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Continuous solutions of a polynomial-like iterative equation with variable coefficients

by Weinian Zhang (Chengdu) and John A. Baker (Waterloo, ON)

Abstract. Using the fixed point theorems of Banach and Schauder we discuss the existence, uniqueness and stability of continuous solutions of a polynomial-like iterative equation with variable coefficients.

I. Introduction. Let $I = [a, b]$ be a given closed bounded interval. Given a continuous $F : I \to I$ such that $F(a) = a$ and $F(b) = b$, and given continuous functions $\lambda_1, \ldots, \lambda_n : I \to [0, 1]$ such that $\sum_{i=1}^{n} \lambda_i(x) = 1$ for all $x \in I$, we wish to find continuous functions $f : I \to I$ such that

$$\lambda_1(x)f(x) + \lambda_2(x)f^2(x) + \ldots + \lambda_n(x)f^n(x) = F(x) \quad \text{for all } x \in I. \tag{1}$$

Here f^i denotes the ith iterate of f (i.e., $f^0(x) = x$ and $f^{i+1}(x) = f(f^i(x))$ for all $x \in I$ and all $i = 0, 1, \ldots$). We suppose that $n \geq 2$.

The case in which the λ_i's are constant was considered in [4]–[7] and [9]–[11] for special choices of F and/or n. Similar equations are discussed on pages 237–240 of [5]. Such problems are related both to problems concerning iterative roots (see [1], [3] and [8]), e.g. finding a function f such that

$$f^n(x) = F(x), \quad \forall x \in I,$$

and to the theory of invariant curves for mappings (see Chapter XI of [5]).

Note that we may assume without loss of generality that $a = 0$ and $b = 1$. Indeed, if $[a, b] \neq [0, 1]$ and (1) holds, define

$$h(t) = a + t(b - a) \quad \text{for } 0 \leq t \leq 1$$

and let

$$g = h^{-1} \circ f \circ h, \quad G = h^{-1} \circ F \circ h, \quad \mu_i = \lambda_i \circ h \quad \text{for } 1 \leq i \leq n$$

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where $\circ$ denotes composition. Since h and h^{-1} are affine and $\sum_{i=1}^{n} \lambda_i(x) = 1$ for all $x \in I$, it follows that

$$\sum_{i=1}^{n} \mu_i(t) g^i(t) = G(t) \quad \text{for all } t \in [0,1]. \tag{2}$$

Conversely, if (2) holds so does (1). Thus assume that $I = [0,1]$.

For economy of exposition we adopt the following notation. Let $C(I)$ denote the real Banach algebra consisting of all continuous maps of I into $\mathbb{R}$ with respect to the uniform norm; for $f \in C(I)$, $\|f\| = \max\{|f(t)| : t \in I\}$. Let

$$X = \{f \in C(I) : 0 = f(0) \le f(t) \le f(1) = 1 \text{ for all } t \in I\}.$$

Note that X is closed under composition and hence under iteration.

For $0 \le m \le 1 \le M$ let

$$X(m,M) = \{f \in X : m(y-x) \le f(y) - f(x) \le M(y-x) \quad \text{whenever } 0 \le x \le y \le 1\}.$$

II. Some lemmas

LEMMA 1. *Suppose* $0 \le m \le 1 \le M$. *Then* $X(m,M)$ *is a compact convex subset of* $C(I)$. *Moreover*, *if* $f, g \in X(m,M)$ *then*

$$\|f^\nu - g^\nu\| \le \sum_{j=0}^{\nu-1} M^j \|f-g\| \quad \textit{for all } \nu = 1, 2, \ldots$$

Proof. It is clear that $X(m,M)$ is a closed, bounded and convex subset of $C(I)$. It is also clear that $X(m,M)$ is uniformly equicontinuous. Thus, by the Ascoli–Arzelà lemma, $X(m,M)$ is a compact convex subset of $C(I)$.

