ANNALES
POLONICI MATHEMATICI
120.3 (2017)

Jordan–Hölder composition series of regular (a, b)-modules

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Abstract. A classical result of singularity theory states that the spectrum of an isolated hypersurface singularity is symmetric with respect to $n/2$, where n is the dimension of the enclosing space. We prove a similar result for the Jordan–Hölder composition series of the (a, b)-module associated to an isolated hypersurface singularity.

1. Introduction. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function with an isolated singularity at the origin. The lattice introduced by E. Brieskorn [Bri70],

$$D := \frac{\Omega_0^{n+1}}{df \wedge \Omega_0^{n-1}},$$

where Ω_0^p are the germs of holomorphic p-forms at the origin, is endowed with several structures: the structure of a free $\mathbb{C}\{a\}$-module and a $\mathbb{C}\{\{b\}\}$-module, where $a := f\cdot$ and $b := df \wedge d^{-1}$, a V-filtration and two different mixed Hodge structures defined by Varchenko [Var82] and Scherk and Steenbrink [SS85].

Many invariants of the hypersurface singularity, such as the spectrum and complex monodromy [Sch03], can be computed using the $\mathbb{C}\{a\}$-module and $\mathbb{C}\{\{b\}\}$-module structures. Following the work of D. Barlet [Bar93] we are therefore interested in the properties of the b-adic completion of the Brieskorn lattice considered as an abstract algebraic structure called an (a, b)-module:

Definition 1.1. An (a, b)-*module* is a free $\mathbb{C}[[b]]$-module E of finite rank over the ring of formal power series in b, endowed with a $\mathbb{C}$-linear endomorphism 'a' which satisfies

$$ab - ba = b^2.$$

The simplest examples of (a, b)-modules have rank 1 and are generated by an element e_λ satisfying $ae_\lambda = \lambda b e_\lambda$ for a complex number λ. We will

2010 *Mathematics Subject Classification*: 32S25, 32S40, 32S50.
Key words and phrases: (a, b)-modules, duality, decomposition.
Received 21 September 2016; revised 8 October 2017 and 29 November 2017.
Published online 15 December 2017.

refer to them as *elementary* (a,b)-modules of parameter λ and denote them by E_λ.

Since all (a,b)-modules E coming from the geometric setting are *regular*, i.e. they are sub-(a,b)-modules of an (a,b)-module $E^\#$ which satisfies $aE^\# \subset bE^\#$, we will be treating only the case of regular (a,b)-modules.

For an (a,b)-module E let us consider a filtration

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = E$$

where the F_i are sub-(a,b)-modules and are *normal*, i.e. such that E/F_i is a free $\mathbb{C}[[b]]$-module and hence is still an (a,b)-module. We call such filtrations *Jordan–Hölder composition series.* The main difference between these filtrations and the Hodge filtration and the V-filtration on an (a,b)-module is that the last two are not normal in general.

The higher residue pairings defined by K. Saito [Sai83] on the Brieskorn lattice may be viewed as polarisations of the mixed Hodge structure on the Brieskorn lattice [Her99]. Such a polarisation induces a non-degenerate hermitian form on the associated (a,b)-module [Kar13].

R. Belgrade [Bel01] uses this form to produce a purely algebraic proof of the symmetry of the spectral numbers of an isolated hypersurface singularity. Since for a regular (a,b)-module all the quotients F_j/F_{j-1} of a Jordan–Hölder composition series are elementary (a,b)-modules E_{λ_i}, $\lambda_i \in \mathbb{C}$, a question may arise whether these λ_i have some symmetry properties.

Unfortunately the quotients of a Jordan–Hölder composition series are not unique up to permutation and vary between series. As showed by D. Barlet [Bar93], the numbers $e^{2\pi i\lambda_i}$ are eigenvalues of the monodromy matrix, so the λ_i are fixed modulo $\mathbb{Z}$.

