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Constrained Gauss variational problems for a condenser with intersecting plates

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Abstract. We study a constrained Gauss variational problem relative to a positive definite kernel on a locally compact space for vector measures associated with a condenser $\mathbf{A} = (A_i)_{i\in I}$ whose oppositely charged plates intersect each other in a set of capacity zero. Sufficient conditions for the existence of minimizers are established, and their uniqueness and vague compactness are studied. Note that the classical (unconstrained) Gauss variational problem would be unsolvable in this formulation. We also analyze continuity of the minimizers in the vague and strong topologies when the condenser and the constraint both vary, describe the weighted equilibrium vector potentials, and single out their characteristic properties. Our approach is based on the simultaneous use of the vague topology and a suitable semimetric structure defined in terms of energy on a set of vector measures associated with $\mathbf{A}$, and on the establishment of completeness results for proper semimetric spaces. The theory developed is valid in particular for the classical kernels, which is important for applications. The study is illustrated by several examples.

1. Introduction. A (function) *kernel* κ on a locally compact space X is a symmetric, lower semicontinuous function on $X \times X$, nonnegative unless X is compact. In the present study we are concerned with a *positive definite* kernel, which means that the energy of any (signed) scalar Radon measure μ on X is nonnegative whenever defined. Our purpose is to investigate *minimum energy problems with an external field*, also known in the literature as *Gauss variational problems* or as *weighted minimum energy problems*, where the infimum is taken over classes of vector measures $\boldsymbol{\mu} = (\mu^i)_{i\in I}$ associated with a generalized condenser $\mathbf{A} = (A_i)_{i\in I}$.

In more detail, a finite ordered collection $\mathbf{A}$ of closed sets $A_i \subset X$, $i \in I$, termed *plates*, with the sign $s_i = \pm 1$ prescribed, is said to be a *generalized condenser* if oppositely signed plates intersect each other in a set of capacity

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zero, while $\boldsymbol{\mu} = (\mu^i)_{i\in I}$ is *associated with* $\mathbf{A}$ if each μ^i, $i \in I$, is a positive scalar Radon measure (charge) supported by A_i.

In accordance with an electrostatic interpretation of a condenser we say that the interaction between the components μ^i, $i \in I$, of $\boldsymbol{\mu}$ is characterized by the matrix $(s_is_j)_{i,j\in I}$, so that the *weighted energy* of $\boldsymbol{\mu}$ is defined by

$$\sum_{i,j\in I} s_is_j \iint \kappa(x,y)\, d\mu^i(x)\, d\mu^j(y) + 2\sum_{i\in I}\int f_i\, d\mu^i,$$

$f_i : X \to [-\infty, \infty]$ being an *external field* acting on the charges on the conductor A_i. The difficulties appearing in the course of our investigation are caused by the fact that a short-circuit between oppositely charged plates of the condenser may occur, for those plates may have points in common; see Theorem 4.6 below providing an example of a condenser for which the classical (unconstrained) Gauss variational problem has *no* solution.

It is therefore meaningful to ask what kinds of additional requirements on the objects under consideration will prevent such a phenomenon, and secure that a minimizer for the corresponding minimum energy problem does exist. We have succeeded in working out a substantive theory by imposing a proper upper *constraint* $\boldsymbol{\sigma} = (\sigma^i)_{i\in I}$ on the vector measures in question (see Section 5 for a formulation of the constrained Gauss variational problem).

The theory developed includes sufficient conditions for the existence of solutions to the constrained Gauss variational problem (see Theorems 6.1 and 7.1, the latter referring to the particular case of the α-Riesz kernel $|x-y|^{\alpha-n}$ of order $\alpha \in (0,n)$ on $\mathbb{R}^n$, $n \geq 3$); sufficient conditions obtained in Theorem 7.1 are shown in Theorem 7.4 to be sharp. Theorems 6.1 and 7.1 are illustrated by Examples 6.2, 6.3 and 7.2. The uniqueness of solutions is analyzed in Lemma 5.4 and Corollary 5.5. We also establish continuity of the solutions in the vague and strong topologies when the condenser $\mathbf{A}$ and the constraint $\boldsymbol{\sigma}$ both vary (Theorem 8.1), describe the weighted vector potentials of the solutions and single out their characteristic properties (Theorem 9.2). This theory is valid, in particular, for the classical kernels on $\mathbb{R}^n$, $n \geq 2$ (cf. Remark 2.2), which is important for applications.

Our approach is mainly based on the simultaneous use of the vague topology and a suitable semimetric structure defined in terms of energy on a set of vector measures associated with $\mathbf{A}$ (see Section 3.2 for the definition of this semimetric structure), and on the establishment of completeness theorems for proper semimetric spaces (Theorems 6.6 and 7.3).

A key observation behind this approach is that, since a (nonzero) positive scalar measure of finite energy does not charge any set of zero capacity, there corresponds to every positive vector measure $\boldsymbol{\mu} = (\mu^i)_{i\in I}$ of finite energy associated with $\mathbf{A}$ a scalar (*signed*) Radon measure $R\boldsymbol{\mu} = \sum_{i\in I} s_i\mu^i$ on X, and the mapping R preserves the corresponding energy semimetric (Theorem 3.8). This approach extends that from [41]–[44] where the oppositely charged plates were assumed to be mutually nonintersecting.

2. Preliminaries. Let X be a locally compact (Hausdorff) space, and let $\mathfrak{M}(X)$ be the linear space of all real-valued scalar Radon measures μ on X equipped with the *vague* topology, i.e. the topology of pointwise convergence on the class $C_0(X)$ of all continuous functions on X with compact support ([1]). The vague topology on $\mathfrak{M}(X)$ is Hausdorff; hence, a vague limit of any sequence (net) in $\mathfrak{M}(X)$ is unique (whenever it exists). These and other notions and results from the theory of measures and integration on a locally compact space, to be used below, can be found in [4, 16] (see also [17] for a short survey).

We denote by μ^+ and μ^- the positive and the negative parts in the Hahn–Jordan decomposition of a measure $\mu \in \mathfrak{M}(X)$, respectively, and by $S_X^\mu = S(\mu)$ its support. A measure μ is said to be *bounded* if $|\mu|(X) < \infty$, where $|\mu| := \mu^+ + \mu^-$. Let $\mathfrak{M}^+(X)$ stand for the (convex, vaguely closed) cone of all positive $\mu \in \mathfrak{M}(X)$, and let $\Psi(X)$ consist of all lower semicontinuous (l.s.c.) functions $\psi : X \to (-\infty, \infty]$, nonnegative unless X is compact. The following well known fact (see e.g. [17, Section 1.1]) will often be used.

LEMMA 2.1. *For any $\psi \in \Psi(X)$ the map $\mu \mapsto \langle \psi, \mu \rangle := \int \psi \, d\mu$ is vaguely l.s.c. on $\mathfrak{M}^+(X)$* ([2]).

A *kernel* $\kappa(x, y)$ on X is defined as a symmetric function from $\Psi(X \times X)$. Given $\mu, \mu_1 \in \mathfrak{M}(X)$, we denote by $\kappa(\mu, \mu_1)$ and $\kappa(\cdot, \mu)$ the *mutual energy* and the *potential* relative to the kernel κ, respectively, i.e. ([3])

$$\kappa(\mu, \mu_1) := \iint \kappa(x, y) \, d\mu(x) \, d\mu_1(y),$$
$$\kappa(x, \mu) := \int \kappa(x, y) \, d\mu(y), \quad x \in X.$$

Observe that $\kappa(x, \mu)$ is well defined provided that $\kappa(x, \mu^+)$ and $\kappa(x, \mu^-)$ are not both infinite, and then $\kappa(x, \mu) = \kappa(x, \mu^+) - \kappa(x, \mu^-)$. In particular, if $\mu \in \mathfrak{M}^+(X)$ then $\kappa(\cdot, \mu)$ is defined everywhere and represents a l.s.c. function on X, bounded from below (see Lemma 2.1). Also note that $\kappa(\mu, \mu_1)$ is well defined and equal to $\kappa(\mu_1, \mu)$ provided that $\kappa(\mu^+, \mu_1^+) + \kappa(\mu^-, \mu_1^-)$ or $\kappa(\mu^+, \mu_1^-) + \kappa(\mu^-, \mu_1^+)$ is finite. For $\mu = \mu_1$ the mutual energy $\kappa(\mu, \mu_1)$ becomes the *energy* $\kappa(\mu, \mu)$. Let $\mathcal{E}_\kappa(X)$ consist of all $\mu \in \mathfrak{M}(X)$ whose energy

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([1]) When speaking of a continuous numerical function we understand that the values are *finite* real numbers.

([2]) Throughout the paper the integrals are understood as *upper* integrals [4].

([3]) When introducing notation of a numerical value, we assume that the corresponding object on the right is well defined (as a finite number or $\pm\infty$).

$\kappa(\mu,\mu)$ is finite, which by definition means that $\kappa(\mu^+,\mu^+)$, $\kappa(\mu^-,\mu^-)$ and $\kappa(\mu^+,\mu^-)$ are all finite, and let $\mathcal{E}^+_\kappa(X) := \mathcal{E}_\kappa(X) \cap \mathfrak{M}^+(X)$.

For a set $Q \subset X$ let $\mathfrak{M}^+(Q)$ consist of all $\mu \in \mathfrak{M}^+(X)$ *concentrated on* (or *carried by*) Q, which means that $Q^c := X \setminus Q$ is locally μ-negligible, or equivalently that Q is μ-measurable and $\mu = \mu|_Q$ where $\mu|_Q = 1_Q \cdot \mu$ is the trace (restriction) of μ on Q [4, Chapter V, Section 5, n° 2, Example]. (Here 1_Q denotes the indicator function of Q.) If Q is closed then μ is concentrated on Q if and only if it is supported by Q, i.e. $S(\mu) \subset Q$. Also note that if either X is *countable at infinity* (i.e. X can be represented as a countable union of compact sets [2, Chapter I, Section 9, n° 9]), or μ is bounded, then the concept of local μ-negligibility coincides with that of μ-negligibility; and hence $\mu \in \mathfrak{M}^+(Q)$ if and only if $\mu^*(Q^c) = 0$, $\mu^*(\cdot)$ being the *outer measure* of a set. Write $\mathcal{E}^+_\kappa(Q) := \mathcal{E}_\kappa(X) \cap \mathfrak{M}^+(Q)$, $\mathfrak{M}^+(Q,q) := \{\mu \in \mathfrak{M}^+(Q) : \mu(Q) = q\}$ and $\mathcal{E}^+_\kappa(Q,q) := \mathcal{E}_\kappa(X) \cap \mathfrak{M}^+(Q,q)$, where $q \in (0,\infty)$.

Among the variety of potential-theoretic principles investigated for example in the comprehensive work by Ohtsuka [33] (see also the references therein), in the present study we shall only need the following three:

(i) A kernel κ is said to satisfy the *continuity principle* (Evans–Vasilesco), or to be *regular* (Choquet [8]) if, for any $\mu \in \mathfrak{M}^+(X)$ with compact support $S(\mu)$, the potential $\kappa(\cdot,\mu)$ is continuous throughout X whenever its restriction to $S(\mu)$ is continuous.

(ii) A kernel κ is said to satisfy *Frostman's maximum principle* if, for any $\mu \in \mathfrak{M}^+(X)$ with compact support $S(\mu)$,

$$\sup_{x\in X} \kappa(x,\mu) = \sup_{x\in S(\mu)} \kappa(x,\mu).$$

(iii) A kernel κ is called *positive definite* if $\kappa(\mu,\mu) \geq 0$ for every (signed) measure $\mu \in \mathfrak{M}(X)$ for which the energy is defined; and κ is said to be *strictly positive definite*, or to satisfy the *energy principle*, if in addition $\kappa(\mu,\mu) > 0$ except for $\mu = 0$.

The above-mentioned principles are not completely independent of one another. In particular, every kernel satisfying the Frostman maximum principle is positive definite [29, 9]. And for a kernel which is finite off the diagonal and continuous in the extended sense on $X \times X$ we have (ii)⇒(i) (see [31], [32], [33, Eq. 1.3], and independently [8]).

Unless explicitly stated otherwise, in all that follows we assume that the kernel κ is positive definite. Then $\mathcal{E}_\kappa(X)$ forms a pre-Hilbert space with the scalar product $\kappa(\mu,\mu_1)$ and the seminorm $\|\mu\|_\kappa := \sqrt{\kappa(\mu,\mu)}$ (see [17]). The topology on $\mathcal{E}_\kappa(X)$ defined by the seminorm $\|\cdot\|_\kappa$ is termed *strong*. Two elements of $\mathcal{E}_\kappa(X)$, μ and μ_1, are said to be *equivalent in $\mathcal{E}_\kappa(X)$* if $\|\mu-\mu_1\|_\kappa = 0$. Note that a (positive definite) kernel satisfies the energy principle if and only if the seminorm $\|\cdot\|_\kappa$ is a norm, or alternatively if and only if two measures in $\mathcal{E}_\kappa(X)$ are equal whenever they are equivalent.

In addition to the strong topology on $\mathcal{E}_\kappa(X)$, it is often useful to consider the so-called *weak* topology on $\mathcal{E}_\kappa(X)$, defined by means of the seminorms $\nu \mapsto |\kappa(\nu,\mu)|$, $\mu \in \mathcal{E}_\kappa(X)$ (see [17]). By the Cauchy–Schwarz inequality

$$|\kappa(\nu,\mu)| \le \|\nu\|_\kappa \cdot \|\mu\|_\kappa \quad \text{for } \nu,\mu \in \mathcal{E}_\kappa(X),$$

the strong topology on $\mathcal{E}_\kappa(X)$ is *finer* than the weak topology.

In contrast to [20, 21] where capacity has been treated as a functional acting on positive numerical functions on X, in the present study we use the (standard) concept of capacity as a set function. Thus the (*inner*) *capacity* of a set $Q \subset X$ relative to the kernel κ, denoted $c_\kappa(Q)$, is

$$c_\kappa(Q) := \Big[\inf_{\mu \in \mathcal{E}^+_\kappa(Q,1)} \kappa(\mu,\mu)\Big]^{-1} \tag{2.1}$$

(see e.g. [17, 33]). Then $0 \le c_\kappa(Q) \le \infty$. (As usual, the infimum over the empty set is taken to be $+\infty$. We also set $1/(+\infty) = 0$ and $1/0 = +\infty$.)

An assertion $\mathcal{U}(x)$ involving a variable point $x \in X$ is said to hold c_κ-*n.e.* on Q if $c_\kappa(N) = 0$ where N consists of all $x \in Q$ for which $\mathcal{U}(x)$ fails to hold. We shall use the short form 'n.e.' (*nearly everywhere*) instead of 'c_κ-n.e.' if this does not cause any misunderstanding.

Following [17, 19], we call a (positive definite) kernel κ *consistent* if it satisfies either of the following two *equivalent* properties:

(C_1) *Every strong Cauchy net in $\mathcal{E}^+_\kappa(X)$ converges strongly to any of its vague cluster points.*

(C_2) *Every strongly bounded and vaguely convergent net in $\mathcal{E}^+_\kappa(X)$ converges weakly to its vague limit.*

A kernel κ is *perfect* if it is consistent and strictly positive definite [17].

Remark 2.2. On $X = \mathbb{R}^n$, $n \ge 3$, the α-Riesz kernel $\kappa_\alpha(x,y) = |x-y|^{\alpha-n}$, $\alpha \in (0,n)$, is strictly positive definite and consistent, hence perfect [10, 11]; thus so is the Newtonian kernel $\kappa_2(x,y) = |x-y|^{2-n}$ [6]. Recently it has been shown by the present authors that if $X = D$ where D is an arbitrary open set in $\mathbb{R}^n$, $n \ge 3$, and G^α_D, $\alpha \in (0,2]$, is the α-Green kernel on D [27, Chapter IV, Section 5], then $\kappa = G^\alpha_D$ is likewise perfect [22, Theorems 4.9, 4.11]. Furthermore, the Green kernel on a planar Greenian set is strictly positive definite by [12, Chapter XIII, Section 7] and it is consistent by [15], hence perfect. The logarithmic kernel $-\log|x-y|$ on a closed disc in $\mathbb{R}^2$ of radius <1 is strictly positive definite, as shown in [36, Section 47], [27, Theorem 1.16]. It is therefore perfect [31], because it satisfies Frostman's maximum principle by [27, Theorem 1.6], and hence is regular by [33, Eq. 1.3]. For analogous results concerning the logarithmic kernel on closed balls of arbitrary finite dimension, see [18].

