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Existence and nonexistence results for quasilinear Schrödinger equations with a general nonlinear term

YAN-FANG XUE (Chongqing and Henan), YING LV (Chongqing) and CHUN-LEI TANG (Chongqing)

Abstract. We study the quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = g(u), \qquad x \in \mathbb{R}^N,$$

where $V(x)$ tends to zero as $|x| \to \infty$ and $g(u)$ satisfies the general hypotheses introduced by Berestycki and Lions. We employ the mountain pass theorem to obtain the existence of a positive ground state solution. Moreover, we prove a nonexistence result by using the Pohozaev manifold.

1. Introduction and main result. In the present paper, we are concerned with the existence and nonexistence results for the quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = g(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $V(x)$ tends to zero as $|x| \to \infty$ and $g \in C(\mathbb{R}, \mathbb{R})$. This problem is related to the standing wave solutions for the quasilinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + W(x)\psi - l(x, |\psi|^2)\psi - \kappa[\Delta\rho(|\psi|^2)]\rho'(|\psi|^2)\psi, \tag{1.2}$$

where $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, κ is a positive constant and l, ρ are real functions. Problem (1.2) appears in different physical models; see [PSW], [LW1] and [LWW1] for an explanation.

Problem (1.1) has been studied by many authors. The classical results [PSW], [LW1], [LWW1] concern problem (1.1) with various potentials, including radially symmetric, coercive and periodic ones. Since then, many

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results for problem (1.1) have appeared, depending on different assumptions on V. See, for example, [W], [AW] for coercive potentials, [ZZZ] for radially symmetric potentials, [AS] for vanishing potentials, [SV1], [SV2], [SC], [FS] for periodic or asymptotically periodic potentials, and [JLW], [MMP], [MM] for constant positive potentials. The steep potential well case was considered in [GT]. The greatest part of the literature focuses on the study of problem (1.1) when V is assumed to be a potential well (see [LWW2], [DMS], [HQZ], [DS1], [LLW3], [LW2], [WYZ], [RS]).

More precisely, in [PSW] and [LW1], by using a constrained minimization argument, a positive ground state solution has been proved for problem (1.1) with $g(u) = \lambda|u|^{q-1}u$, $4 \le q+1 < 2\cdot 2^*$, where $2^* = 2N/(N-2)$ is the Sobolev critical exponent. Then by a change of variables, the quasilinear problem is transformed into a semilinear one (see [LWW1] for an Orlicz space framework and [CJ] for a Sobolev space case). In [LWW2], by utilizing the Nehari method, Liu et al. obtained positive and sign-changing solutions. Recently, a perturbation method was developed in [LLW2] to deal with problem (1.1), which can also be applied to more general quasilinear Schrödinger equations (see [LW2], [LLW1]).

Up to our knowledge, there are few results on problem (1.1) where the general hypotheses of Berestycki and Lions [BL1], [BL2] are assumed on the nonlinearity g. Colin and Jeanjean [CJ] studied the existence of a solution for problem (1.1) with $V(x) \equiv 0$, that is, the autonomous problem

$$-\Delta u - \Delta(u^2)u = g(u), \qquad x \in \mathbb{R}^N. \tag{1.3}$$

These authors show that if:

(g_0) $g(s)$ is locally Hölder continuous on $[0,\infty)$,
(g_1) $-\infty < \liminf_{s\to 0^+} g(s)/s \le \limsup_{s\to 0^+} g(s)/s = -m < 0$,
(g_2') $\limsup_{s\to\infty} |g(s)|/s^{2\cdot 2^*-1} = 0$,
(g_3) there exists $\zeta_0 > 0$ such that $G(\zeta_0) = \int_0^{\zeta_0} g(s)\,ds > 0$,

then problem (1.3) admits a positive radially symmetric solution $u_0 \in H^1(\mathbb{R}^N)$.

We have recently learned that Adachi et al. [ASW] have obtained some existence results for the equation

$$-\Delta u - \kappa\Delta(|u|^\alpha)|u|^{\alpha-2}u = g(u), \qquad x \in \mathbb{R}^N, \tag{1.4}$$

where $N \ge 3$, $\alpha > 1$, $\kappa > 0$ is a parameter and g satisfies (g_0), (g_1), (g_3) and

(g_2^*) $\lim_{s\to\infty} g(s)/s^p = 1$ for some $\frac{N+2}{N-2} < p < \frac{(2\alpha-1)N+2}{N-2}$.

As far as we know, there are no other results concerning (1.1) where exactly the same general hypotheses of Berestycki and Lions [BL1], [BL2] are assumed on g. Theorems 1.1 and 1.6 below are this kind of results.

Colin and Jeanjean [CJ] observed that equation (1.3) is formally the Euler–Lagrange equation associated with the energy functional

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} [(1+2u^2)|\nabla u|^2]\,dx - \int_{\mathbb{R}^N} G(u)\,dx.$$

They made a change of variables $v := f^{-1}(u)$, where f is defined by

$$(1.5)\quad f'(t) = \frac{1}{(1+2f^2(t))^{1/2}},\ t\in[0,\infty);\quad f(t) = -f(-t),\ t\in(-\infty,0].$$

This yields the functional

$$I_0(v) := J_0(f(v)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \int_{\mathbb{R}^N} G(f(v))\,dx.$$

Moreover, they observed that if v is a critical point of I_0, then the function $u = f(v)$ is a solution of problem (1.3). In fact, the critical points of I_0 are weak solutions of the problem

$$-\Delta v = g(f(v))f'(v).$$

Let $l(v) = g(f(v))f'(v)$. Then the above equation becomes a standard semilinear equation

$$-\Delta v = l(v).$$

Colin and Jeanjean proved that $l(v)$ satisfies all the conditions of [BL1, Theorem 1], hence the existence of a critical point follows almost directly from classical results on scalar field equations due to Berestycki and Lions [BL1] when $N\ge 3$. Using a method similar to the one in [CJ], Adachi et al. [ASW] obtained a ground state solution for (1.4).

In the present paper, we suppose that V satisfies the following assumptions:

(V_1) $V\in C^1(\mathbb{R}^N,\mathbb{R})$, $V(x)\ge 0$ for all $x\in\mathbb{R}^N$, and the inequality is strict somewhere.

(V_2) $\|(\nabla V(\cdot),\cdot)^+\|_{N/2} < 2S$, where $(\nabla V(x),x)^+ = \max\{(\nabla V(x),x),0\}$ and S is the best Sobolev constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N)\hookrightarrow L^{2^*}(\mathbb{R}^N)$, that is,

$$S = \inf_{u\in\mathcal{D}^{1,2}\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2},\qquad 2^* = \frac{2N}{N-2}.$$

(V_3) $\lim_{|x|\to\infty} V(x) = 0$.

(V_4) V is radially symmetric.

Now we state our main existence result:

Theorem 1.1. *Suppose that* (V_1)–(V_4) *hold, and the function* $g\in C(\mathbb{R},\mathbb{R})$ *satisfies* (g_1), (g_3) *and*

(g_2) $-\infty < \limsup_{s\to\infty} g(s)/s^{2\cdot 2^*-1} \le 0$.

Then problem (1.1) *has a radial ground state solution.*

REMARK 1.2. Our condition (g_2) is weaker than (g_2'), and the hypotheses on g are almost necessary in the sense specified in [BL1, Subsection 2.2].

