

THE DEFORMATION QUANTISATION MAPPING OF POISSON TO ASSOCIATIVE STRUCTURES IN FIELD THEORY

ARTHEMY V. KISELEV

*Johann Bernoulli Institute for Mathematics and Computer Science
University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands
E-mail: A.V.Kiselev@rug.nl*

Abstract. Let $\{\cdot, \cdot\}_{\mathcal{P}}$ be a variational Poisson bracket in a field model on an affine bundle π over an affine base manifold M^m . Denote by \times the commutative associative multiplication in the Poisson algebra \mathcal{A} of local functionals $\Gamma(\pi) \rightarrow \mathbb{k}$ that take field configurations to numbers. By applying the techniques from geometry of iterated variations, we make well defined the deformation quantization map $\times \mapsto \star = \times + \hbar \{\cdot, \cdot\}_{\mathcal{P}} + \bar{o}(\hbar)$ that produces a noncommutative $\mathbb{k}[[\hbar]]$ -linear star-product \star in \mathcal{A} .

Introduction. Starting from a Poisson bi-vector \mathcal{P} on a given finite-dimensional affine Poisson manifold (N^n, \mathcal{P}) , the Kontsevich graph summation formula [32] yields an explicit deformation $\times \mapsto \star_{\hbar}$ of the commutative product \times in the algebra $\mathcal{A} := C^\infty(N^n)$ of smooth functions. The new operation \star_{\hbar} on the space $\mathcal{A}[[\hbar]] = C^\infty(N)[[[\hbar]]]$ of power series is specified by the Poisson structure on N : namely, $f \star_{\hbar} g = f \times g + \hbar \{f, g\}_{\mathcal{P}} + \bar{o}(\hbar)$ such that all the bi-differential terms at higher powers of the formal parameter \hbar are completely determined by the Poisson bracket $\{\cdot, \cdot\}_{\mathcal{P}}$ in the leading deformation term. The deformed product \star_{\hbar} is no longer commutative if $\mathcal{P} \neq 0$ but it stays associative,

$$\text{Assoc}_{\star_{\hbar}}(f, g, h) := (f \star_{\hbar} g) \star_{\hbar} h - f \star_{\hbar} (g \star_{\hbar} h) \doteq 0 \quad \text{for all } f, g, h \in C^\infty(N)[[[\hbar]]],$$

by virtue of bi-vector's property $\text{Jac}(\mathcal{P}) := \llbracket \mathcal{P}, \mathcal{P} \rrbracket = 0$ to be Poisson, cf. [5].

2010 *Mathematics Subject Classification*: Primary 53D55, 58E30, 81S10; Secondary 53D17, 58Z05, 70S20.

Key words and phrases: Deformation quantization, star-product, field theory, Poisson bracket, associativity.

Max Planck Institute for Mathematics, Vivatsgasse 7, D-53111 Bonn, Germany.

Present address: IHÉS, Le Bois-Marie, 35 route de Chartres, Bures-sur-Yvette, F-91440, France.

The paper is in final form and no version of it will be published elsewhere.

In this paper we extend the Poisson set-up and graph summation technique in the deformation $\times \mapsto \star_{\hbar}$ to the jet-space (super)geometry of N^n -valued fields $\phi \in \Gamma(\pi)$ over another, $(m > 0)$ -dimensional affine base manifold M^m in a given bundle π and secondly, of variational Poisson bi-vectors \mathcal{P} that encode the Poisson brackets $\{ \cdot, \cdot \}_{\mathcal{P}}$ on the space of local functionals taking $\Gamma(\pi) \rightarrow \mathbb{k}$. We explain why an extension of the Kontsevich graph technique [32] is possible and how it is done by using the geometry of iterated variations [19, 22]. For instance, we derive a variational analogue of the Moyal associative \star -product, $f \star g = (f) \exp\left(\overleftarrow{\partial}_i \cdot \hbar \mathcal{P}^{ij} \cdot \overrightarrow{\partial}_j\right) (g)$, in the case where the coefficients \mathcal{P}^{ij} of bi-vector \mathcal{P} are constant (hence the Jacobi identity $[[\mathcal{P}, \mathcal{P}]] = 0$ holds trivially). To process variational Poisson structures with nonconstant coefficients, we analyse (see [5, 6]) the factorization mechanism $\text{Assoc}_{\star_{\hbar}}(f, g, h) = \diamond(\mathcal{P}, \text{Jac}_{\mathcal{P}}(\cdot, \cdot, \cdot))(f, g, h)$ using the Kontsevich graphs at higher powers of the deformation parameter \hbar . We explain why, holding up to $\bar{o}(\hbar^2)$, the associativity of \star_{\hbar} can start leaking at orders $\hbar^{\geq 3}$ in the variational Poisson geometry of field-theoretic models.

Its concept going back to Weyl–Groenewold [16] and Moyal [37], the problem of associativity-preserving deformation quantisation $\times \mapsto \star_{\hbar}$ of commutative product \times in the algebras $C^\infty(N^n)$ of functions on smooth finite-dimensional symplectic manifolds (N^n, ω) was considered by Bayen–Flato–Frönsdal–Lichnerowicz–Sternheimer [2]. Further progress within the symplectic picture was made by De Wilde–Lecomte [10] and independently, Fedosov [14, 13]. To tackle the deformation quantisation problem in the case of finite-dimensional affine Poisson geometries $(N^n, \{ \cdot, \cdot \}_{\mathcal{P}})$ —that is, in absence of the Darboux lemma which guarantees the existence of canonical coordinates on a chart $U_\alpha \subseteq N^n$ in the symplectic case—Kontsevich developed the graph complex technique [29, 28]; it yields an explicit construction of each term in the series $\times \mapsto \star_{\hbar}$, see [32, 30]. We recall this approach and analyse some of its features in Section 1. Specifically, the sum over a suitable set of weighted oriented graphs determines on $N^n \ni \mathbf{u}$ a star-product \star_{\hbar} which (i) contains a given Poisson bracket $\{ \cdot, \cdot \}_{\mathcal{P}}$ in the leading deformation term at \hbar^1 and (ii) is associative modulo the Jacobi identity for $\{ \cdot, \cdot \}_{\mathcal{P}}$,

$$\text{Assoc}_{\star_{\hbar}}(f, g, h) = \diamond(\mathcal{P}, \text{Jac}_{\mathcal{P}}(\cdot, \cdot, \cdot))(f, g, h), \quad f, g, h \in C^\infty(N^n)[[\hbar]], \tag{1}$$

where

$$\text{Jac}_{\mathcal{P}}(a, b, c) = \frac{1}{2} \sum_{\sigma \in S_3} (-)^\sigma \{ \{ \sigma(a), \sigma(b) \}_{\mathcal{P}}, \sigma(c) \}_{\mathcal{P}}, \quad a, b, c \in C^\infty(N^n).$$

The construction of polydifferential operator \diamond has been analysed up to order $\bar{o}(\hbar^4)$ in [5, 6].

A key distinction between associativity mechanisms for the Darboux-symplectic and Poisson cases is a possibility of the star-product self-action on non-constant coefficients $P^{ij}(\mathbf{u})$ of the bracket $\{ \cdot, \cdot \}_{\mathcal{P}}$. It is readily seen that whenever those coefficients are constant, the graph summation formula for \star_{\hbar} then yields the Moyal star-product [37],

$$\star \Big|_{\mathbf{u}=(u^1, \dots, u^n)} = \exp\left(\frac{\overleftarrow{\partial}}{\partial v^i} \Big|_{v^i=u^i} \cdot \hbar P^{ij}(\mathbf{u}) \cdot \frac{\overrightarrow{\partial}}{\partial w^j} \Big|_{w^j=u^j} \right). \tag{2}$$

This formula’s geometric extension to the infinite-dimensional space of N^n -valued fields over a given m -dimensional affine manifold M^m will be obtained in §2.3.1, see Eq. (14).

Valid in finite-dimensional set-up, the result of [32] was known to be not working in the infinite dimension. It could not be applied to field-theoretic models, should one attempt to assemble such geometries via a limiting procedure by first taking infinitely many “zero-dimensional field theories” over the discrete topological space $M^0 = \bigcup_{\mathbf{x} \in M^m} \{\mathbf{x}\}$. In fact, not only is the geometry of N^n -valued fields (here, n internal degrees of freedom attached at every base point $\mathbf{x} \in M^m$) infinite-dimensional if $m > 0$ but also does the mathematical apparatus to encode it become substantially more complex, cf. [19, 22]. Many elements of differential calculus are known to be fragile in the course of transition from finite-dimensional geometry of N^n to the infinite jet spaces $J^\infty(\pi)$ for the bundles π of N^n -valued fields over M^m , or to the infinite jet spaces of maps $J^\infty(M^m \rightarrow N^n)$, cf. [26] vs [43] and [23] vs [1] or contrast [19] vs [40], [22] vs [27], and [32] vs this paper.

The aim of this paper is to develop a tool for regular deformation quantisation $(\mathcal{A}, \times) \mapsto (\mathcal{A}[[\hbar]], \star_\hbar)$ of field theory models. The commutative associative unital algebras (\mathcal{A}, \times) of local functionals equipped with variational Poisson structures $\{\cdot, \cdot\}_{\mathcal{P}}$ are the input data of quantisation algorithm; in the output one obtains the noncommutative products \star_\hbar , associative up to $\bar{o}(\hbar^{\geq 2})$, in the unital algebras $\mathcal{A}[[\hbar]]$.

This paper is structured as follows. In the next section we review the concept of deformation quantisation [32] for finite-dimensional affine Poisson manifolds $(N^n, \{\cdot, \cdot\}_{\mathcal{P}})$: the deformation $\times \mapsto \star_\hbar$ is approached via summation over a class of weighted oriented graphs. In Section 2 we proceed with finite-dimensional affine bundles π over affine manifolds M^m ; the infinitesimal parts of deformations are now specified by the variational Poisson brackets $\{\cdot, \cdot\}_{\mathcal{P}}$ for algebras \mathcal{A} of local functionals taking $\Gamma(\pi) \rightarrow \mathbb{k}$. To extend the deformation quantisation technique to such set-up, we let elements of the Gel’fand calculus of singular linear integral operators enter the game in §2.2. Each graph in the Kontsevich summation formula now encodes a local variational (poly)differential operator. We then inspect in §2.3 the geometric mechanism through which the new star-products can stay associative. Taking star-product (2) by Weyl–Groenewold–Moyal as prototype, we illustrate the algorithm of variational deformation quantisation by presenting this structure’s associative analogue \star for the class of Hamiltonian total differential operators $\|P_\tau^{ij}(\mathbf{x}) d^{|\tau|}/d\mathbf{x}^\tau\|_{i=1, \dots, n}^{j=1, \dots, n}$ whose coefficients do not depend on sections $\mathbf{u} = \phi(\mathbf{x})$. On the other hand, for field-dependent Hamiltonian operators we indicate a channel for the associativity to leak at orders $\bar{o}(\hbar^{\geq 2})$. This effect was altogether suppressed in the seminal picture; originally invisible, it can appear only in the framework of fibre bundles π over the base manifold M^m of positive dimension m .

1. The Kontsevich \star_\hbar -product on finite-dimensional Poisson manifolds. In this section we recall the graph technique [32] for deformation quantisation $\times \mapsto \star_\hbar$ on finite-dimensional affine Poisson manifolds $(N^n, \{\cdot, \cdot\}_{\mathcal{P}})$.

1.1. Let us first consider the direct problem of producing Lie algebra structures from a given associative product in the algebra of functions on N^n .

LEMMA 1. Let \mathcal{A} be an associative algebra. Denote by \star the associative multiplication in \mathcal{A} . Then the bi-linear skew-symmetric operation

$$\{f, g\} := f \star g - g \star f, \quad f, g \in \mathcal{A}, \tag{3}$$

is a Lie bracket satisfying the Jacobi identity.¹

Proof. Indeed, the Jacobiator $\text{Jac}_{\{\cdot, \cdot\}}(f, g, h) = \sum_{\sigma \in \mathcal{O}} \{\{\sigma(f), \sigma(g)\}, \sigma(h)\}$ is assembled by using the sum of associators:

$$\text{Jac}_{\{\cdot, \cdot\}}(f, g, h) = \sum_{\tau \in \mathcal{S}_3} (-)^\tau \text{Assoc}_\star(\sigma(f), \sigma(g), \sigma(h)), \quad f, g, h \in \mathcal{A}.$$

This tells us that the Jacobi identity for the bracket $\{\cdot, \cdot\}$ is an obstruction to the associativity of the product \star . ■

COROLLARY 2. Let $\{\cdot, \cdot\}_\mathcal{P}$ be a Poisson bracket on N^n and \times be the multiplication in the algebra $\mathcal{A} = C^\infty(N^n)$. Suppose that a deformation $\times \mapsto \star = \times + \hbar \{\cdot, \cdot\}_\mathcal{P} + \bar{o}(\hbar)$ of the product in \mathcal{A} to a multiplication in $\mathcal{A}[[\hbar]]$ is such that \star is associative at all orders of the deformation parameter \hbar . Then this deformation $\times \mapsto \star$ yields a transformation $\{\cdot, \cdot\}_\mathcal{P} \mapsto \{\cdot, \cdot\} = \{\cdot, \cdot\}_\mathcal{P} + \bar{o}(1)$ of the Poisson bracket $\{\cdot, \cdot\}_\mathcal{P}$ to a Lie—but not necessarily Poisson—bracket (3).

LEMMA 3 ([32]). Denote by \times the associative multiplication in the algebra $\mathcal{A} = C^\infty(N^n)$. Suppose that a deformation $\times \mapsto \star = \times + \hbar B_1(\cdot, \cdot) + \hbar^2 B_2(\cdot, \cdot) + \bar{o}(\hbar^2)$ is such that $B_1(\cdot, \cdot)$ is a bi-derivation and let $\star \bmod \bar{o}(\hbar^2)$ be associative up to $\bar{o}(\hbar^2)$, that is,

$$\text{Assoc}_\star(f, g, h) = \bar{o}(\hbar^2) \quad \text{for } f, g, h \in \mathcal{A}[[\hbar]].$$

Then the bi-linear skew-symmetric bi-derivation²

$$\{f, g\}_\star := \left. \frac{f \star g - g \star f}{\hbar} \right|_{\hbar=0} = 2B_1^-(f, g)$$

is a Poisson bracket.