THEOREM 2.3 (see [17]). *If a kernel κ on a locally compact space X is perfect, then the cone $\mathcal{E}^+_\kappa(X)$ is strongly complete and the strong topology on $\mathcal{E}^+_\kappa(X)$ is finer than the* (*induced*) *vague topology on $\mathcal{E}^+_\kappa(X)$.*

REMARK 2.4. When speaking of vague convergence, one has to consider *nets* or *filters* in $\mathfrak{M}(X)$ instead of sequences since the vague topology in general does not satisfy the first axiom of countability. We follow Moore and Smith's theory of convergence, based on the concept of nets (see [28]; see also [16, Chapter 0] and [26, Chapter 2]). However, if X is metrizable and countable at infinity, then $\mathfrak{M}^+(X)$ satisfies the first axiom of countability, and the use of nets may be avoided. Indeed, if $\varrho(\cdot,\cdot)$ denotes a metric on such a space X, then a countable base $(\mathcal{V}_n)_{n\in\mathbb{N}}$ of vague neighborhoods of a measure $\mu_0 \in \mathfrak{M}^+(X)$ can be obtained for example as follows after choosing a countable dense sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of X:

$$\mathcal{V}_n = \Big\{\mu \in \mathfrak{M}^+(X) : \int (1 - n\varrho(x_n, x))^+ \, d|\mu - \mu_0|(x) < 1/n\Big\}.$$

The existence of such $\{x_n\}_{n\in\mathbb{N}}$ for the space X in question is ensured by [3, Chapter IX, Section 2, n° 8, Proposition 12, and n° 9, Corollary to Proposition 16].

REMARK 2.5. In contrast to Theorem 2.3, for a perfect kernel κ the whole pre-Hilbert space $\mathcal{E}_\kappa(X)$ is in general strongly *incomplete*, and this is the case even for the α-Riesz kernel of order $\alpha \in (1, n)$ on $\mathbb{R}^n$, $n \geq 3$ (see [6] and [27, Theorem 1.19]). Compare with [39, Theorem 1] where the strong completeness has been established for the metric subspace of all (*signed*) $\nu \in \mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$, $\alpha \in (0, n)$, such that ν^+ and ν^- are supported by closed nonintersecting sets in $\mathbb{R}^n$, $n \geq 3$ (cf. also Theorem 3.9 below). This result from [39] has been proved with the aid of Deny's theorem [10] stating that $\mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ can be completed by making use of tempered distributions on $\mathbb{R}^n$ with finite α-Riesz energy (compare with Remark 2.6 below).

REMARK 2.6. The concept of consistent kernel is an efficient tool in minimum energy problems over classes of *positive scalar* Radon measures with finite energy. Indeed, if Q is closed, $c_\kappa(Q) \in (0, \infty)$, and κ is consistent, then the minimum energy problem in (2.1) has a solution λ [17, Theorem 4.1]; we shall call such a λ an (*inner*) *κ-capacitary measure* on Q ([4]). Later the concept of consistency has been shown to be efficient also in minimum energy problems over classes of *vector measures* associated with a standard condenser [41]–[44] (see also Remarks 4.5 and 5.3 below for a short survey).

([4]) Such a λ is uniquely determined whenever the kernel κ is strictly positive definite.

In contrast to [39, Theorem 1], the approach developed in [41]–[44] substantially used the assumption of the boundedness of the kernel on the product of the oppositely charged plates of a condenser, which made it possible to extend Cartan's proof [6] of the strong completeness of the cone $\mathcal{E}^+_{\kappa_2}(\mathbb{R}^n)$ of all positive measures on $\mathbb{R}^n$ with finite Newtonian energy to an arbitrary consistent kernel κ on a locally compact space X and suitable classes of (*signed*) measures $\mu \in \mathcal{E}_\kappa(X)$.

In the present paper the assumption of the boundedness of the kernel on the product of the oppositely charged plates of a condenser is shown to be essential not only for the proofs in [41]–[44], but also for the validity of the approach developed therein (see Theorem 4.6 below). Nevertheless, we have succeeded in working out a substantive theory by imposing instead a proper upper constraint on the measures under consideration.

3. Generalized condensers. Fix a finite set I of indices $i \in \mathbb{N}$ and an ordered collection $\mathbf{A} := (A_i)_{i\in I}$ of nonempty closed sets $A_i \subset X$, X being a locally compact space, where each A_i, $i \in I$, has the sign $s_i := \operatorname{sign} A_i = \pm 1$ prescribed. Let I^+ consist of all $i \in I$ such that $s_i = +1$, and let $I^- := I \setminus I^+$. The sets A_i, $i \in I^+$, and A_j, $j \in I^-$, are said to be the *positive* and *negative* plates of $\mathbf{A}$. Write

$$A^+ := \bigcup_{i\in I^+} A_i, \qquad A^- := \bigcup_{j\in I^-} A_j, \qquad A := A^+ \cup A^-, \qquad \delta_{\mathbf{A}} := A^+ \cap A^-.$$

DEFINITION 3.1. $(A_i)_{i\in I}$ is said to be a *standard condenser* in X if $\delta_{\mathbf{A}} = \emptyset$.

Fix a (positive definite) kernel κ on X. By relaxing the requirement $\delta_{\mathbf{A}} = \emptyset$, we slightly generalize the notion of standard condenser as follows.

DEFINITION 3.2. $(A_i)_{i\in I}$ is said to be a *generalized condenser* if $c_\kappa(\delta_{\mathbf{A}}) = 0$ ([5]).

A (generalized) condenser $\mathbf{A} = (A_i)_{i\in I}$ is said to be *compact* if all the A_i, $i \in I$, are compact, and *noncompact* if at least one of the A_i is noncompact. A standard condenser can certainly be thought of as a particular case of a generalized condenser. Note that any two equally signed plates of a (generalized) condenser may intersect each other (or even coincide).

In Examples 3.3 and 3.4 below, $X = \mathbb{R}^n$ with $n \geq 3$. Let $B(x, r)$, resp. $\overline{B}(x, r)$, denote the open, resp. closed, n-dimensional ball of radius r centered at $x \in \mathbb{R}^n$. We shall also write $S(x, r) := \partial_{\mathbb{R}^n} B(x, r)$.

([5]) Gonchar and Rakhmanov [24] seem to be the first to consider such a generalization for the kernel $-\log|x - y|$ on $\mathbb{R}^2$.

EXAMPLE 3.3. Let $I^+ := \{1\}$ and $I^- := \{2,3,4\}$. Define $A_1 := \overline{B}(\xi_1, 1)$, $A_2 := \overline{B}(\xi_2, 1)$, $A_3 := \overline{B}(\xi_3, 2)$ and $A_4 := \overline{B}(\xi_4, 1)$ where $\xi_1 = (0, 0, \ldots, 0)$, $\xi_2 = (2, 0, \ldots, 0)$, $\xi_3 = (3, 0, \ldots, 0)$ and $\xi_4 = (-2, 0, \ldots, 0)$. Since $\delta_{\mathbf{A}}$ consists of the points $\xi_5 = (-1, 0, \ldots, 0)$ and $\xi_6 = (1, 0, \ldots, 0)$, the collection $\mathbf{A} = (A_i)_{i\in I}$ forms a generalized condenser in $\mathbb{R}^n$ for any (positive definite) kernel κ on $\mathbb{R}^n$ with the property that $\kappa(x, y) = \infty$ for all $x = y$.

EXAMPLE 3.4. Let $n = 3$, $I^+ := \{1\}$, $I^- := \{2\}$, and let

$$A_i := \{x \in \mathbb{R}^3 : 1 \le x_1 < \infty,\ x_2^2 + x_3^2 = \varrho_i^2(x_1) \text{ with } \varrho_i(x_1) = \exp(-x_1^{r_i})\}$$

where $1 < r_1 < r_2 < \infty$. Then (A_1, A_2) forms a standard condenser with

$$\operatorname{dist}(A_1, A_2) := \inf_{x\in A_1,\, y\in A_2} |x - y| = 0.$$

3.1. Vector measures. Vague topology. In the rest of the paper, we fix a generalized condenser $\mathbf{A} = (A_i)_{i\in I}$ in X, and let $\mathfrak{M}^+(\mathbf{A})$ stand for the Cartesian product $\prod_{i\in I} \mathfrak{M}^+(A_i)$. Then $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ is a positive *vector measure* $(\mu^i)_{i\in I}$ with the components $\mu^i \in \mathfrak{M}^+(A_i)$; such a $\boldsymbol{\mu}$ is said to be *associated* with $\mathbf{A}$. Write $|I| := \operatorname{Card} I$.

DEFINITION 3.5. The *vague topology* on $\mathfrak{M}^+(\mathbf{A})$ is the topology of the product space $\prod_{i\in I} \mathfrak{M}^+(A_i)$, where each of the factors $\mathfrak{M}^+(A_i)$, $i \in I$, is endowed with the vague topology induced from $\mathfrak{M}(X)$. Namely, a net $(\boldsymbol{\mu}_s)_{s\in S} \subset \mathfrak{M}^+(\mathbf{A})$ is said to converge to $\boldsymbol{\mu}$ *vaguely* if, for every $i \in I$, $\mu_s^i \to \mu^i$ vaguely in $\mathfrak{M}(X)$ when s increases along S.

Since all the A_i, $i \in I$, are closed in X, $\mathfrak{M}^+(\mathbf{A})$ is vaguely closed in $\mathfrak{M}^+(X)^{|I|}$. Moreover, since every $\mathfrak{M}^+(A_i)$ is Hausdorff in the vague topology, so is $\mathfrak{M}^+(\mathbf{A})$ [2, Chapter I, Section 8, Proposition 7]. Hence, a vague limit of any net in $\mathfrak{M}^+(\mathbf{A})$ belongs to $\mathfrak{M}^+(\mathbf{A})$ and is unique (whenever it exists).

If $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ and a vector-valued function $\boldsymbol{u} = (u_i)_{i\in I}$ with μ^i-measurable components $u_i : X \to [-\infty, \infty]$ are given, then we write

$$\langle \boldsymbol{u}, \boldsymbol{\mu}\rangle := \sum_{i\in I} \langle u_i, \mu^i\rangle = \sum_{i\in I} \int u_i \, d\mu^i. \tag{3.1}$$

Let $\mathcal{E}_\kappa^+(\mathbf{A})$ consist of all $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A})$ such that $\kappa(\mu^i, \mu^i) < \infty$ for all $i \in I$; or, in other words, $\mathcal{E}_\kappa^+(\mathbf{A}) := \prod_{i\in I} \mathcal{E}_\kappa^+(A_i)$. By [17, Lemma 2.3.1], for any $\boldsymbol{\mu} \in \mathcal{E}_\kappa^+(\mathbf{A})$ we have $\mu^i(\delta_{\mathbf{A}}) = 0$ for all $i \in I$. Hence, each of the components μ^i, $i \in I$, of $\boldsymbol{\mu} \in \mathcal{E}_\kappa^+(\mathbf{A})$ is carried by A_i^δ, where

$$A_i^\delta := A_i \setminus \delta_{\mathbf{A}}, \tag{3.2}$$

though $S(\mu^i)$ may coincide with the whole A_i. Thus we actually have

$$\mathcal{E}_\kappa^+(\mathbf{A}) = \prod_{i\in I} \mathcal{E}_\kappa^+(A_i^\delta).$$

Write $A_\delta^+ := A^+ \setminus \delta_{\mathbf{A}}$ and $A_\delta^- := A^- \setminus \delta_{\mathbf{A}}$. For any $\boldsymbol{\mu} \in \mathcal{E}_\kappa^+(\mathbf{A})$ define

$$R\boldsymbol{\mu} := \sum_{i\in I} s_i \mu^i,$$

the "resultant" of $\boldsymbol{\mu}$. Since $A_\delta^+ \cap A_\delta^- = \emptyset$, $R\boldsymbol{\mu}$ is a (*signed*) scalar Radon measure on X whose positive and negative parts, carried by A_δ^+ and A_δ^- respectively, are defined by

$$(R\boldsymbol{\mu})^+ := \sum_{i\in I^+} \mu^i \quad\text{and}\quad (R\boldsymbol{\mu})^- := \sum_{j\in I^-} \mu^j. \tag{3.3}$$

If $\boldsymbol{\mu} = \boldsymbol{\mu}_1$ where $\boldsymbol{\mu}, \boldsymbol{\mu}_1 \in \mathcal{E}_\kappa^+(\mathbf{A})$, then $R\boldsymbol{\mu} = R\boldsymbol{\mu}_1$, but not the other way around. We call $\boldsymbol{\mu}, \boldsymbol{\mu}_1 \in \mathcal{E}_\kappa^+(\mathbf{A})$ *R-equivalent* if $R\boldsymbol{\mu} = R\boldsymbol{\mu}_1$. For the map $\boldsymbol{\mu} \mapsto R\boldsymbol{\mu}$, $\boldsymbol{\mu} \in \mathcal{E}_\kappa^+(\mathbf{A})$, to be injective it is necessary and sufficient that the A_i^δ, $i \in I$, be mutually nonintersecting.

3.2. A semimetric structure on sets of vector measures. To avoid trivialities, for a given (generalized) condenser $\mathbf{A} = (A_i)_{i\in I}$ and a given (positive definite) kernel κ on X we shall always require that

$$c_\kappa(A_i) > 0 \quad \text{for all } i \in I. \tag{3.4}$$

In accordance with an electrostatic interpretation of a condenser we say that the interaction between the components μ^i, $i \in I$, of $\boldsymbol{\mu} \in \mathcal{E}_\kappa^+(\mathbf{A})$ is characterized by the matrix $(s_i s_j)_{i,j\in I}$, where $s_i := \operatorname{sign} A_i$. Given $\boldsymbol{\mu}, \boldsymbol{\mu}_1 \in \mathcal{E}_\kappa^+(\mathbf{A})$, we define the *mutual energy*

$$\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}_1) := \sum_{i,j\in I} s_i s_j \kappa(\mu^i, \mu_1^j) \tag{3.5}$$

and the *vector potential* $\kappa_{\boldsymbol{\mu}} = (\kappa_{\boldsymbol{\mu}}^i)_{i\in I}$ where

$$\kappa_{\boldsymbol{\mu}}^i(\cdot) := \sum_{j\in I} s_i s_j \kappa(\cdot, \mu^j), \qquad i \in I. \tag{3.6}$$

LEMMA 3.6. *For any $\boldsymbol{\mu}, \boldsymbol{\mu}_1 \in \mathcal{E}_\kappa^+(\mathbf{A})$ we have*

$$\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}_1) = \kappa(R\boldsymbol{\mu}, R\boldsymbol{\mu}_1) \in (-\infty, \infty). \tag{3.7}$$

Proof. This is obtained directly from (3.3) and (3.5). ∎

LEMMA 3.7. *For any $\boldsymbol{\mu} \in \mathcal{E}_\kappa^+(\mathbf{A})$ all the $\kappa_{\boldsymbol{\mu}}^i$, $i \in I$, are well defined and finite n.e. on X.*

Proof. Since $\mu^i \in \mathcal{E}_\kappa^+(X)$ for every $i \in I$, $\kappa(\cdot, \mu^i)$ is finite n.e. on X [17, p. 164]. Furthermore, the set of all $x \in X$ with $\kappa(x, \mu^i) = \infty$ is universally measurable, for $\kappa(\cdot, \mu^i)$ is l.s.c. on X. Combined with the fact that the inner capacity $c_\kappa(\cdot)$ is subadditive on universally measurable sets [17, Lemma 2.3.5], this proves the lemma. ∎

For $\boldsymbol{\mu} = \boldsymbol{\mu}_1 \in \mathcal{E}^+_\kappa(\mathbf{A})$, the mutual energy $\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}_1)$ becomes the *energy* $\kappa(\boldsymbol{\mu}, \boldsymbol{\mu})$ of $\boldsymbol{\mu}$. It follows from Lemma 3.6 that $\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) \geq 0$ for all $\boldsymbol{\mu} \in \mathcal{E}^+_\kappa(\mathbf{A})$, and also that the assertions $\kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) = 0$ and $\boldsymbol{\mu} = \mathbf{0}$ are equivalent if and only if the kernel κ is strictly positive definite.

