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MULTIPLICATIVE LOOPS OF TOPOLOGICAL QUASIFIELDS

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Abstract. Locally compact connected topological non-Desarguesian translation planes have been a popular subject for research in geometry since the seventies of the last century. These planes are coordinatized by locally compact quasifields $(Q,+,\cdot)$ such that the kernel of Q is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. In recent papers we determined the algebraic structure of the multiplicative loops $Q^* = (Q \setminus \{0\}, \cdot)$ of quasifields Q such that Q has dimension 2 over its kernel. Now we compare these cases and give a unified treatment of our results. In particular, we deal with multiplicative loops which either have a one-dimensional normal subloop or contain a compact subgroup.

1. Introduction. The first impulse to study non-associative structures came from the investigation of coordinate systems of non-Desarguesian planes. The translation planes are affine planes with a transitive group of translations. The translation planes are coordinatized by planar quasifields. The finite translation planes and the finite semifields are thoroughly studied in [D], [J], [GN1], [GN2]. The locally compact connected topological non-Desarguesian translation planes and the locally compact quasifields were fruitfully investigated in [Bo], [B1]–[B6], [G], [H1]–[H3], [K], [O], [PS], [S] including classifications.

In [F1] and [F2] we gave loop theoretical characterizations of the algebraic structure of the multiplicative loops $Q^* = (Q \setminus \{0\}, \cdot)$ of locally compact quasifields Q having dimension 2 over its kernel K_r. If $K_r = \mathbb{R}$, then $(Q,+)$ is the vector group $\mathbb{R}^2$ and the topological loop Q^* is homeomorphic to $\mathbb{R} \times S^1$. If $K_r = \mathbb{C}$, then $(Q,+)$ is the vector group $\mathbb{C}^2$ and the loop Q^* is homeomorphic to $\mathbb{R} \times S^3$. In this paper we compare these cases and give a unified treatment of our results.

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In Proposition 3.1 we determine the Lie groups G topologically generated by the left translations of the multiplicative loops Q^* of Q and the continuous sharply transitive sections σ belonging to the loops Q^*. If Q^* is a two-dimensional proper loop, then P. T. Nagy and K. Strambach showed that the group G is the connected component of $\mathrm{GL}_2(\mathbb{R})$ (cf. [NS1], Section 29, p. 345). In this case we give a new characterization for the functions parametrizing the continuous section σ. For $\dim Q = 4$ and $K_r = \mathbb{C}$ the group G is one of the following groups: $\mathrm{Spin}_3(\mathbb{R}) \times \mathbb{R}$, $\mathrm{Spin}_3(\mathbb{R}) \times \mathbb{C}$, $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{R}$, $\mathrm{GL}_2(\mathbb{C})$. We give a new detailed proof of this assertion.

In Section 3 we mostly deal with those properties of the two- and four-dimensional multiplicative loops which emphasize their common features. To do this we give the precise description of the connections between the special algebraic properties of these loops Q^* and the functions parametrizing the continuous section σ corresponding to Q^* (cf. Propositions 3.3, 3.6 and Theorem 3.4). We use here a slight modification of the notion of the decomposable multiplicative loop, the normal subloop decomposition of a multiplicative loop which is a central extension of a one-dimensional normal subgroup by a compact loop and the corresponding proofs comparing with those given in [F1] and [F2].

The last section is devoted to the differences between the two- and four-dimensional multiplicative loops Q^*. Although each locally compact two-dimensional nearfield is the field of complex numbers ([G], XI.12.2 Proposition, p. 348) there are proper Kalscheuer's nearfields of dimension four. Any locally compact two-dimensional semifield is the field of complex numbers (cf. [PS]). In contrast to this we describe a class of four-dimensional proper semifields having the field $\mathbb{C}$ as their kernel. We show that the group generated by all left and right translations of the multiplicative loops Q^* in this class is the group $\mathrm{SL}_4(\mathbb{R}) \times \mathbb{R}$. This fact differs from the two-dimensional proper multiplicative loops since they have infinite-dimensional group as the group generated by all translations (cf. [NS1], Theorem 29.1, p. 345).

2. Preliminaries. For basic facts on loops we refer to [NS1], Section 1. Here we collect the often used notions.

Definition 2.1. A set L with a binary operation $(x, y) \mapsto x \cdot y$ is called a *loop* if there exists an element $1 \in L$ such that $x = 1 \cdot x = x \cdot 1$ holds for all $x \in L$ and for any given $a, b \in L$ the equations $a \cdot y = b$ and $x \cdot a = b$ have unique solutions which are denoted by $y = a \backslash b$ and $x = b/a$. A loop L is *proper* if it is not a group. The kernel of a homomorphism $\alpha : L_1 \to L_2$ from a loop L_1 to a loop L_2 is a *normal* subloop N of L_1, i.e. a subloop of L_1 such that

$$x \cdot N = N \cdot x, \quad (x \cdot N) \cdot y = x \cdot (N \cdot y), \quad (N \cdot x) \cdot y = N \cdot (x \cdot y) \tag{1}$$

hold for all $x, y \in L_1$. The *centre* $Z(L)$ of a loop L consists of all elements z which satisfy the equations $zx \cdot y = z \cdot xy$, $xy \cdot z = x \cdot yz$, $xz \cdot y = x \cdot zy$, $xz = zx$ for all $x, y \in L$. If L has a central subgroup L_1 and the factor loop L/L_1 is isomorphic to the loop L_2 then L is called the *central extension* of L_1 by L_2. A loop L is called *topological*, if it is a topological space and the binary operations $(a, b) \mapsto a \cdot b$, $(a, b) \mapsto b/a$, $(a, b) \mapsto a \backslash b : L \times L \to L$ are continuous.

For all $a \in L$, the left translations $\lambda_a : L \to L$, $x \mapsto a \cdot x$, as well as the right translations $\rho_a : L \to L$, $x \mapsto x \cdot a$, are bijections of L. For topological loops the left and right translations of L are homeomorphisms of L. Every topological connected loop L having a Lie group G as the group topologically generated by the left translations of L corresponds to a sharply transitive continuous section $\sigma : G/H \to G$, where $G/H = \{xH \,|\, x \in G\}$ consists of the left cosets of the stabilizer H of $1 \in L$ such that $\sigma(H) = 1_G$ and $\sigma(G/H)$ generates G. The section σ is *sharply transitive* if the image $\sigma(G/H)$ acts sharply transitively on G/H, which means that to any xH, yH there exists precisely one $z \in \sigma(G/H)$ with $zxH = yH$.