REMARK 1.3. When $N \geq 3$, our Theorem 1.1 extends [CJ, Theorem 1.1], which is the special case of our theorem with $V(x) \equiv 0$.

REMARK 1.4. The fact that the equation is autonomous plays a fundamental role in the proof of [CJ, Theorem 1.1]. In the present paper, problem (1.1) can be nonautonomous ($V(x) \geq 0$), so the arguments in [CJ] do not work any more. We have to find a new method; we make use of an idea introduced in [AP].

REMARK 1.5. The geometrical hypotheses on the potential V do not allow the use of concentration-compactness arguments as in [JT1]; as a consequence, we have to require a symmetry property on V to prevent any possible loss of mass at infinity. If (V_4) is absent, we have the following nonexistence result.

THEOREM 1.6. *Suppose that* (g_1)–(g_3) *hold, and* V *satisfies* (V_1), (V_3) *and*

(V_5) $(\nabla V(x), x) \leq 0$ *for all* $x \in \mathbb{R}^N$;
(V_6) $NV(x) + (\nabla V(x), x) \geq 0$ *for all* $x \in \mathbb{R}^N$, *and the inequality is strict somewhere.*

Then the infimum $p := \inf_{u \in \mathcal{P}} I(u)$ *is not achieved, where* $\mathcal{P}$ *is defined by* (1.9).

The proof of Theorem 1.1 takes some inspiration from the arguments in [CJ] and [AP]. Variational methods cannot be applied directly to find weak solutions of problem (1.1), since the natural associated functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} [(1 + 2u^2)|\nabla u|^2]\, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2\, dx - \int_{\mathbb{R}^N} G(u)\, dx$$

cannot be handled with a standard variational approach. Hence we employ an argument developed in [CJ]. We make a change of variables $v := f^{-1}(u)$, where f is defined in (1.5). Hence J is transformed into

$$I(v) := J(f(v)) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2\, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v)\, dx - \int_{\mathbb{R}^N} G(f(v))\, dx.$$

Then I is well defined on $H^1(\mathbb{R}^N)$ and $I \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ under the hypotheses (V_1), (V_3) and (g_1)–(g_3). From the properties of f, it is clear that to obtain a solution of (1.1) it suffices to obtain a critical point of I (see [CJ]). The critical points of I are weak solutions of the problem

$$-\Delta v + V(x) f(v) f'(v) = g(f(v)) f'(v). \tag{1.6}$$

Let $h(x,v) = g(f(v))f'(v) + V(x)v - V(x)f(v)f'(v)$. Equation (1.6) becomes a standard semilinear equation

$$-\Delta v + V(x)v = h(x,v). \tag{1.7}$$

Naturally, we hope to use the results of Azzollini and Pomponio [AP] to obtain a solution of problem (1.7). Unfortunately, the nonlinear term here is nonautonomous, and the nonlinear term in [AP] is autonomous, so we cannot directly use the conclusions in [AP].

Each solution of (1.6) satisfies the following Pohozaev identity:

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N} V(x)f^2(v)\,dx + \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x),x)f^2(v)\,dx = N\int_{\mathbb{R}^N} G(f(v))\,dx. \tag{1.8}$$

We define the Pohozaev manifold associated with (1.6) by

$$\mathcal{P} := \{v \in H^1(\mathbb{R}^N) : v \neq 0,\ v \text{ satisfies (1.8)}\}. \tag{1.9}$$

The existence of the radial ground state solution is proved with the help of $\mathcal{P}$. The Pohozaev manifold is very effective when the nonlinearity does not satisfy the Ambrosetti–Rabinowitz condition and the monotonicity condition (see [LM], [CLM], [JT2], [J], [AP]). Theorem 1.6 tells us that $p = \inf_{u\in\mathcal{P}} I(u)$ is not achieved under some conditions.

Notation. In this paper, we use the following notation:

- $H^1(\mathbb{R}^N)$ is the usual Hilbert space endowed with the norm
$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx.$$
- $H^1_r(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u \text{ is radial}\}$.
- $L^s(\mathbb{R}^N)$ is the usual Banach space endowed with the norm
$$\|u\|_s^s = \int_{\mathbb{R}^N} |u|^s\,dx, \quad \forall s \in [1,\infty).$$
- $\|u\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^N} |u(x)|$ denotes the usual norm in $L^\infty(\mathbb{R}^N)$.
- $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm
$$\|u\|^2_{\mathcal{D}^{1,2}} = \int_{\mathbb{R}^N} |\nabla u|^2\,dx.$$
- $B_r(y) := \{x \in \mathbb{R}^N : |x-y| < r\}$, $B_r := \{x \in \mathbb{R}^N : |x| < r\}$.
- $u^+ = \max\{u,0\}$, $u^- = \max\{-u,0\}$.
- $|\Omega|$ denotes the Lebesgue measure of the set Ω.
- $C, C_\delta, C_1, C_2, \ldots$ denote various positive constants whose exact value is inessential.

2. Some preliminary results. In this section, using an idea from [BL1], we modify the function g as follows. Set $s_0 := \min\{s \in [\zeta_0, \infty) : g(s) = 0\}$, and $s_0 = \infty$ if $g(s) \neq 0$ for any $s \geq \zeta_0$. Define $\tilde{g} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{g}(s) = \begin{cases} g(s) & \text{on } [0, s_0], \\ 0 & \text{on } \mathbb{R} \setminus [0, s_0]. \end{cases} \tag{2.1}$$

Observe that $\tilde{g}$ satisfies the same conditions as g. Furthermore, by the strong maximum principle, a positive solution of problem (1.1) with $\tilde{g}$ is also a positive solution of (1.1) with g. Indeed, a solution u of problem (1.1) with $\tilde{g}$ satisfies $0 \leq u \leq s_0$, whence $\tilde{g}(u) = g(u)$. So we can suppose that g is as defined in (2.1). With this modification, g satisfies (g_1), (g_3) and

(g_2'') $\lim_{s\to\infty} g(s)/|s|^{2\cdot 2^*-1} = 0$.

Moreover, for $s \geq 0$, we set

$$g_1(s) := (g(s) + ms)^+, \qquad g_2(s) := g_1(s) - g(s),$$

and $g_1(s) = g_2(s) = 0$ if $s < 0$. Then $g_1, g_2 \geq 0$ and

$$\lim_{s\to 0} \frac{g_1(s)}{s} = 0, \tag{2.2}$$

$$\lim_{s\to\infty} \frac{g_1(s)}{|s|^{2\cdot 2^*-1}} = 0, \tag{2.3}$$

$$g_2(s) \geq ms, \qquad \forall s \geq 0. \tag{2.4}$$

By (g_2'') and (2.3), we deduce

$$\lim_{s\to\infty} \frac{g_2(s)}{|s|^{2\cdot 2^*-1}} = 0. \tag{2.5}$$

From (2.2)–(2.5), we infer that for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$g_1(s) \leq C_\delta |s|^{2\cdot 2^*-1} + \delta g_2(s), \tag{2.6}$$

$$G_2(s) \geq \tfrac{1}{2} ms^2, \tag{2.7}$$

$$G_1(s) \leq \frac{C_\delta}{2\cdot 2^*} |s|^{2\cdot 2^*} + \delta G_2(s), \tag{2.8}$$

where $G_i(s) = \int_0^s g(t)\,dt$, $i = 1, 2$. The above inequalities will be used repeatedly in the following.