We show that by carefully choosing a composition series, we may obtain a symmetry property:

Theorem 1.2. *Let E be a regular self-adjoint (a,b)-module. Then it admits a self-adjoint Jordan–Hölder composition series.*

By a *self-adjoint (a,b)-module* we mean a module isomorphic to its own adjoint, and a *self-adjoint Jordan–Hölder composition series* is a composition series

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = E$$

with the following symmetry: the quotients F_{n-i}/F_{n-i-1} and F_i/F_{i-1} are adjoint to each other and the isomorphism of E with its adjoint induces an automorphism on F_{n-i}/F_i.

We are not aware at present whether the sequence of λ_i obtained in the case of isolated singularity of hypersurfaces is related to the spectrum or not. This is trivially so in the case of quasihomogeneous singularities since the associated (a,b)-modules are a direct sum of elementary (a,b)-modules. In more complex cases, such as rank 2 modules $E_{\lambda,\mu}$ described by Barlet, the spectrum and the λ_i differ.

2. Hermitian form on (a,b)-modules. The translation of K. Saito's higher residue pairings into the language of (a,b)-modules suggests the following definitions:

DEFINITION 2.1. Let E be an (a,b)-module. Then the $\mathbb{C}[[b]]$-module $\mathrm{Hom}_{\mathbb{C}[[b]]}(E, E_0)$ endowed with the a-action

$$[a \cdot \varphi](x) = a\varphi(x) - \varphi(ax),$$

where $\varphi \in E^*$ and $x \in E$, is called the *dual* (a,b)-module and denoted by E^*.

DEFINITION 2.2. Let E be an (a,b)-module. The $\mathbb{C}$-vector space E endowed with the action

$$a \cdot_{E^\vee} x = -a \cdot_E x, \qquad b \cdot_{E^\vee} x = -b \cdot_E x,$$

where $x \in E$, is called the *conjugate* (a,b)-module and denoted by $E^\vee$.

The operations of taking the dual and the conjugate are functors (contravariant and covariant) from the category of (a,b)-modules into itself. Their composition is a contravariant functor, called *adjunction*. Keeping this in mind we see that Saito's higher residue pairings are equivalent to the existence of an isomorphism $\Phi : E \to E^{\vee *}$ of (a,b)-modules such that $\Phi^{\vee *} = \Phi$ [Kar13]. We will call such an isomorphism a *hermitian structure* by analogy to the category of complex vector spaces.

If Φ is a hermitian structure, then we can associate to it a map

$$H : E \times E \to E_0, \qquad (x,y) \mapsto \Phi(y)(x),$$

satisfying

$$\begin{aligned} H(bx,y) &= bH(x,y) = H(x,-by), \\ aH(x,y) &= H(ax,y) - H(x,ay), \\ H(x,y) &= S(b)e_0 \;\Rightarrow\; H(y,x) = S(-b)e_0. \end{aligned}$$

We call H the *hermitian form associated to Φ*.

We should remark that not every (a,b)-module admits a hermitian structure. For most (a,b)-modules, E and $E^{\vee *}$ are not isomorphic, and even if they are, there are examples of (a,b)-modules that only admit an anti-hermitian ($\Phi^{\vee *} = -\Phi$) structure. Therefore we call (a,b)-modules satisfying $E \simeq E^{\vee *}$ *self-adjoint*, and those admitting a hermitian structure *hermitian*.

In the general case a regular self-adjoint (a,b)-module can be decomposed (not necessarily in a unique way) into the direct sum of a hermitian (a,b)-module and an anti-hermitian one [Kar13].

Let $\{F_i\}$ be a composition series of a regular hermitian (a,b)-module E:

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = E,$$

and let G_i be the subset of $E^{\vee *}$ of functionals which are zero on F_i. In this way we obtain a composition series of $E^{\vee *}$:

$$0 = G_n \subsetneq G_{n-1} \subsetneq \cdots \subsetneq G_0 = E^{\vee *},$$

and we may provide the following definition:

DEFINITION 2.3. Let $\{F_i\}$ be a composition series of a regular hermitian (a, b)-module E with hermitian structure $\Phi : E \to E^{\vee *}$. We say that $\{F_i\}$ is *self-adjoint* if

$$\Phi(F_i) = G_{n-1},$$

or equivalently $(F_i/F_{i-1})^{\vee *} \simeq F_{n-i+1}/F_{n-i}$ and Φ induces a hermitian structure on F_{n-i}/F_i.