Proof. In the leading order, $\text{Assoc}_{\star \bmod \bar{o}(\hbar^2)}(f, g, h) = \text{Assoc}_\times(f, g, h) + \bar{o}(1) = \bar{o}(1)$. At \hbar^1 , $\text{Assoc}_{\star \bmod \bar{o}(\hbar^2)}(f, g, h) = \hbar [B_1(f, g) \times h + B_1(f \times g, h) - f \times B_1(g, h) - B_1(f, g \times h)] + \bar{o}(\hbar) = \bar{o}(\hbar)$ because B_1 is a derivation with respect to each argument. Next, at \hbar^2 we obtain

$$\begin{aligned} \text{Assoc}_{\star \bmod \bar{o}(\hbar^2)}(f, g, h) &= \hbar^2 [B_1(B_1(f, g), h) - B_1(f, B_1(g, h)) \\ &\quad - (f \times B_2(g, h) - B_2(f, g) \times h + B_2(f, g \times h) - B_2(f \times g, h))] + \bar{o}(\hbar^2). \end{aligned}$$

We are given that this expression’s leading term vanishes. By taking an alternating sum over the group of permutations of three arguments and recalling that the product \times is

¹For example, let \mathcal{A} be the algebra $\mathcal{A} = C^\infty(N^n)$ of smooth functions or the algebra $\mathcal{A}[[\hbar]]$ of formal power series on a given manifold N^n . Then Lie bracket (3) is not necessarily a bi-derivation and its differential order with respect to either of the arguments is not necessarily bounded.

²By assumption, the leading deformation term $\hbar B_1(\cdot, \cdot)$ in \star is a bi-derivation, hence same are its symmetric and skew-symmetric parts, $B_1^+(f, g) = \frac{1}{2}(B_1(f, g) + B_1(g, f))$ and $B_1^-(f, g) = \frac{1}{2}(B_1(f, g) - B_1(g, f))$, respectively.

commutative, we deduce that

$$\begin{aligned} & \sum_{\sigma \in S_3} (-)^\sigma \text{Assoc}_{\star \bmod \bar{o}(\hbar^2)}(\sigma(f), \sigma(g), \sigma(h)) \\ &= \sum_{\sigma \in S_3} (-)^\sigma [B_1(B_1(\sigma(f), \sigma(g)), \sigma(h)) - B_1(\sigma(f), B_1(\sigma(g), \sigma(h)))] + \bar{o}(\hbar^2), \end{aligned}$$

that is, all the four terms containing $B_2(\cdot, \cdot)$ cancel out; in the sum over permutations they are grouped by (1st – 2nd) + (3rd) – (4th). Finally, let us split $B_1(\cdot, \cdot) = B_1^+(\cdot, \cdot) + B_1^-(\cdot, \cdot)$ and infer that in fact, its symmetric part also cancels out in the alternating sum:

$$\sum_{\sigma \in S_3} (-)^\sigma \text{Assoc}_{\star \bmod \bar{o}(\hbar^2)}(\sigma(f), \sigma(g), \sigma(h)) = \text{Jac}_{\{\cdot, \cdot\}_\star}(f, g, h) + \bar{o}(\hbar^2),$$

whence the assertion readily follows. ■

EXAMPLE 1. Let f, g be functions in the Cartesian coordinates p and q on \mathbb{R}^2 . Consider the associative star-product

$$(f \star g)(p, q; \hbar) = f|_{(p,q)} \exp(\overleftarrow{\partial} / \partial p \cdot \hbar \cdot \overrightarrow{\partial} / \partial q) g|_{(p,q)}.$$

We have

$$\frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial q} = \frac{1}{2} \left(\frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial q} + \frac{\partial g}{\partial p} \cdot \frac{\partial f}{\partial q} \right) + \frac{1}{2} \left(\frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial q} - \frac{\partial g}{\partial p} \cdot \frac{\partial f}{\partial q} \right).$$

By construction, we obtain

$$\{f, g\}_\star = (f) \overleftarrow{\partial} / \partial p \cdot \overrightarrow{\partial} / \partial q (g) - (g) \overleftarrow{\partial} / \partial p \cdot \overrightarrow{\partial} / \partial q (f),$$

which is the two functions’ Poisson bracket referred to the canonical Darboux coordinates p and q .

REMARK 1. From now on we shall always assume that the leading deformation term $B_1(\cdot, \cdot)$ at \hbar^1 in \star is skew-symmetric. In the Kontsevich star-product, the symmetric part B_1^+ of a given deformation term B_1 might not be vanishing identically *ab initio* but it can then be trivialised—at the expense of using suitable gauge transformations $f \mapsto f + \hbar D_1(f) + \bar{o}(\hbar)$, $g \mapsto g + \hbar D_1(g) + \bar{o}(\hbar)$ of its arguments (see [32]).

1.2. Now suppose that a Poisson bracket $\{\cdot, \cdot\}_\mathcal{P}$ is given on N^n in advance. Can the commutative associative multiplication \times in the algebra $C^\infty(N^n) \ni f, g$ be deformed to an associative star-product \star_\hbar such that the formal power series $f \star_\hbar g = f \times g + \hbar \cdot \{f, g\}_\mathcal{P} + \sum_{k=2}^{+\infty} \hbar^k B_k(f, g)$ is well defined? More specifically, the bi-linear, not necessarily commutative, star-product $\star_\hbar = \times + \hbar \{\cdot, \cdot\}_\mathcal{P} + \sum_{k>1} \hbar^k B_k(\cdot, \cdot)$ must satisfy the four axioms:

1. it is associative,

$$(f \star_\hbar g) \star_\hbar h \doteq f \star_\hbar (g \star_\hbar h) \quad \text{via } \{\{f, g\}_\mathcal{P}, h\}_\mathcal{P} + \text{c. p.} = 0, \quad f, g, h \in C^\infty(N^n), \quad (1')$$

i.e. modulo the property of bracket $\{\cdot, \cdot\}_\mathcal{P}$ on N^n to be Poisson;

2. the unit function $1 \in C^\infty(N^n)$ remains the neutral element for \star_\hbar ; whatever $f \in C^\infty(N^n)$, one has that $f \star_\hbar 1 = f = 1 \star_\hbar f$;

3. each term $B_k(\cdot, \cdot)$, including the skew-symmetric Poisson bracket $\{\cdot, \cdot\}_\mathcal{P} = B_1(\cdot, \cdot)$ to start with at \hbar , is a bi-linear differential operator of bounded order;

4. the product \star_\hbar is (let to be) $\mathbb{k}[[\hbar]]$ -linear over $C^\infty(N^n)[[\hbar]]$.

THEOREM 4 ([32]). *For every affine n -dimensional Poisson manifold $(N^{n < \infty}, \mathcal{P})$ there exists a star-product $\star_{\hbar} = \times + \hbar \{ \cdot, \cdot \}_{\mathcal{P}} + \sum_{k=2}^{+\infty} \hbar^k B_k(\cdot, \cdot)$ in $\mathcal{A}[[\hbar]]$ satisfying the above four axioms.*

The proof is constructive (cf. [7] and [31, 42]); the graph technique [32, 30, 29, 28] is a convenient way to encode the bi-differential terms $B_k(\cdot, \cdot)$ in perturbation series \star_{\hbar} . Every term in $B_k(f, g)$ at \hbar^k , $k \geq 0$, is encoded by an oriented graph Γ with $k+2$ vertices, of which two sinks contain the respective arguments f and g and each of the remaining k internal vertices is a source for two oriented edges. (In total, there are k such wedges with $2k$ arrows in every such graph Γ .) Neither tadpoles nor multiple edges are permitted (cf. [7]). Next, install a copy of the given Poisson bi-vector \mathcal{P} at each of the k tops of the wedges and decorate every edge in the graph Γ at hand with a summation index running from 1 to $n = \dim N^n$. The two edges issued from each internal vertex are *ordered*, so that the precedent and antecedent edges correspond to the first and second indexes in a copy of the Poisson bi-vector \mathcal{P} .

1.3. To encode multi-vectors $\Xi \in \Gamma(\wedge^* TN^n)$ in a standard way, consider the parity-odd neighbour ΠT^*N^n of cotangent bundle to the manifold N^n and denote by $\xi = (\xi_1, \dots, \xi_n)$ the n -tuple of \mathbb{Z}_2 -parity odd fibre coordinates over a chart $U_\alpha \subseteq N^n$ with an n -tuple $\mathbf{u} = (u^1, \dots, u^n)$ of local coordinates. Whenever the values $\{u^i, u^j\}_{\mathcal{P}}(\mathbf{u}) = P^{ij}(\mathbf{u})$ are given, construct the bi-vector $\mathcal{P} = \frac{1}{2} \langle \xi_i P^{ij} |_{\mathbf{u}} \xi_j \rangle \in \Gamma(\wedge^2 TU_\alpha)$; bi-vectors are Poisson if they satisfy the classical master-equation $[[\mathcal{P}, \mathcal{P}]] = 0$, see footnote 5.

CONVENTION 1. The correspondence between every decorated oriented edge and analytic expressions occurring in $B_k(\cdot, \cdot)$ is established in Fig. 1; at every set of index values, the

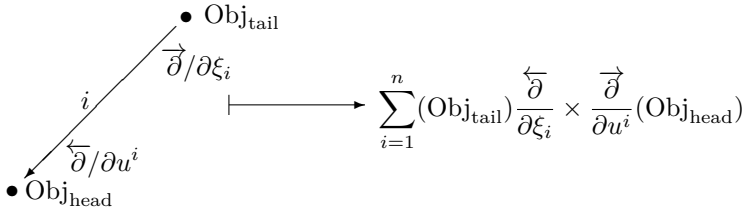


Fig. 1. The matching of indexes in the derivative falling on the arrowhead object and in the Poisson bi-vector stored in the arrow tail vertex is due to the coupling of the two objects' differentials taken with respect to the canonical conjugate variables

respective content of vertices in a connected graph component is multiplied using \times (cf. footnote 4). The expressions determined by different connected components of one graph Γ in their formal sum are also multiplied by using the original product \times .

REMARK 2. Because other arrows may stick into the vertices connected by an edge \xrightarrow{i} in Fig. 1, the objects Obj_{tail} and Obj_{head} contained there can be derivatives (with respect to u^α 's) of the bi-vector \mathcal{P} or, specifically to Obj_{head} but never possible to Obj_{tail} , arguments f and g of the star-product. On the same grounds, because there is another arrow issued from the tail vertex with Obj_{tail} , the formula encoded by a graph Γ does in fact not depend on any of the auxiliary, parity-odd variables ξ_j .

In the Kontsevich star-product, every graph is accompanied with its weight $w(\Gamma) \in \mathbb{R}$; these numbers are obtained by calculating certain explicitly given integrals over the configuration spaces of k distinct points—in fact, the graphs’ vertices containing \mathcal{P} —on the Lobachevsky plane (in its Poincaré model in the upper half-plane), see [32]. The full set of rational values of weights for all graphs in an expansion $\star_{\hbar} \bmod \bar{o}(\hbar^4)$ of the Kontsevich star-product has been obtained in [6].

EXAMPLE 2. For any functions $f, g \in C^\infty(N^n)$, the expansion $f \star_{\hbar} g \bmod \bar{o}(\hbar^2)$ reads as follows:³

$$\begin{aligned}
 & \bullet \star \bullet \\
 & \begin{array}{c} \bullet \\ f \end{array} \star \begin{array}{c} \bullet \\ g \end{array} \\
 = & \begin{array}{c} \bullet \text{---} \bullet \\ f \quad g \end{array} + \frac{\hbar^1}{1!} \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ f \quad g \end{array} + \frac{\hbar^2}{2!} \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ f \quad g \end{array} + \frac{\hbar^2}{3} \left(\begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ f \quad g \end{array} - \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ f \quad g \end{array} \right) + \frac{\hbar^2}{6} \begin{array}{c} \bullet \text{---} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ f \quad g \end{array} + \bar{o}(\hbar^2). \quad (4)
 \end{aligned}$$

Referred to any system of affine local coordinates $\mathbf{u} = (u^1, \dots, u^n)$ on $U_\alpha \subseteq N^n$, for a given Poisson bi-vector $\mathcal{P}|_{\mathbf{u}} = \frac{1}{2} \langle \xi_i P^{ij}(\mathbf{u}) \xi_j \rangle$ this sum of weighted graphs is realised by the formula⁴

$$\begin{aligned}
 f \star_{\hbar} g = & f \times g + \frac{1}{1!} (f) \overleftarrow{\frac{\partial}{\partial u^i}} \cdot \hbar P^{ij} \cdot \overrightarrow{\frac{\partial}{\partial u^j}} (g) + \frac{1}{2!} (f) \overleftarrow{\frac{\partial}{\partial u^{i_1}}} \overleftarrow{\frac{\partial}{\partial u^{i_2}}} \cdot \hbar^2 P^{i_1 j_1} P^{i_2 j_2} \cdot \overrightarrow{\frac{\partial}{\partial u^{j_2}}} \overrightarrow{\frac{\partial}{\partial u^{j_1}}} (g) \\
 & + \frac{1}{3} \left\{ (f) \overleftarrow{\frac{\partial}{\partial u^i}} \overleftarrow{\frac{\partial}{\partial u^k}} \cdot \hbar P^{ij} \cdot \overrightarrow{\frac{\partial}{\partial u^j}} (\hbar P^{k\ell}) \cdot \overrightarrow{\frac{\partial}{\partial u^\ell}} (g) - (f) \overleftarrow{\frac{\partial}{\partial u^k}} \cdot (\hbar P^{k\ell}) \overleftarrow{\frac{\partial}{\partial u^j}} \cdot \hbar P^{ij} \cdot \overrightarrow{\frac{\partial}{\partial u^i}} (g) \right\} \\
 & + \frac{1}{6} (f) \overleftarrow{\frac{\partial}{\partial u^i}} \cdot (\hbar P^{ij}) \overleftarrow{\frac{\partial}{\partial u^k}} \cdot \overrightarrow{\frac{\partial}{\partial u^j}} (\hbar P^{k\ell}) \cdot \overrightarrow{\frac{\partial}{\partial u^\ell}} (g) + \bar{o}(\hbar^2). \quad (5)
 \end{aligned}$$

The values of (derivatives of) both arguments and coefficients of the Poisson bi-vector \mathcal{P} are calculated at $\mathbf{u} \in U_\alpha \subseteq N^n$ in the right-hand side of the above formula.