Below we summarize the properties of the function f of (1.5), which have been proved in [CJ] and [DS].

Lemma 2.1. *The function f has the following properties:*

(1) *f is uniquely defined, C^∞ and invertible;*
(2) *$|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;*
(3) *$|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;*
(4) *$f(t)/t \to 1$ as $t \to 0$;*

(5) $f(t)/\sqrt{t} \to 2^{1/4}$ *as* $t \to \infty$*;*
(6) $f(t)/2 \le tf'(t) \le f(t)$ *for all* $t > 0$*;*
(7) $|f(t)| \le 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$*;*
(8) $f'(t) \to 1$ *as* $t \to 0$*;*
(9) *there exists a positive constant* C *such that* $|f(t)| \ge C|t|$ *for* $|t| \le 1$ *and* $|f(t)| \ge C|t|^{1/2}$ *for* $|t| \ge 1$*.*

REMARK 2.2. By the definition of $g_1(s)$ and $g_2(s)$, we know that $g(s) = 0$ for all $s < 0$. Let v be a critical point of I. Taking $\varphi = -v^-$, one has

$$\int_{\mathbb{R}^N} |\nabla v^-|^2\,dx + \int_{\mathbb{R}^N} V(x)f(v)f'(v)(-v^-)\,dx = 0.$$

Since $f(v)(-v^-) \ge 0$, we get

$$\int_{\mathbb{R}^N} |\nabla v^-|^2\,dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{V(x)f(v)(-v^-)}{\sqrt{1+2f^2(v)}}\,dx = 0.$$

Hence $v^- = 0$ a.e. in $\mathbb{R}^N$ and $v = v^+ \ge 0$. As $u = f(v)$, we conclude that u is a nonnegative solution for problem (1.1).

In order to get a bounded (PS) sequence for the functional I_λ, we make use of the monotone method introduced by Jeanjean [J].

LEMMA 2.3. *Let* $(X, \|\cdot\|)$ *be a Banach space and* $L \subset \mathbb{R}^+$ *be an interval. Consider a family of* C^1 *functionals on* X *defined by*

$$I_\lambda(u) = A(u) - \lambda B(u), \qquad \forall \lambda \in L,$$

with B *nonnegative and either* $A(u) \to \infty$ *or* $B(u) \to \infty$ *as* $\|u\| \to \infty$*. For any* $\lambda \in L$*, set*

$$\Gamma_\lambda = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \ne \gamma(1),\ I_\lambda(\gamma(1)) < 0\}.$$

If for every $\lambda \in L$*, the set* Γ_λ *is nonempty and*

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(\gamma(0)), I_\lambda(\gamma(1))\},$$

then for almost every $\lambda \in L$*, there is a sequence* $\{v_n\} \subset X$ *such that*

(i) $\{v_n\}$ *is bounded;*
(ii) $I_\lambda(v_n) \to c_\lambda$*;*
(iii) $I'_\lambda(v_n) \to 0$ *in the dual* X^* *of* X*.*

We present a variant of the Strauss compactness lemma [BL1, Theorem A.1] which will be used later.

LEMMA 2.4. *Let* $P, Q : \mathbb{R} \to \mathbb{R}$ *be continuous functions satisfying*

$$\lim_{s \to \infty} \frac{P(s)}{Q(s)} = 0,$$

and let u_n, w and φ be measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))\varphi|\,dx < \infty, \qquad P(u_n(x)) \to w(x) \text{ a.e. in } \mathbb{R}^N.$$

Then $\|(P(u_n) - w)\varphi\|_{L^1(B)} \to 0$ for any bounded Borel set B. If we further assume that

$$\lim_{s\to 0} \frac{P(s)}{Q(s)} = 0, \qquad \lim_{|x|\to\infty} \sup_n |u_n(x)| = 0,$$

then $\|(P(u_n) - w)\varphi\|_{L^1(\mathbb{R}^N)} \to 0$.

3. Proof of Theorem 1.1. By the invariance of I under rotations, the principle of symmetric criticality can be used. Hence we will find a critical point of I in $H^1_r(\mathbb{R}^N)$ to obtain a solution of problem (1.1). Using an idea from [J], for almost all λ near 1, we search for bounded Palais–Smale sequences of the perturbed functionals

$$\begin{aligned} I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(u)\,dx + \int_{\mathbb{R}^N} G_2(f(u))\,dx \\ - \lambda \int_{\mathbb{R}^N} G_1(f(u))\,dx, \end{aligned}$$

by using Lemma 2.3. In our case, $X = H^1_r(\mathbb{R}^N)$ and

$$\begin{aligned} A(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(u)\,dx + \int_{\mathbb{R}^N} G_2(f(u))\,dx, \\ B(u) &= \int_{\mathbb{R}^N} G_1(f(u))\,dx. \end{aligned}$$

Obviously, $B(u)$ is nonnegative and $A(u) \to \infty$ as $\|u\| \to \infty$. Now, we just have to define a suitable interval L such that for every $\lambda \in L$, the set Γ_λ is nonempty and $c_\lambda > 0$. Notice that

$$G(\zeta_0) > 0 \text{ for a } \zeta_0 > 0 \;\Leftrightarrow\; \text{there is } \zeta > 0 \text{ such that } G(f(\zeta)) > 0.$$

This is the only place where hypothesis (g_3) is used. For $R > 1$, define

$$w_R(x) = \begin{cases} \zeta & \text{for } |x| \le R, \\ \zeta(R+1-r) & \text{for } r = |x| \in [R, R+1], \\ 0 & \text{for } |x| \ge R+1. \end{cases}$$

Thus $w_R \in H^1_r(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} G(f(w_R))\,dx \ge G(f(\zeta))|B_R| - \Big(\max_{0\le s\le\zeta} |G(f(s))|\Big)|B_{R+1} - B_R|.$$

Hence there exist constants $C_1, C_2 > 0$ such that

$$\int_{\mathbb{R}^N} G_1(f(w_R))\,dx - \int_{\mathbb{R}^N} G_2(f(w_R))\,dx = \int_{\mathbb{R}^N} G(f(w_R))\,dx \geq C_1R^N - C_2R^{N-1} > 0$$

for $R > 0$ large enough. Then there exist $w \in H_r^1(\mathbb{R}^N)$ and $0 < \bar{\delta} < 1$ such that

$$\bar{\delta}\int_{\mathbb{R}^N} G_1(f(w))\,dx - \int_{\mathbb{R}^N} G_2(f(w))\,dx > 0. \tag{3.1}$$

We define L as the interval $[\bar{\delta}, 1]$.