3. Proof of Theorem 1.2. Before proving the theorem we shall introduce a couple of lemmas.

LEMMA 3.1. *Let E be a regular hermitian (a, b)-module, $\Phi : E \to E^{*\vee}$ a hermitian structure and H the associated hermitian form. If there exists a normal sub-(a, b)-module F_1 of rank 1 such that*

$$H(F_1, F_1) = 0,$$

then there exists a normal sub-(a, b)-module F_{n-1} of rank $n - 1$ such that $E/F_{n-1} \simeq F_1^{\vee *}$ and F_{n-1}/F_1 is hermitian.

Proof. Let $F_1 \simeq E_\lambda$, let e_λ be the generator of F_1 and consider the annihilator of this form under H:

$$F_{n-1} := \{x \in E \mid H(e_\lambda, x) = 0\}.$$

We remark that the condition $H(e_\lambda, e_\lambda) = 0$ gives $F_1 \subset F_{n-1}$, and F_{n-1} is normal, because it is the kernel of a morphism.

Consider the exact sequence

$$0 \to F_1 \to E \to E/F_1 \to 0,$$

from which we can pass to the adjoint sequence

$$0 \to (E/F_1)^{\vee *} \to E^{*\vee} \xrightarrow{\pi} F_1^{\vee *} \to 0.$$

Since π is the restriction of forms on E to the sub-(a, b)-module F_1, the kernel of π can be described as follows:

$$K = \{\varphi \in E^{*\vee} \mid \varphi(F_1) = 0\}.$$

The adjoint sequence being exact, we can identify from now on $(E/F_1)^{\vee *}$ with K.

If we consider the restriction

$$\Phi|_{F_{n-1}} : F_{n-1} \to E^{*\vee}$$

and the fact that by definition $\Phi(x)(e_\lambda) = 0$ for all $x \in F_{n-1}$, we find that $\Phi(F_{n-1}) \subset (E/F_1)^{\vee *}$.

On the other hand, for all $\varphi \in (E/F_1)^{\vee *}$ the element $y = \Phi^{-1}(\varphi)$ satisfies $\Phi(y)(e_\lambda) = 0$, therefore we also have $(E/F_1)^{\vee *} \subset \Phi(F_{n-1})$. It follows that $\Phi(F_{n-1}) = (E/F_1)^{\vee *}$, and since Φ is an isomorphism, F_{n-1} is isomorphic to $(E/F_1)^{\vee *}$.

Let us look now at the exact sequence

$$0 \to F_{n-1}/F_1 \to E/F_1 \to E/F_{n-1} \to 0$$

and the adjoint sequence

$$0 \to (E/F_{n-1})^{\vee *} \xrightarrow{i} (E/F_1)^{\vee *} \xrightarrow{\pi} (F_{n-1}/F_1)^{\vee *} \to 0.$$

π designates the restriction map on the forms of $(E/F_1)^{\vee *}$. $\operatorname{Ker} \pi$ is thus the forms of $(E/F_1)^{\vee *}$ that annihilate $(F_{n-1}/F_1)^{\vee *}$ or, with the convention of the previous paragraph, the forms of $E^{\vee *}$ that annihilate F_{n-1} and $F_1 \subset F_{n-1}$:

$$\operatorname{Ker} \pi = \{\varphi \in E^{\vee *} \mid \varphi(F_{n-1}) = 0\}$$

We note that Φ being hermitian gives

$$\Phi(e_\lambda)(F_{n-1}) = \Phi(F_{n-1})(e_\lambda)^\vee = 0,$$

and therefore $\Phi(F_1) \subset \operatorname{Ker} \pi$. An easy calculation shows that $\operatorname{Ker} \pi$ is of rank 1. Since $\Phi(F_1)$ is normal, of rank 1 and is included in $\operatorname{Ker} \pi$, they must be equal.

