^{(\ell)}\|_{\Gamma} \xrightarrow{n\to\infty} 0, \qquad \|S_{\lambda_{\mu_n}}^{(\ell)}(f) - f^{(\ell)}\|_{\Gamma} \xrightarrow{n\to\infty} 0,$$

for every compact set $\Gamma \subset \overline{\Omega}$ and $\ell \in \mathbb{N}$.

Let us state a theorem presented in [4] (see also [7]), which will be used in this section. It is a modification of the Bernstein–Walsh Theorem [20].

Theorem 3.2 (Bernstein–Walsh type theorem). *Let* $(\tau_n)_{n\in\mathbb{N}}$ *and* $(\sigma_n)_{n\in\mathbb{N}}$ *be strictly increasing sequences of positive integers such that* $1 \leq \tau_n/\sigma_n \to \infty$. *Let* K *and* N *be disjoint compact subsets of* $\mathbb{C}$ *with connected complements and with* $0 \in N^o$. *If* f *is a function holomorphic in an open neighborhood of* K, *then there exist* $\theta \in (0,1)$ *and a sequence* $(P_n)_{n\in\mathbb{N}}$ *of polynomials such that, for every* $n \in \mathbb{N}$, $P_n \in \operatorname{span}\{z^{\sigma_n}, z^{\sigma_n+1}, \ldots, z^{\tau_n}\}$ *and*

$$\|f - P_n\|_K \leq \theta^{\tau_n} \quad \text{and} \quad \|P_n\|_N \leq \theta^{\tau_n}.$$

Corollary 3.3. *Let* $(\tau_n)_{n\in\mathbb{N}}$ *and* $(\sigma_n)_{n\in\mathbb{N}}$ *be as before. Let* $h : U \to \mathbb{C}$ *be a holomorphic function on a simply connected, open set* $U \subseteq \mathbb{C}$. *If* $0 \in U$, *then assume that* $h \equiv 0$ *on the connected component of* U *that contains zero. Then there exist a subsequence* $(\tilde{\tau}_n, \tilde{\sigma}_n)_{n\in\mathbb{N}}$ *of* $(\tau_n, \sigma_n)_{n\in\mathbb{N}}$ *and a sequence* $(P_n)_{n\in\mathbb{N}}$ *of polynomials such that, for every* $n \in \mathbb{N}$, $P_n \in \operatorname{span}\{z^{\tilde{\sigma}_n}, z^{\tilde{\sigma}_n+1}, \ldots, z^{\tilde{\tau}_n}\}$ *and*

$$P_n \xrightarrow{n\to\infty} h \quad \text{locally uniformly on } U.$$

Proof. By using Theorem 3.2 for suitable exhausting sequences of compact sets, the result follows easily. ∎

Remark 3.4. Note that since U is open, the approximation is valid for all derivatives.

Theorem 3.5. *Let* Ω *be a simply connected domain such that* $\mathbb{C}_\infty \setminus \overline{\Omega}$ *is connected and* $0 \in \Omega$. *Also, let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a strictly increasing sequence of positive integers with* $\limsup_{n\in\mathbb{N}} \lambda_n/n = \infty$. *Then* $U_{\mathrm{sm}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$ *is non-void.*

Proof. We will prove that $U_{\mathrm{sm}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0) \cap X^\infty(\Omega)$ is G_δ and dense in $X^\infty(\Omega)$, and therefore $U_{\mathrm{sm}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$ is non-void. The plan is to use Baire's Category Theorem, as in various papers concerning universal functions.