In order to introduce a (semi)metric structure on $\mathcal{E}^+_\kappa(\mathbf{A})$, we define

$$\|\boldsymbol{\mu} - \boldsymbol{\mu}_1\|_{\mathcal{E}^+_\kappa(\mathbf{A})} := \|R\boldsymbol{\mu} - R\boldsymbol{\mu}_1\|_\kappa. \tag{3.8}$$

Based on (3.7), we see by straightforward calculation that, in fact,

$$\|\boldsymbol{\mu} - \boldsymbol{\mu}_1\|_{\mathcal{E}^+_\kappa(\mathbf{A})} = \Big[\sum_{i,j\in I} s_i s_j \kappa(\mu^i - \mu^i_1, \mu^j - \mu^j_1)\Big]^{1/2}. \tag{3.9}$$

Hence $\mathcal{E}^+_\kappa(\mathbf{A})$ forms a semimetric space with the semimetric $\|\boldsymbol{\mu} - \boldsymbol{\mu}_1\|_{\mathcal{E}^+_\kappa(\mathbf{A})}$ defined by either of (equivalent) relations (3.8) or (3.9). Similar to the terminology for the pre-Hilbert space $\mathcal{E}_\kappa(X)$, we therefore call the topology of the semimetric space $\mathcal{E}^+_\kappa(\mathbf{A})$ *strong*.

We call $\boldsymbol{\mu}, \boldsymbol{\mu}_1 \in \mathcal{E}^+_\kappa(\mathbf{A})$ *equivalent in* $\mathcal{E}^+_\kappa(\mathbf{A})$ if $\|\boldsymbol{\mu} - \boldsymbol{\mu}_1\|_{\mathcal{E}^+_\kappa(\mathbf{A})} = 0$, or alternatively if $R\boldsymbol{\mu}$ and $R\boldsymbol{\mu}_1$ are equivalent in $\mathcal{E}_\kappa(X)$ (see (3.8)).

We are thus led to the following conclusion.

THEOREM 3.8. *$\mathcal{E}^+_\kappa(\mathbf{A})$ is a semimetric space with the semimetric defined by either of the* (*equivalent*) *relations* (3.8) *or* (3.9), *and this space is isometric to its R-image in $\mathcal{E}_\kappa(X)$. This semimetric is a metric whenever the kernel is strictly positive definite and the A^δ_i, $i \in I$, are mutually disjoint.*

Theorem 1 and Corollary 1 from [39], mentioned briefly in Remark 2.5 above, can now be rewritten as follows ([6]).

THEOREM 3.9. *If $\mathbf{A} = (A_1, A_2)$ is a standard condenser in $\mathbb{R}^n$, $n \geq 3$, with $s_1 s_2 = -1$ and if κ_α is the α-Riesz kernel of an arbitrary order $\alpha \in (0, n)$, then the metric space $\mathcal{E}^+_{\kappa_\alpha}(\mathbf{A})$ is strongly complete, and the strong topology on this space is finer than the vague topology.*

4. Unconstrained f-weighted minimum energy problems. For a (positive definite) kernel κ on X and a (generalized) condenser $\mathbf{A} = (A_i)_{i\in I}$, we shall consider minimum energy problems with an external field over subclasses of $\mathcal{E}^+_\kappa(\mathbf{A})$, to be defined below. Since the admissible measures in those problems are of finite energy, there is no loss of generality in assuming that each A_i coincides with its *κ-reduced kernel* [27, p. 164], which is the set of all $x \in A_i$ such that for every neighborhood U_x of x we have $c_\kappa(A_i \cap U_x) > 0$.

([6]) Note that under the assumptions of Theorem 3.9, κ_α may be unbounded on $A^+ \times A^-$ (compare with Remarks 2.6, 4.5, and 5.3).

Fix a vector-valued function $\mathbf{f} = (f_i)_{i\in I}$, where each $f_i : X \to [-\infty, \infty]$ is μ-measurable for every $\mu \in \mathfrak{M}^+(X)$ and f_i is treated as an *external field* acting on the charges (measures) from $\mathcal{E}^+_\kappa(A_i)$. The $\mathbf{f}$-*weighted vector potential* and the $\mathbf{f}$-*weighted energy* of $\boldsymbol{\mu} \in \mathcal{E}^+_\kappa(\mathbf{A})$ are defined by

$$\mathbf{W}^{\boldsymbol{\mu}}_{\kappa,\mathbf{f}} := \kappa_{\boldsymbol{\mu}} + \mathbf{f}, \tag{4.1}$$

$$G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) := \kappa(\boldsymbol{\mu}, \boldsymbol{\mu}) + 2\langle \mathbf{f}, \boldsymbol{\mu}\rangle, \tag{4.2}$$

respectively (7). Let $\mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A})$ consist of all $\boldsymbol{\mu} \in \mathcal{E}^+_\kappa(\mathbf{A})$ with finite $G_{\kappa,\mathbf{f}}(\boldsymbol{\mu})$.

In order to ensure that $G_{\kappa,\mathbf{f}}(\boldsymbol{\mu})$ is well defined for any $\boldsymbol{\mu} \in \mathcal{E}^+_\kappa(\mathbf{A})$ we shall henceforth tacitly assume that either Case I or Case II holds, where (8):

I. *For every* $i \in I$, $f_i \in \Psi(X)$.

II. *For every* $i \in I$, $f_i = s_i\kappa(\cdot, \zeta)$ *where a (signed) measure* $\zeta \in \mathcal{E}_\kappa(X)$ *is given.*

If Case II takes place then, by (3.3), (3.7) and (4.2),

$$\begin{aligned} G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) &= \|R\boldsymbol{\mu}\|^2_\kappa + 2\sum_{i\in I} s_i\kappa(\zeta, \mu^i) \\ &= \|R\boldsymbol{\mu}\|^2_\kappa + 2\kappa(\zeta, R\boldsymbol{\mu}) = \|R\boldsymbol{\mu} + \zeta\|^2_\kappa - \|\zeta\|^2_\kappa, \end{aligned} \tag{4.3}$$

and consequently

$$-\infty < -\|\zeta\|^2_\kappa \le G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) < \infty \quad \text{for all } \boldsymbol{\mu} \in \mathcal{E}^+_\kappa(\mathbf{A}). \tag{4.4}$$

Fix a numerical vector $\mathbf{a} = (a_i)_{i\in I} \in \mathbb{R}^{|I|}$ with $a_i > 0$, $i \in I$, and a vector-valued function $\mathbf{g} = (g_i)_{i\in I}$ where all the $g_i : X \to (0, \infty)$ are continuous and such that

$$g_{i,\inf} := \inf_{x\in X} g_i(x) > 0. \tag{4.5}$$

Write

$$\begin{aligned} \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) &:= \{\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A}) : \langle g_i, \mu^i\rangle = a_i \text{ for all } i \in I\}, \\ \mathcal{E}^+_\kappa(\mathbf{A}, \mathbf{a}, \mathbf{g}) &:= \mathcal{E}^+_\kappa(\mathbf{A}) \cap \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}). \end{aligned}$$

Because of (4.5),

$$\mu^i(A_i) \le a_i g^{-1}_{i,\inf} < \infty \quad \text{for all } \boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}). \tag{4.6}$$

Since a l.s.c. function is bounded from below on a compact space, we conclude in Case I from (4.6) that there is a positive constant M such that

$$G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) \ge -M > -\infty \quad \text{for all } \boldsymbol{\mu} \in \mathcal{E}^+_\kappa(\mathbf{A}, \mathbf{a}, \mathbf{g}). \tag{4.7}$$

---

(7) $G_{\kappa,\mathbf{f}}(\cdot)$ is also known as the *Gauss functional* (see e.g. [33]). Also note that, when defining $G_{\kappa,\mathbf{f}}(\cdot)$, we have used the notation (3.1).

(8) Then $\mathbf{W}^{\boldsymbol{\mu}}_{\kappa,\mathbf{f}}$ is likewise well defined c_κ-n.e. on X for any $\boldsymbol{\mu} \in \mathcal{E}^+_\kappa(\mathbf{A})$ (see Lemma 3.7).

Also denote

$$\mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) := \mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A}) \cap \mathfrak{M}^+(\mathbf{A},\mathbf{a},\mathbf{g}),$$
$$G_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) := \inf_{\boldsymbol{\mu}\in\mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})} G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}).$$

By (4.4) and (4.7), in either Case I or Case II we thus have

$$G_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) > -\infty. \tag{4.8}$$

If the class $\mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is nonempty, or equivalently if

$$G_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) < \infty, \tag{4.9}$$

then the following (*unconstrained*) **f**-*weighted minimum energy problem*, also known as the *Gauss variational problem* [23, 33], makes sense.

Problem 4.1. *Does there exist* $\boldsymbol{\lambda}_{\mathbf{A}} \in \mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *with*

$$G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_{\mathbf{A}}) = G_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})\,?$$

If $I = I^+ = \{1\}$, $g_1 = 1$, $a_1 = 1$ and $f_1 = 0$, then Problem 4.1 reduces to the problem (2.1) solved by [17, Theorem 4.1] (see Remark 2.6 above).

Remark 4.2. An analysis similar to that for a standard condenser [43, Lemma 6.2] shows that requirement (4.9) is fulfilled if and only if $c_\kappa(\dot{A}^\delta_i) > 0$ for every $i \in I$, where

$$\dot{A}^\delta_i := \{x \in A^\delta_i : |f_i(x)| < \infty\}, \tag{4.10}$$

A^δ_i being defined by (3.2). In view of (3.4), this implies that (4.9) holds automatically whenever Case II takes place, for the potential of $\zeta \in \mathcal{E}_\kappa(X)$ is finite n.e. on X [17, p. 164].

Remark 4.3. If **A** is a compact standard condenser, the kernel κ is continuous on $A^+ \times A^-$, and if Case I holds, then the solvability of Problem 4.1 can be established by exploiting the vague topology only, since then $\mathfrak{M}^+(\mathbf{A},\mathbf{a},\mathbf{g})$ is vaguely compact, while $G_{\kappa,\mathbf{f}}(\cdot)$ is vaguely l.s.c. on $\mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A})$ [33, Theorem 2.30] ([9]). However, these arguments break down if any of the A_i, $i \in I$, is noncompact in X, for then $\mathfrak{M}^+(\mathbf{A},\mathbf{a},\mathbf{g})$ is no longer vaguely compact.

The purpose of the example below is to give an explicit formula for a solution to Problem 4.1 with particular meanings of X, κ, **A**, **a**, **g**, and **f**. Write $S_r := S(0,r)$.

([9]) If κ is (finitely) continuous on $X \times X$, this result by Ohtsuka can be extended to a compact generalized condenser. Such a generalization is established with the aid of Theorem 5.7 in a way similar to that in the proof of Theorem 6.1(ii) below.

EXAMPLE 4.4. Let $\kappa_2(x,y) = |x-y|^{2-n}$ be the Newtonian kernel on $\mathbb{R}^n$ with $n \geq 3$, $I^+ = \{1\}$, $I^- = \{2\}$, $\mathbf{g} = \mathbf{1}$, $\mathbf{a} = \mathbf{1}$, $\mathbf{f} = \mathbf{0}$, $A_1 = S_{r_1}$, and $A_2 = S_{r_2}$, where $0 < r_1 < r_2 < \infty$. Then a solution to Problem 4.1 exists (see Remark 4.3). Let λ_r, $0 < r < \infty$, denote the (unique) κ_2-capacitary measure on S_r (see Remark 2.6); then by symmetry λ_r is the uniformly distributed unit mass over S_r. Based on well known properties of the Newtonian potential of λ_r [27, Chapter II, Section 3, n° 13], we have

$$c_{\kappa_2}(S_r) = r^{n-2}, \tag{4.11}$$

$\kappa_2(\cdot, \lambda_r) = r^{2-n}$ on $\overline{B}(0,r)$ and $\kappa_2(\cdot, \lambda_r) = R^{2-n}$ on S_R, where $R > r$. Thus

$$\kappa_2(\cdot, \lambda_{r_1} - \lambda_{r_2}) = \begin{cases} r_1^{2-n} - r_2^{2-n} & \text{on } S_{r_1}, \\ 0 & \text{on } S_{r_2}. \end{cases} \tag{4.12}$$

Application of [40, Proposition 1(iv)], providing characteristic properties of solutions to Problem 4.1 for a standard condenser, shows that $\boldsymbol{\lambda} := (\lambda_{r_1}, \lambda_{r_2})$ solves Problem 4.1 with X, κ, $\mathbf{A}$, $\mathbf{a}$, $\mathbf{g}$, and $\mathbf{f}$ as indicated above. Hence the corresponding minimum value $G_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ equals $\kappa_2(\boldsymbol{\lambda}, \boldsymbol{\lambda})$ and ([10])

$$\kappa_2(\boldsymbol{\lambda}, \boldsymbol{\lambda}) = \|\lambda_{r_1} - \lambda_{r_2}\|^2_{\kappa_2} = r_1^{2-n} - r_2^{2-n}. \tag{4.13}$$

REMARK 4.5. Let $\mathbf{A}$ still be a standard condenser, though now, in contrast to Remark 4.3, its plates might be noncompact in X. Under the assumption

$$\sup_{x \in A^+,\, y \in A^-} \kappa(x,y) < \infty, \tag{4.14}$$

in [43, 44] an approach has been worked out based on both the vague and the strong topologies on $\mathcal{E}^+_\kappa(\mathbf{A})$ which made it possible to provide a fairly complete analysis of Problem 4.1. In more detail, it has been shown that if the kernel κ is consistent and if all the $g_i|_{A_i}$, $i \in I$, are bounded, then in either Case I or Case II the requirement

$$c_\kappa(A) < \infty \tag{4.15}$$

is sufficient for Problem 4.1 to be solvable for every vector $\mathbf{a}$ (see [43, Theorem 8.1]). However, if (4.15) does not hold then in general there exists a vector $\mathbf{a}'$ such that the problem admits no solution [43] ([11]). Therefore, it

([10]) λ_{r_1} gives in fact the solution to the problem (2.1) for S_{r_1} relative to the classical Green kernel G on $B(0, r_2)$, while $c_G(S_{r_1}) = [r_1^{2-n} - r_2^{2-n}]^{-1}$. This follows from (4.12) and (4.13) by [22, Lemmas 4.4 and 4.5].

([11]) In the case of the α-Riesz kernels of order $1 < \alpha \leq 2$ on $\mathbb{R}^3$, some of the (theoretical) results on the solvability or unsolvability of Problem 4.1 mentioned in [43] have been illustrated in [25, 30] by means of numerical experiments.

was interesting to give a description of the set of all vectors $\mathbf{a}$ for which Problem 4.1 is nevertheless solvable. Such a characterization has been established in [44]. See also footnote 13 below.

Unless explicitly stated otherwise, in all that follows, (4.14) is not assumed to hold. Then the results obtained in [43, 44] and the approach developed are no longer valid. In particular, assumption (4.15) does not guarantee anymore that $G_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is attained among $\boldsymbol{\mu}\in\mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. This can be illustrated by the following assertion.

Theorem 4.6. *Let $X=\mathbb{R}^n$ with $n\geq 3$, $I^+=\{1\}$, $I^-=\{2\}$, $\mathbf{g}=\mathbf{1}$, $\mathbf{a}=\mathbf{1}$, $\mathbf{f}=\mathbf{0}$,*

$$A_1=\bigcup_{k\geq 2}S(x_k,r_{1,k}),\qquad A_2=\bigcup_{k\geq 2}S(x_k,r_{2,k}),$$

where $x_k=(k,0,\dots,0)$, $r_{2,k}^{2-n}=k^2$ and $r_{1,k}^{2-n}=k^2+k^{-q}$ with $q\in(0,\infty)$, and let $\kappa=\kappa_2$ be the Newtonian kernel. Then $G_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ equals 0 and hence cannot be an actual minimum.