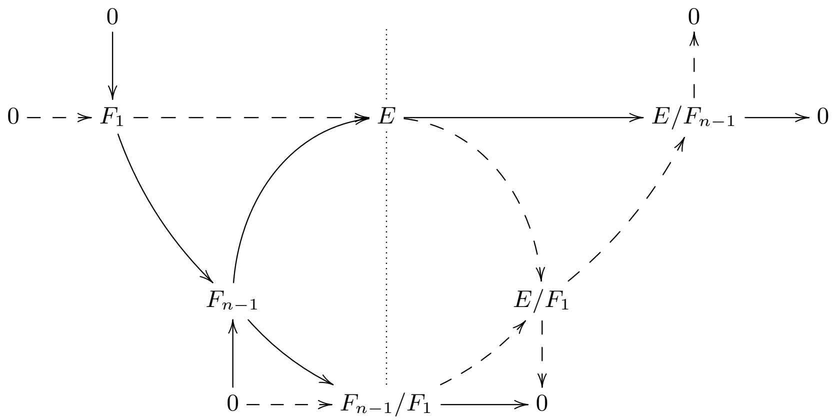

Fig. 1. Modules in symmetric positions with respect to the dotted line are each other's adjoint.

We obtain $(E/F_{n-1})^{\vee *} \simeq \operatorname{Ker} \pi \simeq F_1$. Now we know that Φ sends F_{n-1} onto $(E/F_1)^{\vee *}$ and F_1 onto $\operatorname{Ker} \pi$, so starting with the exact sequence

$$0 \to \operatorname{Ker} \pi \hookrightarrow (E/F_1)^* \xrightarrow{\pi} (F_{n-1}/F_1)^* \to 0$$

we can obtain another by substituting F_1 for $\operatorname{Ker} \pi$ and F_{n-1} for $(E/F_1)^*$:

$$0 \to F_1 \to F_{n-1} \to (F_{n-1}/F_1)^{\vee *} \to 0,$$

so that $(F_{n-1}/F_1)^{\vee *} \simeq (F_{n-1}/F_1)$. Note that the isomorphism is given by $x \mapsto \Phi(x)|_{F_{n-1}}$ and is therefore hermitian.

The proof may be summarized by the graph of interwoven exact sequences presented in Figure 1. ■

REMARK 3.2. If $ae_\lambda = \lambda b e_\lambda$ and $2\lambda \notin \mathbb{N}$, then $H(e_\lambda, e_\lambda) = 0$. In fact $H(e_\lambda, e_\lambda) \in E_0$ must satisfy the equation

$$\begin{aligned} aH(e_\lambda, e_\lambda) &= H(ae_\lambda, e_\lambda) + H(e_\lambda, -ae_\lambda) \\ &= H(\lambda b e_\lambda, e_\lambda) + H(e_\lambda, -\lambda b e_\lambda) = 2\lambda b H(e_\lambda, e_\lambda), \end{aligned}$$

which has non-trivial solutions in E_0 only if $2\lambda \in \mathbb{N}$. The double inversion of signs in the second factor is due to the hermitian nature of the form.

LEMMA 3.3. *Let E be a regular hermitian (a,b)-module with hermitian form H, $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$. If there exist two distinct normal sub-(a,b)-modules $F \simeq E_{\lambda+j}$ and $G \simeq E_\lambda$ then there exists a normal sub-(a,b)-module F_1 such that $H(F_1, F_1) = 0$.*

Proof. Let e_f and e_g be generators of F and G.

By the fundamental property $ab - ba = b^2$ of (a,b)-modules we have $ab^j e_g = (\lambda + j)b \cdot b^j e_g$. Consider now the complex vector space

$$V := \{\alpha e_f + \beta b^j e_g \mid \alpha, \beta \in \mathbb{C}\}$$

Note that every $v \in V$ satisfies $av = (\lambda + j)bv$. The properties of H give

$$(a - 2\lambda + jb)H(v, w) = 0,$$

which has in E_0 only solutions of the form $\alpha b^{2(\lambda+j)} e_0$, $\alpha \in \mathbb{C}$. There exists therefore a $\mathbb{C}$-bilinear B from $V \times V$ to $\mathbb{C}$ such that

$$H(v, w) = B(v, w) b^{2(\lambda+j)} e_0 \quad \forall v, w \in V,$$

which will have an isotropic vector e such that $B(e, e) = 0$, i.e. $H(e, e) = 0$. By taking $F_1 = \langle e \rangle$ we conclude the proof. ■