DEFINITION 2.2. A set Q with two binary operations $+, \cdot : Q \times Q \to Q$ is called a (left) *quasifield* if $(Q,+)$ is an abelian group with neutral element 0, $(Q \setminus \{0\}, \cdot)$ is a loop, $0 \cdot x = x \cdot 0 = 0$, and between these operations the (left) distributive law $x \cdot (y+z) = x \cdot y + x \cdot z$ holds. A quasifield is proper if it is not a (skew) field. The *kernel* K_r of a (left) quasifield Q is a skew field defined by $(x+y) \cdot k = x \cdot k + y \cdot k$ and $(x \cdot y) \cdot k = x \cdot (y \cdot k)$ for all $x, y \in Q$, $k \in K_r$. The *centre* Z of Q is the set $\{z \in K_r \,|\, z \cdot x = x \cdot z \text{ for all } x \in Q\}$. A left quasifield Q is a *semifield*, if in Q also the right distributive law holds. A *nearfield* is a quasifield with associative multiplication. A locally compact connected topological quasifield is a locally compact connected topological space Q such that $(Q,+)$ is a topological group, $(Q \setminus \{0\}, \cdot)$ is a topological loop, the multiplication $\cdot : Q \times Q \to Q$ is continuous and the mappings $\lambda_a : x \mapsto a \cdot x$ and $\rho_a : x \mapsto x \cdot a$ with $0 \neq a \in Q$ are homeomorphisms of Q.

The (left) quasifield Q is a right vector space over K_r and for all $a \in Q$ the map $\lambda_a : Q \to Q$, $x \mapsto a \cdot x$, is K_r-linear. Every locally compact connected nearfield is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or to the nearfield $\mathbb{H}_r = (\mathbb{H}, +, \circ)$ obtained from the skew field $(\mathbb{H}, +, \cdot)$ of quaternions by using the new multiplication $x \circ y = x \cdot \varphi(x)^{-1} \cdot y \cdot \varphi(x)$, where $\varphi(x) = \exp(ir \log |x|)$, for some $r \in \mathbb{R}$. The nearfields $\mathbb{H}_r$, $r \neq 0$, are called proper Kalscheuer's nearfields (cf. [S], 64.19, 64.20, p. 363).

3. Sections corresponding to the multiplicative loops Q^* of Q. Here a locally compact connected topological quasifield Q having dimension 2 over its kernel K_r is treated. Assume that $B = \{e_1, e_2\}$ is a fixed basis of Q as right vector space over K_r with the scalar multiplication induced by K_r^*, where $e_1 \in K_r$ is the identity element of the multiplicative loop Q^* of Q. Hence Q is the vector space of pairs $(x,y)^t \in K_r^2$, K_r is the subspace of pairs $(x,0)^t$ and $(1,0)^t$ is the identity element of Q^*. Since the set Λ_Q of all left translations of Q is a spread set of the vector space Q (cf. Proposition 1.14 in [K], p. 12) the set Λ_Q is a set of (2×2)-matrices such that for any $(\alpha, \gamma)^t \in K_r^2$ there exists a unique matrix of Λ_Q having $(\alpha, \gamma)^t$ as its first column (cf. [F1], p. 2595).

PROPOSITION 3.1. *Let Q^* be the multiplicative loop for a locally compact connected topological proper quasifield Q having dimension two over its kernel K_r. If $K_r = \mathbb{R}$, then the group G topologically generated by the left translations of Q^* is the connected component* $\mathrm{GL}_2^+(\mathbb{R})$ *of the group* $\mathrm{GL}_2(\mathbb{R})$. *If $K_r = \mathbb{C}$, then the group G is one of the following Lie groups*: $\mathrm{Spin}_3(\mathbb{R}) \times \mathbb{R}$, $\mathrm{Spin}_3(\mathbb{R}) \times \mathbb{C}$, $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{R}$, $\mathrm{GL}_2(\mathbb{C})$. *If $G = \mathrm{Spin}_3(\mathbb{R}) \times \mathbb{R}$, then Q is a proper Kalscheuer's nearfield. If Q^* is proper, then it corresponds to a continuous* *sharply transitive section of the form* $\sigma : G/H_{(k,l,s)} \to G$:

$$\begin{pmatrix} ux & -u\overline{y} \\ uy & u\overline{x} \end{pmatrix} H \mapsto \begin{pmatrix} ux & -u\overline{y} \\ uy & u\overline{x} \end{pmatrix} \begin{pmatrix} a(u,x,y) & b(u,x,y) \\ 0 & a^{-1}(u,x,y)e^{ic(u,x,y)} \end{pmatrix} = M_{(u,x,y)} \tag{2}$$

such that $u > 0$, $(x,y) \in \mathbb{C}^2$, $x\overline{x} + y\overline{y} = 1$ *and* $a(u,x,y)$, $b(u,x,y)$, $c(u,x,y)$ *are continuous functions with positive, complex, real values, respectively. If* G *is* $\mathrm{GL}_2(\mathbb{C})$, *then* $a(1,1,0) = 1$, $b(1,1,0) = 0 = c(1,1,0)$. *If* G *is* $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{R}$, *then* $a(1,1,0) = 1$, $b(1,1,0) = 0$, *and* $c(u,x,y)$ *is the constant function* 0. *If* G *is* $\mathrm{Spin}_3(\mathbb{R}) \times \mathbb{C}$, *then* $a(u,x,y)$ *is the constant function* 1, $b(u,x,y)$ *is the constant function* 0 *and* $c(1,1,0) = 0$. *If* G *is* $\mathrm{GL}_2^+(\mathbb{R})$, *then* $x = \cos t$, $y = -\sin t$, $t \in [0, 2\pi)$, $a(u,t) > 0$, $b(u,t) \in \mathbb{R}$, $c(u,t)$ *is the constant function* 0 *with* $a(1,0) = 1$, $b(1,0) = 0$.