Proof. Note that $\mathbf{A}=(A_1,A_2)$ forms a standard condenser in $\mathbb{R}^n$ such that (4.14) fails to hold. Let $\lambda_{k,r}$, $0<r<\infty$, denote the κ_2-capacitary measure on $S(x_k,r)$ (see Remark 2.6). The measures $\boldsymbol{\lambda}_k=(\lambda_{k,r_{1,k}},\lambda_{k,r_{2,k}})$, $k\in\mathbb{N}$, are admissible for Problem 4.1, and therefore, by (3.7),

$$0\leq G_{\kappa_2,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})\leq\kappa_2(\boldsymbol{\lambda}_k,\boldsymbol{\lambda}_k)=\|\lambda_{k,r_{1,k}}-\lambda_{k,r_{2,k}}\|^2_{\kappa_2}.$$

According to (4.13), the right hand side equals

$$r_{1,k}^{2-n}-r_{2,k}^{2-n}=k^{-q},$$

and hence it tends to 0 as $k\to\infty$. This implies that $G_{\kappa_2,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})=0$. By the strict positive definiteness of κ_2, $G_{\kappa_2,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ cannot therefore be an actual minimum, though $c_2(A)<\infty$, which is clear from (4.11) by the countable subadditivity of inner capacity on universally measurable sets [17, Lemma 2.3.5]. ■

Using the electrostatic interpretation, which is possible for the Coulomb kernel $|x-y|^{-1}$ on $\mathbb{R}^3$, we say that *a short-circuit occurs* between the oppositely charged plates A_1 and A_2 from Theorem 4.6, which touch each other at the point at infinity. This certainly may also happen for a generalized condenser (see Definition 3.2). Therefore, it is meaningful to ask what kinds of additional requirements on the vector measures under consideration will prevent this phenomenon, and secure that a solution to the corresponding $\mathbf{f}$-weighted minimum energy problem does exist. The idea below is to find an upper constraint on the measures from $\mathfrak{M}^+(\mathbf{A},\mathbf{a},\mathbf{g})$ which would disallow the blow-up effect.

5. Constrained f-weighted minimum energy problems

5.1. Statement of the problem. Let κ, $\mathbf{A}$, $\mathbf{a}$, $\mathbf{g}$, and $\mathbf{f}$ be as indicated at the beginning of the preceding section, and let $\mathfrak{C}(\mathbf{A})$ consist of all vector measures $\boldsymbol{\sigma} = (\sigma^i)_{i\in I} \in \mathfrak{M}^+(\mathbf{A})$ with the properties $\langle g_i, \sigma^i\rangle > a_i$ and [12]

$$S(\sigma^i) = A_i \quad \text{for all } i \in I; \tag{5.1}$$

these $\boldsymbol{\sigma}$ will serve as *constraints* for $\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Given $\boldsymbol{\sigma} \in \mathfrak{C}(\mathbf{A})$, write

$$\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}) := \{\boldsymbol{\mu} \in \mathfrak{M}^+(\mathbf{A}) : \mu^i \le \sigma^i \text{ for all } i \in I\},$$

where $\mu^i \le \sigma^i$ means that $\sigma^i - \mu^i \in \mathfrak{M}^+(X)$, and also

$$\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}) \cap \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}),$$
$$\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A}) \cap \mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}).$$

Note that $\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A})$ along with $\mathfrak{M}^+(\mathbf{A})$ is vaguely closed, for so is $\mathfrak{M}^+(X)$.

Since $\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, from (4.8) we get

$$-\infty < G_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \le G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \inf_{\boldsymbol{\mu}\in\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})} G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) \le \infty. \tag{5.2}$$

Unless explicitly stated otherwise, in all that follows we assume that the class $\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty, or equivalently that

$$G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty. \tag{5.3}$$

Then the following *constrained* $\mathbf{f}$*-weighted minimum energy problem*, also known as the *constrained Gauss variational problem*, makes sense.

Problem 5.1. *Given $\boldsymbol{\sigma} \in \mathfrak{C}(\mathbf{A})$, does there exist $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with*

$$G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}) = G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})?$$

Lemma 5.2. *For requirement* (5.3) *to hold, it is sufficient that, for every $i \in I$, $\langle g_i, \sigma^i|_{\dot{A}^\delta_i}\rangle > a_i$, where $\dot{A}^\delta_i$ is defined by* (4.10)*, and also that*

$$\kappa(\sigma^i|_K, \sigma^i|_K) < \infty \quad \textit{for any compact } K \subset \dot{A}^\delta_i.$$

Proof. For every $i \in I$ there exists a compact set $K_i \subset \dot{A}^\delta_i$ such that $\langle g_i, \sigma^i|_{K_i}\rangle > a_i$ and $|f_i| \le M_i < \infty$ on K_i for some constant M_i. Then $\boldsymbol{\mu} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ where $\mu^i := a_i\sigma^i|_{K_i}/\langle g_i, \sigma^i|_{K_i}\rangle$, $i \in I$. ■

Remark 5.3. Assume for the moment that (4.14) holds. It has been shown in [42, Theorem 6.2] that if, in addition, the kernel κ is consistent, all the $g_i|_{A_i}$, $i \in I$, are bounded, and if condition (4.15) is satisfied, then in

(12) Assumption (5.1) causes in fact no restriction on the objects in question since, if it does not hold, then Problem 5.1 reduces to the same problem for the generalized condenser $(S^{\sigma^i}_X)_{i\in I}$ in place of $(A_i)_{i\in I}$.

both Cases I and II Problem 5.1 is solvable for any $\boldsymbol{\sigma} \in \mathfrak{C}(\mathbf{A})$ ([13]). But this is no longer valid if requirement (4.14) is dropped.

5.2. On the uniqueness of a solution to Problem 5.1. Let $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ (possibly empty) consist of all solutions to Problem 5.1.

LEMMA 5.4. *For any* $\boldsymbol{\lambda}, \widehat{\boldsymbol{\lambda}} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *we have* $\|\boldsymbol{\lambda} - \widehat{\boldsymbol{\lambda}}\|_{\mathcal{E}^+_\kappa(\mathbf{A})} = 0$.

Proof. Since $\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is convex, (4.2) and (3.7) imply

$$4G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) \le 4G_{\kappa,\mathbf{f}}\Big(\frac{\boldsymbol{\lambda}+\widehat{\boldsymbol{\lambda}}}{2}\Big) = \|R\boldsymbol{\lambda} + R\widehat{\boldsymbol{\lambda}}\|^2_\kappa + 4\langle \mathbf{f}, \boldsymbol{\lambda}+\widehat{\boldsymbol{\lambda}}\rangle.$$

On the other hand, applying the parallelogram identity in $\mathcal{E}_\kappa(X)$ to $R\boldsymbol{\lambda}$ and $R\widehat{\boldsymbol{\lambda}}$ and then adding and subtracting $4\langle \mathbf{f}, \boldsymbol{\lambda}+\widehat{\boldsymbol{\lambda}}\rangle$ we get

$$\|R\boldsymbol{\lambda} - R\widehat{\boldsymbol{\lambda}}\|^2_\kappa = -\|R\boldsymbol{\lambda} + R\widehat{\boldsymbol{\lambda}}\|^2_\kappa - 4\langle \mathbf{f}, \boldsymbol{\lambda}+\widehat{\boldsymbol{\lambda}}\rangle + 2G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}) + 2G_{\kappa,\mathbf{f}}(\widehat{\boldsymbol{\lambda}}).$$

When combined with the preceding relation, this yields

$$0 \le \|R\boldsymbol{\lambda} - R\widehat{\boldsymbol{\lambda}}\|^2_\kappa \le -4G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) + 2G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}) + 2G_{\kappa,\mathbf{f}}(\widehat{\boldsymbol{\lambda}}) = 0,$$

which establishes the lemma because of (3.8). ∎

COROLLARY 5.5. *Let κ be strictly positive definite. Then all the measures in* $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ (*provided that it is nonempty*) *are R-equivalent to one another. Hence* $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *reduces to a single element if, in addition, the* A_i^δ, $i \in I$, *are mutually disjoint,* A_i^δ *being defined by* (3.2).

5.3. Auxiliary results. Based on the definition of the vague topology on $\mathfrak{M}^+(\mathbf{A})$, we call a set $\mathfrak{F} \subset \mathfrak{M}^+(\mathbf{A})$ *vaguely bounded* if for every $i \in I$,

$$\sup_{\boldsymbol{\mu}\in\mathfrak{F}} |\mu^i(\varphi)| < \infty \quad \text{for all } \varphi \in C_0(X).$$

LEMMA 5.6. *If* $\mathfrak{F} \subset \mathfrak{M}^+(\mathbf{A})$ *is vaguely bounded, then it is vaguely relatively compact.*

Proof. It is clear from the above definition that for every $i \in I$ the set

$$\mathfrak{F}^i := \{\mu^i \in \mathfrak{M}^+(A_i) : \boldsymbol{\mu} = (\mu^j)_{j\in I} \in \mathfrak{F}\}$$

is vaguely bounded in $\mathfrak{M}^+(X)$; hence, by [4, Chapter III, Section 2, Proposition 9], it is vaguely relatively compact in $\mathfrak{M}^+(X)$. Since $\mathfrak{F} \subset \prod_{i\in I} \mathfrak{F}^i$, the lemma follows from Tychonoff's theorem on the product of compact spaces [2, Chapter I, Section 9, Theorem 3]. ∎

([13]) Actually, the results described in Remarks 4.5 and 5.3 have been obtained in [42]–[44] even for *infinite* dimensional vector measures.

THEOREM 5.7. *$\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is vaguely compact for any $\boldsymbol{\sigma} \in \mathfrak{C}(\mathbf{A})$ with* ([14])

$$\langle g_i, \sigma^i\rangle < \infty \quad \text{for all } i \in I. \tag{5.4}$$

Proof. It is seen from (4.6) that $\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is vaguely bounded. Hence, by Lemma 5.6, for any net $(\boldsymbol{\mu}_s)_{s\in S} \subset \mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$ there exists a vague cluster point $\boldsymbol{\mu}$, which is in $\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A})$, for $\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A})$ is vaguely closed. Passing to a subnet and changing notation assume that $\boldsymbol{\mu}$ is the vague limit of $(\boldsymbol{\mu}_s)_{s\in S}$. As g_i is positive and continuous, from Lemma 2.1 with $\psi = g_i$ we get

$$\langle g_i, \mu^i\rangle \le \liminf_{s\in S}\, \langle g_i, \mu^i_s\rangle \le a_i \quad \text{for all } i \in I.$$

Thus we only need to show that, under requirement (5.4), $\langle g_i, \mu^i\rangle = a_i$ for every $i \in I$. Consider an exhaustion of A_i by an upper directed family of compact sets $K \subset A_i$. Since the indicator function 1_K of K is upper semicontinuous, Lemma 2.1 with $\psi = -g_i 1_K$ and [17, Lemma 1.2.2] yield

$$\begin{aligned} a_i \ge \langle g_i, \mu^i\rangle &= \lim_{K\uparrow A_i} \langle g_i 1_K, \mu^i\rangle \ge \lim_{K\uparrow A_i}\, \limsup_{s\in S}\, \langle g_i 1_K, \mu^i_s\rangle \\ &= a_i - \lim_{K\uparrow A_i}\, \liminf_{s\in S}\, \langle g_i 1_{A_i\setminus K}, \mu^i_s\rangle. \end{aligned}$$

Hence, the lemma will follow once we show that

$$\lim_{K\uparrow A_i}\, \liminf_{s\in S}\, \langle g_i 1_{A_i\setminus K}, \mu^i_s\rangle = 0. \tag{5.5}$$

Since by (5.4),

$$\infty > \langle g_i, \sigma^i\rangle = \lim_{K\uparrow A_i} \langle g_i 1_K, \sigma^i\rangle,$$

we have

$$\lim_{K\uparrow A_i} \langle g_i 1_{A_i\setminus K}, \sigma^i\rangle = 0.$$

When combined with

$$\langle g_i 1_{A_i\setminus K}, \mu^i_s\rangle \le \langle g_i 1_{A_i\setminus K}, \sigma^i\rangle \quad \text{for every } s \in S,$$

this implies (5.5) as desired. ■

LEMMA 5.8. *The mapping $\boldsymbol{\mu} \mapsto G_{\kappa,\mathbf{f}}(\boldsymbol{\mu})$ is vaguely l.s.c. on $\mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A})$ provided that Case* I *holds, and it is strongly continuous otherwise.*

Proof. The former statement follows from Lemma 2.1, while the latter is obtained directly from (4.3). ■

([14]) For a compact $\mathbf{A}$ relation (5.4) holds automatically.

Definition 5.9. A net $(\boldsymbol{\mu}_s)_{s\in S} \subset \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is said to be *minimizing* in Problem 5.1 if

$$\lim_{s\in S} G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}_s) = G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}). \tag{5.6}$$

Let $\mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ consist of all those $(\boldsymbol{\mu}_s)_{s\in S}$; it is nonempty because of (5.3).

Lemma 5.10. *For any $(\boldsymbol{\mu}_s)_{s\in S}$ and $(\boldsymbol{\nu}_t)_{t\in T}$ in $\mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ we have*

$$\lim_{(s,t)\in S\times T} \|\boldsymbol{\mu}_s - \boldsymbol{\nu}_t\|_{\mathcal{E}^+_\kappa(\mathbf{A})} = 0, \tag{5.7}$$

where $S\times T$ is the upper directed product (15) *of the upper directed sets S and T.*

Proof. In the same manner as in the proof of Lemma 5.4 we get

$$0 \le \|R\boldsymbol{\mu}_s - R\boldsymbol{\nu}_t\|^2_\kappa \le -4G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) + 2G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}_s) + 2G_{\kappa,\mathbf{f}}(\boldsymbol{\nu}_t),$$

which gives (5.7) when combined with (3.8), (5.2), (5.3) and (5.6). ∎

Corollary 5.11. *Every $(\boldsymbol{\mu}_s)_{s\in S} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is strong Cauchy in $\mathcal{E}^+_\kappa(\mathbf{A})$.*

6. On the solvability of Problem 5.1. Theorem 6.1 below establishes sufficient conditions for the existence of solutions to Problem 5.1 for a (positive definite) kernel κ on a locally compact space X and a (generalized) condenser $\mathbf{A}$. It is inspired partly, in cases (i) and (ii), by [42] and [1], respectively.

Theorem 6.1. *Given κ, $\mathbf{A}$, $\mathbf{g}$, and $\boldsymbol{\sigma}$, let any of the following cases hold:*

- (i) *κ is consistent and both (4.14) and (5.4) are fulfilled;*
- (ii) *κ is regular, the (generalized) condenser $\mathbf{A}$ is compact, and each of the $\kappa(\cdot,\sigma^i)$, $i\in I$, is (finitely) continuous on the (compact) set A_i.*

Then in either Case I *or Case* II, *Problem* 5.1 *is solvable for any vector $\mathbf{a}$, and the class $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ of all its solutions is vaguely compact. Furthermore, every minimizing net $(\mu_s)_{s\in S} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ converges to any $\boldsymbol{\lambda} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ strongly in $\mathcal{E}^+_\kappa(\mathbf{A})$; and hence also vaguely provided that κ is strictly positive definite and the A^δ_i, $i\in I$, are mutually disjoint* (16).

The proof of Theorem 6.1 is given in Section 6.2 below. It is based on Theorem 6.6 on the completeness of the semimetric space $\mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A},\mathbf{a},\mathbf{g}) := \mathcal{E}^+_\kappa(\mathbf{A}) \cap \mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$, which is the subject of Section 6.1.

(15) See e.g. [26, Chapter 2, Section 3].

(16) Then a solution $\boldsymbol{\lambda}$ to Problem 5.1 is unique by Corollary 5.5.