Lemma 3.1. *For all $\lambda \in L$, $\Gamma_\lambda \neq \emptyset$.*

Proof. Let $\theta > 0$ be sufficiently large and $\bar{w} = w(\cdot/\theta)$. Define $\gamma : [0,1] \to H_r^1(\mathbb{R}^N)$ by

$$\gamma(t) = \begin{cases} \bar{w}^t = \bar{w}(\cdot/t) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0. \end{cases}$$

Obviously, γ is a continuous path from 0 to $\bar{w}$. Moreover,

$$\begin{aligned} I_\lambda(\gamma(1)) = I_\lambda\biggl(w\Bigl(\frac{x}{\theta}\Bigr)\biggr) &= \frac{\theta^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla w|^2\,dx + \frac{\theta^N}{2}\int_{\mathbb{R}^N} V(\theta x)f^2(w)\,dx \\ &\quad + \theta^N\int_{\mathbb{R}^N} G_2(f(w))\,dx - \lambda\theta^N\int_{\mathbb{R}^N} G_1(f(w))\,dx \\ &\leq \frac{\theta^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla w|^2\,dx + \frac{\theta^N}{2}\int_{\mathbb{R}^N} V(\theta x)f^2(w)\,dx \\ &\quad - \theta^N\Bigl(\bar{\delta}\int_{\mathbb{R}^N} G_1(f(w))\,dx - \int_{\mathbb{R}^N} G_2(f(w))\,dx\Bigr). \end{aligned}$$

Using (3.1), (V_3) and the Lebesgue theorem, for a suitable choice of θ we conclude that $I_\lambda(\gamma(1)) < 0$, hence $\gamma \in \Gamma_\lambda$. ■

Lemma 3.2. *For all $\lambda \in L$, $c_\lambda > 0$.*

Proof. Thanks to Lemma 2.1(9), we can deduce that there is $C_3 > 0$ such that

$$f^2(t) \geq C_3(t^2 - |t|^{2^*}). \tag{3.2}$$

By (2.7), (2.8), (3.2), Lemma 2.1(3) and the Sobolev inequality, we have

$$
\begin{aligned}
I_\lambda(u) &\ge \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)f^2(u)\,dx+\int_{\mathbb{R}^N}G_2(f(u))\,dx\\
&\quad-\int_{\mathbb{R}^N}G_1(f(u))\,dx\\
&\ge \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)f^2(u)\,dx+(1-\delta)\frac{m}{2}\int_{\mathbb{R}^N}f^2(u)\,dx\\
&\quad-\frac{C_\delta}{2\cdot 2^*}\int_{\mathbb{R}^N}f^{2\cdot 2^*}(u)\,dx\\
&\ge \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx+(1-\delta)\frac{m}{2}\int_{\mathbb{R}^N}C_3(u^2-|u|^{2^*})\,dx\\
&\quad-\frac{C_\delta}{2\cdot 2^*}2^{2^*/2}\int_{\mathbb{R}^N}|u|^{2^*}\,dx\\
&\ge \min\Big\{\frac{1}{2},(1-\delta)\frac{m}{2}C_3\Big\}\|u\|^2\\
&\quad-\Big((1-\delta)\frac{m}{2}C_3+\frac{C_\delta}{2\cdot 2^*}2^{2^*/2}\Big)S^{-2^*/2}\|\nabla u\|_2^{2^*}\\
&\ge C_4\|u\|^2-C_5\|u\|^{2^*},
\end{aligned}
$$

for any $0<\delta<1$. Hence there is $\rho>0$ small enough such that $I_\lambda(u)>0$ whenever $\|u\|\le\rho$, $u\ne 0$. Moreover, for any $\|u\|=\rho$, one has $I_\lambda(u)\ge C_6>0$. Now fix $\lambda\in L$ and $\gamma\in\Gamma_\lambda$. Since $\gamma(0)=0$ and $I_\lambda(\gamma(1))<0$, certainly $\|\gamma(1)\|>\rho$. Thanks to the continuity of γ, we deduce that there is $t_\gamma\in(0,1)$ such that $\|\gamma(t_\gamma)\|=\rho$. Hence, for any $\lambda\in L$,

$$c_\lambda\ge\inf_{\gamma\in\Gamma_\lambda}I_\lambda(\gamma(t_\gamma))\ge C_6>0. \ \blacksquare$$

LEMMA 3.3. *Assume that $\{u_n\}$ is a bounded (PS) sequence of the functional I_λ for $\lambda\in L$. Then $\{u_n\}$ has a convergent subsequence.*

Proof. Since $\{u_n\}$ is a bounded (PS) sequence for I_λ, that is, $I_\lambda(u_n)$ is bounded and $I'_\lambda(u_n)\to 0$ in $(H_r^1(\mathbb{R}^N))'$, we infer that there exists u in $H_r^1(\mathbb{R}^N)$ such that $u_n\rightharpoonup u$ in $H_r^1(\mathbb{R}^N)$, $u_n\to u$ in $L^p(\mathbb{R}^N), 2<p<2^*$ and $u_n(x)\to u(x)$ a.e. in $\mathbb{R}^N$.

We apply Lemma 2.4 for $P(s)=g_i(f(s))f'(s)$, $i=1,2$, $Q(s)=|f(s)|^{2\cdot 2^*-1}$, $w=g_i(f(u))f'(u)$ and $\varphi\in C_0^\infty(\mathbb{R}^N)$. It follows from Lemma 2.1(2)(3), (2.3) and (2.5) that

$$\lim_{s\to\infty}\frac{P(s)}{Q(s)}=\lim_{s\to\infty}\frac{g_i(f(s))f'(s)}{|f(s)|^{2\cdot 2^*-1}}=0,$$

and

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))\varphi|\,dx = \sup_n \int_{\mathbb{R}^N} |f^{2\cdot 2^*-1}(u_n)\varphi|\,dx$$
$$\leq \sup_n \int_{\mathbb{R}^N} 2^{(2^*-1)/2}|u_n|^{2^*}|\varphi|\,dx$$
$$\leq 2^{(2^*-1)/2}\|\varphi\|_\infty \sup_n \int_{\mathbb{R}^N} |u_n|^{2^*}\,dx < \infty.$$

Hence we deduce that

$$\int_{\mathbb{R}^N} g_i(f(u_n))f'(u_n)\varphi \to \int_{\mathbb{R}^N} g_i(f(u))f'(u)\varphi.$$

As a consequence,

$$0 = \langle I'_\lambda(u_n), \varphi\rangle + o(1) = \langle I'_\lambda(u), \varphi\rangle,$$

and so

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} V(x)f(u)f'(u)u\,dx = \lambda \int_{\mathbb{R}^N} g_1(f(u))f'(u)u\,dx - \int_{\mathbb{R}^N} g_2(f(u))f'(u)u\,dx. \tag{3.3}$$

We apply Lemma 2.4 again for

$$P(s) = g_1(f(s))f'(s)s, \qquad Q(s) = f(s)f'(s)s + |f(s)|^{2\cdot 2^*-1}f'(s)s,$$
$$w = g_1(f(u))f'(u)u, \qquad \varphi = 1.$$

By the Strauss radial lemma,

$$\lim_{|x|\to\infty} \sup_n |u_n(x)| \leq \lim_{|x|\to\infty} \sup_n \frac{C\|u_n\|_{H_1}}{|x|^{(N-2)/2}} = 0.$$

Hence we can deduce that

$$\int_{\mathbb{R}^N} g_1(f(u_n))f'(u_n)u_n \to \int_{\mathbb{R}^N} g_1(f(u))f'(u)u. \tag{3.4}$$

By (2.4) and Lemma 2.1(6), we find that

$$g_2(f(u))f'(u)u \geq mf(u)f'(u)u \geq \frac{m}{2}f^2(u) \geq 0.$$

It follows from the Fatou lemma that

$$\int_{\mathbb{R}^N} g_2(f(u))f'(u)u \leq \liminf_n \int_{\mathbb{R}^N} g_2(f(u_n))f'(u_n)u_n. \tag{3.5}$$