EXAMPLE 3 (Moyal–Weyl–Groenewold). Suppose that all coefficients P^{ij} of the Poisson bi-vector \mathcal{P} are constant, which is a well defined property with respect to all local coordinate systems on the affine manifold N^n at hand. In effect, the graphs with at least one arrow arriving at a vertex containing \mathcal{P} make no contribution to the star-product \star . The only contributing graphs are portrayed in this figure,

$$f \star g = \begin{array}{c} \bullet \text{---} \bullet \\ f \quad g \end{array} + \frac{\hbar^1}{1!} \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ f \quad g \end{array} + \frac{\hbar^2}{2!} \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ f \quad g \end{array} + \frac{\hbar^3}{3!} \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ \swarrow \quad \searrow \\ \bullet \text{---} \bullet \\ f \quad g \end{array} + \dots + \frac{\hbar^k}{k!} \begin{array}{c} \bullet \\ \vdots \\ \bullet \text{---} \bullet \\ f \quad g \end{array} + \dots$$

³The precedence-antecedence of edges is given by the ordering of indexes $i < j, i_1 < j_1, i_2 < j_2$, and $k < \ell$ in the analytic formula, see (5).

⁴Note that a graph itself suggests the easiest-to-read way to write down the respective differential operator’s formula; this will be particularly convenient in the variational setting of Section 2, see p. 233.

These graphs are such that their weights in the power series combine it to the Moyal exponent,

$$(f \star g)(\mathbf{u}; \hbar) = \left[(f(\mathbf{u})) \exp\left(\overleftarrow{\frac{\partial}{\partial u^i}} \cdot \hbar P^{ij} \cdot \overrightarrow{\frac{\partial}{\partial v^j}}\right) (g(\mathbf{v})) \right] \Big|_{\mathbf{u}=\mathbf{v}}. \tag{2}$$

Here we accept that the use of every next copy of the bi-vector \mathcal{P} creates a new pair of summation indexes.

REMARK 3. The introduction of two identical copies, $\mathbf{u} \in U_\alpha \subseteq N^n$ and $\mathbf{v} \in U_\alpha \subseteq N^n$, of the geometry where the objects f and g are defined reveals an idea that will be used heavily in what follows.

PROPOSITION 5. *The associativity of Moyal star-product (2) is established by the a posteriori congruence mechanism.*

Proof (see [8]). From the identity

$$(f(u) \times g(v)|_{u=v}) \overleftarrow{\frac{\partial}{\partial u}} = \left[(f(u) \times g(v)) \left(\overleftarrow{\frac{\partial}{\partial u}} + \overleftarrow{\frac{\partial}{\partial v}} \right) \right] \Big|_{u=v}$$

we infer that

$$\begin{aligned} & ((f \star g) \star h - f \star (g \star h))(\mathbf{u}; \hbar) \\ &= \left[\left[(f|_{\mathbf{u}}) \exp\left(\overleftarrow{\frac{\partial}{\partial u^i}} \hbar P^{ij} \overrightarrow{\frac{\partial}{\partial v^j}}\right) (g|_{\mathbf{v}}) \right] \Big|_{\mathbf{u}=\mathbf{v}} \exp\left(\overleftarrow{\frac{\partial}{\partial u^k}} \hbar P^{k\ell} \overrightarrow{\frac{\partial}{\partial w^\ell}}\right) (h|_{\mathbf{w}}) \right] \Big|_{\mathbf{u}=\mathbf{w}} \\ &\quad - \left[(f|_{\mathbf{u}}) \exp\left(\overleftarrow{\frac{\partial}{\partial u^i}} \hbar P^{ij} \overrightarrow{\frac{\partial}{\partial v^j}}\right) \left[(g|_{\mathbf{v}}) \exp\left(\overleftarrow{\frac{\partial}{\partial v^k}} \hbar P^{k\ell} \overrightarrow{\frac{\partial}{\partial w^\ell}}\right) (h|_{\mathbf{w}}) \right] \Big|_{\mathbf{v}=\mathbf{w}} \right] \Big|_{\mathbf{u}=\mathbf{v}} \\ &= \left[(f|_{\mathbf{u}}) \exp\left(\overleftarrow{\frac{\partial}{\partial u^i}} \hbar P^{ij} \overrightarrow{\frac{\partial}{\partial v^j}}\right) (g|_{\mathbf{v}}) \exp\left(\left(\overleftarrow{\frac{\partial}{\partial u^k}} + \overleftarrow{\frac{\partial}{\partial v^k}}\right) \cdot \hbar P^{k\ell} \overrightarrow{\frac{\partial}{\partial w^\ell}}\right) (h|_{\mathbf{w}}) \right] \Big|_{\mathbf{u}=\mathbf{v}=\mathbf{w}} \\ &\quad - \left[(f|_{\mathbf{u}}) \exp\left(\overleftarrow{\frac{\partial}{\partial u^i}} \hbar P^{ij} \cdot \left(\overrightarrow{\frac{\partial}{\partial v^j}} + \overrightarrow{\frac{\partial}{\partial w^j}}\right)\right) (g|_{\mathbf{v}}) \exp\left(\overleftarrow{\frac{\partial}{\partial v^k}} \hbar P^{k\ell} \overrightarrow{\frac{\partial}{\partial w^\ell}}\right) (h|_{\mathbf{w}}) \right] \Big|_{\mathbf{u}=\mathbf{v}=\mathbf{w}} \equiv 0, \end{aligned}$$

which is due to the Baker–Campbell–Hausdorff formula for the exponent of sums of commuting derivatives, and by having indexes relabelled. ■

1.4. Whenever the coefficients $P^{ij}(\mathbf{u})$ are not constant on the domain $U_\alpha \subseteq N^n$, the classical master-equation⁵ $[[\mathcal{P}, \mathcal{P}]] = 0$ is a nontrivial constraint for the bi-vector \mathcal{P} . Where is the Jacobi identity for the Poisson bracket $\{\cdot, \cdot\}_{\mathcal{P}}$ hidden in the associator $(f \star_{\hbar} g) \star_{\hbar} h - f \star_{\hbar} (g \star_{\hbar} h)$ for the full star-product?

EXAMPLE 4. It is easy to see that $\text{Assoc}_{\star_{\hbar}}(f, g, h) = \frac{2}{3} \text{Jac}_{\mathcal{P}}(f, g, h) + \bar{o}(\hbar^2)$.

⁵ The Jacobi identity for Poisson bracket $\{\cdot, \cdot\}_{\mathcal{P}}$ is equivalent to the zero-value condition $[[\mathcal{P}, \mathcal{P}]](f, g, h) = 0$ for all Hamiltonians f, g, h ; the tri-vector $[[\mathcal{P}, \mathcal{P}]]$ is viewed here as a tri-linear totally antisymmetric mapping and we denote by $[[\cdot, \cdot]]$ the Schouten bracket (i.e. parity-odd Poisson bracket); in coordinates, one proves that $[[\mathcal{P}, \mathcal{P}]] \stackrel{\text{Th.}}{=} (\mathcal{P}) \overleftarrow{\frac{\partial}{\partial u^i}} \cdot \overrightarrow{\frac{\partial}{\partial \xi_i}} (\mathcal{P}) - (\mathcal{P}) \overleftarrow{\frac{\partial}{\partial \xi_i}} \cdot \overrightarrow{\frac{\partial}{\partial u^i}} (\mathcal{P})$.

By definition, we put

$$\begin{array}{c} \boxed{\bullet \bullet} \\ \downarrow \quad \downarrow \quad \downarrow \\ 1 \quad 2 \quad 3 \end{array} := \begin{array}{c} \bullet \\ \swarrow \quad \downarrow \quad \searrow \\ i \quad j \quad k \\ \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \end{array} - \begin{array}{c} \bullet \\ \swarrow \quad \downarrow \quad \searrow \\ i \quad j \quad k \\ \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \end{array} - \begin{array}{c} \bullet \\ \swarrow \quad \downarrow \quad \searrow \\ i \quad j \quad k \\ \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \end{array} = 0. \tag{6}$$

In formulae, by ascribing the index ℓ to the unlabeled edge, the identity reads

$$(\partial_\ell \mathcal{P}^{ij} \mathcal{P}^{\ell k} + \partial_\ell \mathcal{P}^{jk} \mathcal{P}^{\ell i} + \partial_\ell \mathcal{P}^{ki} \mathcal{P}^{\ell j}) \partial_i(1) \partial_j(2) \partial_k(3) = 0.$$

The coefficient of $\partial_i \otimes \partial_j \otimes \partial_k$ is the familiar form of the Jacobi identity.

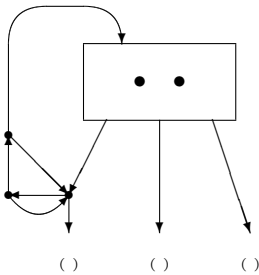
To understand how sums of graphs can vanish by virtue of differential consequences of Jacobi identity (6), let us note that for a given Poisson bi-vector \mathcal{P} and for every derivation ∂_i falling on the Jacobiator $\text{Jac}_{\mathcal{P}}(a, b, c)$, the Leibniz rule yields

$$\partial_i(\text{Jac}_{\mathcal{P}}(a, b, c)) = (\partial_i(\text{Jac}(\mathcal{P}))) (a, b, c) + \text{Jac}_{\mathcal{P}}(\partial_i a, b, c) + \text{Jac}_{\mathcal{P}}(a, \partial_i b, c) + \text{Jac}_{\mathcal{P}}(a, b, \partial_i c).$$

The last three terms in the right-hand side of the above formula amount to a redefinition of Jacobiator’s arguments; hence every such term vanishes. Consequently, the first term in which the derivation ∂_i acts on the two internal vertices of the Jacobiator itself is equal to zero: $(\partial_i(\text{Jac}(\mathcal{P}))) (\cdot, \cdot, \cdot) = 0$. One now proceeds recursively over arbitrarily large finite set of derivations that simultaneously fall on the Jacobiator, then acting independently from each other according to the Leibniz rule.

DEFINITION 1. A *Leibniz graph* is a graph whose vertices are either sinks, or the sources for two arrows, or the Jacobiator (which is a source for three arrows); there must be at least one Jacobiator vertex. The three arrows originating from a Jacobiator vertex must land on three distinct vertices (and not on the Jacobiator itself).⁶

EXAMPLE 5. An example of a Leibniz graph is given in Fig. 2.



- There is a cycle,
- there is a loop,
- there are no tadpoles in this graph,
- an arrow falls back on $\text{Jac}(\mathcal{P})$,
- and $\text{Jac}(\mathcal{P})$ does not stand on all of the three sinks.

Fig. 2. An example of Leibniz graph

Leibniz graphs encode (poly)differential operators $\diamond(\mathcal{P}, \text{Jac}(\mathcal{P}))$ whose arguments are at least one copy of the tri-vector $\text{Jac}(\mathcal{P})$ and possibly, the bi-vector \mathcal{P} itself.⁷

PROPOSITION 6. For every Poisson bi-vector \mathcal{P} the value—at the Jacobiator $\text{Jac}(\mathcal{P})$ —of every (poly)differential operator encoded by the Leibniz graph(s) is zero.

⁶Each edge falling on a Jacobiator works by the Leibniz rule on the two internal vertices in it. Combined with expansion (6) of the Jacobiator using graphs, this tells us that every Leibniz graph expands to a sum of Kontsevich graphs which were introduced before.

⁷In Example 4 the Leibniz graph amounts to just one tri-vector vertex and no extra copies of the Poisson bi-vector in other internal vertices, of which there are none.

COROLLARY 7. *To prove that the associator for the Kontsevich star-product \star_{\hbar} vanishes for every Poisson structure contained in each internal vertex within a graph expansion of \star_{\hbar} , it suffices to realize the associator as a sum of Leibniz graphs:*⁸

$$\text{Assoc}_{\star_{\hbar}}(f, g, h) = \diamond(\mathcal{P}, \text{Jac}_{\mathcal{P}}(\cdot, \cdot, \cdot))(f, g, h). \tag{1}$$

From Example 4 we already know that the factorizing polydifferential operator in (1) is $\diamond = \frac{2}{3} \mathbf{1} + \bar{o}(1)$.

EXAMPLE 6. The assembly of factorizing operator $\diamond \bmod \bar{o}(\hbar)$, i.e. at order 3 in the expansion $\text{Assoc}_{\star_{\hbar}}(\cdot, \cdot, \cdot) \bmod \bar{o}(\hbar^3)$, is explained in [5]; linear in its argument at \hbar^1 , the operator $\diamond \bmod \bar{o}(\hbar)$ has differential order one with respect to the Jacobiator.

The next step $\diamond \bmod \bar{o}(\hbar^2)$ in factorization (1), now at order 4 with respect to \hbar in the associator, is achieved in [6].

PROPOSITION 8 ([6]). *No solution $\diamond \bmod \bar{o}(\hbar^2)$ of factorization problem (1) can have differential order less than two with respect to the Jacobiator $\text{Jac}(\mathcal{P})$; conversely, there always exists a Leibniz graph at \hbar^2 in the polydifferential operator \diamond such that at least two arrows fall on the Jacobiator.*

2. Deformation quantisation $\times \mapsto \star_{\hbar}$ in the algebras \mathcal{A} of local functionals for field models.