Example 6.2. Let $\mathbf{A} = (A_i)_{i\in I}$ be as in Example 3.3, and let $\kappa_\alpha(x, y) = |x - y|^{\alpha-n}$ be the α-Riesz kernel of order $\alpha \in (0, 2)$ on $\mathbb{R}^n$, $n \geq 3$, which is regular by [27, Theorem 1.7]. Also assume that $\mathbf{g} = \mathbf{1}$ and that either Case II holds or $f_i(x) < \infty$ κ_α-n.e. on A_i, $i = 1, 2$. Let λ_i denote the (unique) κ_α-capacitary measure on A_i (see Remark 2.6); then $\kappa_\alpha(\cdot, \lambda_i)$ is continuous on $\mathbb{R}^n$ and $S^{\lambda_i}_{\mathbb{R}^n} = A_i$ [27, Chapter II, Section 3, n° 13]. For any $\mathbf{a} = (a_i)_{i\in I}$ define $\sigma^i := c_i\lambda_i$, $i \in I$, where $a_i < c_i < \infty$. As $\boldsymbol{\sigma} = (\sigma^i)_{i\in I}$ clearly has finite α-Riesz energy, relation (5.3) holds by Lemma 5.2, and since $\mathbf{A}$ is compact, Problem 5.1 admits a solution according to Theorem 6.1(ii). Thus, no short-circuit occurs between the oppositely charged plates of the condenser $\mathbf{A}$, though they intersect each other over the set $\delta_{\mathbf{A}} = \{\xi_5, \xi_6\}$ with $c_\alpha(\delta_{\mathbf{A}}) = 0$ (see Example 3.3).

Example 6.3. Let $X = \mathbb{R}^3$, $I^+ = \{1\}$, $I^- = \{2\}$, and let $\kappa_2(x, y) = |x - y|^{-1}$ be the Coulomb kernel. Define

$$A_1 := \{x \in \mathbb{R}^3 : 1 \leq x_1 < \infty,\ x_2^2 + x_3^2 = \exp(-2x_1)\},$$
$$A_2 := \{x \in \mathbb{R}^3 : -\infty < x_1 \leq -1,\ x_2^2 + x_3^2 = \exp(2x_1)\}.$$

Also assume that $\mathbf{g} = \mathbf{1}$ and that either Case II holds or $f_i(x) < \infty$ κ_2-n.e. on A_i, $i = 1, 2$. For any $\mathbf{a} = (a_i)_{i\in I}$ define $\sigma^i := c_i m_2|_{A_i}$, $i \in I$, where $a_i < c_i < \infty$ and m_2 is the 2-dimensional Lebesgue measure. Then (5.3) holds by Lemma 5.2, and Problem 5.1 has a solution according to Theorem 6.1(i). Note that under the stated assumptions the solvability of Problem 5.1 cannot be obtained from [42, Theorem 6.2] (see also Remark 5.3 above), because $c_2(A_i) = \infty$ for $i = 1, 2$ by [43, Example 8.2].

6.1. On the strong completeness of $\mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A}, \mathbf{a}, \mathbf{g})$. The first of the following two lemmas is partly inspired by [34, 14].

Lemma 6.4. *Assume that the hypotheses of case* (ii) *of Theorem* 6.1 *hold. Then for every $\boldsymbol{\mu} \in \mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A})$ all the $\kappa(\cdot, \mu^i)$, $i \in I$, are continuous on X.*

Proof. As $\kappa(\cdot, \sigma^i)$ is continuous on the (compact) set A_i, it follows from the regularity of the kernel κ that $\kappa(\cdot, \sigma^i)$ is continuous on all of X. Since $\kappa(\cdot, \mu^i)$ is l.s.c. and since $\kappa(\cdot, \mu^i) = \kappa(\cdot, \sigma^i) - \kappa(\cdot, \sigma^i - \mu^i)$ with $\kappa(\cdot, \sigma^i)$ continuous and $\kappa(\cdot, \sigma^i - \mu^i)$ l.s.c., it follows that $\kappa(\cdot, \mu^i)$ is also upper semicontinuous, hence continuous. ∎

Lemma 6.5. *Suppose that* (4.14) *holds. If a net $(\boldsymbol{\mu}_s)_{s\in S} \subset \mathcal{E}^+_\kappa(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is strongly bounded, then each of its vague cluster points $\boldsymbol{\mu}$ has finite energy.*

Proof. It is clear from (3.7) that the net $(R\boldsymbol{\mu}_s)_{s\in S} \subset \mathcal{E}_\kappa(X)$ of scalar measures is strongly bounded. We proceed by showing that

$$\sup_{s\in S} \|R\boldsymbol{\mu}^\pm_s\|_\kappa < \infty. \tag{6.1}$$

Indeed, by (4.6) we have $R\boldsymbol{\mu}_s^{\pm}(X) \le M < \infty$ for all $s \in S$. Together with (4.14), this implies that $\kappa(R\boldsymbol{\mu}_s^+, R\boldsymbol{\mu}_s^-)$ remains bounded as s ranges through S, and (6.1) follows.

If now $(\boldsymbol{\mu}_t)_{t\in T}$ is a subnet of $(\boldsymbol{\mu}_s)_{s\in S}$ that converges vaguely to $\boldsymbol{\mu}$, then $(R\boldsymbol{\mu}_t^+)_{t\in T}$ and $(R\boldsymbol{\mu}_t^-)_{t\in T}$ converge vaguely to $R\boldsymbol{\mu}^+$ and $R\boldsymbol{\mu}^-$, respectively. Using the fact that the map $(\nu_1,\nu_2)\mapsto \nu_1\otimes\nu_2$ from $\mathfrak{M}^+(X)\times\mathfrak{M}^+(X)$ into $\mathfrak{M}^+(X\times X)$ is vaguely continuous [4, Chapter 3, Section 5, Exercise 5] and applying Lemma 2.1 to $X\times X$ and $\psi=\kappa$, we conclude from (6.1) that $R\boldsymbol{\mu}^+$ and $R\boldsymbol{\mu}^-$ are both of finite energy. Hence $\boldsymbol{\mu}\in\mathcal{E}_\kappa^+(\mathbf{A})$. ∎

THEOREM 6.6. *Assume that the hypotheses of either of cases* (i) *or* (ii) *of Theorem* 6.1 *hold. Then for any given* $\mathbf{a}$ *the semimetric space* $\mathcal{E}_\kappa^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *is strongly complete. In more detail, any strong Cauchy net* $(\boldsymbol{\mu}_s)_{s\in S}\subset\mathcal{E}_\kappa^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *converges strongly to any of its vague cluster points* (*such points exist*)*. If, moreover, the kernel* κ *is strictly positive definite and the* A_i^δ, $i\in I$, *are mutually disjoint, then the strong topology on the space* $\mathcal{E}_\kappa^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *is finer than the induced vague topology.*

Proof. Fix a strong Cauchy net $(\boldsymbol{\mu}_s)_{s\in S}\subset\mathcal{E}_\kappa^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$. By Theorem 5.7, any of its vague cluster points $\boldsymbol{\mu}$ (such points exist) belongs to $\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$. Choose a subnet $(\boldsymbol{\mu}_t)_{t\in T}$ of $(\boldsymbol{\mu}_s)_{s\in S}$ such that

$$\boldsymbol{\mu}_t\to\boldsymbol{\mu}\quad\text{vaguely (as } t \text{ increases along } T). \tag{6.2}$$

Application of (3.7) and (3.8) then shows that the net of scalar measures $R\boldsymbol{\mu}_t$, $t\in T$, is strong Cauchy in the pre-Hilbert space $\mathcal{E}_\kappa(X)$, and therefore it is strongly bounded.

Assume first that the hypotheses of case (i) of Theorem 6.1 hold. By Lemma 6.5, then $\boldsymbol{\mu}\in\mathcal{E}_\kappa^+(\mathbf{A})$ and hence $\boldsymbol{\mu}\in\mathcal{E}_\kappa^{\boldsymbol{\sigma}}(\mathbf{A},\mathbf{a},\mathbf{g})$. We proceed by proving that

$$\lim_{t\in T}\|\boldsymbol{\mu}_t-\boldsymbol{\mu}\|_{\mathcal{E}_\kappa^+(\mathbf{A})}=0. \tag{6.3}$$

The nets $(R\boldsymbol{\mu}_t^+)_{t\in T}$ and $(R\boldsymbol{\mu}_t^-)_{t\in T}$ converge vaguely to $R\boldsymbol{\mu}^+$ and $R\boldsymbol{\mu}^-$, respectively. Since by (6.1) these nets are strongly bounded in $\mathcal{E}_\kappa^+(X)$, property (C_2) (see Section 2) shows that they approach $R\boldsymbol{\mu}^+$ and $R\boldsymbol{\mu}^-$, respectively, also in the weak topology, and so $R\boldsymbol{\mu}_t\to R\boldsymbol{\mu}$ weakly. Thus

$$\|\boldsymbol{\mu}_t-\boldsymbol{\mu}\|^2_{\mathcal{E}_\kappa^+(\mathbf{A})}=\|R\boldsymbol{\mu}_t-R\boldsymbol{\mu}\|^2_\kappa=\lim_{t'\in T}\kappa(R\boldsymbol{\mu}_t-R\boldsymbol{\mu},R\boldsymbol{\mu}_t-R\boldsymbol{\mu}_{t'}),$$

and hence, by the Cauchy–Schwarz inequality,

$$\|\boldsymbol{\mu}_t-\boldsymbol{\mu}\|^2_{\mathcal{E}_\kappa^+(\mathbf{A})}\le\|\boldsymbol{\mu}_t-\boldsymbol{\mu}\|_{\mathcal{E}_\kappa^+(\mathbf{A})}\liminf_{t'\in T}\|\boldsymbol{\mu}_t-\boldsymbol{\mu}_{t'}\|_{\mathcal{E}_\kappa^+(\mathbf{A})},$$

which leads to (6.3) by letting t increase along T, because $\|\boldsymbol{\mu}_t-\boldsymbol{\mu}_{t'}\|_{\mathcal{E}_\kappa^+(\mathbf{A})}$ becomes arbitrarily small when $t,t'\in T$ are sufficiently large.

Since a strong Cauchy net converges strongly to any of its strong cluster points (even in the present case of a semimetric space), it follows from the above that $\boldsymbol{\mu}_s \to \boldsymbol{\mu}$ strongly in $\mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Thus, $(\boldsymbol{\mu}_s)_{s\in S}$ converges strongly to any of its vague cluster points. Assume now that κ is strictly positive definite and the A^δ_i, $i \in I$, are mutually disjoint. Then the vague cluster set of $(\boldsymbol{\mu}_s)_{s\in S}$ has to reduce to a single measure $\boldsymbol{\mu}$. As $\mathfrak{M}^+(\mathbf{A})$ is Hausdorff in the vague topology, $\boldsymbol{\mu}_s \to \boldsymbol{\mu}$ also vaguely [2, Chapter I, Section 9, n° 1, Corollary]. This proves the theorem in case (i).

Assume next that the hypotheses of case (ii) of Theorem 6.1 hold. Since by Lemma 6.4 each of $\kappa(\cdot, \mu^i_t)$ and $\kappa(\cdot, \mu^i)$, $t \in T$, $i \in I$, is continuous and hence bounded on the (compact) set A_j, $j \in I$, relation (6.2) yields

$$\lim_{t\in T}\lim_{t'\in T} \kappa(\mu^i_t, \mu^j_{t'}) = \lim_{t\in T}\lim_{t'\in T}\int \kappa(\cdot, \mu^i_t)\, d\mu^j_{t'} = \lim_{t\in T}\int \kappa(\cdot, \mu^i_t)\, d\mu^j$$
$$= \lim_{t\in T}\int \kappa(\cdot, \mu^j)\, d\mu^i_t = \kappa(\mu^j, \mu^i) < \infty \quad \text{for all } i, j \in I.$$

Hence, $\boldsymbol{\mu} \in \mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and moreover $\boldsymbol{\mu}_t \to \boldsymbol{\mu}$ strongly. To complete the proof of the theorem, it remains to apply the arguments of the preceding paragraph. ∎

Remark 6.7. As the semimetric space $\mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is isometric to its R-image, Theorem 6.6 has singled out strongly complete topological subspaces of the pre-Hilbert space $\mathcal{E}_\kappa(X)$, whose elements are (*signed*) measures. See also Theorem 7.3 below, pertaining to the case of the α-Riesz kernel with $\alpha \in (0, n)$ on $\mathbb{R}^n$, $n \geq 3$. This is of independent interest since, by Cartan, the whole pre-Hilbert space $\mathcal{E}_\kappa(X)$ is in general strongly incomplete.

6.2. Proof of Theorem 6.1. Fix any net $(\boldsymbol{\mu}_s)_{s\in S} \in \mathbb{M}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$; it exists by assumption (5.3), and it is strong Cauchy in the semimetric space $\mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A}, \mathbf{a}, \mathbf{g})$ by Corollary 5.11. By Theorem 5.7 there is a subnet $(\boldsymbol{\mu}_t)_{t\in T}$ of $(\boldsymbol{\mu}_s)_{s\in S}$ converging vaguely to some $\boldsymbol{\mu}$ in $\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$, while according to Theorem 6.6 we actually have $\boldsymbol{\mu} \in \mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and

$$\lim_{s\in S} \|\boldsymbol{\mu}_s - \boldsymbol{\mu}\|_{\mathcal{E}^+_\kappa(\mathbf{A})} = 0. \tag{6.4}$$

Also note that, by (5.3), (5.6) and Lemma 5.8,

$$-\infty < G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) \leq \lim_{t\to T} G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}_t) = G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty,$$

the first inequality following from (4.4) and (4.7). Thus $\boldsymbol{\mu} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and therefore $G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) \geq G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. All this combined shows that $\boldsymbol{\mu} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$.

To verify that $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is vaguely compact, fix a net $(\boldsymbol{\lambda}_s)_{s\in S}$ of its elements. By Lemma 5.4, it is strong Cauchy in $\mathcal{E}^{\boldsymbol{\sigma}}_\kappa(\mathbf{A}, \mathbf{a}, \mathbf{g})$, and the same arguments as above show that the (nonempty) vague cluster set of $(\boldsymbol{\lambda}_s)_{s\in S}$ is contained in $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$.

If $\boldsymbol{\lambda}$ is any element of $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$, then by Lemma 5.4, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are equivalent in $\mathcal{E}^+_\kappa(\mathbf{A})$, which in view of (6.4) implies that $\boldsymbol{\mu}_s \to \boldsymbol{\lambda}$ strongly in $\mathcal{E}^+_\kappa(\mathbf{A})$. Assuming now that the kernel κ is strictly positive definite and the A_i^δ, $i \in I$, are mutually disjoint, we see from the last assertion of Theorem 6.6 that $\boldsymbol{\mu}_s \to \boldsymbol{\lambda}$ also vaguely. □

7. On the solvability of Problem 5.1 for Riesz kernels. Throughout this section consider the α-Riesz kernel $\kappa_\alpha(x,y) = |x-y|^{\alpha-n}$ of an arbitrary order $\alpha \in (0,n)$ on $\mathbb{R}^n$, $n \geq 3$. We shall write simply α instead of κ_α if κ_α serves as an index. The α-Riesz kernel is strictly positive definite and moreover perfect (see [10, 11]); hence, the metric space $\mathcal{E}^+_\alpha(\mathbb{R}^n)$ is complete in the induced strong topology. However, by Cartan [6] (see also [27, Theorem 1.19]), the whole pre-Hilbert space $\mathcal{E}_\alpha(\mathbb{R}^n)$ for $\alpha \in (1,n)$ is strongly incomplete; compare with Theorems 3.9, 6.6 and Remark 6.7 above, as well as with Theorem 7.3 below.

Let $\overline{Q}$ be the closure of $Q \subset \mathbb{R}^n$ in $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\omega_{\mathbb{R}^n}\}$, the one-point compactification of $\mathbb{R}^n$. The following theorem provides sufficient conditions for the solvability of Problem 5.1 for a noncompact (generalized) condenser $\mathbf{A}$ (compare with Theorem 6.1) ([17]).