If $\nu = 1$ then the inequality is trivial. Suppose it holds when $1 \le \nu \le k$ for some $k \ge 1$. Then, for all $x \in I$,

$$\begin{aligned} |f^{k+1}(x) - g^{k+1}(x)| &= |f(f^k(x)) - g(g^k(x))| \\ &\le |f(f^k(x)) - f(g^k(x))| + |f(g^k(x)) - g(g^k(x))| \\ &\le M\|f^k - g^k\| + \|f-g\| \\ &\le M\Big(\sum_{j=0}^{k-1} M^j\Big)\|f-g\| + \|f-g\| \\ &= \Big(\sum_{j=0}^{k} M^j\Big)\|f-g\|. \end{aligned}$$

Thus, by induction, the inequality is true for all $\nu \ge 1$. ∎

LEMMA 2. *Suppose* $0 < m \le 1 \le M$ *and* $f, g \in X(m, M)$. *Then*

(i) $f^{-1} \in X(M^{-1}, m^{-1})$,
(ii) $\|f - g\| \le M\|f^{-1} - g^{-1}\|$, *and*
(iii) $\|f^{-1} - g^{-1}\| \le m^{-1}\|f - g\|$.

Proof. Since $m > 0$, f is a strictly increasing homeomorphism of I onto itself and, for $0 \le x < y \le 1$,

$$M^{-1} \le \frac{y' - x'}{f(y') - f(x')} \le m^{-1}$$

where $y' = f^{-1}(y)$ and $x' = f^{-1}(x)$. Thus (i) holds.

To prove (ii) note that for all $x \in I$,

$$\begin{aligned}|f(x) - g(x)| &= |f(x) - f((f^{-1} \circ g)(x))| \le M|x - f^{-1}(g(x))| \\ &= M|g^{-1}(g(x)) - f^{-1}(g(x))| \le M\|g^{-1} - f^{-1}\|.\end{aligned}$$

It follows that $\|f - g\| \le M\|g^{-1} - f^{-1}\| = M\|f^{-1} - g^{-1}\|$.

Property (iii) follows easily from (i) and (ii). ∎

These lemmas are essentially Lemmas 2.2 and 2.5 of [11]. Also note that, by (iii), the *inversion map* $\mathcal{I} : X(m, M) \to X(M^{-1}, m^{-1})$ (defined by $\mathcal{I}f = f^{-1}$ for $f \in X(m, M)$) is a Lipschitz mapping.

LEMMA 3. *If* $f \in X(m, M)$ *and* $g \in X(s, S)$ *with* $0 \le m \le 1 \le M$ *and* $0 \le s \le 1 \le S$, *then* $f \circ g \in X(ms, MS)$ *and*

$$f^k \in X(m^k, M^k) \quad \textit{for all } k = 0, 1, \ldots$$

Proof. It suffices to note that, for $0 \le x \le y \le 1$,

$$f(g(y)) - f(g(x)) \le M(g(y)) - g(x)) \le MS(y - x)$$

and, similarly,

$$f(g(y)) - f(g(x)) \ge ms(y - x). \ \blacksquare$$

III. Existence. Our main result is the following

THEOREM 1. *Suppose that* $\lambda_1(x) \ge c$ *for all* $x \in I$ *and*

$$\operatorname{Lip} \lambda_k := \sup\left\{\frac{|\lambda_k(y) - \lambda_k(x)|}{y - x} : 0 \le x < y \le 1\right\} \le \beta \quad \textit{for } k = 1, 2, \ldots$$

where c *and* β *are real constants such that*

$$0 < c < 1 \quad \textit{and} \quad 0 \le n\beta \le 1.$$

Also suppose that $F \in X(\delta, M)$ *with*

$$n\beta \le \delta \le 1 \le M.$$

Then (1) *has a solution* f *in* $X(0, (M + n\beta)/c)$.

Proof. Let $L = (M + n\beta)/c$ and note that $L > 1$ since $0 < c < 1 \le M$.
For $x \in I$ and $f \in X(0, L)$ define $f_x : I \to \mathbb{R}$ by

$$f_x(t) = \sum_{i=1}^{n} \lambda_i(x) f^{i-1}(t) \quad \text{for } t \in I.$$

Our task is to prove that, for some $f \in X(0, L)$,

$$(1)' \qquad f_x(f(x)) = F(x) \quad \text{for all } x \in I.$$

The idea behind our proof is based on the observation that if every f_x were a bijection of I then $(1)'$ would be equivalent to

$$(1)'' \qquad f(x) = (f_x)^{-1}(F(x)) \quad \text{for all } x \in I;$$

i.e. recasting the problem as a fixed point problem.