In this section we lift the Kontsevich graph technique from a quantisation $\times \mapsto \star_{\hbar}$ of the product \times in the algebra \mathcal{A} of smooth functions on a finite-dimensional affine Poisson manifold (N^n, \mathcal{P}) to the deformation $\times \mapsto \star_{\hbar}$ of the product of local functionals in the geometry of N^n -valued fields over an affine base manifold M^m . We shall analyse the construction of local variational polydifferential operators which are encoded by the Kontsevich graphs (in particular, by the Leibniz graphs), now containing at each internal vertex a copy of the variational Poisson structure $\{\cdot, \cdot\}_{\mathcal{P}}$. It is the Gel'fand formalism of singular linear integral operators supported on the diagonal [15] that becomes our working language.

2.1. Field model geometry. To extend the affine geometry of Section 1, let us list the ingredients of the affine bundle set-up.⁹

Let $(M^m, \text{dvol}(\cdot))$ be an m -dimensional oriented affine real manifold equipped with a volume element.¹⁰ Let $\pi : E^{m+n} \rightarrow M^m$ be an affine bundle with n -dimensional fibres N^n over the base M^m . Denote by $\mathbf{u} = (u^1, \dots, u^n)$ an n -tuple of local coordinates in the fibre N^n .

Denote by $J^\infty(\pi)$ the total space of the bundle π_∞ of infinite jets $j^\infty(\mathbf{s})(\cdot)$ for sections $\mathbf{s} \in \Gamma(\pi)$ of the bundle π over M^m ; the infinite jet space $J^\infty(\pi)$ is the projective limit $\text{proj} \lim_{k \rightarrow +\infty} J^k(\pi)$ of the sequence of finite jet spaces $J^k(\pi)$,

$$M^m \xleftarrow{\pi} E^{m+n} = J^0(\pi) \leftarrow J^1(\pi) \leftarrow \dots \leftarrow J^k(\pi) \leftarrow \dots \leftarrow J^\infty(\pi).$$

⁸The same technique, showing the vanishing of a sum of Kontsevich graphs by writing it as a sum of Leibniz graphs, has been used in [4] to solve another problem in the graph calculus.

⁹In retrospect, the construction in Section 1 can be viewed as a special case of such “bundles” over a point M^0 .

¹⁰Not excluding the case where the volume element $\text{dvol}(\mathbf{x})$ can nontrivially depend on the jets $j_{\mathbf{x}}^\infty(\phi)$ of sections $\phi \in \Gamma(\pi)$ over points $\mathbf{x} \in M^m$, for the sake of brevity let us not write such admissible second argument in $\text{dvol}(\cdot, j_\infty(\phi)(\cdot))$.

It is clear that affine reparametrisations $\tilde{\mathbf{x}}(\mathbf{x})$ of local coordinates on the base M^m induce linear transformations of smooth sections' derivatives up to positive order k for all $k > 0$. By definition, we put $[\mathbf{u}]$ for an object's dependence on sections \mathbf{s} and their derivatives up to arbitrarily large but still finite order, which is well defined by the above.

Denote by $\bar{H}^m(\pi)$ the vector space of integral functionals $\Gamma(\pi) \rightarrow \mathbb{k}$ of form $F = \int f(\mathbf{x}_1, [\mathbf{u}]) \cdot \text{dvol}(\mathbf{x}_1)$ such that $F(\mathbf{s}) = \int_{M^m} f(\mathbf{x}_1, j_{\mathbf{x}_1}^\infty(\mathbf{s})) \cdot \text{dvol}(\mathbf{x}_1)$. Viewed as functionals $\Gamma(\pi) \rightarrow \mathbb{k}$ that take sections $\phi \in \Gamma(\pi)$ over M^m to numbers, the integral objects $F, G, H \in \bar{H}^m(\pi)$ can be shifted by using the null functionals $Z : \Gamma(\pi) \rightarrow 0 \in \mathbb{k}$. Those can be of topological nature,¹¹ $Z \in H^m(\pi)$. We always quotient them out in this paper by taking the factorgroup $\bar{H}^m(\pi)/H^m(\pi)$. Secondly, null integral functionals $\Gamma(\pi) \rightarrow 0 \in \mathbb{k}$ can mark the zero class $\int \text{d}_h(\Theta) \cong \int 0 \in \bar{H}^m(\pi)$ in the top-degree horizontal cohomology group¹² for $J^\infty(\pi)$ over M^m .

By brute force, introduce the multiplication $\times : F \otimes G \mapsto F \times G = \int f(\mathbf{x}_1, [\mathbf{u}]) \cdot \text{dvol}(\mathbf{x}_1) \times \int g(\mathbf{x}_2, [\mathbf{u}]) \cdot \text{dvol}(\mathbf{x}_2) = \iint f(\mathbf{x}_1, [\mathbf{u}]) \times g(\mathbf{x}_2, [\mathbf{u}]) \cdot \text{dvol}(\mathbf{x}_1) \text{dvol}(\mathbf{x}_2) : \Gamma(\pi) \rightarrow \mathbb{k}$ for $G = \int g(\mathbf{x}_2, [\mathbf{u}]) \cdot \text{dvol}(\mathbf{x}_2)$. This yields the algebra \mathcal{A} of local functionals,¹³ also denoted by $\bar{\mathfrak{M}}^m(\pi)$ in [19, 22].

Referring only to the fibre's local portrait but not to its global organisation, we introduce the \mathbb{Z}_2 -parity odd coordinates $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$ in the reversed-parity cotangent spaces $\Pi T_{(\mathbf{x}, \mathbf{s}(\mathbf{x}))}^* N^n$ to the fibres $N^n \simeq \pi^{-1}(\mathbf{x})$ of the bundle π , see [22, §2.1] and [19, §2.1]. Let us note that for vector spaces $N^n = \mathbb{R}^n$, the vector space isomorphism $T_{(\mathbf{x}, \mathbf{s}(\mathbf{x}))} N^n \simeq N^n$ reduces this construction of Kupershmidt's variational cotangent bundle [33] over $\pi_\infty : J^\infty(\pi) \rightarrow M^m$ to the Whitney sum $J^\infty(\pi \times_M \Pi\hat{\pi})$.

CONVENTION 2. The notation $\pi \times_M \Pi\hat{\pi}$ will be used in what follows to avoid an agglomeration of formulae. Indeed, the case of affine bundle π already impels the construction of horizontal jet bundle $\overline{J_{\pi_\infty}^\infty}(\Pi T^*\pi)$ over the space $J^\infty(\pi)$.

The variational bi-vectors $\mathcal{P} \in \bar{H}^m(\pi \times_M \Pi\hat{\pi})$ are integral functionals of the form

$$\mathcal{P} = \frac{1}{2} \int \langle \boldsymbol{\xi} \cdot A|_{(\mathbf{x}, [\mathbf{u}])}(\boldsymbol{\xi}) \rangle = \frac{1}{2} \int \xi_i P_\tau^{ij}(\mathbf{x}, [\mathbf{u}]) \xi_{j,\tau} \cdot \text{dvol}(\mathbf{x}),$$

¹¹For instance, set $m = 1$, let $M^m := \mathbb{S}^1 \cup \mathbb{S}^1$, take the usual angle variables $\varphi_1, \varphi_2 : \mathbb{R}^1 \rightarrow \mathbb{S}^1$ on the two circles, and consider the null Lagrangian $\mathcal{L} = \int \text{d}\varphi_1 - \int \text{d}\varphi_2$ that takes every section of an affine bundle π over such M^1 to $2\pi - 2\pi = 0 \in \mathbb{k}$. Obviously, the cohomology class \mathcal{L} in $H^1(\pi)$ is nonzero for the top-degree form $\text{d}\varphi_1 - \text{d}\varphi_2$; for it is only locally but not globally exact.

¹²The integrations by parts \cong over M^m are nominally present in the construction of horizontal cohomology groups $\bar{H}^i(\pi)$ for the jet space $J^\infty(\pi)$ over the bundle π . By default, let us technically assume that no boundary terms would ever appear from ∂M^m in any formulae. For that, either let the base M^m be a closed manifold (hence $\partial M^m = \emptyset$; for instance, take $M^1 = \mathbb{S}^1$) or choose the class $\Gamma(\pi)$ of admissible sections for the bundle π in such a way that they decay so rapidly towards the boundary ∂M^m that all the integrands under study also vanish at ∂M^m . For instance, suppose that $M^1 = \mathbb{R}$ and all the field profiles $\mathbf{u} = \phi(\mathbf{x})$ are Schwarz.

¹³We recall that in the (graded-)commutative set-up one has $(F \overset{\bar{\mathfrak{M}}^m(\pi)}{\times} G)(\mathbf{s}) = F(\mathbf{s}) \overset{\mathbb{k}}{\times} G(\mathbf{s})$ but a known mechanism destroys this algebra homomorphism in a larger setting of formal non-commutative variational symplectic geometry and its calculus of cyclic words ([22], cf. [27, 39]).

where the linear total differential operators $A = \parallel P_\tau^{ij} \cdot \left(\frac{d}{d\mathbf{x}}\right)^\tau \parallel_{i=1, \dots, n}^{j=1, \dots, n}$ are skew-adjoint (to make the object \mathcal{P} well defined); for all multi-indexes τ , the parity-odd symbols $\xi_{j, \emptyset} = \xi_j$, $\xi_{j, x^k}, \xi_{j, x^k x^\ell}, \dots, \xi_{j, \tau}, \dots$ are the respective jet fibre coordinates.

The construction of variational k -vectors with $k \geq 0$ is alike, see [22]. Due to the introduction of parity-odd variables ξ as canonical conjugates of the n -tuples \mathbf{u} , the vector space of all variational multivectors is naturally endowed with the parity-odd variational Poisson bracket, or variational *Schouten bracket* $[[\cdot, \cdot]]$. Its construction— as descendent structure with respect to the Batalin–Vilkovisky Laplacian Δ —was recalled in [19, 22].

DEFINITION 2. A variational bi-vector \mathcal{P} is called *Poisson* if it satisfies the classical master-equation $[[\mathcal{P}, \mathcal{P}]] \cong 0$. The horizontal cohomology class equivalence $\cong 0$ means, in particular, that the variational tri-vector $[[\mathcal{P}, \mathcal{P}]]$, viewed as an integral functional, takes $\Gamma(\pi) \rightarrow 0 \in \mathbb{k}$.

Every variational Poisson bi-vector \mathcal{P} induces the respective variational Poisson bracket $\{\cdot, \cdot\}_\mathcal{P} : \bar{H}^m(\pi) \times \bar{H}^m(\pi) \rightarrow \bar{H}^m(\pi)$ on the space of integral functionals $\Gamma(\pi) \rightarrow \mathbb{k}$. An axiomatic construction of $\{\cdot, \cdot\}_\mathcal{P}$ is explained in Definition 3; it is the *derived bracket* $[[[\mathcal{P}, \cdot], \cdot]]$ of two Hamiltonians (see [22, §3]).¹⁴

The bracket $\{\cdot, \cdot\}_\mathcal{P}$ is extended, via the Leibniz rule, from the vector space $\bar{H}^m(\pi)$ of integral functionals H_1, H_2, \dots to the Poisson structure on the algebra \mathcal{A} of (sums of) such functionals' formal products $H_1 \times \dots \times H_\ell : \Gamma(\pi) \rightarrow \mathbb{k}$.

REMARK 4. The value of $\{\cdot, \cdot\}_\mathcal{P}$ in $\bar{H}^m(\pi)$ at two integral functionals does not depend on a choice of representatives for the two arguments and for the variational Poisson bi-vector $\mathcal{P} \in \bar{H}^m(\pi \times_M \Pi\hat{\pi})$, taken modulo those integral functionals which map all sections of the respective (super)bundle to $0 \in \mathbb{k}$. This is no longer necessarily so for the higher-order terms, beyond the variational Poisson bracket $\{\cdot, \cdot\}_\mathcal{P}$ at \hbar^1 , in expansions (4).

REMARK 5. Let us remember that every integral functional—e.g., taken as a building block in a local functional—does carry its own integration variable which runs through that integral functional's own copy of the base M^m for the respective (super)bundle. For a given model over $(M^m, \text{dvol}(\cdot))$, the variational Poisson bi-vector

$$\mathcal{P} = \frac{1}{2} \int \xi_i P_\tau^{ij}(\mathbf{x}, [\mathbf{u}]) \left(\frac{d}{d\mathbf{x}}\right)^\tau (\xi_j) \cdot \text{dvol}(\mathbf{x})$$

and two Hamiltonians, $F = \int f(\mathbf{x}_1, [\mathbf{u}]) \cdot \text{dvol}(\mathbf{x}_1)$ and $G = \int g(\mathbf{x}_2, [\mathbf{u}]) \cdot \text{dvol}(\mathbf{x}_2)$, are integral functionals defined at sections of the bundles $\pi \times_M \Pi\hat{\pi}$ and π , respectively. In total, these three objects carry *three* copies of the given volume element $\text{dvol}(\cdot)$ on M^m .

On the other hand, the variational Poisson bracket $\{F, G\}_\mathcal{P}$ of F and G with respect to \mathcal{P} is an integral functional $\Gamma(\pi) \rightarrow \mathbb{k}$ that carries *one* copy of the volume element. Why and where to have the two copies of $\text{dvol}(\cdot)$ gone? We now recall an answer to this question.

¹⁴Note that an attempt to modify the volume element $\text{dvol}(\cdot)$ on M^m can affect the output of $\{\cdot, \cdot\}_\mathcal{P}$.