LEMMA 3.4. *Let E be a regular (a,b)-module and*

$$0 \subsetneq \cdots \subsetneq F_{i-1} \subsetneq F_i \subsetneq F_{i+1} \subsetneq \cdots \subsetneq E$$

be a Jordan–Hölder composition series with $F_i/F_{i-1} \simeq E_{\lambda_i}$ for all i and suppose there is a j such that $\lambda_{j+1} \neq \lambda_j \bmod \mathbb{Z}$. Then we can find another Jordan–Hölder composition series that differs only in the jth term F'_j such that $F'_j/F_{j-1} \simeq E_{\lambda'_j}$ and $F_{j+1}/F'_j \simeq E_{\lambda'_{j+1}}$ with $\lambda_j = \lambda'_{j+1} \bmod \mathbb{Z}$ and $\lambda_{j+1} = \lambda'_j \bmod \mathbb{Z}$, i.e. we can permute the quotients up to an integer shift of the parameters.

Proof. Consider the (a,b)-module $G := F_{j+1}/F_{j-1}$ and the canonical projection $\pi : E \to E/F_{j-1}$. Then G is a rank two module. Using the classification of regular (a,b)-modules of rank 2 given by D. Barlet [Bar93] we see that the only two possibilities for G are:

$$G \simeq E_{\lambda_j} \oplus E_{\lambda_{j+1}},$$

in which case we take $F'_j = \pi^{-1}(E_{\lambda_{j+1}})$, or

$$G \simeq E_{\lambda_{j+1}+1,\lambda_j}$$

generated by y and t satisfying

$$ay = \lambda_j by, \quad at = \lambda_{j+1}bt + y,$$

which also has another set of generators: t and $x := y + (\lambda_{j+1} - \lambda_j + 1)bt$, which satisfy

$$ax = (\lambda_j + 1)bx, \quad at = (\lambda_j - 1)bt + x.$$

In this case we take $F'_j = \pi^{-1}(\langle x\rangle)$. ∎

LEMMA 3.5. *Let E be a regular hermitian (a,b)-module. Suppose there is a unique normal elementary sub-(a,b)-module $F_1 \simeq E_\lambda$ with $\lambda \in \mathbb{Z}$ (resp. $\lambda \in \mathbb{Z} + 1/2$), and every composition series that starts with F_1 contains at least one more elementary quotient E_μ with μ an integer (resp. an integer $+1/2$). Then there exists a normal sub-(a,b)-module F_{n-1} of rank $n-1$ such that $(E/F_{n-1})^{\vee *} \simeq F_1$ and F_{n-1}/F_1 is hermitian.*

Proof. Let F_1 be the elementary sub-(a,b)-module of the hypothesis and $\{F_i\}$ a J-H sequence beginning with F_1 and such that $E/F_{n-1} \simeq E_\mu$ with μ an integer (resp. an integer $+1/2$). We can find such a sequence by repeatedly using the previous lemma.

Consider the exact sequence

$$0 \to F_{n-1} \to E \to E/F_{n-1} \to 0$$

and the adjoint sequence

$$0 \to (E/F_{n-1})^{\vee *} \xrightarrow{i} E^{\vee *} \xrightarrow{\pi} F_{n-1}^{\vee *} \to 0.$$

The image of i is a normal elementary sub-(a,b)-module of $E^{\vee *}$ isomorphic to $E_{-\mu}$. But there is only one such sub-(a,b)-module: $\Phi(F_1)$. Thus $(F/F_{n-1})^{\vee *} \simeq F_1$.