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive integers such that $\limsup_{n\in\mathbb{N}} \lambda_n/n = \infty$, and let $(f_j)_{j\in\mathbb{N}}$ be an enumeration of all polynomials having coefficients in $\mathbb{Q} + i\mathbb{Q}$. Let $(K_m)_{m\in\mathbb{N}}$ be a sequence of compact sets with connected complements and with $K_m \cap \overline{\Omega} = \emptyset$ such that every non-empty compact set $K \subset \mathbb{C} \setminus \overline{\Omega}$ with connected complement is contained in some K_m. For the existence of such a sequence we refer to [12, p. 198] and [9, Chapter 2.2]. Also, consider the sequence $(\Gamma_k)_{k\in\mathbb{N}}$ of compact subsets of $\overline{\Omega}$ described at the beginning of this subsection (for example take $(\overline{D(0,k)} \cap \overline{\Omega})_{k\in\mathbb{N}}$, where $D(0,k)$ denotes the open disk of center 0 and radius k).

For every $m_1, m_2, j_1, j_2, k, s, n \in \mathbb{N}$, we define

$$E(m_1, m_2, j_1, j_2, s, n) = \{f \in A^\infty(\Omega) : \|S_n(f) - f_{j_1}\|_{K_{m_1}} < 1/s \text{ and } \|S_{\lambda_n}(f) - f_{j_2}\|_{K_{m_2}} < 1/s\},$$

$$\Xi(k, s, n) = \{f \in A^\infty(\Omega) : \|S_n^{(\ell)}(f) - f^{(\ell)}\|_{\Gamma_k} < 1/s \text{ and } \|S_{\lambda_n}^{(\ell)}(f) - f^{(\ell)}\|_{\Gamma_k} < 1/s,\ \ell = 0, 1, \ldots, s\}.$$

In view of Mergelyan's Theorem,

$$U_{\mathrm{sm}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0) = \bigcap_{m_1, m_2, j_1, j_2, k, s \in \mathbb{N}} \bigcup_{n\in\mathbb{N}} (E(m_1, m_2, j_1, j_2, s, n) \cap \Xi(k, s, n)),$$

and for every $m_1, m_2, j_1, j_2, k, s \in \mathbb{N}$,

$$\bigcup_{n\in\mathbb{N}} (E(m_1, m_2, j_1, j_2, s, n) \cap \Xi(k, s, n))$$

is open in $A^\infty(\Omega)$ (see [10]).

The next step is to prove that for every $m_1, m_2, j_1, j_2, k, s \in \mathbb{N}$, the set

$$\bigcup_{n\in\mathbb{N}} (E(m_1, m_2, j_1, j_2, s, n) \cap \Xi(k, s, n)) \cap X^\infty(\Omega)$$

is dense in $X^\infty(\Omega)$. Then using Baire's Category Theorem we find that $U_{\mathrm{sm}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0) \cap X^\infty(\Omega)$ is dense in $X^\infty(\Omega)$, which completes the proof.

Fix $m_1, m_2, j_1, j_2, k, s \in \mathbb{N}$. Let $\epsilon > 0$, $\Gamma \subset \overline{\Omega}$ compact and $g \in X^\infty(\Omega)$. Without loss of generality, we may assume that g is a polynomial and $\{0\} \cup \Gamma_k \subseteq \Gamma^o$.

We will find $f \in H(\Omega)$ and $n \in \mathbb{N}$ such that for every $\ell = 0, 1, \ldots, s$,

$$\sup_{z\in K_{m_1}} |S_n(f)(z) - f_{j_1}(z)| < 1/s, \qquad \sup_{z\in K_{m_2}} |S_{\lambda_n}(f)(z) - f_{j_2}(z)| < 1/s,$$

$$\sup_{z\in\Gamma_k} |S_n^{(\ell)}(f)(z) - f^{(\ell)}(z)| < 1/s, \qquad \sup_{z\in\Gamma_k} |S_{\lambda_n}^{(\ell)}(f)(z) - f^{(\ell)}(z)| < 1/s,$$

$$\sup_{z\in\Gamma} |f^{(\ell)}(z) - g^{(\ell)}(z)| < \epsilon.$$

Consider simply connected open sets U_0, U_1 such that $\Gamma \subset U_0$, $K_{m_1} \subset U_1$ and $U_0 \cap U_1 = \emptyset$ (see [6] and [9]). Also let

$$H(z) = \begin{cases} g(z), & z \in U_0, \\ f_{j_1}(z), & z \in U_1. \end{cases}$$