Proof. The first assertion was proved in [NS1], Theorem 29.1, p. 345. If $K_r = \mathbb{C}$ then the group G is a connected closed subgroup of $\mathrm{GL}_2(\mathbb{C})$. As the loop Q^* is homeomorphic to $S^3 \times \mathbb{R}$ the group G operates transitively on the sphere S^3 of oriented lines through 0 in $\mathbb{R}^4$ and it has a four-dimensional subgroup $\mathrm{Spin}_3(\mathbb{R}) \times \mathbb{R}$. According to [V], p. 24, the group G is the product ST such that S is conjugate to the group $\mathrm{SL}_2(\mathbb{C})$ or to $SU_2(\mathbb{C}) = \mathrm{Spin}_3(\mathbb{R})$ and T is a Lie subgroup of the centralizer of S in $\mathrm{GL}_4(\mathbb{R})$. Since $SU_2(\mathbb{C})$ is a maximal compact subgroup of $\mathrm{SL}_2(\mathbb{C})$ the group T is a connected subgroup of the group $\left\{ \left(\begin{smallmatrix} z & 0 \\ 0 & \overline{z} \end{smallmatrix} \right) \mid z \in \mathbb{C} \setminus \{0\} \right\}$ and the group G is one of the groups listed in the assertion. In these cases $\dim Q^* = 4$. If G is the group $\mathrm{Spin}_3(\mathbb{R}) \times \mathbb{R}$, then Q^* is isomorphic to G and hence Q is a proper Kalscheuer's nearfield. If Q^* is proper, then we may assume that the stabilizer H of the identity of Q^* is the subgroup

$$H_{(k,l,s)} = \left\{ \begin{pmatrix} k & l \\ 0 & k^{-1}e^{is} \end{pmatrix} \;\middle|\; k > 0,\ l \in \mathbb{C},\ s \in \mathbb{R} \right\} \tag{3}$$

if G is $\mathrm{GL}_2(\mathbb{C})$, the subgroup $H_{(k,l,0)}$ if G is $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{R}$, the subgroup $H_{(1,0,s)}$ if G is $\mathrm{Spin}_3(\mathbb{R}) \times \mathbb{C}$. The elements g of G have a unique decomposition as the product

$$g = \begin{pmatrix} ux & -u\overline{y} \\ uy & u\overline{x} \end{pmatrix} h \quad \text{with } x, y \in \mathbb{C},\ x\overline{x} + y\overline{y} = 1,\ u > 0,\ h \in H_{(k,l,s)}. \tag{4}$$

In particular if G is the group $\mathrm{GL}_2^+(\mathbb{R})$, then $\dim Q^* = 2$ and the stabilizer H of the identity of Q^* may be chosen as the subgroup (3) with $k > 0$, $l \in \mathbb{R}$, $s = 0$. The elements g of $G = \mathrm{GL}_2^+(\mathbb{R})$ can be written uniquely as the product (4) such that $x = \cos t$, $y = -\sin t$, $t \in [0, 2\pi)$. Hence a continuous section corresponding to a loop Q^* has form (2) satisfying the properties as in the assertion. ■

Description for the functions a, b of the section σ for a multiplicative loop Q^* with $\dim Q^* = 2$: Using the sharply transitivity property we characterized in [F2] the continuous functions a, b, c defining the section σ belonging to a four-dimensional multiplicative loop Q^*. Now we give a similar description for a two-dimensional multiplicative loop Q^*. The section σ corresponding to Q^* is sharply transitive precisely if for all pairs (u_1, t_1),

(u_2, t_2) in $\mathbb{R}_{>0} \times [0, 2\pi)$ there is precisely one $(u, t) \in \mathbb{R}_{>0} \times [0, 2\pi)$ and $k > 0$, $l \in \mathbb{R}$ such that

$$\begin{pmatrix} u\cos t & u\sin t \\ -u\sin t & u\cos t \end{pmatrix} \begin{pmatrix} a(u,t) & b(u,t) \\ 0 & a^{-1}(u,t) \end{pmatrix} \begin{pmatrix} u_1\cos t_1 & u_1\sin t_1 \\ -u_1\sin t_1 & u_1\cos t_1 \end{pmatrix} = \begin{pmatrix} u_2\cos t_2 & u_2\sin t_2 \\ -u_2\sin t_2 & u_2\cos t_2 \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}. \tag{5}$$

The determinants of the matrices on both sides of (5) are equal. Hence $u = u_1^{-1}u_2$ and matrix equation (5) reduces to

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(u_1^{-1}u_2,t) & b(u_1^{-1}u_2,t) \\ 0 & a^{-1}(u_1^{-1}u_2,t) \end{pmatrix} = \begin{pmatrix} \cos t_2 & \sin t_2 \\ -\sin t_2 & \cos t_2 \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix} \begin{pmatrix} \cos t_1 & -\sin t_1 \\ \sin t_1 & \cos t_1 \end{pmatrix}. \tag{6}$$

Comparing in both sides of matrix equation (6) the elements in the first column we have

$$\cos t a(u_1^{-1}u_2,t) = \cos t_2 \cos t_1 k + \cos t_2 \sin t_1 l + \sin t_2 \sin t_1 k^{-1}, \tag{7}$$

$$-\sin t a(u_1^{-1}u_2,t) = -\sin t_2 \cos t_1 k - \sin t_2 \sin t_1 l + \cos t_2 \sin t_1 k^{-1}. \tag{8}$$

Taking the square of both sides of equations (7), (8) and adding these equations we obtain

$$a(u_1^{-1}u_2,t) = \sqrt{k^2\cos^2 t_1 + (l^2+k^{-2})\sin^2 t_1 + 2kl\cos t_1 \sin t_1}\,, \tag{9}$$

$$\cos t = \frac{\cos t_2 \cos t_1 k + \cos t_2 \sin t_1 l + \sin t_1 \sin t_2 k^{-1}}{\sqrt{k^2\cos^2 t_1 + (l^2+k^{-2})\sin^2 t_1 + 2kl\cos t_1 \sin t_1}}\,, \tag{10}$$

$$\sin t = \frac{\sin t_2 \cos t_1 k + \sin t_1 \sin t_2 l - \sin t_1 \cos t_2 k^{-1}}{\sqrt{k^2\cos^2 t_1 + (l^2+k^{-2})\sin^2 t_1 + 2kl\cos t_1 \sin t_1}}\,. \tag{11}$$