Since $\langle I'_\lambda(u_n), u_n\rangle \to 0$, by (3.3)–(3.5) one has

$$\begin{aligned}
&\limsup_n \Big[\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^N} V(x) f(u_n) f'(u_n) u_n\,dx \Big] \\
&\quad = \limsup_n \Big[\lambda \int_{\mathbb{R}^N} g_1(f(u_n)) f'(u_n) u_n\,dx - \int_{\mathbb{R}^N} g_2(f(u_n)) f'(u_n) u_n\,dx \Big] \\
&\quad = \lambda \int_{\mathbb{R}^N} g_1(f(u)) f'(u) u\,dx - \liminf_n \int_{\mathbb{R}^N} g_2(f(u_n)) f'(u_n) u_n\,dx \\
&\quad \le \lambda \int_{\mathbb{R}^N} g_1(f(u)) f'(u) u\,dx - \int_{\mathbb{R}^N} g_2(f(u)) f'(u) u\,dx \\
&\quad = \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} V(x) f(u) f'(u) u\,dx.
\end{aligned}$$

By the Fatou lemma,

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx \le \liminf_n \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx,$$

$$\int_{\mathbb{R}^N} V(x) f(u) f'(u) u\,dx \le \liminf_n \int_{\mathbb{R}^N} V(x) f(u_n) f'(u_n) u_n\,dx.$$

These last three inequalities imply that, up to a subsequence,

$$\begin{aligned}
&\lim_n \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx = \int_{\mathbb{R}^N} |\nabla u|^2\,dx, \\
&\lim_n \int_{\mathbb{R}^N} V(x) f(u_n) f'(u_n) u_n\,dx = \int_{\mathbb{R}^N} V(x) f(u) f'(u) u\,dx.
\end{aligned} \tag{3.6}$$

Hence

$$\lim_n \int_{\mathbb{R}^N} g_2(f(u_n)) f'(u_n) u_n = \int_{\mathbb{R}^N} g_2(f(u)) f'(u) u.$$

From (2.4), we infer that there exists a positive and continuous function $k(s)$ such that

$$g_2(f(s)) f'(s) s = m f(s) f'(s) s + k(s).$$

Thanks to the Fatou lemma,

$$\int_{\mathbb{R}^N} k(u)\,dx \le \liminf_n \int_{\mathbb{R}^N} k(u_n)\,dx,$$

$$\int_{\mathbb{R}^N} f(u) f'(u) u\,dx \le \liminf_n \int_{\mathbb{R}^N} f(u_n) f'(u_n) u_n\,dx.$$

Hence

$$\lim_n \int_{\mathbb{R}^N} f(u_n) f'(u_n) u_n\,dx = \int_{\mathbb{R}^N} f(u) f'(u) u\,dx. \tag{3.7}$$

Set $w_n := u_n - u$. Then $w_n \rightharpoonup 0$ in $H_r^1(\mathbb{R}^N)$. Similar to the proof of (3.7), we can show that

$$\lim_n \int_{\mathbb{R}^N} f(w_n)f'(w_n)w_n\,dx = 0. \tag{3.8}$$

Since the embedding $H_r^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ $(2 < p < 2^*)$ is compact, one has

$$\lim_n \int_{\mathbb{R}^N} |w_n|^p\,dx = 0. \tag{3.9}$$

Let us show that

$$\lim_n \int_{\mathbb{R}^N} w_n^2\,dx = \lim_n \int_{\mathbb{R}^N} f(w_n)f'(w_n)w_n\,dx. \tag{3.10}$$

On the one hand, by Lemma 2.1(4)(8), for any $\epsilon > 0$ there is $\sigma > 0$ such that

$$\left|1 - \frac{f(s)}{s}f'(s)\right| < \epsilon, \qquad \forall |s| \le \sigma.$$

Thus

$$\begin{aligned}
\limsup_n &\int_{\{x\in\mathbb{R}^N:\,|w_n(x)|\le\sigma\}} |w_n^2 - f(w_n)f'(w_n)w_n|\,dx \\
&= \limsup_n \int_{\{x\in\mathbb{R}^N:\,|w_n(x)|\le\sigma\}} w_n^2\left|1 - \frac{f(w_n)}{w_n}f'(w_n)\right|dx \\
&\le \epsilon \limsup_n \int_{\mathbb{R}^N} w_n^2 \to 0 \qquad (\text{as } \epsilon \to 0).
\end{aligned}$$

On the other hand, using Lemma 2.1(2)(3) and (3.9), we deduce that

$$\begin{aligned}
&\int_{\{x\in\mathbb{R}^N:\,|w_n(x)|>\sigma\}} |w_n^2 - f(w_n)f'(w_n)w_n|\,dx \\
&\quad\le \int_{\{x\in\mathbb{R}^N:\,|w_n(x)|>\sigma\}} w_n^2\,dx + \int_{\{x\in\mathbb{R}^N:\,|w_n(x)|>\sigma\}} |f(w_n)f'(w_n)w_n|\,dx \\
&\quad\le 2\int_{\{x\in\mathbb{R}^N:\,|w_n(x)|>\sigma\}} w_n^2\,dx \le \frac{2}{\sigma^{p-2}} \int_{\mathbb{R}^N} w_n^p\,dx \;\to 0 \qquad (\text{as } n \to \infty).
\end{aligned}$$

Therefore (3.10) holds. From (3.8) and (3.10), up to a subsequence, we have

$$\lim_n \int_{\mathbb{R}^N} w_n^2\,dx = \lim_n \int_{\mathbb{R}^N} (u_n - u)^2\,dx = 0.$$

Hence

$$\lim_n \int_{\mathbb{R}^N} u_n^2\,dx = \int_{\mathbb{R}^N} u^2\,dx.$$

Combining this with (3.6), we conclude that $u_n \to u$ strongly in $H_r^1(\mathbb{R}^N)$. ∎

LEMMA 3.4. *For almost every $\lambda \in L$, there is $u^\lambda \in H_r^1(\mathbb{R}^N)$, $u^\lambda \neq 0$, such that $I'_\lambda(u^\lambda) = 0$ and $I_\lambda(u^\lambda) = c_\lambda$.*

Proof. It follows from Lemmas 3.1, 3.2 and 2.3 that for almost every $\lambda \in L$, there exists a bounded sequence $\{u_n^\lambda\} \subset H_r^1(\mathbb{R}^N)$ such that $I_\lambda(u_n^\lambda) \to c_\lambda$ and $I'_\lambda(u_n^\lambda) \to 0$ in $(H_r^1(\mathbb{R}^N))'$. Up to a subsequence, by Lemma 3.3, there exists $u^\lambda \in H_r^1(\mathbb{R}^N)$ such that $u_n^\lambda \to u^\lambda$. By Lemma 3.2, $c_\lambda > 0$, hence $u^\lambda \neq 0$, and we get the conclusion. ∎

Proof of Theorem 1.1. The proof is in two steps.