2.2. Elements of the geometry of iterated variations

2.2.1. Let $(\mathbf{s}, \mathbf{s}^\dagger)$ be a two-component section of the Whitney sum $\pi \times_M \Pi \hat{\pi}$ of bundles. Suppose that this section undergoes an infinitesimal shift along the direction

$$(\delta \mathbf{s}, \delta \mathbf{s}^\dagger)(\mathbf{x}, \mathbf{s}(\mathbf{x}), \mathbf{s}^\dagger(\mathbf{x})) = \sum_{i=1}^n (\delta s^i(\mathbf{x}) \cdot \vec{e}_i(\mathbf{x}) + \delta s_i^\dagger(\mathbf{x}) \cdot \vec{e}^{\dagger,i}(\mathbf{x})),$$

which we decompose with respect to the adapted basis $(\vec{e}_i, \vec{e}^{\dagger,i})$ in the tangent space $T_{(\mathbf{x}, \mathbf{s}(\mathbf{x}))} \pi^{-1}(\mathbf{x}) \oplus T_{\mathbf{s}^\dagger(\mathbf{x})} T_{(\mathbf{x}, \mathbf{s}(\mathbf{x}))}^* \pi^{-1}(\mathbf{x})$. At their attachment point, the vectors \vec{e}_i and $\vec{e}^{\dagger,j}$ are—by definition—tangent to the respective coordinate lines for variables u^i and ξ_j . By construction, these vectors \vec{e}_i and $\vec{e}^{\dagger,j}$ are dual; at every i running from 1 to n , the two *ordered* couplings of (co)vectors attached over $\mathbf{x} \in M^m$ at the fibres' points—with values $\mathbf{s}(\mathbf{x})$ and $\mathbf{s}^\dagger(\mathbf{x})$ of the respective coordinates—are

$$\langle \overset{\text{first}}{\vec{e}_i}, \overset{\text{second}}{\vec{e}^{\dagger,i}} \rangle = +1 \quad \text{and} \quad \langle \overset{\text{first}}{\vec{e}^{\dagger,i}}, \overset{\text{second}}{\vec{e}_i} \rangle = -1. \tag{7}$$

Likewise, the coefficients $\delta s^i(\cdot, \mathbf{s}(\cdot))$ and $\delta s_i^\dagger(\cdot, \mathbf{s}(\cdot), \mathbf{s}^\dagger(\cdot))$ of the virtual shifts along the i th coordinate lines u^i and ξ_i are normalised via

$$\delta s^i(\mathbf{x}, \mathbf{s}(\mathbf{x})) \cdot \delta s_i^\dagger(\mathbf{x}, \mathbf{s}(\mathbf{x}), \mathbf{s}^\dagger(\mathbf{x})) \equiv 1 \quad (\text{no summation!}) \tag{8}$$

over all internal points $\mathbf{x} \in \text{supp } \delta s^i \subseteq M^m$.

CONVENTION 3. The differentials of functionals' densities are expanded with respect to the bases $\vec{e}^{\dagger,j}, \vec{e}_i$ in the fibres tangent spaces; the plus or minus signs in the sections' shifts are chosen in such a way that the couplings always evaluate to +1.

The directed variations $\overrightarrow{\delta \mathbf{s}}$ and $\overleftarrow{\delta \mathbf{s}^\dagger}$, as well as $\overleftarrow{\delta \mathbf{s}}$ and $\overrightarrow{\delta \mathbf{s}^\dagger}$, are singular linear integral operators supported, due to (7), on the diagonal. Each variation contains n copies of Dirac's δ -distribution weighted by the respective coefficients δs^i and δs_i^\dagger . We have

$$\begin{aligned} \overrightarrow{\delta \mathbf{s}} &= \int d\mathbf{y} \left\langle (\delta s^i) \left(\frac{\overleftarrow{\partial}}{\partial \mathbf{y}} \right)^\sigma (\mathbf{y}) \cdot \overbrace{\vec{e}_i(\mathbf{y}) | \vec{e}^{\dagger,i}(\cdot)}^{\text{first second}} \right\rangle \frac{\overrightarrow{\partial}}{\partial u_\sigma^i}, \\ \overleftarrow{\delta \mathbf{s}^\dagger} &= \int d\mathbf{z} \left\langle (\delta s_i^\dagger) \left(\frac{\overleftarrow{\partial}}{\partial \mathbf{z}} \right)^\sigma (\mathbf{z}) \cdot \overbrace{(-\vec{e}^{\dagger,i})(\mathbf{z}) | \vec{e}_i(\cdot)}^{\text{first second}} \right\rangle \frac{\overrightarrow{\partial}}{\partial \xi_{i,\sigma}}, \\ \overleftarrow{\delta \mathbf{s}} &= \int d\mathbf{y} \frac{\overleftarrow{\partial}}{\partial u_\sigma^i} \left\langle \overbrace{\vec{e}^{\dagger,i}(\cdot) | \vec{e}_i(\mathbf{y})}^{\text{second first}} \cdot \left(\frac{\overrightarrow{\partial}}{\partial \mathbf{y}} \right)^\sigma (\delta s^i)(\mathbf{y}) \right\rangle, \\ \overrightarrow{\delta \mathbf{s}^\dagger} &= \int d\mathbf{z} \frac{\overleftarrow{\partial}}{\partial \xi_{i,\sigma}} \left\langle \overbrace{\vec{e}_i(\cdot) | (-\vec{e}^{\dagger,i})(\mathbf{z})}^{\text{second first}} \cdot \left(\frac{\overrightarrow{\partial}}{\partial \mathbf{z}} \right)^\sigma (\delta s_i^\dagger)(\mathbf{y}) \right\rangle, \end{aligned}$$

see [19, §2.2–3] for details; for brevity, the indication of fibre points for given $\mathbf{s}(\cdot)$ and $\mathbf{s}^\dagger(\cdot)$ is omitted in such formulae. Whenever acting on local functionals, these linear operators yield those functionals' responses to infinitesimal shifts of their arguments, i.e. of the sections at which the functionals are evaluated.

Let us now recall the mechanism of integration by parts (see [19] and [22, §2.5]).

LEMMA 9. *In absence of boundary terms, the on-the-diagonal integration by parts converts derivatives along one copy of the integration domain M^m into $(-1) \times$ derivatives with respect to the same variables, now referred to another copy of the base.*

Proof. Consider a point \mathbf{y} of the affine manifold M^n and denote by $\mathbf{y} + \delta\mathbf{y} \in M^n$ a near-by point with coordinates $y^i + \delta y^i$, here and immediately below $1 \leq i, \alpha \leq n$; the notation $\lim_{\delta\mathbf{y} \rightarrow 0}$ makes obvious sense. For the sake of brevity, put $\sigma := \{x^\alpha\}$. We have that, due to the absence of boundary terms and then by definition,¹⁵

$$\begin{aligned} & \int d\mathbf{y} \left\langle (\delta s^i) \overleftarrow{\frac{\partial}{\partial y^\alpha}}(\mathbf{y}) \cdot \overrightarrow{e_i(\mathbf{y}), e^{\dagger,i}(\mathbf{x})} \overrightarrow{\frac{\partial}{\partial u_{x^\alpha}^i}} f(\mathbf{x}, [\mathbf{u}], [\boldsymbol{\xi}]) \Big|_{j_\infty^\sigma(s, s^\dagger)} \right\rangle \\ &= \int d\mathbf{y} \delta s^i(\mathbf{y}) \left(-\overrightarrow{\frac{\partial}{\partial y^\alpha}} \right) \left\langle \overrightarrow{e_i(\mathbf{y}), e^{\dagger,i}(\mathbf{x})} \overrightarrow{\frac{\partial}{\partial u_{x^\alpha}^i}} f(\mathbf{x}, [\mathbf{u}], [\boldsymbol{\xi}]) \Big|_{j_\infty^\sigma(s, s^\dagger)} \right\rangle \\ &\stackrel{\text{def}}{=} - \int d\mathbf{y} \delta s^i(\mathbf{y}) \lim_{\delta y^\alpha \rightarrow +0} \frac{1}{\delta y^\alpha} \left\{ \begin{array}{l} \left\langle \overrightarrow{e_i(\mathbf{y} + \delta y^\alpha), e^{\dagger,i}(\mathbf{x})} \overrightarrow{\frac{\partial}{\partial u_{x^\alpha}^i}} f(\mathbf{x}, [\mathbf{u}], [\boldsymbol{\xi}]) \Big|_{j_\infty^\sigma(s, s^\dagger)} \right\rangle \\ \quad +1 \text{ if } \mathbf{x} = \mathbf{y} + \delta y^\alpha \\ - \left\langle \overrightarrow{e_i(\mathbf{y}), e^{\dagger,i}(\mathbf{x})} \overrightarrow{\frac{\partial}{\partial u_{x^\alpha}^i}} f(\mathbf{x}, [\mathbf{u}], [\boldsymbol{\xi}]) \Big|_{j_\infty^\sigma(s, s^\dagger)} \right\rangle \\ \quad +1 \text{ if } \mathbf{x} = \mathbf{y} \end{array} \right\} \\ &\stackrel{\text{def}}{=} \int d\mathbf{y} \delta s^i(\mathbf{y}) \left\langle \overrightarrow{e_i(\mathbf{y}), e^{\dagger,i}(\mathbf{x})} \left(-\overrightarrow{\frac{\partial}{\partial x^\alpha}} \right) \overrightarrow{\frac{\partial}{\partial u_{x^\alpha}^i}} f(\mathbf{x}, [\mathbf{u}], [\boldsymbol{\xi}]) \Big|_{j_\infty^\sigma(s, s^\dagger)} \right\rangle \\ &\stackrel{\text{def}}{=} \int d\mathbf{y} \delta s^i(\mathbf{y}) \left\langle \overrightarrow{e_i(\mathbf{y}), e^{\dagger,i}(\mathbf{x})} \cdot \left(\left(-\overrightarrow{\frac{d}{dx^\alpha}} \right) \overrightarrow{\frac{\partial}{\partial u_{x^\alpha}^i}} f(\mathbf{x}, [\mathbf{u}], [\boldsymbol{\xi}]) \right) \Big|_{j_\infty^\sigma(s, s^\dagger)} \right\rangle. \end{aligned}$$

For multi-indexes σ longer than $\{x^\alpha\}$ the powers $(\overleftarrow{\partial}/\partial\mathbf{y})^\sigma$ are processed by repeated integrations by parts; this yields $(-\overrightarrow{d}/d\mathbf{x})^\sigma$. In the course of derivation of densities with respect to not a_σ^i but $b_{j,\tau}$ and so, in the course of using the other of two (co)vectors' couplings, all reasonings are still performed in exactly the same way. ■

COROLLARY 10. *The derivatives with respect to base variables are transported along an edge to the arrowhead according to the scenarios drawn in Fig. 3; each derivative is referred to the copy of base manifold M^m over which the object or structure it acts on is defined.*

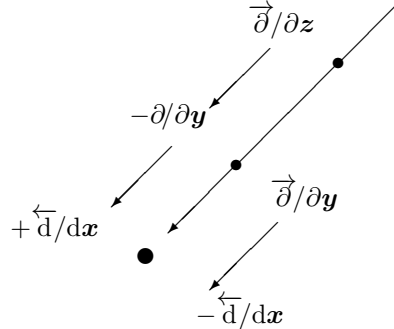


Fig. 3. Each on-the-diagonal push of a derivative along the edge creates an extra minus sign

2.2.2. Let us now explain how the edges in Kontsevich's graphs get oriented; this mechanism is unseparable from the integration by parts. Every edge is realised by the linking of variations—with respect to the canonical conjugate variables u^i and ξ_i —of objects

¹⁵The definition of total derivative d/dx , which is

$$\left(j_\infty^\sigma(s)^* \left(\left(\frac{d}{dx} f \right) (x, [u]) \right) \right) (x_0) \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x} \left(j_\infty^\sigma(s)^* \left(f(x, [u]) \right) \right) \right) (x_0),$$

explains why the partial derivatives $\partial/\partial x$ reshape into d/dx as soon as they arrive to the graph's vertices and there, they act on the objects f which are defined over jet bundles and which are evaluated at the infinite jets $j_\infty^\sigma(s)$ of sections s .

that are contained in the two adjacent vertices. The orientation of such edge is the ordering $\delta s^\dagger \prec \delta s$ of singular linear integral operators; initially, they act as shown in Fig. 4.

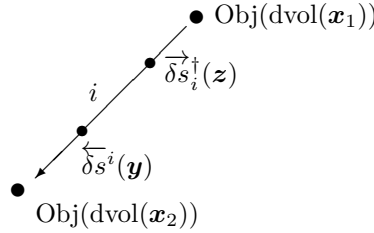


Fig. 4. Decorated by i , such oriented edge appeared in the operation \bullet in [32]; here we extend the formula which the edge encodes to the set-up of affine bundles over M^m by letting the points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}$ and \mathbf{z} run through the integration domain M^m of positive dimension m .

Every edge in an oriented graph Γ contributes to the summand (which Γ encodes in the star-product \star_{\hbar}) by the linking of variations and by the linking of differentials of objects contained in the vertices. Novel with respect to the classical set-up of Section 1, the variations δs and δs^\dagger absorb the derivatives—previously, non-existent—along the base manifold M^m .

The singular linear integral operators δs and δs^\dagger now act not only on the spaces of local functionals but also on elements of their native space of singular distributions. At the same time, regular integral functionals—like \mathcal{P} and F or G from \mathcal{A} , which are contained in vertices of a graph Γ at hand—themselves can discard the copy of volume elements $\text{dvol}(\cdot)$ which they are equipped with and by this, reshape into singular integral operators. The linking of normalised variations yields singular linear integral operators that act via multiplication by ± 1 . At the end of the day, the linking of every two neighbouring objects converts one of them into a singular linear integral operator such that the (co)vectors contained in it act on their duals, resulting in the multiples $+1$ or -1 .