The elements in the second column in both sides of matrix equation (6) are

$$\cos t b(u_1^{-1}u_2,t) + \sin t a^{-1}(u_1^{-1}u_2,t) = -\cos t_2 \sin t_1 k + \cos t_2 \cos t_1 l + \sin t_2 \cos t_1 k^{-1}, \tag{12}$$

$$-\sin t b(u_1^{-1}u_2,t) + \cos t a^{-1}(u_1^{-1}u_2,t) = \sin t_2 \sin t_1 k - \sin t_2 \cos t_1 l + \cos t_2 \cos t_1 k^{-1}. \tag{13}$$

Multiplying equation (12) by $\cos t$ and (13) by $-\sin t$ and adding the obtained equations we have

$$b(u_1^{-1}u_2,t) = \frac{-\sin t_1 \cos t_1 k^2 + (\cos^2 t_1 - \sin^2 t_1)kl + (l^2+k^{-2})\cos t_1 \sin t_1}{\sqrt{k^2\cos^2 t_1 + (l^2+k^{-2})\sin^2 t_1 + 2kl\cos t_1 \sin t_1}}. \tag{14}$$

If $\dim Q^* = 2$, then for a continuous sharply transitive section σ given by (5) $u = u_1^{-1}u_2 > 0$, t is given by (10), (11), and for all fixed u the functions $a(u_1^{-1}u_2,t)$, $b(u_1^{-1}u_2,t)$ are given by (9), (14). Any continuous sharply transitive section σ belonging to a two-dimensional multiplicative loop Q^* has this form with suitable $k > 0$, $l \in \mathbb{R}$.

The left translation with an element $(s,z)^t$ of the multiplicative loop Q^* of a left quasifield Q is a linear transformation $M_{(u,x,y)} \in \sigma(G/H_{(k,l,s)})$ defined by

$$\begin{pmatrix} s \\ z \end{pmatrix} \cdot \begin{pmatrix} v \\ w \end{pmatrix} = M_{(u,x,y)} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} ux & -u\overline{y} \\ uy & u\overline{x} \end{pmatrix} \begin{pmatrix} a(u,x,y) & b(u,x,y) \\ 0 & a^{-1}(u,x,y)e^{ic(u,x,y)} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}, \tag{15}$$

where $s = uxa(u,x,y)$, $z = uya(u,x,y)$. The elements of the kernel K_r of Q are $(0,0)^t$, $(s,0)^t$, $s \in \mathbb{C} \setminus \{0\}$ if $\dim Q = 4$, and $s \in \mathbb{R} \setminus \{0\}$ if $\dim Q = 2$. The matrix representation of the left translations with the elements of the one-dimensional connected subgroup $\{(r,0)^t \,|\, r > 0\}$ of the kernel K_r of Q is

$$M_{(u,1,0)} = \left\{ \begin{pmatrix} ua(u,1,0) & ub(u,1,0) \\ 0 & ua^{-1}(u,1,0)e^{ic(u,1,0)} \end{pmatrix} \,\middle|\, u > 0 \right\} \tag{16}$$

with $r = ua(u,1,0)$ and $c(u,1,0) = 0$ if $\dim Q = 2$. The subset

$$\mathcal{T}_{\mathbb{R}} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(1,t) \\ 0 & a^{-1}(1,t) \end{pmatrix} \,\middle|\, t \in [0,2\pi) \right\} \tag{17}$$

of the image of the section (2) consisting of elliptic elements acts sharply transitively on the oriented lines through $(0,0)^t$ in $\mathbb{R}^2$. Therefore $\mathcal{T}_{\mathbb{R}}$ is the set of all left translations of a one-dimensional compact loop. Similarly the subset

$$\mathcal{T}_{\mathbb{C}} = \left\{ \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} a(1,x,y) & b(1,x,y) \\ 0 & a^{-1}(1,x,y)e^{ic(1,x,y)} \end{pmatrix} \,\middle|\, x,y \in \mathbb{C},\ x\bar{x} + y\bar{y} = 1 \right\} \tag{18}$$

of the image of (2) seen as a set of (4×4)-real matrices acts sharply transitively on the oriented lines through $(0,0,0,0)^t$ in $\mathbb{R}^4$. Hence $\mathcal{T}_{\mathbb{C}}$ is the set of all left translations of a loop homeomorphic to S^3.

Definition 3.2. Let Q be a locally compact connected topological quasifield having dimension two over its kernel. The multiplicative loop Q^* of Q is called *decomposable*, if the set of all left translations of Q^* is a product $\mathcal{T}\mathcal{K}$, where $\mathcal{T}$ is the set of all left translations of a compact loop given by (17), respectively (18), and $\mathcal{K}$ is the set (16) of all left translations of Q^* belonging to the subgroup $\{(r,0)^t \,|\, r > 0\}$ of the kernel K_r of Q.

Proposition 3.3. *Let Q be a locally compact connected topological quasifield having dimension two over its kernel. The multiplicative loop Q^* of Q is decomposable if and only if for all $u > 0$, $x,y \in \mathbb{C}$, $x\overline{x} + y\overline{y} = 1$ one has $a(u,x,y) = a(1,x,y)a(u,1,0)$, $c(u,x,y) = c(u,1,0) + c(1,x,y)$ and $b(u,x,y) = a(1,x,y)b(u,1,0) + a^{-1}(u,1,0)e^{ic(u,1,0)}b(1,x,y)$ if $\dim Q^* = 4$ or for all $u > 0$, $t \in [0,2\pi)$, one has $a(u,t) = a(1,t)a(u,0)$, $b(u,t) = a(1,t)b(u,0) + a^{-1}(u,0)b(1,t)$ if $\dim Q^* = 2$.*