STEP 1: *The existence of a radially symmetric solution.* By Lemma 3.4, we can find $\lambda_n \nearrow 1$ such that for any $n \geq 1$, there is $v_n \in H_r^1(\mathbb{R}^N)$, $v_n \neq 0$, satisfying $I_\lambda(v_n) = c_{\lambda_n}$ and $I'_{\lambda_n}(v_n) = 0$ in $(H_r^1(\mathbb{R}^N))'$, that is,

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v_n)\,dx + \int_{\mathbb{R}^N} G_2(f(v_n))\,dx - \lambda_n \int_{\mathbb{R}^N} G_1(f(v_n))\,dx = c_{\lambda_n}, \tag{3.11}$$

$$\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \int_{\mathbb{R}^N} V(x)f(v_n)f'(v_n)v_n\,dx + \int_{\mathbb{R}^N} g_2(f(v_n))f'(v_n)v_n\,dx - \lambda_n \int_{\mathbb{R}^N} g_1(f(v_n))f'(v_n)v_n\,dx = 0. \tag{3.12}$$

Next, we would like to prove that $\{v_n\}$ is a bounded (PS) sequence for I on the level $c := c_1$. Since $H_r^1(\mathbb{R}^N)$ is a natural constraint, it follows that v_n is a weak solution of the problem

$$-\Delta w + V(x)f(w)f'(w) + g_2(f(w))f'(w) - \lambda_n g_1(f(w))f'(w) = 0,$$

and it satisfies the Pohozaev equality

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N} V(x)f^2(v_n)\,dx + \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x), x)f^2(v_n)\,dx + N\int_{\mathbb{R}^N} G_2(f(v_n))\,dx - N\lambda_n \int_{\mathbb{R}^N} G_1(f(v_n))\,dx = 0. \tag{3.13}$$

It follows from (3.11) and (3.13) that

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx - \frac{1}{2N}\int_{\mathbb{R}^N} (\nabla V(x), x)f^2(v_n)\,dx = c_{\lambda_n}. \tag{3.14}$$

By Lemma 2.1(3), the Hölder inequality, (V_2) and the Sobolev inequality,

$$\begin{aligned}\int_{\mathbb{R}^N} (\nabla V(x), x) f^2(v_n)\,dx &\le \int_{\mathbb{R}^N} (\nabla V(x), x)^+ v_n^2\,dx \\ &\le \|(\nabla V(\cdot), \cdot)^+\|_{N/2} \Big(\int_{\mathbb{R}^N} v_n^{2N/(N-2)}\,dx\Big)^{(N-2)/N} \\ &< 2S\Big(\int_{\mathbb{R}^N} v_n^{2^*}\,dx\Big)^{(N-2)/N} < 2\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx.\end{aligned}$$

Hence there is a small constant $\beta > 0$ such that

$$\int_{\mathbb{R}^N} (\nabla V(x), x) f^2(v_n)\,dx \le (2-\beta)\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$c_{\lambda_n} \ge \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx - \frac{1}{2N}(2-\beta)\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx = \frac{\beta}{2N}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx.$$

Hence there exists a constant $C_7 > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx \le C_7 \quad \text{for all } n \ge 1. \tag{3.16}$$

By (3.12) and Lemma 2.1(6),

$$\begin{aligned}\int_{\mathbb{R}^N} g_2(f(v_n)) f'(v_n) v_n\,dx &- \lambda_n \int_{\mathbb{R}^N} g_1(f(v_n)) f'(v_n) v_n\,dx \\ &= -\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx - \int_{\mathbb{R}^N} V(x) f(v_n) f'(v_n) v_n\,dx \\ &\le -\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(v_n)\,dx \le 0.\end{aligned}$$

By (2.6), Lemma 2.1(6)(7) and the Sobolev inequality,

$$\begin{aligned}\int_{\mathbb{R}^N} g_2(f(v_n)) f'(v_n) v_n\,dx &\le \lambda_n \int_{\mathbb{R}^N} g_1(f(v_n)) f'(v_n) v_n\,dx \\ &\le \int_{\mathbb{R}^N} \big(C_\delta |f(v_n)|^{2\cdot 2^*-1} + \delta g_2(f(v_n))\big) f'(v_n) v_n\,dx \\ &\le C_\delta 2^{2^*/2} \int_{\mathbb{R}^N} |v_n|^{2^*}\,dx + \delta \int_{\mathbb{R}^N} g_2(f(v_n)) f'(v_n) v_n\,dx \\ &\le C_\delta (2/S)^{2^*/2} \Big(\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx\Big)^{2^*/2} + \delta\int_{\mathbb{R}^N} g_2(f(v_n)) f'(v_n) v_n\,dx.\end{aligned}$$

Therefore

$$(1-\delta)\int_{\mathbb{R}^N} g_2(f(v_n))f'(v_n)v_n\,dx \le C_\delta(2/S)^{2^*/2}\Big(\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx\Big)^{2^*/2}.$$

It follows from (3.16) that $\int_{\mathbb{R}^N} g_2(f(v_n))f'(v_n)v_n\,dx$ is bounded. By (2.4) and Lemma 2.1(6),

$$\int_{\mathbb{R}^N} g_2(f(v_n))f'(v_n)v_n\,dx \ge \int_{\mathbb{R}^N} mf(v_n)f'(v_n)v_n\,dx \ge \frac{m}{2}\int_{\mathbb{R}^N} f^2(v_n)\,dx.$$

Hence $\int_{\mathbb{R}^N} f^2(v_n)\,dx$ is bounded. Combining this with (3.16), we can show that $\int_{\mathbb{R}^N} v_n^2\,dx$ is bounded. In fact, by Lemma 2.1(9), we observe that

$$\begin{aligned}\int_{\{x\in\mathbb{R}^N:\,|v_n(x)|\le 1\}} v_n^2\,dx &\le \frac{1}{C^2}\int_{\{x\in\mathbb{R}^N:\,|v_n(x)|\le 1\}} f^2(v_n)\,dx\\ &\le \frac{1}{C^2}\int_{\mathbb{R}^N} f^2(v_n)\,dx < C_8.\end{aligned}$$

Moreover, by the Sobolev inequality and Lemma 2.1(9),

$$\begin{aligned}\int_{\{x\in\mathbb{R}^N:\,|v_n(x)|>1\}} v_n^2\,dx &\le \int_{\{x\in\mathbb{R}^N:\,|v_n(x)|>1\}} v_n^{2^*}\,dx\\ &\le C_8\Big(\int_{\{x\in\mathbb{R}^N:\,|v_n(x)|>1\}}|\nabla v_n|^2\,dx\Big)^{2^*/2}\\ &\le C_8\Big(\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx\Big)^{2^*/2} < C_9.\end{aligned}$$

Hence $\int_{\mathbb{R}^N} v_n^2\,dx$ is bounded.