EXAMPLE 7. The edge $\mathcal{P} \xrightarrow{i} F$ encodes the formula¹⁶

$$\iint d\mathbf{x}_1 \cdot \left(\frac{1}{2} \xi_\alpha P_\sigma^{\alpha\beta} \Big|_{(\mathbf{x}_1, [\mathbf{u}])} \xi_{\beta, \sigma} \right) \frac{\overrightarrow{\partial}}{\partial \xi_{i, \tau}} \left\langle \underline{\underline{\overleftarrow{e}_i(\mathbf{x}_1)}} \Big| \iint d\mathbf{y}_1 d\mathbf{y}_2 \left\langle \underline{\underline{\overleftarrow{e}_i(\mathbf{y}_1)}} \cdot \delta s_i^\dagger(\mathbf{y}_1) \Big| \delta s^i(\mathbf{y}_2) \cdot \underline{\underline{\overleftarrow{e}_i(\mathbf{y}_2)}} \right\rangle \underline{\underline{\overleftarrow{e}_i(\mathbf{x}_2)}} \right\rangle \left[\left(+ \frac{\overrightarrow{d}}{d\mathbf{x}_2} \right)^\tau \left(- \frac{\overrightarrow{d}}{d\mathbf{x}_2} \right)^{\sigma_2} \Big| \frac{\overrightarrow{\partial}}{\partial u_{\sigma_2}^i} (f)(\mathbf{x}_2, [\mathbf{u}]) \cdot \text{dvol}(\mathbf{x}_2) \right]. \quad (9)$$

The singular distributions work out the diagonal $\mathbf{x}_1 = \mathbf{y}_1 = \mathbf{y}_2 = \mathbf{x}_2$; both couplings evaluate to $+1$, and normalisation (8) makes the edge’s cargo invisible (indeed, it contributes via multiplication by $+1$).

REMARK 6. The three singular operators in (9) can be directed in the opposite way, i.e. against the edge orientation along which the derivatives are transported in any case. This would keep the volume element at the arrow tail.

¹⁶The notation $[\dots]$, not yet essential in the one-time variations here, will be explained in Convention 4 on p. 236.

EXAMPLE 8. Consider the edge $\mathcal{P} \xrightarrow{j} G$ encoding the formula

$$\begin{aligned} & \iint \text{dvol}(\mathbf{x}_1) \, \text{d}\mathbf{x}_2 \cdot \left(\frac{1}{2} \xi_\alpha P_{\sigma}^{\alpha\beta} \Big|_{(\mathbf{x}_1, [\mathbf{u}])} \xi_{\beta, \sigma} \right) \overleftarrow{\partial}_{\xi_{j, \tau}} \\ & \left\langle \overleftarrow{\underline{\underline{e}}}_j(\mathbf{x}_1) \Big| \iint \text{d}\mathbf{y}_1 \, \text{d}\mathbf{y}_2 \left\langle \overleftarrow{\underline{\underline{e}}}_j^{\text{second}}(\mathbf{y}_1) \cdot \delta s_j^\dagger(\mathbf{y}_j) \Big| \delta s^j(\mathbf{y}_2) \cdot \overleftarrow{\underline{\underline{e}}}_j(\mathbf{y}_2) \right\rangle \Big| \overleftarrow{\underline{\underline{e}}}_j^{\text{first}}(\mathbf{x}_2) \right\rangle \\ & \left[\left(+ \frac{\overrightarrow{\text{d}}}{\text{d}\mathbf{x}_2} \right)^\tau \left(- \frac{\overrightarrow{\text{d}}}{\text{d}\mathbf{x}_2} \right)^{\sigma_2} \right] \overrightarrow{\partial}_{\partial u_{\sigma_2}^j} (g)(\mathbf{x}_2, [\mathbf{u}]). \end{aligned}$$

In this case, both couplings evaluate to -1 by (7) still their values' product is $+1$; the diagonal-making and normalisation mechanism remain the same as before.

Summarising, the ordering in (7) is the only mechanism that creates sign factors; the direction in which the operators act along the edge does not necessarily coincide with that edge's orientation in a graph Γ . The arrow specifies the direction to transport the derivatives by using the integrations by parts.

DEFINITION 3. The *variational Poisson bracket* $\{F, G\}_{\mathcal{P}}$ of two integral functionals F and G with respect to a given variational Poisson bi-vector \mathcal{P} is the graph



By using two pairs of normalised variations and by letting the volume element stay in the vertex containing G , we realise the geometry of singular distributions encoded by this picture via the formula

$$\begin{aligned} & \iiint \text{d}\mathbf{x}_1 \, \text{d}\mathbf{x} (f|_{(\mathbf{x}_1, [\mathbf{u}])}) \overleftarrow{\partial}_{\partial u_{\sigma_1}^i} \left[\left(- \frac{\overleftarrow{\text{d}}}{\text{d}\mathbf{x}_1} \right)^{\sigma_1} \left(+ \frac{\overleftarrow{\text{d}}}{\text{d}\mathbf{x}_1} \right)^{\tau_1} \right] \\ & \left\langle \overleftarrow{\underline{\underline{e}}}_j^{\text{first}}(\mathbf{x}_1) \Big| \iint \text{d}\mathbf{y}_1 \, \text{d}\mathbf{y}_2 \left\langle \overleftarrow{\underline{\underline{e}}}_j^{\text{first}}(\mathbf{y}_2) \cdot \delta s^i(\mathbf{y}_2) \Big| \delta s_i^\dagger(\mathbf{y}_1) \cdot \overleftarrow{\underline{\underline{e}}}_j^{\text{second}}(\mathbf{y}_1) \right\rangle \Big| \overleftarrow{\underline{\underline{e}}}_j(\mathbf{x}) \right\rangle \cdot \\ & \cdot \overrightarrow{\partial}_{\xi_{i, \tau_1}} \left(\frac{1}{2} \xi_\alpha P_{\zeta}^{\alpha\beta} \Big|_{(\mathbf{x}, [\mathbf{u}])} \xi_{\beta, \zeta} \right) \overleftarrow{\partial}_{\xi_{j, \tau_2}} \cdot \\ & \cdot \left\langle \overleftarrow{\underline{\underline{e}}}_j^{\text{first}}(\mathbf{x}) \Big| \iint \text{d}\mathbf{z}_1 \, \text{d}\mathbf{z}_2 \left\langle \overleftarrow{\underline{\underline{e}}}_j^{\text{first}}(\mathbf{z}_1) \cdot \delta s_j^\dagger(\mathbf{z}_1) \Big| \delta s^j(\mathbf{z}_2) \cdot \overleftarrow{\underline{\underline{e}}}_j^{\text{second}}(\mathbf{z}_2) \right\rangle \Big| \overleftarrow{\underline{\underline{e}}}_j^{\text{second}}(\mathbf{x}_2) \right\rangle \\ & \left[\left(+ \frac{\overrightarrow{\text{d}}}{\text{d}\mathbf{x}_2} \right)^{\tau_2} \left(- \frac{\overrightarrow{\text{d}}}{\text{d}\mathbf{x}_2} \right)^{\sigma_2} \right] \overrightarrow{\partial}_{\partial u_{\sigma_2}^j} (g|_{(\mathbf{x}_2, [\mathbf{u}])}) \cdot \text{dvol}(\mathbf{x}_2). \end{aligned}$$

The two pairs of couplings evaluate to $\underline{(-1)} \cdot (-1) \times (+1) \cdot \underline{(+1)} = +1$. The algorithm's output is (cf. [33, 38])

$$\{F, G\}_{\mathcal{P}} = \frac{1}{2} \int \left\langle \frac{\delta F}{\delta \mathbf{u}} \cdot \overrightarrow{A} \left(\frac{\delta G}{\delta \mathbf{u}} \right) \right\rangle - \frac{1}{2} \int \left\langle \left(\frac{\delta F}{\delta \mathbf{u}} \right) \overleftarrow{A} \cdot \frac{\delta G}{\delta \mathbf{u}} \right\rangle, \tag{11}$$

where A is the Hamiltonian operator built into the variational Poisson bi-vector $\mathcal{P} = \frac{1}{2} \int \langle \xi \cdot \overrightarrow{A}(\xi) \rangle$.

REMARK 7. Because the operator A is skew-adjoint, one could now integrate by parts, obtaining an even shorter expression,

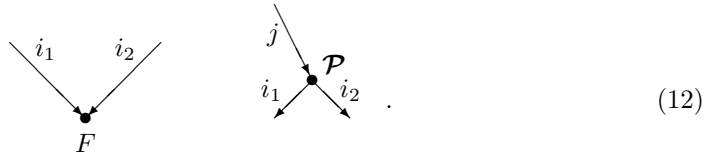
$$\cong \int \left\langle \frac{\delta F}{\delta \mathbf{u}} \cdot \overrightarrow{A} \left(\frac{\delta G}{\delta \mathbf{u}} \right) \right\rangle.$$

Still let it be remembered that it is the wedge graph in (10) that does define the variational Poisson bracket, whereas this formula is its remote consequence.

REMARK 8. When the variational Poisson bracket of two given functionals is assembled by Definition 3—to be evaluated at a section $\mathbf{s} \in \Gamma(\pi)$ —the total derivatives $d/d\mathbf{x}$ immediately follow the partial derivative $\partial/\partial u_\sigma^i$ in the construction of variational derivatives $\delta/\delta \mathbf{u}$. Such inseparability of the horizontal and vertical derivations referring to their own geometries M^m and N^n , respectively, is specific only to one-step reasonings (for instance, derivation of the Euler–Lagrange equations of motion from a given action functional). However, a necessity to iterate the virtual shifts of a section $\mathbf{s} \in \Gamma(\pi)$ reveals a difficulty of the classical jet bundle geometry (e. g., this was acknowledged in [17, §15.1]).

We shall now explain how the graded permutability of iterated variations is achieved in the course of both all the couplings evaluation and transporting the derivatives along M^m to their final positions.

2.2.3. Consider a vertex where two or more arrows arrive—or a vertex that contains \mathcal{P} (so that two partial derivatives, $\overleftarrow{\partial}/\partial \xi_{i_1, \tau_1}$ and $\overleftarrow{\partial}/\partial \xi_{i_2, \tau_2}$, retro-act on its content) and that serves as the head for another arrow (hence, bringing the partial derivative $\partial/\partial u_\sigma^i$ followed by $(-d/d\mathbf{x})^\sigma$ and possibly, by the total derivative(s) $(+d/d\mathbf{x})^\tau$ specified by that arrow’s tail)



In which consecutive order are those partial and total derivatives, related to different edges, applied to the content of a vertex?

Furthermore, the associativity of Kontsevich star-product \star_{\hbar} is achieved in particular due to many cancellations of similar terms in the associator $(\cdot \star_{\hbar} \cdot) \star_{\hbar} \cdot - \cdot \star_{\hbar} (\cdot \star_{\hbar} \cdot)$. Consider those three-sink graphs which can be built using at least two pairs of weighted graphs in the inner and outer products, respectively. Their cancellation prescribes that the resulting analytic expressions, encoded by every such non-contributing graph with three sinks, must not depend on a scenario to compose that graph. We conclude that the action of total derivatives $d/d\mathbf{x}_\ell$ in the *inner* star-products in the associator $(F \star_{\hbar} G) \star_{\hbar} H - F \star_{\hbar} (G \star_{\hbar} H)$ is delayed until all the partial derivatives $\partial/\partial u_\sigma$ would have finished acting in the *outer* star-products.¹⁷

¹⁷This allowed an intrinsic regularisation of the Laplacian Δ and variational Schouten bracket $[\cdot, \cdot]$ in the Batalin–Vilkovisky formalism (see [19, 20, 21] and [7, 17]), the concept was furthered to the formal noncommutative symplectic supergeometry and calculus of cyclic words (see [22] and [27]).

PROPOSITION 11 (see [19, 20, 21, 22]). *The vertical derivations $\partial/\partial u_\sigma^i$ and (the lifts d/dx_ℓ of) horizontal derivations $\partial/\partial x_\ell$ are performed at different stages. First, the vertical derivations $\partial/\partial \mathbf{u}_\sigma$ along N^n , together with their counterparts $\partial/\partial \xi_\tau$ from the parity-odd symplectic dual, frame the edges of entire graph Γ . In the meantime, the derivatives along the base M^m are stored inside the variations δs by using $\partial/\partial \mathbf{y}_k$. Lastly, all the horizontal derivatives $(\pm\partial/\partial \mathbf{y}_k)^\sigma$ are channelled from δs^i to $(\mp\partial/\partial x_\ell)^\sigma$, at the end of the day acting on the objects which are targets of $\partial/\partial u_\sigma^i$.*

CONVENTION 4. To indicate the delayed arrival of total derivatives to their final places (where we write them at once), let us embrace these operators by using $[\dots]$ in all formulas (e.g., see Definition 3).

REMARK 9. We emphasize that by the definition of total derivative (see footnote 15), the derivatives $\partial/\partial \mathbf{y}_k(\delta s^i)$ of virtual shifts δs^i for sections $u^i = s^i(\mathbf{x}_\ell)$ reshape, under integration by parts, into the derivatives $-\partial/\partial x_\ell(s^i)$ of those sections but they do not affect the parity-odd variables $\xi_{j,\zeta}$ which parametrise the fibres of another bundle. Consequently, the *total* derivatives $(+\overrightarrow{d}/d\mathbf{x}_\ell)^\tau \circ (-\overrightarrow{d}/d\mathbf{x}_\ell)^\sigma$ refer only to the jet space $J^\infty(\pi)$ where they act on the respective fibre variables \mathbf{u}_σ in a vertex' content.

EXAMPLE 9. The first graph in (12) corresponds to the formula

$$(f|_{(\mathbf{x}_1, [\mathbf{u}]})} \frac{\overleftarrow{\partial}}{\partial u_{\sigma_1}^{i_1}} \frac{\overleftarrow{\partial}}{\partial u_{\sigma_2}^{i_2}} \left[\left(-\frac{\overleftarrow{d}}{d\mathbf{x}_1} \right)^{\sigma_1 \cup \sigma_2} \circ \left(+\frac{\overleftarrow{d}}{d\mathbf{x}_1} \right)^{\tau_1 \cup \tau_2} \right], \tag{13}$$

where the multi-indexes τ_1 and τ_2 arrive from the respective arrow tails.