Theorem 7.1. *Suppose that $\overline{A^+} \cap \overline{A^-}$ consists of at most one point, i.e.*

$$\text{either } \overline{A^+} \cap \overline{A^-} = \emptyset, \quad \text{or } \overline{A^+} \cap \overline{A^-} = \{x_0\} \quad \text{where } x_0 \in \overline{\mathbb{R}^n}, \tag{7.1}$$

and let $\mathbf{g}$ *and* $\boldsymbol{\sigma} \in \mathfrak{C}(\mathbf{A})$ *satisfy* (5.4). *Then in either Case* I *or Case* II *Problem* 5.1 *is solvable for any given* $\mathbf{a}$*, and the class of all its solutions is vaguely compact. Furthermore, every minimizing sequence converges to every $\boldsymbol{\lambda} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\alpha,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ strongly in $\mathcal{E}^+_\alpha(\mathbf{A})$; and hence also vaguely provided that the $A_i \setminus \{x_0\}$, $i \in I$, are mutually disjoint.*

Theorem 7.1 is sharp in the sense that it is no longer valid if assumption (5.4) is omitted from its hypotheses (see Theorem 7.4 below).

We omit the proof of Theorem 7.1 since it resembles that of Theorem 6.1, the only difference being in referring to Theorem 7.3 on strong completeness (see below) instead of Theorem 6.6.

Example 7.2. Let $\mathbf{A} = (A_i)_{i\in I}$ be as in Example 3.4 and let $\alpha = 2$; then $c_2(A_i) < \infty$, $i = 1,2$ [43, Example 8.2], and hence there exists a (unique) κ_2-capacitary measure λ_i on A_i. Let $\mathbf{g} = \mathbf{1}$, and suppose that either Case II holds or $f_i(x) < \infty$ c_2-n.e. on A_i, $i = 1,2$. For any $\mathbf{a} = (a_1,a_2)$ define

([17]) In a particular case of a condenser with two oppositely charged plates some of the results obtained in this section have been presented earlier in [45].

$\sigma^i := c_i\lambda_i$, $i = 1, 2$, where $a_i < c_i < \infty$ ([18]). Then $\sigma^i(A_i) = c_i < \infty$, hence (5.4) is fulfilled. As $\boldsymbol{\sigma} = (\sigma^1, \sigma^2)$ has finite Newtonian energy, relation (5.3) holds by Lemma 5.2, and Problem 5.1 has a solution according to Theorem 7.1 which is unique by Lemma 5.4. Thus, no short-circuit occurs between A_1 and A_2, though these oppositely charged conductors touch each other at $\omega_{\mathbb{R}^3}$. Note that although $c_2(A) < \infty$, Theorem 6.2 from [42] on the solvability of Problem 5.1 for a standard condenser cannot be applied, due to the unboundedness of the Coulomb kernel κ_2 on $A_1 \times A_2$.

7.1. A strong completeness theorem for a semimetric subspace of $\mathcal{E}^+_\alpha(\mathbf{A})$. Theorem 7.3 below provides a strong completeness result for the semimetric space $\mathcal{E}^{\boldsymbol{\sigma}}_\alpha(\mathbf{A}, \mathbf{a}, \mathbf{g})$ in the case where the (generalized) condenser $\mathbf{A}$ is not necessarily compact (compare with Theorem 6.6(ii)). In turn, its proof substantially uses Theorem 3.9 on the strong completeness of $\mathcal{E}^+_\alpha(\mathbf{A})$ for a standard condenser.

THEOREM 7.3. *Suppose that for $\mathbf{A}$, $\mathbf{g}$ and $\boldsymbol{\sigma} \in \mathfrak{C}(\mathbf{A})$ assumptions* (5.4) *and* (7.1) *both hold. Then the semimetric space $\mathcal{E}^{\boldsymbol{\sigma}}_\alpha(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is strongly complete. In more detail, any strong Cauchy sequence $\{\boldsymbol{\mu}_k\}_{k\in\mathbb{N}} \subset \mathcal{E}^{\boldsymbol{\sigma}}_\alpha(\mathbf{A}, \mathbf{a}, \mathbf{g})$ converges strongly to any of its vague cluster points $\boldsymbol{\mu}$ (which exist). If, moreover, the $A_i \setminus \{x_0\}$, $i \in I$, are mutually disjoint, then the strong topology on $\mathcal{E}^{\boldsymbol{\sigma}}_\alpha(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is finer than the induced vague topology.*

Proof. Fix a strong Cauchy sequence $\{\boldsymbol{\mu}_k\}_{k\in\mathbb{N}} \subset \mathcal{E}^{\boldsymbol{\sigma}}_\alpha(\mathbf{A}, \mathbf{a}, \mathbf{g})$. According to Theorem 5.7, any of its vague cluster points $\boldsymbol{\mu}$ (which exist) belongs to $\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. As $\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is sequentially vaguely closed by Remark 2.4, one can choose a (strong Cauchy) subsequence $\{\boldsymbol{\mu}_{k_m}\}_{m\in\mathbb{N}}$ of $\{\boldsymbol{\mu}_k\}_{k\in\mathbb{N}}$ converging vaguely to the measure $\boldsymbol{\mu}$, i.e.

$$\mu^i_{k_m} \to \mu^i \quad \text{vaguely in } \mathfrak{M}(X),\ i \in I. \tag{7.2}$$

We proceed by showing that $\kappa_\alpha(\boldsymbol{\mu}, \boldsymbol{\mu})$ is finite, hence

$$\boldsymbol{\mu} \in \mathcal{E}^{\boldsymbol{\sigma}}_\alpha(\mathbf{A}, \mathbf{a}, \mathbf{g}), \tag{7.3}$$

and moreover that $\boldsymbol{\mu}_{k_m} \to \boldsymbol{\mu}$ strongly as $m \to \infty$, i.e.

$$\lim_{m\to\infty} \|\boldsymbol{\mu}_{k_m} - \boldsymbol{\mu}\|_{\mathcal{E}^+_\alpha(\mathbf{A})} = 0. \tag{7.4}$$

Assume first that $\overline{A^+} \cap \overline{A^-}$ either is empty or coincides with $\{\omega_{\mathbb{R}^n}\}$. Then $\mathbf{A}$ forms a standard condenser in $\mathbb{R}^n$, and hence, by (7.2), $R\boldsymbol{\mu}_{k_m} \to R\boldsymbol{\mu}$ (as $m \to \infty$) in the vague topology of $\mathfrak{M}(\mathbb{R}^n)$. Noting that $\{R\boldsymbol{\mu}_{k_m}\}_{m\in\mathbb{N}}$ is a strong Cauchy sequence in $\mathcal{E}_\alpha(\mathbb{R}^n)$, we conclude from [39, Theorem 1

([18]) Under the assumptions of Example 7.2 requirement (5.1) holds since for the given A_i, $i = 1, 2$, and $\kappa = \kappa_2$ we have $S^{\lambda_i}_{\mathbb{R}^3} = A_i$, which can be seen from the construction of the κ_2-capacitary measure described in [27, Theorem 5.1].

and Corollary 1] (see also Theorem 3.9 above) that there exists a unique $\eta \in \mathcal{E}_\alpha(\mathbb{R}^n)$ such that

$$R\boldsymbol{\mu}_{k_m} \to \eta \quad \text{strongly and vaguely as } m \to \infty.$$

As the vague topology is Hausdorff, we thus have $\eta = R\boldsymbol{\mu}$, which in view of (3.7), (3.8) and the last display results in (7.3) and (7.4).

We next proceed by analyzing the case

$$\overline{A^+} \cap \overline{A^-} = \{x_0\} \quad \text{where } x_0 \in \mathbb{R}^n. \tag{7.5}$$

Consider the inversion I_{x_0} with respect to $S(x_0, 1)$; namely, each point $x \neq x_0$ is mapped to the point x^* on the ray through x which issues from x_0, uniquely determined by

$$|x - x_0| \cdot |x^* - x_0| = 1.$$

This is a self-homeomorphism of $\mathbb{R}^n \setminus \{x_0\}$; furthermore,

$$|x^* - y^*| = \frac{|x - y|}{|x - x_0| \cdot |y - x_0|}. \tag{7.6}$$

Extend it to a self-homeomorphism of $\overline{\mathbb{R}^n}$ by setting $I_{x_0}(x_0) = \omega_{\mathbb{R}^n}$ and $I_{x_0}(\omega_{\mathbb{R}^n}) = x_0$.

To each (signed scalar) measure $\nu \in \mathfrak{M}(\mathbb{R}^n)$ with $\nu(\{x_0\}) = 0$, in particular to every $\nu \in \mathcal{E}_\alpha(\mathbb{R}^n)$, there corresponds the *Kelvin transform* $\nu^* \in \mathfrak{M}(\mathbb{R}^n)$ by means of the formula

$$d\nu^*(x^*) = |x - x_0|^{\alpha - n}\, d\nu(x), \qquad x^* \in \mathbb{R}^n$$

(see [35] or [27, Chapter IV, Section 5, n° 19]). In consequence of (7.6),

$$\kappa_\alpha(x^*, \nu^*) = |x - x_0|^{n-\alpha} \kappa_\alpha(x, \nu), \qquad x^* \in \mathbb{R}^n,$$

and therefore

$$\kappa_\alpha(\nu^*, \nu_1^*) = \kappa_\alpha(\nu, \nu_1) \tag{7.7}$$

for every $\nu_1 \in \mathfrak{M}(\mathbb{R}^n)$ that does not have an atomic mass at x_0. It is clear that the Kelvin transformation is additive and that it is an involution, i.e.

$$(\nu + \nu_1)^* = \nu^* + \nu_1^*, \tag{7.8}$$

$$(\nu^*)^* = \nu. \tag{7.9}$$

Write $A_i^* := I_{x_0}(\overline{A_i}) \cap \mathbb{R}^n$ and $\operatorname{sign} A_i^* := \operatorname{sign} A_i = s_i$ for every $i \in I$; then $\mathbf{A}^* = (A_i^*)_{i \in I}$ forms a *standard* condenser in $\mathbb{R}^n$, which is seen from (7.5) in view of the properties of I_{x_0}.

Applying the Kelvin transformation to each of the components ν^i, $i \in I$, of any given $\boldsymbol{\nu} = (\nu^i)_{i \in I} \in \mathcal{E}_\alpha^+(\mathbf{A})$ we get $\boldsymbol{\nu}^* := (\nu^i)_{i \in I}^* \in \mathfrak{M}^+(\mathbf{A}^*)$. Based on Lemma 3.6, identity (3.8) and relations (7.7) and (7.8), we also see that the α-Riesz energy of $\boldsymbol{\nu}^*$ is finite, and furthermore

$$\|\boldsymbol{\nu}_1^* - \boldsymbol{\nu}_2^*\|_{\mathcal{E}_\alpha^+(\mathbf{A}^*)} = \|\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2\|_{\mathcal{E}_\alpha^+(\mathbf{A})} \quad \text{for all } \boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \in \mathcal{E}_\alpha^+(\mathbf{A}). \tag{7.10}$$

Summarizing the above, because of (7.9) we arrive at the following observation: *the Kelvin transformation is an injective isometric mapping of* $\mathcal{E}^+_\alpha(\mathbf{A})$ *onto* $\mathcal{E}^+_\alpha(\mathbf{A}^*)$.

Let $\boldsymbol{\mu}_{k_m}$, $m \in \mathbb{N}$, and $\boldsymbol{\mu}$ be the measures chosen at the beginning of the proof. In view of (4.6) and (7.2), for each $i \in I$ one can apply [27, Lemma 4.3] to $\mu^i_{k_m}$, $m \in \mathbb{N}$, and μ^i, and consequently

$$\boldsymbol{\mu}^*_{k_m} \to \boldsymbol{\mu}^* \quad \text{vaguely as } m \to \infty. \tag{7.11}$$

But $\{\boldsymbol{\mu}^*_{k_m}\}_{m\in\mathbb{N}}$ is a strong Cauchy sequence in $\mathcal{E}^+_\alpha(\mathbf{A}^*)$, which is clear from (7.10). This together with (7.11) implies with the aid of Theorem 3.9 that $\boldsymbol{\mu}^* \in \mathcal{E}^+_\alpha(\mathbf{A}^*)$ and also that

$$\lim_{m\to\infty} \|\boldsymbol{\mu}^*_{k_m} - \boldsymbol{\mu}^*\|_{\mathcal{E}^+_\alpha(\mathbf{A}^*)} = 0.$$

Another application of the above observation then leads to (7.3) and (7.4), as was to be proved.

In turn, (7.4) implies that $\boldsymbol{\mu}_k \to \boldsymbol{\mu}$ strongly in $\mathcal{E}^+_\alpha(\mathbf{A})$ as $k \to \infty$, for $\{\boldsymbol{\mu}_k\}_{k\in\mathbb{N}}$ is strong Cauchy and hence converges strongly to any of its strong cluster points. It has thus been established that $\{\boldsymbol{\mu}_k\}_{k\in\mathbb{N}}$ converges strongly to any of its vague cluster points, which is the first assertion of the theorem. Assume now that all the $A_i \setminus \{x_0\}$, $i \in I$, are mutually disjoint. Then $\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|_{\mathcal{E}^+_\alpha(\mathbf{A})}$ defines a metric on $\mathcal{E}^+_\alpha(\mathbf{A})$, and hence $\boldsymbol{\mu}$ has to be the unique vague cluster point of $\{\boldsymbol{\mu}_k\}_{k\in\mathbb{N}}$. Since the vague topology is Hausdorff, $\boldsymbol{\mu}$ is actually the vague limit of $\{\boldsymbol{\mu}_k\}_{k\in\mathbb{N}}$ [2, Chapter I, Section 9, n° 1]. ∎

7.2. On the sharpness of Theorem 7.1. Our next aim is to show that Theorem 7.1 is no longer valid if (5.4) is omitted from its hypotheses.

Assume for simplicity that $\mathbf{g} = \mathbf{1}$, $\mathbf{a} = \mathbf{1}$ and $\mathbf{f} = \mathbf{0}$. Furthermore, let $0 < \alpha \le 2$, $I^+ = \{1\}$, $I^- = \{2\}$, and let $A_2 \subset \mathbb{R}^n$ be a closed set with $c_\alpha(A_2) = \infty$, but α-thin at $\omega_{\mathbb{R}^n}$ ([19]). Assume moreover that the (open) set $D := A_2^c$ is connected and A_1 is a compact subset of D with $c_\alpha(A_1) > 0$; then $\mathbf{A} := (A_1, A_2)$ forms a standard condenser in $\mathbb{R}^n$. For any constraint $\boldsymbol{\sigma} \in \mathfrak{C}(\mathbf{A})$, let $\mathcal{E}^{\boldsymbol{\sigma}}_\alpha(\mathbf{A}, \mathbf{1})$ stand for the class of vector measures admissible in Problem 5.1 with these data.

([19]) By definition, which goes back to Brelot [5, Theorem VII.13], a closed set $F \subset \mathbb{R}^n$ is *α-thin at $\omega_{\mathbb{R}^n}$* if either F is compact, or the inverse of F relative to $S(0,1)$ has $x = 0$ as an α-irregular boundary point [27, Theorem 5.10]. The existence of a closed set Q with $c_\alpha(Q) = \infty$, but α-thin at $\omega_{\mathbb{R}^n}$ has been proved by Landkof [27, Corollary, p. 283, and Lemma 5.5] (see also [7, pp. 276–277]). In the case $n = 3$ and $\alpha = 2$, an example of such a Q can be given as follows [43, Example 8.2]:

$$Q := \{x \in \mathbb{R}^3 : 0 \le x_1 < \infty,\ x_2^2 + x_3^2 \le \exp(-2x_1^r), \text{ where } 0 < r \le 1\};$$

note that Q thus defined would have finite $c_2(\cdot)$ if r in its definition were > 1.