Proof. The set

$$\{M_{(u,x,y)} \,|\, u > 0,\ x,y \in \mathbb{C},\ x\overline{x} + y\overline{y} = 1 \text{ or } x = \cos t,\ y = -\sin t,\ c(u,x,y) = 0,\ t \in [0,2\pi)\}$$

acts sharply transitively on Q^*. Hence any point $(v,w)^t \setminus \{(0,0)\}$ is the image of the point $(1,0)^t$ under a unique linear mapping $M_{(u,x,y)}$ given by (15). For all $s > 0$, $m,n \in \mathbb{C}$, $m\overline{m} + n\overline{n} = 1$ or $m = \cos\varphi$, $n = -\sin\varphi$, $\varphi \in [0,2\pi)$ the matrix equation

$$\mathcal{T}_{\mathbb{C}}\left[M_{(u,1,0)} \begin{pmatrix} sma(s,m,n) \\ sna(s,m,n) \end{pmatrix} \right] = \begin{pmatrix} ux & -u\overline{y} \\ uy & u\overline{x} \end{pmatrix} \begin{pmatrix} a(u,x,y) & b(u,x,y) \\ 0 & a^{-1}(u,x,y)e^{ic(u,x,y)} \end{pmatrix} \begin{pmatrix} sma(s,m,n) \\ sna(s,m,n) \end{pmatrix} \tag{19}$$

holds if and only if the identities of the assertion are satisfied. ∎

The multiplicative loop Q^* is homeomorphic either to $S^3 \times \mathbb{R}$ or to $S^1 \times \mathbb{R}$. We wish to study under which circumstances the loop Q^* has a connected normal subloop N^* such that the factor loop Q^*/N^* is homeomorphic either to the sphere S^3 or to S^1.

THEOREM 3.4. *Let Q be a locally compact connected topological quasifield having dimension two over its kernel. The multiplicative loop Q^* of Q has a one-dimensional connected normal subloop N^* consisting of real elements such that the factor loop Q^*/N^* is homeomorphic to S^3 or to S^1 if and only if N^* is the group $\{(u,0)^t \,|\, u > 0\}$ isomorphic to $\mathbb{R}$ and $a(u,1,0) = 1$, $b(u,1,0) = 0 = c(u,1,0)$, $a(u,x,y) = a(1,x,y)$, $b(u,x,y) = b(1,x,y)$, $c(u,x,y) = c(1,x,y)$ for all $u > 0$, $x, y \in \mathbb{C}$, $x\overline{x} + y\overline{y} = 1$, or $a(u,0) = 1$, $b(u,0) = 0$, $a(u,t) = a(1,t)$, $b(u,t) = b(1,t)$, $c(u,t) = 0$ for all $t \in [0, 2\pi)$. Then Q^* is a central extension of the normal subgroup N^* by a loop homeomorphic to S^3 or to S^1.*

Proof. The left translations of a normal subloop N^* of Q^* generate a normal subgroup N of the group G topologically generated by all left translations of Q^* (cf. Lemma 1.7 in [NS1], p. 19). Since Q^*/N^* is homeomorphic to S^3 or to S^1 the subloop N^* is homeomorphic to $\mathbb{R}$. The group topologically generated by the left translations of a proper loop homeomorphic to $\mathbb{R}$ is the universal covering $\widetilde{PSL_2(\mathbb{R})}$ of $PSL_2(\mathbb{R})$ (cf. [NS1], Section 18, p. 235). But $\widetilde{PSL_2(\mathbb{R})}$ is not a subgroup of G listed in Proposition 3.1. Hence N^* is a group isomorphic to $\mathbb{R}$ and the set Λ_{Q^*} of all left translations of Q^* must contain the group $\left\{ \left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right),\ 0 < u \in \mathbb{R} \right\}$ as a normal subgroup. N^* has the form $\{(u,0)^t \,|\, u > 0\}$ which is a central subgroup of Q^* such that the intersection of a compact subloop (18) as well as of (17) of Q^* with N^* is 1. Hence Q^* is a central extension as in the assertion. According to (2) for all $u > 0$ one has $a(u,1,0) = 1$, $b(u,1,0) = 0$, $c(u,1,0) = 0$ if $\dim Q^* = 4$ and $a(u,0) = 1$, $b(u,0) = 0$ if $\dim Q^* = 2$. To obtain the necessary and sufficient conditions under which N^* is normal in Q^* we often use the fact that by (15) the element

$$\begin{pmatrix} ux & -u\overline{y} \\ uy & u\overline{x} \end{pmatrix} \begin{pmatrix} a(u,x,y) & b(u,x,y) \\ 0 & a^{-1}(u,x,y)e^{ic(u,x,y)} \end{pmatrix}$$

belongs to the left translation of $(uxa(u,x,y), uya(u,x,y))^t$ with $u > 0$, $x, y \in \mathbb{C}$, $x\overline{x} + y\overline{y} = 1$ if $\dim Q^* = 4$ and $x = \cos t$, $y = -\sin t$, the function c is constant 0 if $\dim Q^* = 2$. For all elements $q_1 := (x,y)^t$ of S^3 or S^1, $q_2 := (v,w)^t$ of Q^* the condition $(N^* \cdot q_1) \cdot q_2 = N^* \cdot (q_1 \cdot q_2)$ of (1) holds if and only if

$$\left[\begin{pmatrix} u \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right] \cdot \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} u' \\ 0 \end{pmatrix} \cdot \left[\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} v \\ w \end{pmatrix} \right]$$

for all $x, y \in \mathbb{C}$, $x\bar{x} + y\bar{y} = 1$, $(v,w) \in \mathbb{C}^2 \setminus \{(0,0)\}$, or for all $x = \cos t$, $y = -\sin t$, $(v,w) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with suitable $u, u' > 0$. This is equivalent to

$$\begin{pmatrix} ra(r,m,n)mv + rb(r,m,n)mw - ra^{-1}(r,m,n)w\bar{n}e^{ic(r,m,n)} \\ ra(r,m,n)nv + rb(r,m,n)nw + ra^{-1}(r,m,n)w\bar{m}e^{ic(r,m,n)} \end{pmatrix}$$
$$= \begin{pmatrix} u'(a(1,x,y)xv + b(1,x,y)xw - a^{-1}(1,x,y)w\bar{y}e^{ic(1,x,y)}) \\ u'(a(1,x,y)yv + b(1,x,y)yw + a^{-1}(1,x,y)w\bar{x}e^{ic(1,x,y)}) \end{pmatrix}$$

such that $ux = ra(r,m,n)m$, $uy = ra(r,m,n)n$ with $m\bar{m} + n\bar{n} = 1$. As $u^2(x\bar{x} + y\bar{y}) = r^2a^2(r,m,n)(m\bar{m} + n\bar{n})$ we obtain $m = x$, $n = y$, $u = ra(r,x,y)$. Using this for all (x,y) of S^3 or S^1 and (v,w) of $\mathbb{C}^2 \setminus \{(0,0)\}$ or $\mathbb{R}^2 \setminus \{(0,0)\}$ we obtain

$$\begin{aligned}
&\big[(x(va(r,x,y) + wb(r,x,y)) - \bar{y}wa^{-1}(r,x,y)e^{ic(r,x,y)})\big] \\
&\quad \times \big[y(va(1,x,y) + wb(1,x,y)) + \bar{x}wa^{-1}(1,x,y)e^{ic(1,x,y)}\big] \\
&\qquad = \big[(y(va(r,x,y) + wb(r,x,y)) + \bar{x}wa^{-1}(r,x,y)e^{ic(r,x,y)})\big] \\
&\qquad\quad \times \big[x(va(1,x,y) + wb(1,x,y)) - \bar{y}wa^{-1}(1,x,y)e^{ic(1,x,y)}\big].
\end{aligned}$$