Up to a subsequence, there exists $v\in H_r^1(\mathbb{R}^N)$ such that $v_n\rightharpoonup v$ in $H_r^1(\mathbb{R}^N)$. Similar to the proof of (3.4), we can show that for any $\varphi\in H_r^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} g_1(f(v_n))f'(v_n)\varphi \to \int_{\mathbb{R}^N} g_1(f(v))f'(v)\varphi.$$

By the Banach–Steinhaus theorem, $g_1(f(v_n))f'(v_n)$ is bounded in $(H_r^1(\mathbb{R}^N))'$. Since $\lambda_n\nearrow 1$ and $I'_{\lambda_n}(v_n)=0$ in $(H_r^1(\mathbb{R}^N))'$, we get

$$I'(v_n)=I'_{\lambda_n}(v_n)+(\lambda_n-1)g_1(f(v_n))f'(v_n)\to 0, \tag{3.17}$$

where we also use the boundedness of $g_1(f(v_n))f'(v_n)$. Moreover, from $I_\lambda(v_n)=c_{\lambda_n}$ and the boundedness of $\{v_n\}$, we deduce that

$$I(v_n)=I_{\lambda_n}(v_n)+(\lambda_n-1)\int_{\mathbb{R}^N} G_1(f(v_n))\,dx\to c. \tag{3.18}$$

It follows from (3.17) and (3.18) that $\{v_n\}$ is a (PS) sequence for the functional I. By Lemma 3.3, v is a nontrivial mountain pass type solution for problem (1.1). We can now use the strong maximum principle to get $v > 0$ by a standard argument.

STEP 2: *The existence of a radial ground state solution.* A radial ground state solution is a solution minimizing the action I among all the nontrivial radial solutions. Set

$$S_r := \{u \in H_r^1(\mathbb{R}^N) : u \neq 0,\, I'(u) = 0\}, \qquad c_r = \inf_{u \in S_r} I(u).$$

Let $\{u_n\} \subset S_r$ be such that $I(u_n) \to c_r$. Similar to the proof of Step 1, we find that the sequence $\{u_n\}$ is bounded. By Lemma 3.3, there is $u \in H_r^1(\mathbb{R}^N)$ such that $u_n \to u$ in $H_r^1(\mathbb{R}^N)$. We only need to prove that $c_r > 0$. In fact, by (2.6), Lemma 2.1(6)(7) and the Sobolev inequality, for any $u \in S_r$,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2\,dx &\le \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} V(x) f(u) f'(u) u\,dx \\ &\quad + (1-\delta)\int_{\mathbb{R}^N} g_2(f(u)) f'(u) u\,dx \\ &= \int_{\mathbb{R}^N} g_1(f(u)) f'(u) u\,dx - \delta \int_{\mathbb{R}^N} g_2(f(u)) f'(u) u\,dx \\ &\le C_\delta \int_{\mathbb{R}^N} |f(u)|^{2\cdot 2^*-1} f'(u) u\,dx \le C_\delta 2^{2^*/2} \int_{\mathbb{R}^N} u^{2^*}\,dx \\ &\le C_\delta (2/S)^{2^*/2} \Big(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\Big)^{2^*/2}.\end{aligned}$$

Hence there is a constant $C_{10} > 0$ such that

$$\inf_{u \in S_r} \int_{\mathbb{R}^N} |\nabla u|^2\,dx \ge C_{10} > 0.$$

Similar to the proof of (3.15), we can get

$$\int_{\mathbb{R}^N} (\nabla V(x), x) f^2(u)\,dx \le (2-\beta) \int_{\mathbb{R}^N} |\nabla u|^2\,dx.$$

Consequently, for any $u \in S_r \subset \mathcal{P}$,

$$\begin{aligned} I(u) &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2N}\int_{\mathbb{R}^N} (\nabla V(x), x) f^2(u)\,dx \\ &\ge \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2N}(2-\beta)\int_{\mathbb{R}^N} |\nabla u|^2\,dx \ge \beta C_{10} > 0.\end{aligned}$$

So we see that $c_r > 0$, which completes the proof. ∎

4. Proof of Theorem 1.6. As mentioned in Section 1, Colin and Jeanjean [CJ] proved the existence of a ground state solution for the problem

$$-\Delta u - (\Delta u^2)u = g(u),$$

where $g(u)$ satisfies the general hypotheses introduced by Berestycki and Lions. After the change of variables $v := f^{-1}(u)$, they obtain the semilinear equation

$$-\Delta v = g(f(v))f'(v). \tag{4.1}$$

Equation (4.1) is associated with the energy functional

$$I_0(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \int_{\mathbb{R}^N} G(f(v))\,dx.$$

Set

$$S_0 := \{v \in H^1(\mathbb{R}^N) : v \neq 0,\ I_0'(v) = 0\},$$
$$\mathcal{P}_0 := \left\{v \in H^1(\mathbb{R}^N) : v \neq 0,\ \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx = N\int_{\mathbb{R}^N} G(f(v))\,dx\right\}.$$

Using a similar method to [JT2], we can deduce

$$\inf_{u\in\mathcal{P}_0} I_0(u) = \inf_{u\in\mathcal{S}_0} I_0(u).$$

Notice that the Pohozaev manifold associated with problem (1.6) is

$$\mathcal{P} := \{v \in H^1(\mathbb{R}^N) : v \neq 0,\ v \text{ satisfies } (1.8)\}.$$

LEMMA 4.1. *For all $u \in \mathcal{P}$, $I(u) > 0$.*

Proof. It follows from (1.8) and (V_5) that, for any $u \in \mathcal{P}$,

$$I(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2N}\int_{\mathbb{R}^N} (\nabla V(x), x)f^2(u)\,dx > 0. \tag{4.2}$$ ■

LEMMA 4.2. *Suppose that $u \in H^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} G(f(u))\,dx > 0$. Then there is $t_u > 0$ such that*

$$u^{t_u} := u(\cdot/t_u) \in \mathcal{P}.$$

Proof. Define $\gamma(t) := I(u^t) = I(u(x/t))$. Then

$$\gamma(t) = \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{t^N}{2}\int_{\mathbb{R}^N} V(tx)f^2(u)\,dx - t^N\int_{\mathbb{R}^N} G(f(u))\,dx.$$

Hence we can show that $\gamma(t) > 0$ whenever $t > 0$ is small enough. It follows from (V_3) and the Lebesgue theorem that

$$\lim_{t\to\infty}\int_{\mathbb{R}^N} V(tx)f^2(u)\,dx = 0;$$

combining this with $\int_{\mathbb{R}^N} G(f(u))\,dx > 0$, we deduce

$$\lim_{t\to\infty} \gamma(t) = -\infty.$$

So there exists $t_u > 0$ such that $\gamma'(t_u) = 0$. Therefore $u^{t_u} = u(\cdot/t_u) \in \mathcal{P}$. ∎

LEMMA 4.3. *Let $u \in \mathcal{P}_0$, and for any $y \in \mathbb{R}^N$, set $u_y := u(\cdot - y) \in \mathcal{P}_0$. Choose $t_y > 0$ such that $\tilde{u}_y = u_y(\cdot/t_y) \in \mathcal{P}$. Then*

$$\lim_{|y|\to\infty} t_y = 1.$$

Proof. We claim that $\limsup_{|y|\to\infty} t_y < \infty$.