By replacing $E^{\vee *}$ by E and $(E/F_{n-1})^{\vee *}$ by F_1 in the sequence we obtain

$$0 \to F_1 \xrightarrow{i} E \to F_{n-1}^{\vee *} \to 0,$$

which is exact and i is the inclusion of sub-(a,b)-modules, so $F_{n-1}^{\vee *} \simeq E/F_1$ or equivalently $F_{n-1} \simeq (E/F_1)^{\vee *}$. Note that the first isomorphism is given by Φ^{-1}, while the second by the restriction of Φ.

Consider the following sequence and its adjoint:

$$0 \to F_{n-1}/F_1 \to E/F_1 \to E/F_{n-1} \to 0,$$
$$0 \to (E/F_{n-1})^{\vee *} \to (E/F_1)^{\vee *} \to (F_{n-1}/F_1)^{\vee *} \to 0;$$

by replacing $(E/F_{n-1})^{\vee *}$ and $(E/F_1)^{\vee *}$ with F_1 and F_{n-1} we obtain

$$0 \to F_1 \xrightarrow{\varphi} F_{n-1} \xrightarrow{\pi} (F_{n-1}/F_1)^{\vee *} \to 0.$$

By the uniqueness of F_1, φ can only be (up to multiplication by a complex number) the inclusion $F_1 \subset F_{n-1}$, and hence $(F_{n-1}/F_1)^{\vee *} \simeq (F_{n-1}/F_1)$. Note that π is the restriction of Φ to F_{n-1}, so the isomorphism is hermitian. ∎

We can now prove a reduced version of Theorem 1.2.

THEOREM 3.6. *If E is a hermitian (a,b)-module, then it admits a self-adjoint Jordan–Hölder composition series.*

Proof. We use induction on the rank of the (a,b)-module. For rank 0 and 1 the assertion is obvious.

Suppose we have proved the assertion for every rank $< n$. Find $F_1 \subset F_{n-1}$ of rank 1 and $n-1$ such that $(E/F_{n-1})^{\vee *} \simeq F_1$ and F_{n-1}/F_1 is hermitian. We present an exhaustive list of cases that might occur:

(i) We can find $G \simeq E_\lambda$, a normal elementary sub-(a,b)-module of E, with $2\lambda \notin \mathbb{Z}$. Then $\Phi(G)(G) = 0$ by Remark 3.2 and we can apply Lemma 3.1.

We still need to prove the induction step for (a,b)-modules whose only normal elementary sub-(a,b)-modules are isomorphic to E_λ, with $2\lambda \in \mathbb{Z}$.

(ii) There are two distinct normal elementary sub-(a,b)-modules isomorphic to E_n and E_m, where m, n are integers or half-integers. We apply Lemmas 3.3 and 3.1.

The (a,b)-modules that were not included in the previous items have a unique normal elementary sub-(a,b)-module isomorphic to E_m, where m is an integer or a half-integer.

(iii) There is only one normal elementary sub-(a,b)-module of parameter λ mod $\mathbb{Z}$, where $\lambda = 0$ or $1/2$, but at least two quotients of a J-H sequence are of parameter λ mod $\mathbb{Z}$. We apply Lemma 3.5.

Only modules of rank at most 2 (one for each possible value of λ) still need to be checked.

(iv) The rank of E is 2 and one quotient of a J-H sequence is 0 mod $\mathbb{Z}$, and the other $1/2$ mod $\mathbb{Z}$. By the classification of rank 2 modules this case is impossible. In fact, with the notation of [Bar93],

$$(E_\lambda \oplus E_\mu)^{\vee *} \simeq E_{-\lambda} \oplus E_{-\mu}, \qquad E_{\lambda,\mu}^{\vee *} \simeq E_{1-\lambda,1-\mu},$$

so if $\lambda = 0 \bmod \mathbb{Z}$ and $\mu = 1/2 \bmod \mathbb{Z}$, the (a,b)-module is not self-adjoint.

By induction hypothesis, F_{n-1}/F_1 has a J-H composition series that satisfies the conclusion, and by taking the inverse image by the canonical morphism $F_{n-1} \to F_{n-1}/F_1$ and adding 0 and E we find the desired J-H sequence of E. ∎

Since for an anti-hermitian form A we have $A(e,e) = 0$ for every $e \in E$, by using an anti-hermitian version of Lemma 3.1 alone and proceeding by induction, we can prove Theorem 1.2 in the anti-hermitian case.