The last equation holds if and only if for all (v,w) in $\mathbb{C}^2 \setminus \{(0,0)\}$ or in $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\begin{aligned}
&\big(a(r,x,y)a^{-1}(1,x,y)e^{ic(1,x,y)} - a^{-1}(r,x,y)a(1,x,y)e^{ic(r,x,y)}\big)(x\bar{x} + y\bar{y})vw \\
&\quad + \big(b(r,x,y)a^{-1}(1,x,y)e^{ic(1,x,y)} - a^{-1}(r,x,y)b(1,x,y)e^{ic(r,x,y)}\big)(x\bar{x} + y\bar{y})w^2 = 0
\end{aligned}$$

and hence

$$\begin{aligned}
a(r,x,y)a^{-1}(1,x,y)e^{ic(1,x,y)} - a^{-1}(r,x,y)a(1,x,y)e^{ic(r,x,y)} &= 0, \\
b(r,x,y)a^{-1}(1,x,y)e^{ic(1,x,y)} - a^{-1}(r,x,y)b(1,x,y)e^{ic(r,x,y)} &= 0.
\end{aligned}$$

Multiplying the last two equations by $e^{-ic(1,x,y)}$ one obtains

$$\begin{aligned}
a(r,x,y)a^{-1}(1,x,y) - a^{-1}(r,x,y)a(1,x,y)e^{ic(r,x,y)-ic(1,x,y)} &= 0, \\
b(r,x,y)a^{-1}(1,x,y) - a^{-1}(r,x,y)b(1,x,y)e^{ic(r,x,y)-ic(1,x,y)} &= 0.
\end{aligned}$$

Since $a(r,x,y)$ is positive for all $r > 0$ we get $c(r,x,y) = c(1,x,y)$ and hence $a(r,x,y) = a(1,x,y)$, $b(r,x,y) = b(1,x,y)$ for all $r > 0$, $x,y \in \mathbb{C}$, $x\bar{x} + y\bar{y} = 1$ or for all $x = \cos t$, $y = -\sin t$. If we take into account the obtained restrictions for the functions $a(r,x,y)$, $b(r,x,y)$, $c(r,x,y)$, a straightforward computation shows that the condition $(q_1 \cdot N^*) \cdot q_2 = q_1 \cdot (N^* \cdot q_2)$ of (1) holds for all elements $q_1 := (x,y)^t$ of S^3 or S^1, $q_2 := (v,w)^t$ of Q^*. This proves the assertion. ∎

If the loop Q^* has the central subgroup $N^* = \{(u,0) \,|\, u > 0\}$, then it has also the central subgroup $N^* \times Z_2$, where Z_2 is the group of order 2. If $\dim Q^* = 4$, then the factor loop $Q^*/(N^* \times Z_2)$ is homeomorphic to the three-dimensional real projective space, if $\dim Q^* = 2$, then $Q^*/(N^* \times Z_2)$ is homeomorphic to S^1.

Theorem 3.5. *If the multiplicative loop Q^* of a locally compact connected topological quasifield Q having dimension two over its kernel has the one-dimensional normal subgroup $N^* = \{(u,0)^t \,|\, u > 0\}$, then Q^* is decomposable.*

Proof. According to Theorem 3.4 the loop Q^* has N^* as a normal subgroup if and only if for all $u > 0$, $x,y \in \mathbb{C}$, $x\bar{x} + y\bar{y} = 1$ one has $a(u,1,0) = 1$, $b(u,1,0) = 0 = c(u,1,0)$, $a(u,x,y) = a(1,x,y)$, $b(u,x,y) = b(1,x,y)$, $c(u,x,y) = c(1,x,y)$ if $\dim Q^* = 4$, or for all $u > 0$, $t \in [0,2\pi)$ one has $a(u,0) = 1$, $b(u,0) = 0$, $a(u,t) = a(1,t)$, $b(u,t) = b(1,t)$ if $\dim Q^* = 2$. Hence the identities given in Proposition 3.3 hold. ∎

Proposition 3.6. *Let Q be a locally compact connected topological quasifield having dimension two over its kernel. The set Λ_{Q^*} of all left translations of the multiplicative* *loop Q^* of Q contains the group* $\mathrm{Spin}_3(\mathbb{R})$ *if* $\dim Q = 4$ *or the group* $\mathrm{SO}_2(\mathbb{R})$ *if* $\dim Q = 2$ *precisely if* Λ_{Q^*} *has the form*

$$\left\{ \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \begin{pmatrix} ua(u,1,0) & ub(u,1,0) \\ 0 & ua^{-1}(u,1,0)e^{ic(u,1,0)} \end{pmatrix} \;\middle|\; u > 0,\ x, y \in \mathbb{C},\ x\bar{x} + y\bar{y} = 1 \quad \text{or} \quad x = \cos t,\ y = -\sin t,\ c(u,1,0) = 0 \right\} \tag{20}$$

with the continuous functions $a(u,1,0) > 0$, $b(u,1,0) \in \mathbb{C}$ *or* $\mathbb{R}$, $c(u,1,0) \in \mathbb{R}$ *such that* $ua(u,1,0)$ *is strictly monotone. In this case* Q^* *is decomposable.*

Proof. If $\dim Q = 4$, then the assertion is proved in Proposition 10 in [F2], whereas if $\dim Q = 2$, then the proof of the assertion is given in Proposition 15 of [F1]. ∎

4. Applications. Although the group topologically generated by the left translations of any two-dimensional proper multiplicative loop is $\mathrm{GL}_2^+(\mathbb{R})$ (cf. [NS1], Section 29, p. 345), here we show that the groups $\mathrm{GL}_2(\mathbb{C})$, $\mathrm{Spin}_3(\mathbb{R}) \times \mathbb{C}$ and $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{R}$ are realized as the group generated by the left translations of a four-dimensional multiplicative loop. The given examples consist of loops which are multiplicative loops of semifields, central extensions of $\mathbb{R}$ by a loop defined on S^3 or which contain the group $\mathrm{Spin}_3(\mathbb{R})$ illustrating Theorem 3.4 and Proposition 3.6.