In view of Runge's Theorem, H can be approximated by polynomials, uniformly on compact subsets of $U_0 \cup U_1$. Applying the Weierstraß Theorem, we may find a polynomial p such that for every $\ell = 0, 1, \ldots, s$,

$$\sup_{z\in\Gamma} |p^{(\ell)}(z) - g^{(\ell)}(z)| < \epsilon/2 \quad \text{and} \quad \sup_{z\in K_{m_1}} |p^{(\ell)}(z) - f_{j_1}^{(\ell)}(z)| < 1/s.$$

Since $\limsup_{n\in\mathbb{N}} \lambda_n/n = \infty$, there exists a strictly increasing sequence $(\mu_n)_{n\in\mathbb{N}}$ of positive integers such that $\lambda_{\mu_n}/\mu_n \to \infty$. Thus, $\lambda_{\mu_n}/(\mu_n+1) \to \infty$.

Without loss of generality, we may consider a simply connected, open set U_2 such that $K_{m_2} \subset U_2$ and $U_2 \cap U_0 = \emptyset$ (otherwise, we choose a new simply connected open set $\tilde{U}_0$, disjoint from U_1, U_2, with $\Gamma \subset \tilde{U}_0$).

Applying Corollary 3.3 for

$$(\sigma_n)_{n\in\mathbb{N}} = (\mu_n+1)_{n\in\mathbb{N}}, \qquad (\tau_n)_{n\in\mathbb{N}} = (\lambda_{\mu_n})_{n\in\mathbb{N}},$$
$$h(z) = \begin{cases} f_{j_2}(z) - p(z), & z \in U_2, \\ 0, & z \in U_0, \end{cases}$$

we may fix a polynomial $P \in \operatorname{span}\{z^{\mu_{n_0}+1}, \ldots, z^{\lambda_{\mu_{n_0}}}\}$ such that

$$\mu_{n_0} \geq \deg(p),$$
$$\sup_{z\in K_{m_2}} |P^{(\ell)}(z) - f_{j_2}^{(\ell)}(z) + p^{(\ell)}(z)| < 1/s \quad \text{for } \ell = 0,1,\ldots,s,$$
$$\sup_{z\in\Gamma} |P^{(\ell)}(z)| < \min\{1/s, \epsilon/2\} \quad \text{for } \ell = 0,1,\ldots,s.$$

We set $f(z) = P(z) + p(z)$. Then $S_{\mu_{n_0}}(f)(z) = p(z)$ and $S_{\lambda_{\mu_{n_0}}}(f)(z) = P(z) + p(z) = f(z)$. Consequently, for $\ell = 0,1,\ldots,s$, we have

$$\sup_{z\in K_{m_1}} |S_{\mu_{n_0}}(f)(z) - f_{j_1}(z)| = \sup_{z\in K_{m_1}} |p(z) - f_{j_1}(z)| < 1/s,$$
$$\sup_{z\in K_{m_2}} |S_{\lambda_{\mu_{n_0}}}(f)(z) - f_{j_2}(z)| = \sup_{z\in K_{m_2}} |f(z) - f_{j_2}(z)|$$
$$= \sup_{z\in K_{m_2}} |P(z) + p(z) - f_{j_2}(z)| < 1/s,$$
$$\sup_{z\in\Gamma_k} |S_{\mu_{n_0}}^{(\ell)}(f)(z) - f^{(\ell)}(z)| = \sup_{z\in\Gamma_k} |p^{(\ell)}(z) - (P^{(\ell)}(z) + p^{(\ell)}(z))|$$
$$\leq \sup_{z\in\Gamma} |P^{(\ell)}(z)| < 1/s,$$
$$\sup_{z\in\Gamma_k} |S_{\lambda_{\mu_0}}^{(\ell)}(f)(z) - f^{(\ell)}(z)| = 0 < 1/s,$$
$$\sup_{z\in\Gamma} |f^{(\ell)}(z) - g^{(\ell)}(z)| \leq \sup_{z\in\Gamma} |P^{(\ell)}(z)| + \sup_{z\in\Gamma} |p^{(\ell)}(z) - g^{(\ell)}(z)| < \epsilon.$$