Suppose $f \in X(0, L)$ and $x \in I$. Then $f_x(0) = 0$, $f_x(1) = 1$, $f_x(t) \in I$ for all $t \in I$ and f_x is continuous. Moreover, if $0 \le t \le u \le 1$ then, by Lemma 3,

$$\begin{aligned} f_x(u) - f_x(t) &= \sum_{i=1}^{n} \lambda_i(x)(f^{i-1}(u) - f^{i-1}(t)) \\ &\le \sum_{i=1}^{n} \lambda_i(x) L^{i-1}(u - t) \le \Big(\sum_{i=1}^{n} L^{i-1}\Big)(u - t) \end{aligned}$$

and

$$f_x(u) - f_x(t) \ge \lambda_1(x)(u - t) \ge c(u - t).$$

Thus

$$(3) \qquad f_x \in X(c, C) \quad \text{for } x \in I \text{ and } f \in X(0, L)$$

where $C = \sum_{i=1}^{n} L^{i-1}$.

If $f \in X(0, L)$, $0 \le x < y \le 1$ and $t \in I$ then

$$|f_y(t) - f_x(t)| = \Big|\sum_{i=1}^{n} (\lambda_i(y) - \lambda_i(x)) f^{i-1}(t)\Big| \le n\beta(y - x).$$

Thus

$$(4) \qquad \|f_y - f_x\| \le n\beta|y - x| \quad \text{for } f \in (0, L) \text{ and } x, y \in I.$$

Now suppose that $f \in X(0, L)$, $0 \le x < y \le 1$ and $t \in I$. By (3) and (4),

$$\begin{aligned} 0 = t - t &= f_y(f_y^{-1}(t)) - f_x(f_x^{-1}(t)) \\ &= f_y(f_y^{-1}(t)) - f_y(f_x^{-1}(t)) + f_y(f_x^{-1}(t)) - f_x(f_x^{-1}(t)) \\ &\ge c(f_y^{-1}(t) - f_x^{-1}(t)) - n\beta(y - x) \end{aligned}$$

and, similarly,

$$0 \le C(f_y^{-1}(t) - f_x^{-1}(t)) + n\beta(y - x)$$

so that

$$(5) \qquad -n\beta C^{-1} \le (f_y^{-1}(t) - f_x^{-1}(t))/(y - x) \le n\beta c^{-1}.$$

Thus, for $f \in X(0,L)$,

$$\|f_y^{-1} - f_x^{-1}\| \le n\beta c^{-1}|y-x| \qquad \text{for all } x, y \in I \tag{6}$$

since $0 < c < 1 < L < C$.

Now for $f \in X(0,L)$ define $Tf : I \to \mathbb{R}$ by

$$Tf(x) = f_x^{-1}(F(x)) \qquad \text{for } x \in I;$$

notice that $Tf(0) = 0$, $Tf(1) = 1$ and $Tf(x) \in I$ for all $x \in I$.

Suppose that $f \in X(0,L)$ and $0 \le x < y \le 1$. By (5) and (i) of Lemma 2,

$$\begin{aligned} Tf(y) - Tf(x) &= f_y^{-1}(F(y)) - f_x^{-1}(F(x)) \\ &= f_y^{-1}(F(y)) - f_x^{-1}(F(y)) + f_x^{-1}(F(y)) - f_x^{-1}(F(x)) \\ &\le n\beta c^{-1}(y-x) + c^{-1}(F(y) - F(x)) \\ &\le (n\beta + M)c^{-1}(y-x) = L(y-x). \end{aligned}$$

Similarly,

$$\begin{aligned} Tf(y) - Tf(x) &= f_y^{-1}(F(y)) - f_x^{-1}(F(y)) + f_x^{-1}(F(y)) - f_x^{-1}(F(x)) \\ &\ge (-n\beta C^{-1})(y-x) + C^{-1}(F(y) - F(x)) \\ &\ge (-n\beta + \delta)C^{-1}(y-x) \ge 0 \end{aligned}$$

since $n\beta \le \delta \le 1$. Thus $Tf \in X(0,L)$. We conclude that T maps $X(0,L)$ into itself.