There are two classes of four-dimensional semifields Q having the field $\mathbb{C}$ as their kernel (cf. [K], Section 6). In the first class are the Rees algebras which are fully characterized in [NS1], Section 29.2. The multiplicative loop Q^* of a semifield Q in the second class is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} * \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda_{(x_1,x_2)} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & -c\bar{x_2} - x_2 \\ x_2 & \bar{x_1} + r\bar{x_2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{21}$$

where $(x_1, x_2)^t$, $(y_1, y_2)^t \in \mathbb{C}^2 \setminus \{(0,0)^t\}$, $r \geq 0$ and $c = c_1 + ic_2 \in \mathbb{C}$, $c_2 \geq 0$ are constants such that for all $v \in \mathbb{R}$ one has $0 < P_{r,c}(v) = v^4 + (2\,\mathrm{Re}\,c - r^2)v^2 - 2rv + |c|^2 - 1$ (cf. [K], p. 83), $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$. The kernel K_r of the semifield $Q_{(r,c)}$ defined by (21) is $K_r = \{(k,0)^t \mid k \in \mathbb{C}\}$ and the centre Z of $Q_{(r,c)}$ is $\{(k,0)^t \mid k \in \mathbb{R}\}$.

Proposition 4.1. *The multiplicative loop* $Q^*_{(r,c)}$ *is the direct product of the group* $\mathbb{R}$ *and a loop* $L_{(r,c)}$ *realized on* S^3 *and having the multiplication*

$$\begin{pmatrix} x_1 & -c\bar{x_2} - x_2 \\ x_2 & \bar{x_1} + r\bar{x_2} \end{pmatrix} \circ \begin{pmatrix} y_1 & -c\bar{y_2} - y_2 \\ y_2 & \bar{y_1} + r\bar{y_2} \end{pmatrix} = \begin{pmatrix} z_1 & -c\bar{z_2} - z_2 \\ z_2 & \bar{z_1} + r\bar{z_2} \end{pmatrix}, \tag{22}$$

where $z_1 = x_1y_1 - c\bar{x_2}y_2 - x_2y_2$, $z_2 = x_2y_1 + \bar{x_1}y_2 + r\bar{x_2}y_2$, $|\det(\lambda_{(x_1,x_2)})| = |\det(\lambda_{(y_1,y_2)})| = 1 = |\det(\lambda_{(z_1,z_2)})|$. *The group generated by all left translations of* $Q^*_{(r,c)}$, *respectively of* $L_{(r,c)}$, *is the group* $\mathrm{GL}_2(\mathbb{C})$, *respectively the group of complex* (2×2)*-matrices the determinants of which have absolute value* 1. *The group generated by all translations of* $L_{(r,c)}$, *respectively of* $Q^*_{(r,c)}$, *is the group* $\mathrm{SL}_4(\mathbb{R})$, *respectively the direct product* $\mathbb{R} \times \mathrm{SL}_4(\mathbb{R})$.

Proof. Let $\lambda_{(x_1,x_2)}$ be a matrix in (21). If $x_2 \neq 0$ then

$$\det(\lambda_{(x_1,x_2)}) \in \mathbb{C} \quad \text{and} \quad \lambda_{(x_1,x_2)}\overline{\lambda_{(x_1,x_2)}}^t \notin \mathbb{R} \cdot I,$$

where I is the identity matrix. Hence the group $G_{Q^*_{(r,c)}}$ generated by the left translations of $Q^*_{(r,c)}$ is the group $\mathrm{GL}_2(\mathbb{C})$. The loop $Q^*_{(r,c)}$ has a central subgroup $Z_0^* = \{(k,0)^t \mid k > 0\} \cong \mathbb{R}$. The set $S_{(r,c)}$ of matrices

$$\lambda_{(x_1,x_2)} = \begin{pmatrix} x_1 & -c\bar{x_2} - x_2 \\ x_2 & \bar{x_1} + r\bar{x_2} \end{pmatrix}, \qquad |\det(\lambda_{(x_1,x_2)})| = 1,$$

topologically generates the group Δ of complex matrices A with $|\det(A)| = 1$ and the map $S_{(r,c)} \to S_{(r,c)}Z_0/Z_0$, where Z_0 is the group of the left translations by the elements of Z_0^*, is bijective. The product $\circ : S_{(r,c)} \times S_{(r,c)} \to S_{(r,c)}$ given by (22) in the assertion yields a loop $L_{(r,c)}$ diffeomorphic to S^3 because $L_{(r,c)}$ is a system of representatives with respect to the subgroup $\left\{\left(\begin{smallmatrix} k & l \\ 0 & k^{-1}e^{is} \end{smallmatrix}\right),\ k > 0,\ l \in \mathbb{C},\ s \in \mathbb{R}\right\}$ in the group Δ. Hence the multiplicative loop $Q^*_{(r,c)}$ of $Q_{(r,c)}$ is isomorphic to the direct product of $\mathbb{R}$ and $L_{(r,c)}$. The group generated by all translations of the compact loop $L_{(r,c)}$ is $\mathrm{SL}_4(\mathbb{R})$ (cf. [F2], Proposition 11). Since $Q^*_{(r,c)}$ is the direct product of $\mathbb{R}$ and $L_{(r,c)}$, the group generated by all translations of $Q^*_{(r,c)}$ is the direct product $\mathbb{R} \times \mathrm{SL}_4(\mathbb{R})$. This proves the assertion. ■