As f satisfies all the requirements, the set $\bigcup_{n\in\mathbb{N}}(E(m_1,m_2,j_1,j_2,s,n) \cup \Xi(k,s,n)) \cap X^\infty(\Omega)$ is dense in $X^\infty(\Omega)$, which gives the result. ∎

To complete the picture we will prove that $U_{\mathrm{sm}}(\Omega,(\lambda_n)_{n\in\mathbb{N}},0)$ is empty if $\limsup_{n\in\mathbb{N}} \lambda_n/n < \infty$. We use some ideas of [4] and [7].

Lemma 3.6. *Let $\Omega \subset \mathbb{C}$ be a domain such that $\mathbb{C} \setminus \overline{\Omega}$ is unbounded. There exists an increasing sequence $(E_n)_{n\in\mathbb{N}}$ of compact subsets of $\mathbb{C} \setminus \overline{\Omega}$*

with connected complements such that $E := \bigcup_{n\in\mathbb{N}} E_n$ is closed and non-thin at ∞.

Proof. First, we will treat the case of Ω unbounded. Let U be an unbounded, connected component of $\mathbb{C}\setminus\overline{\Omega}$. Then U is path-connected. Now fix a sequence $(z_n)_{n\in\mathbb{N}}$ of distinct points in U with $|z_n|\to\infty$.

Let γ_1 be a path joining z_1 to z_2 in U, and set $F_1=\{\gamma_1(t): t\in[0,1]\}$. Then F_1 is a compact subset of $\mathbb{C}\setminus\overline{\Omega}$. If U_1 is the unbounded component of F_1^c, then set $E_1=U_1^c$. Note that E_1 is a compact subset of $\mathbb{C}\setminus\overline{\Omega}$ with connected complement. (Also the bounded components of F_1^c lie in $\mathbb{C}\setminus\overline{\Omega}$ because Ω is connected and unbounded.)

Now, let γ_2 be a path from z_2 to z_3 in U and $F_2=E_1\cup\{\gamma_2(t): t\in[0,1]\}$. Again, if U_2 is the unbounded component of F_2^c, then set $E_2=U_2^c$. As before, E_2 is a compact subset of $\mathbb{C}\setminus\overline{\Omega}$ with connected complement and $E_1\subseteq E_2$.

Proceeding inductively, for every $n\in\mathbb{N}$, let γ_n be a path from z_n to z_{n+1} in U and $F_n=E_{n-1}\cup\{\gamma_n(t): t\in[0,1]\}$. Moreover, let U_n be the unbounded component of F_n^c and set $E_n=U_n^c$. It is easy to see that $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of compact subsets of $\mathbb{C}\setminus\overline{\Omega}$ with connected complements.

Since $E=\bigcup_{n\in\mathbb{N}} E_n$ is the union of an increasing sequence of compact sets, it is easy to see that E is a closed, connected and unbounded set. From [20, Theorem 3.8.3], a connected set containing more than one point is non-thin at every point of its closure. So E is non-thin at ∞, which completes the proof.

Now, if Ω is bounded, then $\mathbb{C}\setminus\Omega$ contains a half-line and the lemma follows immediately using the arguments presented above. ■

THEOREM 3.7. *Let Ω be a simply connected domain such that $\mathbb{C}\setminus\overline{\Omega}$ is unbounded and $0\in\Omega$, and let $f\in H(\Omega)$. If for a sequence $(E_n)_{n\in\mathbb{N}}$ as in Lemma 3.6 and for sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of positive integers with $a_n\ge b_n$ for all $n\in\mathbb{N}$, we have*

$$S_{a_n}(f)\xrightarrow{n\to\infty}1 \quad \textit{and} \quad S_{b_n}(f)\xrightarrow{n\to\infty}0, \qquad \textit{locally uniformly on } E,$$

then the sequence $(a_n/b_n)_{n\in\mathbb{N}}$ is unbounded.