We now wish to extend the result to all regular self-adjoint (a, b)-modules. We have proven in [Kar13] that every regular (a, b)-module E can be decomposed into a direct sum of hermitian or anti-hermitian (a, b)-modules.

Proof of Theorem 1.2. Decompose E into

$$E = \bigoplus_{i=1}^{m} H_i$$

where m is an integer, while the H_i are either indecomposable self-adjoint or of the form $G \oplus G^{\vee *}$, where G is an indecomposable non-self-adjoint (a, b)-module.

Each term of this sum admits a self-adjoint composition series. In fact if H_i is indecomposable self-adjoint, then it is hermitian or anti-hermitian. We can therefore apply Theorem 3.6.

On the other hand, if $H_i = G \oplus G^{\vee *}$, we can easily find a self-adjoint Jordan–Hölder composition series. Take in fact any Jordan–Hölder series of G,

$$0 = G_0 \subsetneq \cdots \subsetneq G_n = G,$$

and consider the adjoint series

$$0 = (G/G_n)^{\vee *} \subsetneq (G/G_{n-1})^{\vee *} \subsetneq \cdots \subsetneq (G/G_0)^{\vee *} = G^{\vee *}.$$

Then the following composition series of $G \oplus G^{\vee *}$ is self-adjoint:

$$\begin{aligned} 0 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G = G \oplus (G/G_n)^{\vee *} \subsetneq G \oplus (G/G_{n-1})^{\vee *} \\ \subsetneq \cdots \subsetneq G \oplus (G/G_0)^{\vee *} = G \oplus G^{\vee *}. \end{aligned}$$

We will now prove the theorem by induction on m. The case $m = 1$ was already proven.

Suppose now $m \geq 2$ and let $E' := H_1$ and $F := \sum_{i=2}^{m} H_i$. Then $E = E' \oplus F$, and E' and F are both self-adjoint. By the remark above we can find a self-adjoint composition series of E':

$$0 = E'_0 \subsetneq \cdots \subsetneq E'_r = E',$$

while by induction hypothesis, we can find a self-adjoint composition series of F:

$$0 = F_0 \subsetneq \cdots \subsetneq F_s = F.$$

Then the following composition series is self-adjoint:

$$\begin{aligned} 0 = E'_0 \subsetneq E'_1 \subsetneq \cdots \subsetneq E'_{[r/2]} \subsetneq E'_{[r/2]} \oplus F_1 \subsetneq \cdots \subsetneq E'_{[r/2]} \oplus F_{[s/2]} [\cdots] \\ E'_{[(r+1)/2]} \oplus F_{[(s+1)/2]} \subsetneq E'_{[(r+1)/2]} \oplus F_{[(s+1)/2]+1} \subsetneq \cdots \subsetneq E'_{[(r+1)/2]} \oplus F \\ \subsetneq E'_{[(r+1)/2]+1} \oplus F \subsetneq \cdots \subsetneq E' \oplus F, \end{aligned}$$

where depending on the parity of r and s, $[\cdots]$ stands for

(i) the $=$ sign if r and s are both even;
(ii) the $\subsetneq$ sign if one is even and the other odd;
(iii) the subsequence $\subsetneq E'_{[r/2]} \oplus F_{[(s+1)/2]} \subsetneq$ if r and s are both odd.

The last case needs a short verification. If r and s are odd, then the two central quotients of the series are isomorphic to $E'_{[(r+1)/2]}/E'_{[r/2]}$ and $F_{[(s+1)/2]}/F_{[s/2]}$. Since E'_i and F_i are self-adjoint, both quotients are self-adjoint (a,b)-modules of rank 1. They are therefore isomorphic to E_0. ∎

Acknowledgements. Partially funded by grant NCN 2013/10/E/ST1/00688 "Rozmaitości o trywialnej pierwszej klasie Cherna".

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