Let $\varphi : \mathbb{R}_{>0} \to \mathrm{Spin}_3(\mathbb{R})$, $\varphi(1) = 1$, be a continuous mapping and $\mathbb{H} = (\mathbb{R}^4, +, \cdot)$ be the skew field of quaternions. Then $\mathbb{H}_\varphi = (\mathbb{R}^4, +, \circ)$ with the multiplication $\circ$ given by $0 \circ x = 0$ and for $m \neq 0$ by $m \circ x = m \cdot x^{\varphi(|m|)} = m \cdot \varphi(|m|)^{-1} \cdot x \cdot \varphi(|m|)$ is a four-dimensional topological quasifield. The kernel K_r of $\mathbb{H}_\varphi$ is isomorphic to the field $\mathbb{C}$ precisely if $\varphi(\mathbb{R}_{>0})$ lies in a subfield of $\mathbb{H}$ isomorphic to $\mathbb{C}$ (cf. [H1], pp. 234–238). The multiplicative loop $\mathbb{H}^*_\varphi$ of $\mathbb{H}_\varphi$ is defined by the multiplication

$$m \circ x = M_{(m_1,m_2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} m_1 & -\overline{m_2}(\varphi(|m|))^2 \\ m_2 & \overline{m_1}(\varphi(|m|))^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{23}$$

with $x = x_1 + jx_2$, $m = m_1 + jm_2$, $x_1, x_2, m_1, m_2 \in \mathbb{C}$.

Proposition 4.2. *The set $\Lambda_{\mathbb{H}^*_\varphi}$ of all left translations of the loop $\mathbb{H}^*_\varphi$ contains the group* $\mathrm{Spin}_3(\mathbb{R})$ *and the group topologically generated by $\Lambda_{\mathbb{H}^*_\varphi}$ is* $\mathrm{Spin}_3(\mathbb{R}) \times \mathbb{C}$.

Proof. For each matrix $M_{(m_1,m_2)}$ one has $M_{(m_1,m_2)} \cdot \overline{M}^t_{(m_1,m_2)} = (m_1\overline{m_1} + m_2\overline{m_2}) \cdot I \in \mathbb{R}I$, where I is the identity matrix, and $\det(M_{(m_1,m_2)}) = (m_1\overline{m_1} + m_2\overline{m_2})(\varphi(|m|))^2 \in \mathbb{C}^*$. Hence the group $G_{\mathbb{H}^*_\varphi}$ generated by the left translations of $\mathbb{H}^*_\varphi$ is the group $\mathrm{Spin}_3(\mathbb{R}) \times \mathbb{C}$ and $a(u,x,y) = 1$, $b(u,x,y) = 0$ (cf. Proposition 3.1). Since the matrix $M_{(u,x,y)}$ in (2) coincides with $M_{(m_1,m_2)}$ given by (23) and $\det(M_{(u,x,y)}) = u^2 e^{ic(u,x,y)}$, we obtain $u = \sqrt[4]{|\det(M_{(m_1,m_2)})|^2} = |m|$ and $e^{ic(u,x,y)} = \frac{\det(M_{(m_1,m_2)})}{u^2} = (\varphi(|m|))^2$. Hence the function c depends only on the variable $u = |m|$. According to Proposition 3.6 the set $\Lambda_{\mathbb{H}^*_\varphi}$ contains the group $\mathrm{Spin}_3(\mathbb{R})$ and the assertion is proved. ■

Let Q be the quasifield given by formula (2) in [H2], p. 87. For $a_2 \neq 0$ the multiplication of Q is given by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1x_1 - a_2\overline{x_2} + \dfrac{a_2\varrho(a_2/|a_2|)}{\sqrt{1 + |\varrho(a_2/|a_2|)|^2}}(\overline{x_1} - x_1) \\ a_1x_2 + a_2\overline{x_1} - \dfrac{a_2\varrho(a_2/|a_2|)}{\sqrt{1 + |\varrho(a_2/|a_2|)|^2}}(x_2 - \overline{x_2}) \end{pmatrix},$$

where $a_i, x_i \in \mathbb{C}$, $i = 1,2$, and $\varrho : S^1 \to \{il \mid l \in \mathbb{R}\}$ is a continuous non-constant function having pure imaginary values, $\binom{a_1}{0} \circ \binom{x_1}{x_2} = \binom{a_1 x_1}{a_1 x_2}$ for $a_2 = 0$. The kernel K_1 of the quasifield Q_ϱ is $K_1 = \left\{\binom{x}{y} \mid x,y \in \mathbb{R}\right\}$ such that $\binom{a_1}{a_2} \circ \binom{x}{y} = \binom{a_1 x - a_2 y}{a_1 y + a_2 x}$. The coordinate change $T : \mathbb{C}^2 \to \mathbb{C}^2$, $(r+si, u+vi)^t \mapsto (r+ui, s+vi)^t$ transforms the kernel K_1 to $K_2 = \{(x+iy,0)^t \mid x,y \in \mathbb{R}\} = \{(z,0)^t \mid z \in \mathbb{C}\}$ and the multiplication of the loop Q_ϱ^* is given by

$$\begin{pmatrix} a_{11}+ia_{12} \\ a_{21}+ia_{22} \end{pmatrix} \circ \begin{pmatrix} x_{11}+ix_{12} \\ x_{21}+ix_{22} \end{pmatrix}$$
$$= T^{-1}\left(\begin{pmatrix} a_{11}+ia_{21} & -a_{12}+ia_{22}+\dfrac{2a_{21}\operatorname{Im}(\varrho(a_2/|a_2|))}{\sqrt{1+(\operatorname{Im}\varrho(a_2/|a_2|))^2}} \\ a_{12}+ia_{22} & a_{11}-ia_{21}+\dfrac{2a_{22}\operatorname{Im}(\varrho(a_2/|a_2|))}{\sqrt{1+(\operatorname{Im}\varrho(a_2/|a_2|))^2}} \end{pmatrix}\begin{pmatrix} x_{11}+ix_{21} \\ x_{12}+ix_{22} \end{pmatrix}\right).$$

Proposition 4.3. *The group topologically generated by the left translations of Q_ϱ^* is the group* $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{R}$. *The loop Q_ϱ^* is a central extension of $Z_0^* = \{(c,0)^t \mid c > 0\} \cong \mathbb{R}$ by a loop homeomorphic to S^3.*

Proof. The assertion is proved in Proposition 13 of [F2]. ■

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