To reach a contradiction, suppose that there exists a sequence $y_n \in \mathbb{R}^N$ such that $t_{y_n} \to \infty$ as $|y_n| \to \infty$. Then

$$\begin{aligned} I(\tilde{u}_{y_n}) = \frac{t_{y_n}^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{t_{y_n}^{N}}{2} \int_{\mathbb{R}^N} V(t_{y_n}x) f^2(u(x - y_n/t_{y_n}))\,dx \\ - t_{y_n}^N \int_{\mathbb{R}^N} G(f(u))\,dx. \end{aligned}$$

By Lemma 2.1(3), we deduce that

$$\begin{aligned} &\int_{\mathbb{R}^N} V(t_{y_n}x) f^2(u(x - y_n/t_{y_n}))\,dx \\ &\le \int_{B_r} V(t_{y_n}x) u^2(x - y_n/t_{y_n})\,dx + \int_{B_r^c} V(t_{y_n}x) u^2(x - y_n/t_{y_n})\,dx \\ &\le \sup_{x\in\mathbb{R}^N} V(x) \int_{B_r(-y_n/t_{y_n})} u^2\,dx + \sup_{x\in B_r^c} V(t_{y_n}x) \int_{\mathbb{R}^N} u^2\,dx. \end{aligned} \tag{4.3}$$

By the absolute continuity of the Lebesgue integral, for any $\epsilon > 0$ there is $\tilde{r} > 0$ such that for any $r < \tilde{r}$,

$$\sup_{x\in\mathbb{R}^N} V(x) \int_{B_r(-y_n/t_{y_n})} u^2\,dx \le \epsilon. \tag{4.4}$$

Since we suppose that $t_{y_n} \to \infty$ as $|y_n| \to \infty$, by (4.3), (4.4) and (V_3) we have

$$\lim_{|y_n|\to\infty} \int_{\mathbb{R}^N} V(t_{y_n}x) f^2(u(x - y_n/t_{y_n}))\,dx = 0.$$

Hence $I(\tilde{u}_{y_n}) \to -\infty$ as $|y_n| \to \infty$. We get a contradiction with Lemma 4.1, so the claim holds.

Next, we prove that $\lim_{|y|\to\infty} t_y = 1$.

For every $u \in \mathcal{P}_0$,

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2\,dx = N \int_{\mathbb{R}^N} G(f(u))\,dx. \tag{4.5}$$

Since $\tilde{u}_y = u\big(\frac{\cdot - y}{t_y}\big) \in \mathcal{P}$, one has

$$\frac{N-2}{2} t_y^{N-2} \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{N}{2} t_y^N \int_{\mathbb{R}^N} V(t_y x + y) f^2(u)\,dx + \tfrac{1}{2} t_y^N \int_{\mathbb{R}^N} (\nabla V(t_y x + y), t_y x + y) f^2(u)\,dx = N t_y^N \int_{\mathbb{R}^N} G(f(u))\,dx. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$N(t_y^2 - 1) \int_{\mathbb{R}^N} G(f(u))\,dx = \frac{t_y^2}{2} \int_{\mathbb{R}^N} \big(NV(t_y x + y) + (\nabla V(t_y x + y), t_y x + y)\big) f^2(u)\,dx. \tag{4.7}$$

By using (V_3), (V_5), (V_6), the Lebesgue theorem and the claim above, we deduce that the right hand side in (4.7) goes to zero as $|y| \to \infty$. Hence $\lim_{|y|\to\infty} t_y = 1$. ∎

LEMMA 4.4. *Set* $p_0 := \inf_{u\in\mathcal{P}_0} I_0(u) = \inf_{u\in\mathcal{S}_0} I_0(u)$. *Then* $p \le p_0$.

Proof. Let $u \in H^1(\mathbb{R}^N)$ be a ground state solution of (4.1). Then $u \in \mathcal{P}_0$ and $I_0(u) = p_0$. By translation invariance, we have $u_y := u(\cdot - y) \in \mathcal{P}_0$ for any $y \in \mathbb{R}^N$. It follows from Lemma 4.2 that there is $t_y > 0$ such that $\tilde{u}_y = u_y(\cdot/t_y) \in \mathcal{P}$; hence

$$\begin{aligned}|I(\tilde{u}_y) - p_0| &= |I(\tilde{u}_y) - I_0(u_y)| \\ &\le \frac{|t_y^{N-2} - 1|}{2} \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{t_y^N}{2} \int_{\mathbb{R}^N} V(t_y x + y) f^2(u)\,dx \\ &\quad + |t_y^N - 1| \int_{\mathbb{R}^N} G(f(u))\,dx.\end{aligned}$$

By Lemma 4.3 and (V_3), one has $I(\tilde{u}_y) \to p_0$ as $|y| \to \infty$. Hence

$$p = \inf_{u\in\mathcal{P}} I(u) \le p_0. \ \blacksquare$$

LEMMA 4.5. *Assume that* $v \in H^1(\mathbb{R}^N)$ *and* $\int_{\mathbb{R}^N} G(f(v))\,dx > 0$. *Then there is* $t_v > 0$ *such that* $v^{t_v} = v(\cdot/t_v) \in \mathcal{P}_0$. *If* $v \in \mathcal{P}$, *then there exists* $t_v > 0$ *such that* $v(\cdot/t_v) \in \mathcal{P}_0$ *and* $t_v \le 1$.

Proof. (i) Since $\int_{\mathbb{R}^N} G(f(v))\,dx > 0$, for any $v \in H^1(\mathbb{R}^N)$ there is $t_v > 0$ such that

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2\,dx = N t_v^2 \int_{\mathbb{R}^N} G(f(v))\,dx. \tag{4.8}$$

Hence $v(\cdot/t_v) \in \mathcal{P}_0$.

(ii) If $v \in \mathcal{P}$, then v satisfies

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N} V(x)f^2(v)\,dx + \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x),x)f^2(v)\,dx = N\int_{\mathbb{R}^N} G(f(v))\,dx. \tag{4.9}$$

By using (V_6), we have $\int_{\mathbb{R}^N} G(f(v))\,dx > 0$, so we can use the conclusion of (i): there is $t_v > 0$ such that (4.8) holds. It follows from (4.8) and (4.9) that

$$\frac{1}{2}\int_{\mathbb{R}^N} [NV(x) + (\nabla V(x),x)]f^2(v)\,dx = N(1-t_v^2)\int_{\mathbb{R}^N} G(f(v))\,dx. \tag{4.10}$$

Combining this with (V_6), we obtain $t_v \le 1$. ∎

Proof of Theorem 1.6. For contradiction, suppose there is $v \in H^1(\mathbb{R}^N)$ such that $I(v) = p$ and $I'(v) = 0$. Then $v \in \mathcal{P}$. By Lemma 4.5, there exists $t_v > 0$ such that $v^t := v(\cdot/t_v) \in \mathcal{P}_0$ and $t_v \le 1$. By using the strong maximum principle, we can assume that $v > 0$. It follows from (V_6) and (4.10) that $0 < t_v < 1$. From (4.2) and (V_5), we deduce that

$$\begin{aligned} p = I(v) &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \frac{1}{2N}\int_{\mathbb{R}^N} (\nabla V(x),x)f^2(v)\,dx \\ &> \frac{t_v^{N-2}}{N}\int_{\mathbb{R}^N} |\nabla v|^2\,dx = I_0(v^t) \ge p_0, \end{aligned}$$

which contradicts Lemma 4.4. ∎

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Yan-Fang Xue
School of Mathematics and Statistics
Southwest University
400715 Chongqing, China
and
School of Mathematics and Statistics
Xinyang Normal University
464000 Henan, China
E-mail: xueyanfang2015@163.com

Ying Lv, Chun-Lei Tang (corresponding author)
School of Mathematics and Statistics
Southwest University
400715 Chongqing, China
E-mail: ly0904@swu.edu.cn
tangcl@swu.edu.cn