THEOREM 7.4. *Under these requirements, there exists $\boldsymbol{\sigma} \in \mathfrak{C}(\mathbf{A})$ with $1 < \sigma^1(A_1) < \infty$ and $\sigma^2(A_2) = \infty$ such that, for every $\boldsymbol{\nu} \in \mathcal{E}_\alpha^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{1})$,*

$$\kappa_\alpha(\boldsymbol{\nu}, \boldsymbol{\nu}) > \inf_{\boldsymbol{\mu} \in \mathcal{E}_\alpha^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{1})} \kappa_\alpha(\boldsymbol{\mu}, \boldsymbol{\mu}) =: w_\alpha^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{1}).$$

Proof. Denote by $G = G_D^\alpha$ the α-Green kernel on the locally compact space D, defined by

$$G_D^\alpha(x, y) := \kappa_\alpha(x, \varepsilon_y) - \kappa_\alpha(x, \varepsilon_y^{D^c}), \qquad x, y \in D,$$

ε_y being the unit Dirac measure at y and $\varepsilon_y^{D^c} \in \mathfrak{M}^+(D^c)$ the α-Riesz balayage of ε_y onto D^c [22, 27]. Since $c_\alpha(A_1)$ is (strictly) positive, so is $c_G(A_1)$ by [13, Lemma 2.6], and hence, by the compactness of A_1, there exists a measure $\lambda \in \mathcal{E}_G^+(A_1, 1)$ with

$$G(\lambda, \lambda) = c_G(A_1)^{-1} < \infty.$$

In fact, such a λ is unique, for the α-Green kernel is strictly positive definite by [22, Theorem 4.9]. As S_D^λ is compact, it is seen from [22, Lemma 4.4] that both λ and λ^{D^c} have finite α-Riesz energy and moreover

$$\|\lambda\|_G = \|\lambda - \lambda^{D^c}\|_\alpha.$$

Finally, since $D^c = A_2$ is α-thin at $\omega_{\mathbb{R}^n}$, from [22, Theorem 3.21] (see also the earlier papers [37, Theorem B] and [38, Theorem 4]) we get

$$q := \lambda(A_1) - \lambda^{D^c}(A_2) > 0. \tag{7.12}$$

Consider an exhaustion of A_2 by an increasing sequence of compact sets K_ℓ, $\ell \in \mathbb{N}$. Since $c_\alpha(A_2) = \infty$, it follows from the strict positive definiteness of the α-Riesz kernel and from the subadditivity of $c_\alpha(\cdot)$ on universally measurable sets that $c_\alpha(A_2 \setminus K_\ell) = \infty$ for all $\ell \in \mathbb{N}$. Hence, for every ℓ one can choose a measure $\tau_\ell \in \mathcal{E}_\alpha^+(A_2 \setminus K_\ell, q)$ with compact support so that

$$\lim_{\ell \to \infty} \|\tau_\ell\|_\alpha = 0.$$

Certainly, there is no loss of generality in assuming $K_\ell \cup S_{\mathbb{R}^n}^{\tau_\ell} \subset K_{\ell+1}$.

Choose a constraint

$$\sigma^1 := \lambda + \delta_1, \qquad \sigma^2 := \lambda^{D^c} + \delta_2 + \sum_{\ell \in \mathbb{N}} \tau_\ell,$$

where δ_i, $i = 1, 2$, is a positive nonzero bounded Radon measure whose support coincides with A_i ([20]). We assert that the problem of minimizing $\kappa_\alpha(\boldsymbol{\mu}, \boldsymbol{\mu})$ over the class $\mathcal{E}_\alpha^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{1})$ with the constraint $\boldsymbol{\sigma}$ thus defined is unsolvable.

([20]) In particular, such a δ_i can be constructed as follows. Consider a sequence of points x_j of A_i which is dense in A_i and define $\delta_i = \sum_{j \in \mathbb{N}} 2^{-j} \varepsilon_{x_j}$.

It follows from the above that the sequence $\{\boldsymbol{\mu}_\ell\}_{\ell\in\mathbb{N}}$ with $\mu_\ell^1 = \lambda$ and $\mu_\ell^2 = \lambda^{D^c} + \tau_\ell$, $\ell \in \mathbb{N}$, belongs to $\mathcal{E}_\alpha^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{1})$, so that

$$\kappa_\alpha(\boldsymbol{\mu}_\ell, \boldsymbol{\mu}_\ell) \ge w_\alpha^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{1}) \quad \text{for all } \ell \in \mathbb{N}, \tag{7.13}$$

and moreover

$$\lim_{\ell\to\infty} \kappa_\alpha(\boldsymbol{\mu}_\ell, \boldsymbol{\mu}_\ell) = \lim_{\ell\to\infty} \|\lambda - \lambda^{D^c} - \tau_\ell\|_\alpha^2 = \|\lambda - \lambda^{D^c}\|_\alpha^2 = \|\lambda\|_G^2. \tag{7.14}$$

On the other hand, for any $\theta \in \mathcal{E}_\alpha^+(\mathbb{R}^n)$ the α-Riesz balayage θ^{D^c} is in fact the orthogonal projection of θ onto the convex cone $\mathcal{E}_\alpha^+(D^c)$, i.e.

$$\|\theta - \theta^{D^c}\|_\alpha < \|\theta - \nu\|_\alpha \quad \text{for all } \nu \in \mathcal{E}_\alpha^+(D^c),\ \nu \ne \theta^{D^c}$$

([21, Theorem 4.12] or [22, Theorem 3.1]). For an arbitrary $\boldsymbol{\mu} \in \mathcal{E}_\alpha^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{1})$ we therefore obtain

$$\kappa_\alpha(\boldsymbol{\mu}, \boldsymbol{\mu}) = \|\mu^1 - \mu^2\|_\alpha^2 \ge \|\mu^1 - (\mu^1)^{D^c}\|_\alpha^2 = \|\mu^1\|_G^2 \ge \|\lambda\|_G^2, \tag{7.15}$$

which yields $w_\alpha^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{1}) \ge \|\lambda\|_G^2$. Since the converse inequality holds in consequence of (7.13) and (7.14), we actually have

$$w_\alpha^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{1}) = \|\lambda\|_G^2. \tag{7.16}$$

To complete the proof, assume on the contrary that the extremal problem under consideration is solvable, i.e. there is $\boldsymbol{\mu} \in \mathcal{E}_\alpha^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{1})$ with $\kappa_\alpha(\boldsymbol{\mu}, \boldsymbol{\mu}) = w_\alpha^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{1})$. Substituting (7.16) into (7.15), we see that then all the inequalities in (7.15) have to be equalities. But this is possible only provided that both $\lambda = \mu^1$ and $\lambda^{D^c}(D^c) = 1 = \lambda(A_1)$ hold, which contradicts (7.12). ∎

8. Continuity of the minimizers $\boldsymbol{\lambda}_{\mathbf{A}}^{\boldsymbol{\sigma}}$ with respect to $(\mathbf{A}, \boldsymbol{\sigma})$. We now return again to the case of an arbitrary positive definite kernel κ on a locally compact space X. Given a (generalized) condenser $\mathbf{A} = (A_i)_{i\in I}$, fix a net of (generalized) condensers $\mathbf{A}_t := (A_i^t)_{i\in I}$, $t \in T$, with $\operatorname{sign} A_i^t = \operatorname{sign} A_i$ such that, for each $i \in I$, $A_i^{t_2} \subset A_i^{t_1}$ whenever $t_2 \ge t_1$, and also

$$A_i = \bigcap_{t\in T} A_i^t.$$

Furthermore, fix constraints $\boldsymbol{\sigma} = (\sigma^i)_{i\in I} \in \mathfrak{C}(\mathbf{A})$ and $\boldsymbol{\sigma}_t = (\sigma_t^i)_{i\in I} \in \mathfrak{C}(\mathbf{A}_t)$ such that $\sigma_{t_1}^i \ge \sigma_{t_2}^i \ge \sigma^i$ for all $t_2 \ge t_1$ and $i \in I$, and also

$$\boldsymbol{\sigma}_t \to \boldsymbol{\sigma} \quad \text{vaguely as } t \text{ increases along } T. \tag{8.1}$$

Then the following statement on continuity holds.

Theorem 8.1. *Assume in addition that for a certain $t_0 \in T$ all the hypotheses of either Theorem* 6.1 *or Theorem* 7.1 *hold for $\mathbf{A}_{t_0}$ and $\boldsymbol{\sigma}_{t_0}$ in place of* $\mathbf{A}$ *and* $\boldsymbol{\sigma}$. *Then in either Case* I *or Case* II *we have*

$$G_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) = \lim_{t\in T} G_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}_t}(\mathbf{A}_t, \mathbf{a}, \mathbf{g}). \tag{8.2}$$

Fix $\boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}} \in \mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$, *and for every* $t \geq t_0$ *fix* $\boldsymbol{\lambda}^{\boldsymbol{\sigma}_t}_{\mathbf{A}_t} \in \mathfrak{S}^{\boldsymbol{\sigma}_t}_{\kappa,\mathbf{f}}(\mathbf{A}_t,\mathbf{a},\mathbf{g})$; *such solutions to Problem* 5.1 *with the corresponding data exist. Then the* (*nonempty*) *vague cluster set of* $(\boldsymbol{\lambda}^{\boldsymbol{\sigma}_t}_{\mathbf{A}_t})_{t\geq t_0}$ *is contained in* $\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. *Furthermore,* $\boldsymbol{\lambda}^{\boldsymbol{\sigma}_t}_{\mathbf{A}_t} \to \boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}$ *strongly in* $\mathcal{E}^+_\kappa(\mathbf{A}_{t_0})$, *i.e.*

$$\lim_{t\in T} \|\boldsymbol{\lambda}^{\boldsymbol{\sigma}_t}_{\mathbf{A}_t} - \boldsymbol{\lambda}^{\boldsymbol{\sigma}}_{\mathbf{A}}\|_{\mathcal{E}^+_\kappa(\mathbf{A}_{t_0})} = 0,$$

and hence also vaguely provided that the kernel κ *is strictly positive definite and the* A_i^δ, $i \in I$, *are mutually disjoint.*

Proof. From the monotonicity of $\mathbf{A}_t$ and $\boldsymbol{\sigma}_t$ we get

$$\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) \subset \mathcal{E}^{\boldsymbol{\sigma}_{t_2}}_{\kappa,\mathbf{f}}(\mathbf{A}_{t_2},\mathbf{a},\mathbf{g}) \subset \mathcal{E}^{\boldsymbol{\sigma}_{t_1}}_{\kappa,\mathbf{f}}(\mathbf{A}_{t_1},\mathbf{a},\mathbf{g}) \tag{8.3}$$

whenever $t_2 \geq t_1 \geq t_0$, and therefore

$$-\infty < G^{\boldsymbol{\sigma}_{t_0}}_{\kappa,\mathbf{f}}(\mathbf{A}_{t_0},\mathbf{a},\mathbf{g}) \leq \lim_{t\in T} G^{\boldsymbol{\sigma}_t}_{\kappa,\mathbf{f}}(\mathbf{A}_t,\mathbf{a},\mathbf{g}) \leq G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) < \infty, \tag{8.4}$$

where the first inequality is valid by (5.2) with $\mathbf{A}_{t_0}$ and $\boldsymbol{\sigma}_{t_0}$ in place of $\mathbf{A}$ and $\boldsymbol{\sigma}$, respectively, while the last one holds by the (standing) assumption (5.3).

According to Theorems 6.1 and 7.1, under the stated hypotheses for every $t \geq t_0$ there exists a minimizer $\boldsymbol{\lambda}_t := \boldsymbol{\lambda}^{\boldsymbol{\sigma}_t}_{\mathbf{A}_t} \in \mathfrak{S}^{\boldsymbol{\sigma}_t}_{\kappa,\mathbf{f}}(\mathbf{A}_t,\mathbf{a},\mathbf{g})$. In consequence of (8.4), $\lim_{t\in T} G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_t)$ exists and

$$-\infty < \lim_{t\in T} G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_t) \leq G^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}) < \infty. \tag{8.5}$$

For an arbitrary fixed $\ell \in T$ such that $\ell \geq t_0$, we also see from (8.3) that

$$\boldsymbol{\lambda}_t \in \mathcal{E}^{\boldsymbol{\sigma}_\ell}_{\kappa,\mathbf{f}}(\mathbf{A}_\ell,\mathbf{a},\mathbf{g}) \quad \text{for all } t \geq \ell.$$

We next proceed by showing that

$$\|\boldsymbol{\lambda}_{t_2} - \boldsymbol{\lambda}_{t_1}\|^2_{\mathcal{E}^+_\kappa(\mathbf{A}_\ell)} \leq G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_{t_2}) - G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_{t_1}) \quad \text{whenever } \ell \leq t_1 \leq t_2. \tag{8.6}$$

For any $\tau \in (0,1]$ we have $\boldsymbol{\mu} := (1-\tau)\boldsymbol{\lambda}_{t_1} + \tau\boldsymbol{\lambda}_{t_2} \in \mathcal{E}^{\boldsymbol{\sigma}_{t_1}}_{\kappa,\mathbf{f}}(\mathbf{A}_{t_1},\mathbf{a},\mathbf{g})$, hence $G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) \geq G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_{t_1})$. Evaluating $G_{\kappa,\mathbf{f}}(\boldsymbol{\mu})$ and then letting $\tau \to 0$, we get

$$-\kappa(\boldsymbol{\lambda}_{t_1},\boldsymbol{\lambda}_{t_1}) + \kappa(\boldsymbol{\lambda}_{t_1},\boldsymbol{\lambda}_{t_2}) - \langle \mathbf{f},\boldsymbol{\lambda}_{t_1}\rangle + \langle \mathbf{f},\boldsymbol{\lambda}_{t_2}\rangle \geq 0,$$

and (8.6) follows. Noting that, by (8.5), the net $G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_t)$, $t \geq \ell$, is Cauchy in $\mathbb{R}$, we see from (8.6) that $(\boldsymbol{\lambda}_t)_{t\geq\ell}$ is strong Cauchy in $\mathcal{E}^{\boldsymbol{\sigma}_\ell}_\kappa(\mathbf{A}_\ell,\mathbf{a},\mathbf{g})$.

In accordance with Theorem 5.7, under the stated assumptions the set $\mathfrak{M}^{\boldsymbol{\sigma}_\ell}(\mathbf{A}_\ell,\mathbf{a},\mathbf{g})$ is vaguely compact. Hence there is a (strong Cauchy) subnet $(\boldsymbol{\lambda}_s)_{s\in S}$ of $(\boldsymbol{\lambda}_t)_{t\geq\ell}$ such that

$$\boldsymbol{\lambda}_s \to \boldsymbol{\lambda} \quad \text{vaguely as } s \text{ increases along } S, \tag{8.7}$$

where $\boldsymbol{\lambda}$ is some element of $\mathfrak{M}^{\boldsymbol{\sigma}_\ell}(\mathbf{A}_\ell, \mathbf{a}, \mathbf{g})$. Since the vague limit is unique, λ^i is carried by A_i^ℓ for every $\ell \geq t_0$, and hence by $A_i = \bigcap_{\ell \geq t_0} A_i^\ell$. Since the vague limit of the (positive) measures $\sigma_s^i - \lambda_s^i$ is likewise the positive measure $\sigma^i - \lambda^i$ (see (8.1) and (8.7)), we thus altogether get

$$\boldsymbol{\lambda} \in \mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}). \tag{8.8}$$

Assume first that X, κ, $\mathbf{g}$, $\mathbf{A}_{t_0}$ and $\boldsymbol{\sigma}_{t_0}$ satisfy the assumptions of Theorem 6.1. Since $(\boldsymbol{\lambda}_t)_{t \geq \ell}$ is strong Cauchy in $\mathcal{E}_\kappa^{\boldsymbol{\sigma}_\ell}(\mathbf{A}_\ell, \mathbf{a}, \mathbf{g})$ and since (8.7) holds, Theorem 6.6 shows that $\kappa(\boldsymbol{\lambda}, \boldsymbol{\lambda}) < \infty$, hence

$$\boldsymbol{\lambda} \in \mathcal{E}_\kappa^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \tag{8.9}$$

(see (8.8)), and moreover

$$\boldsymbol{\lambda}_t \to \boldsymbol{\lambda} \quad \text{strongly in } \mathcal{E}_\kappa^+(\mathbf{A}_\ell) \text{ as } t \text{ increases along } T. \tag{8.10}$$

Applying Lemma 5.8 with $\mathbf{A}_\ell$ in place of $\mathbf{A}$, from relations (8.4), (8.5), (8.7) and (8.10) we get

$$\begin{aligned} -\infty < G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}) \leq \lim_{s \in S} G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_s) &= \lim_{t \in T} G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_t) \\ &= \lim_{t \in T} G_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}_t}(\mathbf{A}_t, \mathbf{a}, \mathbf{g}) \leq G_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty, \end{aligned}$$

the first inequality being valid by (4.4) and (4.7). Thus $\boldsymbol{\lambda} \in \mathcal{E}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ (see (8.9)), and therefore $G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}) \geq G_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Combined with the last display, this proves (8.2) and also

$$\boldsymbol{\lambda} \in \mathfrak{S}_{\kappa,\mathbf{f}}^{\boldsymbol{\sigma}}(\mathbf{A}, \mathbf{a}, \mathbf{g}). \tag{8.11}$$

Assume next that X, κ, $\mathbf{g}$, $\mathbf{A}_{t_0}$ and $\boldsymbol{\sigma}_{t_0}$ satisfy the assumptions of Theorem 7.1. Based on (8.7) and (8.8), we then arrive at both (8.9) and (8.10), now with the aid of Theorem 7.3, and hence again at (8.2) and (8.11) by arguments similar to those applied just above.