The second graph in (12) contributes with the expression

$$\left[\left(+\frac{\overleftarrow{d}}{d\mathbf{x}_k} \right)^{\tau_1} \left| \frac{\overrightarrow{\partial}}{\partial \xi_{i_1, \tau_1}} \int \frac{1}{2} \xi_\alpha \left\{ \left[\left(+\frac{\overrightarrow{d}}{d\mathbf{x}} \right)^\tau \left(-\frac{\overrightarrow{d}}{d\mathbf{x}} \right)^\sigma \right] \frac{\overrightarrow{\partial}}{\partial u_\sigma^j} (P_\zeta^{\alpha\beta}|_{(\mathbf{x}, [\mathbf{u}]})} \right\} \xi_{\beta, \zeta} \operatorname{dvol}(\mathbf{x}) \right. \right. \\ \left. \left. \frac{\overleftarrow{\partial}}{\partial \xi_{i_2, \tau_2}} \left[\left(+\frac{\overrightarrow{d}}{d\mathbf{x}_\ell} \right)^{\tau_2} \right] \right],$$

where the multi-index τ arrives from the tail of arrow decorated with j and where the copies of base M^m for objects at the heads of arrows that carry i_1 and i_2 are marked using k and ℓ , respectively.

Summarising, the local portrait of oriented edges around every vertex in a given graph Γ determines the vertex-incoming partial derivatives with respect to variables \mathbf{u}_σ , in-coming graded partial derivatives with respect to the parity-odd variables ξ_τ , and (powers of) delayed $(\pm 1) \times$ total derivatives. All these derivations act on the object contained in a vertex at hand, that is, on either a Hamiltonian density or structure constants $P_\zeta^{\alpha\beta}(\mathbf{x}, [\mathbf{u}])$ of the variational Poisson bracket. Note that in both cases, the arguments are referred to the geometry of $J^\infty(\pi)$, hence those objects are expressed in terms of sections of the parity-even jet bundle $\pi_\infty : J^\infty(\pi) \rightarrow M^m$.

Globally, each oriented graph Γ in the Kontsevich summation formula (4) encodes a singular linear integral operator that acts on a local functional contained in one of the sinks.

2.3. On the associativity of star-product

2.3.1. Let $\mathcal{P} = \frac{1}{2} \int \langle \xi \cdot P_\tau |_{\mathbf{x}}(\xi) \rangle$ be a variational Poisson bi-vector such that its coefficients $P_\tau^{ij}(\mathbf{x})$ do not depend explicitly on the jet fibre variables \mathbf{u}_σ in the bundle π_∞ over $M^m \ni \mathbf{x}$. Let $F = \int f(\mathbf{x}_1, [\mathbf{u}]) \cdot \text{dvol}(\mathbf{x}_1)$ and $G = \int g(\mathbf{x}_2, [\mathbf{v}]) \cdot \text{dvol}(\mathbf{x}_2)$ be integral functionals referred to two identical copies of the jet space $J^\infty(\pi)$. A variational generalisation $F \star G$ of the Moyal–Groenewold–Weyl star-product \star for F and G is the local functional which is constructed from (2) for the variational Poisson bracket $\{\cdot, \cdot\}_\mathcal{P}$ by using the techniques from §2.2. The variational Moyal product of two integral functionals is expressed by the formula

$$F \star G = \int \left(f|_{(\mathbf{x}_1, [\mathbf{u}])} \exp \left(\overleftarrow{\frac{\partial}{\partial u_\sigma^i}} \left[\left(-\overleftarrow{\frac{d}{d\mathbf{x}_1}} \right)^\sigma \left(\overleftarrow{\frac{d}{d\mathbf{x}_1}} \right)^\tau \right] \cdot \overrightarrow{\frac{\partial}{\partial \xi_{i,\tau}}} \left(\frac{\hbar}{2} \xi_\alpha P_\lambda^{\alpha\beta}(\mathbf{x}) \xi_{\beta,\lambda} \right) \overleftarrow{\frac{\partial}{\partial \xi_{j,\zeta}}} \cdot \left[\left(\overrightarrow{\frac{d}{d\mathbf{x}_2}} \right)^\zeta \left(-\overrightarrow{\frac{d}{d\mathbf{x}_2}} \right)^\chi \right] \overrightarrow{\frac{\partial}{\partial v_\chi^j}} \right) g|_{(\mathbf{x}_2, [\mathbf{v}])} \Big|_{\substack{\mathbf{x}_1 = \mathbf{x} = \mathbf{x}_2 \\ [\mathbf{u}] = [\mathbf{v}]}} \cdot \text{dvol}(\mathbf{x}). \tag{14}$$

The angular brackets [...] in (14) embrace the total derivatives whose action—in every term of the towered wedge graph expansion of \star —antecedes¹⁸ the action of partial derivatives with respect to u_σ^i and v_χ^j . For instance, the expansion starts as follows:

$$F \star G = F \times G + \frac{\hbar^1}{1!} \{F, G\}_\mathcal{P} + \frac{\hbar^2}{2!} \int (f|_{(\mathbf{x}, [\mathbf{u}])}) \frac{\overleftarrow{\partial}}{\partial u_{\sigma_1}^{i_1}} \frac{\overleftarrow{\partial}}{\partial u_{\sigma_2}^{j_2}} \left(-\overleftarrow{\frac{d}{d\mathbf{x}}} \right)^{\sigma_1 \cup \sigma_2} \cdot \left(\overleftarrow{\frac{d}{d\mathbf{x}}} \right)^{\tau_1 \cup \tau_2} \frac{\overrightarrow{\partial}}{\partial \xi_{i_1, \tau_1}} \left(\frac{1}{2} \xi_{\alpha_1} P_{\lambda_1}^{\alpha_1 \beta_1}(\mathbf{x}) \xi_{\beta_1, \lambda_1} \right) \frac{\overleftarrow{\partial}}{\partial \xi_{j_1, \zeta_1}} \cdot \frac{\overrightarrow{\partial}}{\partial \xi_{i_2, \tau_2}} \left(\frac{1}{2} \xi_{\alpha_2} P_{\lambda_2}^{\alpha_2 \beta_2}(\mathbf{x}) \xi_{\beta_2, \lambda_2} \right) \cdot \frac{\overleftarrow{\partial}}{\partial \xi_{j_2, \zeta_2}} \left(\overrightarrow{\frac{d}{d\mathbf{x}}} \right)^{\zeta_1 \cup \zeta_2} \left(-\overrightarrow{\frac{d}{d\mathbf{x}}} \right)^{\chi_1 \cup \chi_2} \frac{\overrightarrow{\partial}}{\partial u_{\chi_2}^{j_2}} \frac{\overrightarrow{\partial}}{\partial u_{\chi_1}^{i_1}} (g|_{(\mathbf{x}, [\mathbf{u}])}) \cdot \text{dvol}(\mathbf{x}) + \bar{o}(\hbar^2).$$

Let $\tilde{\mathbf{x}} = I_{\alpha\beta} \mathbf{x} + \tilde{\boldsymbol{\mu}}$, $\tilde{\mathbf{u}} = J_{\alpha\beta} \mathbf{u} + \tilde{\boldsymbol{\nu}}(\mathbf{x})$ be an affine change of variables in π such that the Jacobian matrix J is locally constant on the intersection $V_\alpha \cap V_\beta$ of two charts $V_\alpha, V_\beta \subseteq M^m$. Then formula (14) is invariant with respect to such coordinate reparametrisation. The associativity of (14) is proved in a standard way (see the proof of Proposition 5). The associator $(F \star G) \star H - F \star (G \star H)$ of three given integral functionals over $J^\infty(\pi)$ itself is an *integral* functional whose density is identically zero at all points of $J^\infty(\pi)$ over M^m .

PROPOSITION 12. *Formula (14) provides the deformation quantisation of first and, via factorisation by using the junior Poisson bracket for the modified system, of second variational Poisson structures for the Drinfel’d–Sokolov hierarchies.*

EXAMPLE 10 (root system A_1). Consider the Korteweg–de Vries equation

$$w_t = -\frac{1}{2} w_{xxx} + 6ww_x = \left(-\frac{1}{2} D_x^3 + 2wD_x + 2D_x \circ w \right) \left(\frac{\delta}{\delta w} \int \frac{1}{2} w^2 dx \right)$$

¹⁸We recall that the action of total derivatives contained, e.g., in $F \star G$ itself constituting a part of the object $(F \star G) \star H - F \star (G \star H)$ is also delayed until all the partial derivatives would have acted on the densities f, g , or h .

realised by using its second, field-dependent variational Poisson structure.¹⁹ Consider the Miura substitution [35, 36] $w = \frac{1}{2}(u_x^2 - u_{xx})$; let us explain in advance that the conserved current $w \, dx = \frac{1}{2}(u_x^2 - u_{xx}) \, dx$ stems—via the First Noether theorem—from the Noether symmetry $\varphi_1 = u_x$ of the action $\mathcal{L} = \iint (\frac{1}{2}u_x u_y + \frac{1}{2}e^{2u}) \, dx \wedge dy$ for the Liouville equation $\mathcal{E}_{\text{Liou}} = \{u_{xy} = \exp(2u)\}$.

The mapping $w = w(x, [u])$ is determined by the *integral* $w \in \ker \frac{d}{dy} \Big|_{\mathcal{E}_{\text{Liou}}}$; we recall that the coefficient “2” in the right-hand side of $u_{xy} = \exp(2u)$ is the only entry of the Cartan matrix $K = \|2\|_1^1$ for Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

By definition, put $\vartheta = \frac{1}{2}u_x$ so that $\ell_\vartheta^{(u)} = \frac{1}{2} \frac{d}{dx}$ is the first Hamiltonian operator \hat{B}_1^{mKdV} of modified KdV hierarchy and so that $w = 2\vartheta^2 - \vartheta_x$. Denote by $\square = 4\vartheta + D_x = 2u_x + D_x$ the adjoint $(\ell_w^{(\vartheta)})^\dagger$ of linearisation $\ell_w^{(\vartheta)} = 4\vartheta - D_x$. By using the chain rules $\delta/\delta\vartheta = (\ell_w^{(\vartheta)})^\dagger \circ \delta/\delta w$ and $\delta/\delta u = (\ell_\vartheta^{(u)})^\dagger \circ (\ell_w^{(\vartheta)})^\dagger \circ \delta/\delta w$, we cast the (potential) modified KdV equations,

$$u_t = -\frac{1}{2}u_{xxx} + u_x^3 = \square(w), \quad \vartheta_t = -\frac{1}{2}\vartheta_{xxx} + 12\vartheta^2\vartheta_x,$$

into their canonical De Donder–Weyl’s representation [9, 11]

$$u_t = \frac{\delta H[w[\vartheta]]}{\delta \vartheta}, \quad \vartheta_t = -\frac{\delta H[w[\vartheta[u]]]}{\delta u} \quad \text{with } H = \int \frac{1}{2}w^2 \, dx.$$

Clearly, we then recover the KdV evolution

$$w_t = (\ell_w^{(\vartheta)}(\vartheta_t))[w] = (-\frac{1}{2}D_x^3 + 4wD_x + 2w_x)(\delta H(x, [w])/\delta w).$$

This factorisation pattern involving the Fréchet derivatives (or *linearisations*),

$$\hat{A}_2^{\text{KdV}} = \ell_w^{(\vartheta)} \circ \ell_\vartheta^{(u)} \circ (\ell_w^{(\vartheta)})^\dagger,$$

is common to all the root systems of ranks $r \geq 1$, that is, for the (modified) Drinfel’d–Sokolov hierarchies [12]. It is seen that the hierarchy for respective analogue of potential modified KdV equation for \mathbf{u} constitutes the maximal commutative subalgebra in the Lie algebra of Noether symmetries for Leznov–Saveliev’s nonperiodic 2D Toda chains [34] $u_{xy}^i = \exp(\sum_{j=1}^r \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \cdot u^j)$. The algorithm for construction of r integrals [44] w^1, \dots, w^r is known from [41], see [24] for an illustration. The De Donder–Weyl formalism [9, 11] furthers the approach: the variables $\vartheta_1, \dots, \vartheta_r$ are the canonical conjugate *momenta*, $\vartheta_i = \partial L/\partial u_y^i$, for the genuine *coordinates* u^1, \dots, u^r satisfying the 2D Toda equations. The Lagrangian density is $L = \frac{1}{2}\kappa_{ij}u_x^i u_y^j + \langle a_i, \exp(K_j^i u^j) \rangle$, where each row of the Cartan matrix $K = \|K_j^i\|$ is symmetrised to $\kappa = \|a_i \cdot K_j^i\|_{i=1, \dots, r}^{j=1, \dots, r}$ by using the root lengths, $\alpha_i := 1/\langle \alpha_i, \alpha_i \rangle$ at every i . Consequently, the junior variational Poisson structure for the modified Drinfel’d–Sokolov hierarchy is

$$\hat{B}_1 = \left\| \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle \langle \alpha_j, \alpha_j \rangle} \frac{d}{dx} \right\|_{i=1, \dots, r}^{j=1, \dots, r}$$

for every root system $\alpha_1, \dots, \alpha_r$. Its coefficients are constants $\in \mathbb{k}$.

¹⁹Through the Fourier transform, the Hamiltonian operator \hat{A}_2^{KdV} encodes the Virasoro algebra, cf. [3].

Having thus factorised a higher, field-dependent variational Poisson structure through the junior variational Poisson structures whose coefficients do not depend explicitly on the new fields $\phi \in \Gamma(\pi)$, we reduce the large deformation quantisation problem for functionals $F[\mathbf{w}], G[\mathbf{w}], H[\mathbf{w}] : \Gamma(\tilde{\pi}) \rightarrow \mathbb{k}$ to a much smaller Moyal–Groenewold–Weyl case (14) of the same functionals $F[\mathbf{w}[\mathbf{u}]], G[\mathbf{w}[\mathbf{u}]], H[\mathbf{w}[\mathbf{u}]] : \Gamma(\pi) \rightarrow \mathbb{k}$, now referred to the jet bundle π_∞ of (potential) modified hierarchies.