Aiming to prove that T is continuous, suppose that $f, g \in X(0,L)$. By the lemmas, for any $x \in I$ we have

$$\begin{aligned} |Tf(x) - Tg(x)| &= |f_x^{-1}(F(x)) - g_x^{-1}(F(x))| \le \|f_x^{-1} - g_x^{-1}\| \\ &\le c^{-1}\|f_x - g_x\| \le c^{-1} \max_{t \in I} \sum_{i=2}^{n} \lambda_i(x)|f^{i-1}(t) - g^{i-1}(t)| \\ &\le c^{-1} \sum_{i=2}^{n} \lambda_i(x)\|f^{i-1} - g^{i-1}\| \\ &\le c^{-1} \sum_{i=2}^{n} \lambda_i(x)\Big(\sum_{j=0}^{i-2} L^j\Big)\|f-g\| \\ &\le c^{-1}\Big(\sum_{j=0}^{n-2} L^j\Big)\Big(\sum_{i=2}^{n} \lambda_i(x)\Big)\|f-g\| \\ &= c^{-1}\Big(\sum_{j=0}^{n-2} L^j\Big)(1 - \lambda_1(x))\|f-g\| \\ &\le c^{-1}(1-c)\Big(\sum_{j=0}^{n-2} L^i\Big)\|f-g\|; \end{aligned}$$

recall that $0 < c < 1$ and $c \le \lambda_1(x)$ for all $x \in I$. We have proved that

$$\|Tf - Tg\| \le \gamma \|f - g\| \quad \text{for all } f, g \in X(0, L) \tag{7}$$

where

$$\gamma = c^{-1}(1 - c) \sum_{j=0}^{n-2} L^j. \tag{8}$$

Thus T is continuous. By Schauder's fixed point theorem T has a fixed point, i.e., $(1)''$ holds for some $f \in X(0, L)$. ■

IV. Uniqueness and stability. If $\gamma < 1$ then T is a contraction, in which case Banach's fixed point theorem implies that our problem has a unique solution.

THEOREM 2. *If, in addition to the assumptions of Theorem* 1, c *is so close to* 1 *that*

$$(1 - c) \sum_{j=1}^{n-1} (M + n\beta)^{j-1}/c^j < 1$$

then (1) *has a unique solution* f *in* $X(0, (M + n\beta)/c)$.

Proof. It suffices to note (7) and (8) and recall that $L = (M + n\beta)/c$. ■

Under the assumptions of Theorem 2, the solution to our problem depends continuously upon the given data in the sense of

THEOREM 3. *In addition to the assumptions of Theorem* 2, *suppose that* $\mu_1, \ldots, \mu_n : I \to I$ *are continuous*, $\sum_{i=1}^n \mu_i(x) = 1$ *for all* $x \in I$, $\mu_1(x) \ge c$ *for all* $x \in I$,

$$|\mu_k(y) - \mu_k(x)| \le \beta |y - x| \quad \text{for } x, y \in I \text{ and } 1 \le k \le n$$

and $G \in X(\delta, M)$. *Let* g *be that member of* $X(0, L)$ *satisfying*

$$\sum_{k=1}^n \mu_k(x) g^k(x) = G(x) \quad \text{for all } x \in I \tag{9}$$

(*whose existence and uniqueness is guaranteed by Theorem* 2). *Then*

$$\|f - g\| \le (1 - \gamma)^{-1} c^{-1} \Big(\sum_{i=1}^n \|\lambda_i - \mu_i\| + \|F - G\| \Big). \tag{10}$$

Proof. To indicate the dependence of the relevant operators on the given data, let us write $\lambda_x \varphi$ instead of φ_x for $\varphi \in X(0, L)$ and write T_λ instead of T. For $\varphi \in X(0, L)$ and $x \in I$ define $\mu_x \varphi(t) = \sum_{i=1}^n \mu_i(x) \varphi^{i-1}(t)$ for $t \in I$. For $\varphi \in X(0, L)$ let

$$T_\mu \varphi(x) = (\mu_x \varphi)^{-1}(G(x)) \quad \text{for } x \in I.$$

Suppose then that $f, g \in X(0, L)$, (1) and (9) hold and $x \in I$. Then

$$\begin{aligned}|f(x) - g(x)| &= |(\lambda_x f)^{-1}(F(x)) - (\mu_x g)^{-1}(G(x))| \\ &\le |(\lambda_x f)^{-1}(F(x)) - (\mu_x g)^{-1}(F(x))| \\ &\quad + |(\mu_x g)^{-1}(F(x)) - (\mu_x g)^{-1}(G(x))| \\ &\le \|(\lambda_x f)^{-1} - (\mu_x g)^{-1}\| + c^{-1}|F(x) - G(x)| \\ &\le c^{-1}\{\|\lambda_x f - \mu_x g\| + \|F - G\|\}\end{aligned}$$