It has thus been shown that, under the hypotheses of Theorem 8.1, relation (8.2) holds and $(\boldsymbol{\lambda}_{\mathbf{A}_t}^{\boldsymbol{\sigma}_t})_{t \geq t_0}$, being strong Cauchy in $\mathcal{E}_\kappa^+(\mathbf{A}_{t_0})$, converges strongly in $\mathcal{E}_\kappa^+(\mathbf{A}_{t_0})$ to any of its vague cluster points $\boldsymbol{\lambda}$, and also this $\boldsymbol{\lambda}$ solves Problem 5.1 for the condenser $\mathbf{A}$ and the constraint $\boldsymbol{\sigma}$. If the kernel κ is strictly positive definite and the A_i^δ, $i \in I$, are mutually disjoint, such a solution is uniquely determined by Corollary 5.5, so that the vague cluster set of $(\boldsymbol{\lambda}_{\mathbf{A}_t}^{\boldsymbol{\sigma}_t})_{t \geq t_0}$ reduces to the given $\boldsymbol{\lambda}$. Hence $\boldsymbol{\lambda}_{\mathbf{A}_t}^{\boldsymbol{\sigma}_t} \to \boldsymbol{\lambda}$ also vaguely [2, Chapter I, Section 9, n° 1, Corollary]. ∎

9. The f-weighted vector potentials of the solutions. Theorem 9.2 below establishes a description of the **f**-weighted vector potentials $\mathbf{W}_{\kappa,\mathbf{f}}^{\boldsymbol{\lambda}} = (W_{\kappa,\mathbf{f}}^{\boldsymbol{\lambda},i})_{i \in I}$ (see (4.1)) of the solutions to Problem 5.1 (provided they exist), and it also singles out their characteristic properties.

LEMMA 9.1. *For* $\boldsymbol{\lambda} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ *to solve Problem* 5.1, *it is necessary and sufficient that*

$$(9.1)\qquad \sum_{i\in I}\langle W^{\boldsymbol{\lambda},i}_{\kappa,\mathbf{f}},\nu^i-\lambda^i\rangle \ge 0 \quad \textit{for all } \boldsymbol{\nu}\in\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g}).$$

Proof. By direct calculation, for any $\boldsymbol{\mu},\boldsymbol{\nu}\in\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ and any $h\in(0,1]$ we obtain

$$G_{\kappa,\mathbf{f}}(h\boldsymbol{\nu}+(1-h)\boldsymbol{\mu})-G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) = 2h\sum_{i\in I}\langle W^{\boldsymbol{\mu},i}_{\kappa,\mathbf{f}},\nu^i-\mu^i\rangle + h^2\|\boldsymbol{\nu}-\boldsymbol{\mu}\|^2_{\mathcal{E}^+_\kappa(\mathbf{A})}.$$

If $\boldsymbol{\mu}=\boldsymbol{\lambda}$ solves Problem 5.1, then the left (and hence the right) side of this display is ≥ 0, for the class $\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$ is convex, which leads to (9.1) by letting $h\to 0$. Conversely, if (9.1) holds, then the preceding formula with $\boldsymbol{\mu}=\boldsymbol{\lambda}$ and $h=1$ implies that $G_{\kappa,\mathbf{f}}(\boldsymbol{\nu})\ge G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda})$ for all $\boldsymbol{\nu}\in\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$, hence $\boldsymbol{\lambda}\in\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$. ∎

THEOREM 9.2. *Given a generalized condenser* $\mathbf{A}$ *and a constraint* $\boldsymbol{\sigma}\in\mathfrak{C}(\mathbf{A})$, *assume that for every* $i\in I$, $\kappa(\cdot,\sigma^i)$ *is upper semicontinuous* (*hence continuous*) *on* X *and upper bounded on* $A_i\setminus K$ *for a sufficiently large compact set* $K\subset X$. *Suppose that each* g_i, $i\in I$, *is upper bounded on* $A_i\setminus K$ *with* K *as indicated just above, and*

$$(9.2)\qquad \langle g_i,\sigma^i|_{\dot A_i^\delta}\rangle > a_i,$$

$\dot A_i^\delta$ *being defined by* (4.10). *If, moreover, each* f_i, $i\in I$, *is lower bounded on* A_i *then, for any given* $\boldsymbol{\lambda}\in\mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$, *the following two assertions are equivalent* ([21]):

- (i) $\boldsymbol{\lambda}\in\mathfrak{S}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A},\mathbf{a},\mathbf{g})$.
- (ii) *There exists* $(w^i_{\boldsymbol{\lambda}})_{i\in I}\in\mathbb{R}^{|I|}$ *such that for all* $i\in I$,

$$(9.3)\qquad W^{\boldsymbol{\lambda},i}_{\kappa,\mathbf{f}} \ge w^i_{\boldsymbol{\lambda}} g_i \quad (\sigma^i-\lambda^i)\text{-a.e.},$$

$$(9.4)\qquad W^{\boldsymbol{\lambda},i}_{\kappa,\mathbf{f}} \le w^i_{\boldsymbol{\lambda}} g_i \quad \lambda^i\text{-a.e.}$$

Proof. We first observe that since the potential $\kappa(\cdot,\sigma^i)$ is bounded on A_i, so is $\kappa(\cdot,\mu^i)$, where $(\mu^i)_{i\in I}$ is any measure from $\mathfrak{M}^{\boldsymbol{\sigma}}(\mathbf{A})$. Furthermore, since $\kappa(\cdot,\sigma^i)$ is continuous on X, so is $\kappa(\cdot,\mu^i)$ (see the proof of Lemma 6.4). Moreover, obviously

$$(9.5)\qquad \sigma^i|_K\in\mathcal{E}^+_\kappa(K) \quad \text{for any compact } K\subset A_i.$$

([21]) In Case I the requirement of the lower boundedness of f_i, $i\in I$, is fulfilled automatically. Furthermore, in Case I relation (9.4) is equivalent to the following apparently stronger assertion: $W^{\boldsymbol{\lambda},i}_{\kappa,\mathbf{f}}\le w^i_{\boldsymbol{\lambda}} g_i$ everywhere on $S^{\lambda^i}_X$.

We proceed with the proof that assertions (i) and (ii) are equivalent. Suppose first that (i) holds, i.e. $\boldsymbol{\lambda} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ solves Problem 5.1. To verify (ii), fix any $i \in I$. For every $\boldsymbol{\mu} = (\mu^\ell)_{\ell \in I} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ write $\boldsymbol{\mu}_i := (\mu^\ell_i)_{\ell \in I}$ where $\mu^\ell_i := \mu^\ell$ for all $\ell \neq i$ and $\mu^i_i = 0$; then $\boldsymbol{\mu}_i \in \mathcal{E}^+_{\kappa,\mathbf{f}}(\mathbf{A})$. Also define $\tilde{f}_i := f_i + \kappa^i_{\boldsymbol{\lambda}_i}$; then by substituting (3.6) we obtain

$$\tilde{f}_i(x) = f_i(x) + s_i \sum_{\ell \in I,\, \ell \neq i} s_\ell \kappa(x, \lambda^\ell), \qquad x \in X. \tag{9.6}$$

Since f_i is lower bounded on A_i, we conclude from the properties of $\kappa(\cdot, \lambda^\ell)$, $\ell \in I$, established above that the function

$$W^{\lambda^i}_{\kappa,\tilde{f}_i} := \kappa(\cdot, \lambda^i) + \tilde{f}_i, \qquad i \in I, \tag{9.7}$$

is likewise lower bounded on A_i. Also observe that $W^{\lambda^i}_{\kappa,\tilde{f}_i}$ is finite everywhere on $\dot{A}^\delta_i$, which is clear from (4.10).

Furthermore, by (3.5) and (4.2), for any $\boldsymbol{\mu} \in \mathcal{E}^{\boldsymbol{\sigma}}_{\kappa,\mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with the additional property that $\boldsymbol{\mu}_i = \boldsymbol{\lambda}_i$ (in particular, for $\boldsymbol{\mu} = \boldsymbol{\lambda}$) we get

$$G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) = G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda}_i) + G_{\kappa,\tilde{f}_i}(\mu^i).$$

Combined with $G_{\kappa,\mathbf{f}}(\boldsymbol{\mu}) \geq G_{\kappa,\mathbf{f}}(\boldsymbol{\lambda})$, this yields $G_{\kappa,\tilde{f}_i}(\mu^i) \geq G_{\kappa,\tilde{f}_i}(\lambda^i)$; hence λ^i minimizes $G_{\kappa,\tilde{f}_i}(\nu)$ where ν ranges over $\mathcal{E}^{\sigma^i}_{\kappa,\tilde{f}_i}(A_i, a_i, g_i)$. This enables us to show that there exists $w_{\lambda^i} \in \mathbb{R}$ such that

$$W^{\lambda^i}_{\kappa,\tilde{f}_i} \geq w_{\lambda^i} g_i \quad (\sigma^i - \lambda^i)\text{-a.e.}, \tag{9.8}$$

$$W^{\lambda^i}_{\kappa,\tilde{f}_i} \leq w_{\lambda^i} g_i \quad \lambda^i\text{-a.e.} \tag{9.9}$$

Indeed, (9.8) holds with

$$w_{\lambda^i} := L_i := \sup\{t \in \mathbb{R} : W^{\lambda^i}_{\kappa,\tilde{f}_i} \geq t g_i \ (\sigma^i - \lambda^i)\text{-a.e.}\}.$$

In turn, (9.8) with $w_{\lambda^i} = L_i$ implies that $L_i < \infty$, because

$$\widetilde{W}^{\lambda^i}_{\kappa,\tilde{f}_i}(x) := \frac{W^{\lambda^i}_{\kappa,\tilde{f}_i}(x)}{g_i(x)} < \infty$$

for all $x \in \dot{A}^\delta_i$ (see (4.5)), hence $(\sigma^i - \lambda^i)$-a.e. on $\dot{A}^\delta_i$, while $(\sigma^i - \lambda^i)(\dot{A}^\delta_i) > 0$ by (9.2). Also, $L_i > -\infty$ since, by the upper boundedness of g_i on A_i, $\widetilde{W}^{\lambda^i}_{\kappa,\tilde{f}_i}$ is lower bounded on A_i.

We next establish (9.9) with $w_{\lambda^i} = L_i$. To this end, for any $w \in \mathbb{R}$ write

$$A^+_i(w) := \{x \in A_i : W^{\lambda^i}_{\kappa,\tilde{f}_i}(x) > w g_i(x)\},$$

$$A^-_i(w) := \{x \in A_i : W^{\lambda^i}_{\kappa,\tilde{f}_i}(x) < w g_i(x)\}.$$

Assume, on the contrary, that (9.9) with $w_{\lambda^i} = L_i$ does not hold, i.e. $\lambda^i(A_i^+(L_i)) > 0$. Since $\widetilde{W}^{\lambda^i}_{\kappa,\tilde f_i}$ is λ^i-measurable, one can choose $w_i \in (L_i, \infty)$ so that $\lambda^i(A_i^+(w_i)) > 0$. At the same time, as $w_i > L_i$, relation (9.8) with $w_{\lambda^i} = L_i$ yields $(\sigma^i - \lambda^i)(A_i^-(w_i)) > 0$. Therefore, there exist compact sets $K_1 \subset A_i^+(w_i)$ and $K_2 \subset A_i^-(w_i)$ such that

$$0 < \langle g_i, \lambda^i|_{K_1}\rangle < \langle g_i, (\sigma^i - \lambda^i)|_{K_2}\rangle. \tag{9.10}$$

Write $\tau^i := (\sigma^i - \lambda^i)|_{K_2}$; then $\kappa(\tau^i, \tau^i) < \infty$ by (9.5). Since $\langle W^{\lambda^i}_{\kappa,\tilde f_i}, \tau^i\rangle \le \langle w_i g_i, \tau^i\rangle < \infty$, we get $\langle \tilde f_i, \tau^i\rangle < \infty$ in view of (9.7). Define

$$\theta^i := \lambda^i - \lambda^i|_{K_1} + c_i\tau^i, \quad \text{where} \quad c_i := \langle g_i, \lambda^i|_{K_1}\rangle / \langle g_i, \tau^i\rangle.$$

Noting that $c_i \in (0, 1)$ by (9.10), we see by straightforward verification that $\langle g_i, \theta^i\rangle = a_i$ and $\theta^i \le \sigma^i$, hence $\theta^i \in \mathcal{E}^{\sigma^i}_{\kappa,\tilde f_i}(A_i, a_i, g_i)$. On the other hand,

$$\begin{aligned}\langle W^{\lambda^i}_{\kappa,\tilde f_i}, \theta^i - \lambda^i\rangle &= \langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_i g_i, \theta^i - \lambda^i\rangle \\ &= -\langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_i g_i, \lambda^i|_{K_1}\rangle + c_i\langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_i g_i, \tau^i\rangle < 0,\end{aligned}$$

which is impossible in view of the scalar version of Lemma 9.1. The contradiction obtained establishes (9.9).

Substituting (9.6) into (9.7) and then comparing the result obtained with (3.6) and (4.1), we see that

$$W^{\lambda^i}_{\kappa,\tilde f_i} = W^{\boldsymbol\lambda,i}_{\kappa,\mathbf f}. \tag{9.11}$$

Combined with (9.8) and (9.9), this proves (9.3) and (9.4) with $w^i_{\boldsymbol\lambda} := w_{\lambda^i}$, $i \in I$. This completes the proof that (i) implies (ii).

Conversely, suppose (ii) holds. On account of (9.11), for every $i \in I$ relations (9.8) and (9.9) are then fulfilled with $w_{\lambda^i} := w^i_{\boldsymbol\lambda}$ and $\tilde f_i$ defined by (9.6). This yields

$$\lambda^i(A_i^+(w_{\lambda^i})) = 0 \quad \text{and} \quad (\sigma^i - \lambda^i)(A_i^-(w_{\lambda^i})) = 0.$$

For any $\boldsymbol\nu \in \mathcal{E}^{\boldsymbol\sigma}_{\kappa,\mathbf f}(\mathbf A, \mathbf a, \mathbf g)$ we therefore get

$$\begin{aligned}\langle W^{\boldsymbol\lambda,i}_{\kappa,\mathbf f}, \nu^i - \lambda^i\rangle &= \langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_{\lambda^i} g_i, \nu^i - \lambda^i\rangle \\ &= \langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_{\lambda^i} g_i, \nu^i|_{A_i^+(w_{\lambda^i})}\rangle + \langle W^{\lambda^i}_{\kappa,\tilde f_i} - w_{\lambda^i} g_i, (\nu^i - \sigma^i)|_{A_i^-(w_{\lambda^i})}\rangle \ge 0.\end{aligned}$$

Summing up these inequalities over all $i \in I$, we conclude from Lemma 9.1 that $\boldsymbol\lambda$ is a solution to Problem 5.1. ∎

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