Indeed, let π and $\tilde{\pi}$ be two affine bundles over the base M^m . Consider a jet space morphism $\mathbf{w}^{(\infty)} : J^\infty(\pi) \rightarrow J^\infty(\tilde{\pi})$ specified by a Miura-type substitution $\mathbf{w} = \mathbf{w}(\mathbf{x}, [\mathbf{u}]) : \Gamma(\pi_\infty) \rightarrow \Gamma(\tilde{\pi})$ of positive differential order. The Hamiltonian differential operators factorise via²⁰

$$A|_{(\mathbf{x}, [\mathbf{w}])} = \overrightarrow{\ell}_{\mathbf{w}}^{(\mathbf{u})} \circ B|_{(\mathbf{x}, [\mathbf{u}])} \circ \overrightarrow{\ell}_{\mathbf{w}}^{(\mathbf{u})\dagger}, \tag{15}$$

where $\{\cdot, \cdot\}_{\frac{1}{2}} \int \langle \chi, A(\chi) \rangle$ is the variational Poisson bracket *induced* for functionals $H[\mathbf{w}] : \Gamma(\tilde{\pi}) \rightarrow \mathbb{k}$ from a *given* variational Poisson structure $\{\cdot, \cdot\}_{\frac{1}{2}} \int \langle \xi, B(\xi) \rangle$ for the pull-backs $H[\mathbf{w}[\mathbf{u}]] : \Gamma(\pi) \rightarrow \mathbb{k}$.

Formula (15) correlates senior Poisson structures $\{\cdot, \cdot\}_{\frac{1}{2}} \int \langle \chi, A_{i+1}(\chi) \rangle$ for multi-Hamiltonian hierarchies with the junior Hamiltonian operators B_i for the respective modified hierarchies of completely integrable PDE systems (see [18, 25] and references therein). Solutions $(\pi, \mathbf{w}(\mathbf{x}, [\mathbf{u}]), B)$ are “good” if the coefficients of differential operator B , referred to (\mathbf{x}, u_σ^j) , do not contain the jet variables u_σ^j explicitly (that is, the star-product for the Hamiltonian operator B amounts to formula (14)). In this case the output of deformation quantisation procedure $\times \mapsto \star$ is associative at all orders of the deformation parameter \hbar . However, for a given Hamiltonian differential operator A over $J^\infty(\tilde{\pi})$, its factorisation problem can be very hard.

2.3.2. Finally, let us take a generic variational Poisson brackets $\{\cdot, \cdot\}_{\mathcal{P}}$ with field-dependent coefficients $P_\tau^{ij}(\mathbf{x}, [\mathbf{u}])$. For instance, suppose that factorisation (15), reducing a given Hamiltonian differential operator \widehat{A}_2 on $J^\infty(\tilde{\pi})$ to the Moyal case for \widehat{B}_1 on $J^\infty(\pi)$, is not yet known.

It is readily seen that the splitting of differential consequences from the Jacobi identity into the separately vanishing homogeneous components, see Proposition 6 in §1.4, no longer takes place without reservations in the variational setting. This is because not only the vertical derivatives along the fibre of $J^\infty(\pi)$ work by the Leibniz rule over the five vertices in every Jacobiator (6) but also do the total derivatives in their trail, as in (13), convert the Jacobiator on its three arguments into an indivisible object. Therefore, in the variational picture only those Kontsevich graph expansions of Leibniz graphs can vanish

²⁰Integrating by parts,

$$\begin{aligned} \frac{1}{2} \int \left\langle (\xi) \left(\overleftarrow{\ell}_{\mathbf{w}}^{(\mathbf{u})} \right)^\dagger \cdot \left(P_\tau|_{(\mathbf{x}, [\mathbf{u}])} \left(\frac{d}{d\mathbf{x}} \right)^\tau \circ \overrightarrow{\ell}_{\mathbf{w}}^{(\mathbf{u})\dagger} \right) (\xi) \right\rangle \\ \cong \frac{1}{2} \int \left\langle \xi \cdot \left(\overrightarrow{\ell}_{\mathbf{w}}^{(\mathbf{u})} \circ P_\tau|_{(\mathbf{x}, [\mathbf{u}])} \left(\frac{d}{d\mathbf{x}} \right)^\tau \circ \overrightarrow{\ell}_{\mathbf{w}}^{(\mathbf{u})\dagger} \right) (\xi) \right\rangle = \frac{1}{2} \int \langle \xi, \overrightarrow{A}(\xi) \rangle, \end{aligned}$$

we construct the Hamiltonian differential operator in total derivatives that takes variational covectors to (the generating sections of) evolutionary vector fields.

in which the Jacobiator is not split. Yet we remember from [5] that this is already not the case at \hbar^3 in the associator for \star_{\hbar} , see §1.4.

Secondly, for every triple of arguments, the Jacobiator $\text{Jac}_{\mathcal{P}}(\cdot, \cdot, \cdot) \cong 0 : \Gamma(\pi) \rightarrow 0 \in \mathbb{k}$ is the map which, in terms of [19, 21, 22], can be a *synonym of zero*. Namely, if the density of this cohomologically trivial integral functional is not vanishing over all points of M^m , the local variational polydifferential operator \diamond in

$$\text{Assoc}_{\star_{\hbar}}(F, G, H) = \diamond(\mathcal{P}, \text{Jac}_{\mathcal{P}}(\cdot, \cdot, \cdot))(F, G, H), \quad F, G, H \in \mathcal{A}[[\hbar]], \quad (1)$$

can produce a nonzero integral functional from its zero-value argument $\text{Jac}(\mathcal{P})$. Indeed, whenever two or more arrows arrive at a vertex in the argument of \diamond , see (12), the order in which partial and then total derivatives act is (13). Therefore, the mechanism $\delta/\delta \mathbf{u} \circ d/d\mathbf{x} \equiv 0$ that guarantees the vanishing of the first variation for a cohomologically trivial argument is stepped over.

COROLLARY 13. *In the field-theoretic setting, the associativity of star-product \star_{\hbar} can start leaking at order $\hbar^{\geq 3}$ for a variational Poisson structure $\{\cdot, \cdot\}_{\mathcal{P}}$ with field-dependent coefficients in the leading deformation term, so that $\text{Assoc}_{\star_{\hbar}}(\cdot, \cdot, \cdot) \doteq \bar{o}(\hbar^{\geq 2})$.*

Acknowledgements. The author is grateful to M. Kontsevich for advice and R. Buring for discussion. The author also thanks the organizers of 50th Sophus Lie Seminar (26–30 September 2016 in Będlewo, Poland) for their hospitality. This research was supported in part by JBI RUG project 106552 (Groningen) and by the IHÉS and MPIM (Bonn), to which the author is grateful for their warm hospitality.

References

- [1] M. Alexandrov, A. Schwarz, O. Zaboronsky, M. Kontsevich, *The geometry of the master equation and topological quantum field theory*, Internat. J. Modern Phys. A 12 (1997), 1405–1429.
- [2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation theory and quantization. I. Deformations of symplectic structures, II. Physical applications*, Ann. Physics 111 (1978), 61–110, 111–151.
- [3] A. A. Belavin, *KdV-type equations and W-algebras*, in: Integrable Systems in Quantum Field Theory and Statistical Mechanics, Adv. Stud. Pure Math. 19, Academic Press, Boston MA, 1989, 117–125.
- [4] A. Bouisaghoulane, R. Buring, A. V. Kiselev, *The Kontsevich tetrahedral flow revisited*, J. Geom. Phys. 119 (2017), 272–285.
- [5] R. Buring, A. V. Kiselev, *On the Kontsevich \star -product associativity mechanism*, Phys. Particles Nuclei Lett. 14 (2017), 403–407.
- [6] R. Buring, A. V. Kiselev, *Software modules and computer-assisted proof schemes in the Kontsevich deformation quantization*, Preprint IHÉS/M/17/05, 2017.
- [7] A. S. Cattaneo, G. Felder, *A path integral approach to the Kontsevich quantization formula*, Comm. Math. Phys. 212 (2000), 591–611.
- [8] A. S. Cattaneo, D. Indelicato, *Formality and star products*, in: Poisson Geometry, Deformation Quantisation and Group Representations, London Math. Soc. Lecture Note Ser. 323, Cambridge Univ. Press, Cambridge, 2005, 79–144.

- [9] T. De Donder, *Théorie invariante du calcul des variations*, Gauthier–Villars, Paris, 1930.
- [10] M. De Wilde, P. B. A. Lecomte, *Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys. 7 (1983), 487–496.
- [11] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Grad. Sch. Sci. Monogr. Ser. 2, Academic Press, New York, 1967.
- [12] V. G. Drinfel’d, V. V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, in: Current Problems in Mathematics 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, VINITI, Moscow, 1984, 81–180.
- [13] B. V. Fedosov, *Formal quantization*, in: Some Problems in Modern Mathematics and their Applications to Problems in Mathematical Physics, MFTI, Moscow, 1985, 129–136.
- [14] B. V. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom. 40 (1994), 213–238.
- [15] I. M. Gel’fand, G. E. Shilov, *Generalized Functions*, 1. *Properties and Operations*, 2. *Spaces of Fundamental and Generalized Functions*, Academic Press, New York–London, 1964, 1968.
- [16] H. J. Groenewold, *On the principles of elementary quantum mechanics*, Physica 12 (1946), 405–460.
- [17] M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems*, Princeton Univ. Press, Princeton, NJ, 1992.
- [18] A. V. Kiselev, *Hamiltonian flows on Euler-type equations*, Teoret. Mat. Fiz. 144 (2005), no. 1, 83–93; English transl.: Theoret. and Math. Phys. 144 (2005), 952–960.
- [19] A. V. Kiselev, *The geometry of variations in Batalin–Vilkovisky formalism*, J. Phys. Conf. Ser. 474 (2013), Paper 012024.
- [20] A. V. Kiselev, *The Jacobi identity for graded-commutative variational Schouten bracket revisited*, Phys. Particles Nuclei Lett. 11 (2014), 950–953.
- [21] A. V. Kiselev, *The right-hand side of the Jacobi identity: to be naught or not to be?* in: Integrable Systems and Quantum Symmetries, J. Phys. Conf. Ser. 670 (2016), Paper 012030.
- [22] A. V. Kiselev, *The calculus of multivectors on noncommutative jet spaces*, arXiv: 1210.0726 (v5) [math.DG].
- [23] A. V. Kiselev, A. O. Krutov, *Non-Abelian Lie algebroids over jet spaces*, J. Nonlinear Math. Phys. 21 (2014), 188–213.
- [24] A. V. Kiselev, J. W. van de Leur, *A geometric derivation of KdV-type hierarchies from root systems*, in: Group Analysis of Differential Equations and Integrable Systems (Protaras, 2008), 2009, 87–106.
- [25] A. V. Kiselev, J. W. van de Leur, *Symmetry algebras of Lagrangian Liouville-type systems*, Teoret. Mat. Fiz. 162 (2010), no. 2, 179–195; English transl.: Theoret. Math. Phys. 162 (2010), 149–162.
- [26] A. V. Kiselev, J. W. van de Leur, *Variational Lie algebroids and homological evolutionary vector fields*, Teoret. Mat. Fiz. 167 (2011), 432–447; English transl.: Theoret. Math. Phys. 167 (2011), 772–784.
- [27] M. Kontsevich, *Formal (non)commutative symplectic geometry*, in: The Gel’fand Mathematical Seminars, 1990–1992, Birkhäuser, Boston MA, 1993, 173–187.
- [28] M. Kontsevich, *Feynman diagrams and low-dimensional topology*, in: First European Congress of Mathematics (Paris, 1992) 2, Progr. Math. 120, Birkhäuser, Basel, 1994, 97–121.

- [29] M. Kontsevich, *Homological algebra of mirror symmetry*, in: Proceedings of International Congress of Mathematicians (Zürich, 1994), vol. 1, Birkhäuser, Basel, 1995, 120–139.
- [30] M. Kontsevich, *Formality conjecture*, in: Deformation Theory and Symplectic Geometry (Ascona, 1996), Math. Phys. Stud. 20, Kluwer, Dordrecht, 1997, 139–156.
- [31] M. Kontsevich, *Operads and motives in deformation quantization*, Lett. Math. Phys. 48 (1999), 35–72.
- [32] M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66 (2003), 157–216.
- [33] B. A. Kupershmidt, *Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalisms*, in: Geometric Methods in Mathematical Physics (Lowell, Mass., 1979), Lecture Notes in Math. 775, Springer, Berlin, 1980, 162–218.
- [34] A. N. Leznov, M. V. Saveliev, *Group-Theoretical Methods for Integration of Nonlinear Dynamical Systems*, Progr. Phys. 15, Birkhäuser, Basel, 1992.
- [35] R. M. Miura, *Korteweg–de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation*, J. Mathematical Phys. 9 (1968), 1202–1204.
- [36] R. M. Miura, C. S. Gardner, M. D. Kruskal, *Korteweg–de Vries equation and generalizations. II. Existence of conservation laws and constants of motion*, J. Mathematical Phys. 9 (1968), 1204–1209.
- [37] J. E. Moyal, *Quantum mechanics as a statistical theory*, Math. Proc. Cambridge Philos. Soc. 45 (1949), 99–124.
- [38] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Grad. Texts in Math. 107 (2nd ed.), Springer, New York, 1993.
- [39] P. J. Olver, V. V. Sokolov, *Integrable evolution equations on associative algebras*, Comm. Math. Phys. 193 (1998), 245–268.
- [40] A. Schwarz, *Geometry of Batalin–Vilkovisky quantization*, Comm. Math. Phys. 155 (1993), 249–260.
- [41] A. B. Shabat, *Higher symmetries of two-dimensional lattices*, Phys. Lett. A 200 (1995), 121–133.
- [42] D. E. Tamarkin, *Operadic proof of M. Kontsevich’s formality theorem*, Ph.D. Thesis, The Pennsylvania State Univ., 1999.
- [43] A. Yu. Vaintrob, *Lie algebroids and homological vector fields*, Uspekhi Mat. Nauk 52 (1997), no. 2(314), 161–162; English transl.: Russian Math. Surveys 52 (1997), 428–429.
- [44] A. V. Zhiber, V. V. Sokolov, *Exactly integrable hyperbolic equations of Liouville type*, Uspekhi Mat. Nauk 56 (2001), no. 1(337), 63–106; English transl.: Russian Math. Surveys 56 (2001), 61–101.