by Lemma 2 since $\lambda_x f, \mu_x g \in X(c, C)$. By using Lemma 1 several times we find that, for all $t \in I$,

$$\begin{aligned}|\lambda_x f(t) - \mu_x g(t)| &= \Big|\sum_{i=1}^{n} \lambda_i(x) f^{i-1}(t) - \mu_i(x) g^{i-1}(t)\Big| \\ &\le \sum_{i=1}^{n} |\lambda_i(x) - \mu_i(x)|\,|f^{i-1}(t)| + \sum_{i=1}^{u} \mu_i(x)|f^{i-1}(t) - g^{i-1}(t)| \\ &\le \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + \sum_{i=2}^{n} \mu_i(x)\|f^{i-1} - g^{i-1}\| \\ &\le \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + \sum_{i=2}^{n} \mu_i(x)\Big(\sum_{j=0}^{i-2} L^j\Big)\|f - g\| \\ &\le \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + \sum_{i=2}^{n} \mu_i(x)\Big(\sum_{j=0}^{n-2} L^j\Big)\|f - g\| \\ &= \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + (1 - \mu_1(x))c(1 - c)^{-1}\gamma\|f - g\| \\ &\le \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + (1 - c)\gamma c(1 - c)^{-1}\|f - g\|\end{aligned}$$

by the definition (8) of γ. It follows that

$$\|f - g\| \le c^{-1}\Big\{\sum_{i=1}^{n} \|\lambda_i - \mu_i\| + \gamma c\|f - g\| + \|F - G\|\Big\},$$

i.e., (10) holds. ∎

V. Remarks and questions. The *normalization* assumption that $\sum_{i=1}^{n} \lambda_i(x) = 1$ is not severe. Instead one could suppose that $\lambda_i : I \to [0, \infty)$ is continuous for $1 \le i \le n$ and $\sum_{i=1}^{n} \lambda_i(x) > 0$ for all $x \in I$. Then the equation can be normalized by dividing by $\sum_{i=1}^{n} \lambda_i(x)$; of course, the assumptions on F would have to be altered appropriately.

We conclude the paper with some questions for possible future discussion.

1. How can (1) be treated without the assumption that $\lambda_1(x) \geq c > 0$ for all $x \in I$?

2. What more can be said in case the given functions $\lambda_1, \ldots, \lambda_n$ and F are smooth?

3. What can be said in case $F(0) = 1$ and $F(1) = 0$?

References

[1] N. H. Abel, *Oeuvres complètes*, Vol. II, Christiania, 1981, 36–39.
[2] J. G. Dhombres, *Itération linéaire d'ordre deux*, Publ. Math. Debrecen 24 (1977), 277–287.
[3] J. M. Dubbey, *The Mathematical Work of Charles Babbage*, Cambridge Univ. Press, 1978.
[4] M. Kuczma, *Functional Equations in a Single Variable*, Monograf. Mat. 46, PWN, Warszawa, 1968.
[5] M. Kuczma, B. Choczewski, and R. Ger, *Iterative Functional Equations*, Encyclopedia Math. Appl. 32, Cambridge Univ. Press, 1990.
[6] A. Mukherjea and J. S. Ratti, *On a functional equation involving iterates of a bijection on the unit interval*, Nonlinear Anal. 7, (1983), 899–908.
[7] S. Nabeya, *On the function equation $f(p+qx+rf(x)) = a+bx+cf(x)$*, Aequationes Math. 11 (1974), 199–211.
[8] J. Z. Zhang and L. Yang, *Discussion on iterative roots of continuous and piecewise monotone functions*, Acta Math. Sinica 26 (1983), 398–412 (in Chinese).
[9] W. N. Zhang, *Discussion on the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$*, Chinese Sci. Bull. 32 (1987), 1444–1451.
[10] —, *Stability of the solution of the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$*, Acta Math. Sci. 8 (1988), 421–424.
[11] —, *Discussion on the differentiable solutions of the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$*, Nonlinear Anal. 15 (1990), 387–398.

Department of Mathematics
Sichuan Union University
Chengdu 610064
P.R. China

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, Canada
N2L 3G1
E-mail: jabaker@math.uwaterloo.ca

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