Proof. Assume that there exists $C>0$ such that $a_n/b_n\le C$ for all $n\in\mathbb{N}$. Fix $\xi\in E_1\subseteq\mathbb{C}\setminus\overline{\Omega}$.

Now consider the following sequence of polynomials:

$$P_n(z)=\frac{\xi^{b_n}}{z^{b_n}}(S_{a_n}(f)(z)-S_{b_n}(f)(z)),$$

with $\deg(P_n)=a_n-b_n\le Cb_n-b_n<Cb_n$.

Since E is closed and non-thin at ∞, the set $\tilde{E}:=E\cap D(0,|\xi|+1)^c$ is also closed and non-thin at ∞ (as thinness is a local property).

From our assumptions on f, for every $z \in \tilde{E}$, we have

$$\lim_{n\to\infty} |P_n(z)|^{1/C\mu_n} = \lim_{n\to\infty} \left|\frac{\xi}{z}\right|^{1/C} |S_{\lambda_{\mu_n}}(f)(z) - S_{\mu_n}(f)(z)|^{1/C\mu_n} < 1. \tag{3.1}$$

Fix $\Gamma \subset \tilde{E}$ connected, compact, and containing more than one point. Then $\|S_{a_n}(f) - S_{b_n}(f)\|_\Gamma \le A$ for all $n \in \mathbb{N}$, for some $A > 0$. Thus,

$$\limsup_{n\to\infty} \|P_n\|_\Gamma^{1/Cb_n} \le \limsup_{n\to\infty} \left\|\frac{\xi}{z}\right\|_\Gamma^{1/C} A^{1/Cb_n} \le \left(\frac{|\xi|}{|\xi|+1}\right)^{1/C} < 1. \tag{3.2}$$

By applying the Müller–Yavrian Theorem 2.2 for $d_n = Cb_n$, the inequalities (3.1), (3.2) show that the conditions of the theorem are fulfilled. Therefore, $P_n \to 0$ uniformly on compact subsets of $\mathbb{C}$. In particular, $P_n(\xi) \to 0$ as $n \to \infty$.

However, since $\xi \in E_1$, we have $P_n(\xi) = S_{a_n}(f)(\xi) - S_{b_n}(f)(\xi) \to 1$ as $n \to \infty$, which is a contradiction. As a result, the sequence $(a_n/b_n)_{n\in\mathbb{N}}$ is unbounded. ∎

Remark 3.8. In the proof of the previous theorem, pointwise convergence of $(S_{a_n}(f))_{n\in\mathbb{N}}$ and $(S_{b_n}(f))_{n\in\mathbb{N}}$ on E would have been enough if the sequence of polynomials $(S_{a_n}(f) - S_{b_n}(f))_{n\in\mathbb{N}}$ was supposed to be uniformly bounded on a suitable compact set Γ.

Corollary 3.9. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain such that $\mathbb{C}_\infty \setminus \overline{\Omega}$ is connected and $0 \in \Omega$. If $(\lambda_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive integers such that $\limsup_{n\in\mathbb{N}} \lambda_n/n < \infty$, then $U_{\mathrm{sm}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0) = \emptyset$.*

Proof. Let E be as in Lemma 3.6. If $f \in U_{\mathrm{sm}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0)$, then there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of positive integers such that

$$S_{\lambda_{\mu_n}}(f) \xrightarrow{n\to\infty} 1 \quad \text{and} \quad S_{\mu_n}(f) \xrightarrow{n\to\infty} 0, \qquad \text{locally uniformly on } E.$$

In view of the previous theorem, $(\lambda_n/n)_{n\in\mathbb{N}}$ is unbounded, which is a contradiction. So $U_{\mathrm{sm}}(\Omega, (\lambda_n)_{n\in\mathbb{N}}, 0) = \emptyset$. ∎

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