

A viscoelastic contact problem with normal damped response and friction

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Abstract. We study an evolution problem which describes the quasistatic contact of a viscoelastic body with a foundation. We model the contact with normal damped response and a local friction law. We derive a variational formulation of the model and we establish the existence of a unique weak solution to the problem. The proof is based on monotone operators and fixed point arguments. We also establish the continuous dependence of the solution on the contact boundary conditions.

1. Introduction. Frictional contact between deformable bodies can be frequently found in industry and everyday life. Because of the importance in metal forming and automotive industry, a considerable effort has been made with the modeling and numerical simulations of contact problems, and the engineering literature concerning this topic is rather extensive.

An early attempt to study contact problems for linear elastic and viscoelastic materials within the framework of variational inequalities was made in [5]. An excellent reference on the field of contact problems with or without friction is [7]. The mathematical, mechanical and numerical state of the art can be found in the proceedings [11]. The quasistatic approximation of contact problems is obtained when the applied forces in the system vary slowly with time and therefore the inertial terms in the equation of motion can be neglected. There has been considerable interest in the study of quasistatic contact problems recently (see, e.g., [1, 2, 4, 8, 12, 13] and the references therein).

Quasistatic contact problems for viscoelastic materials with a general damped response condition and friction have been studied recently in [14]. There the friction has been modeled with a version of Coulomb's law and

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therefore the model was set as a family of variational inequalities for the velocity field, parametrized with time. The existence of a unique weak solution of the model has been proved using results on elliptic variational inequalities and fixed point arguments.

In this paper we investigate the process of quasistatic frictional contact between a deformable body and a foundation. The body is assumed to be viscoelastic with a nonlinear constitutive law. The contact is modeled with normal damped response, which represents the possible behavior of a layer lubricant on the contact surface. Both the viscoelastic constitutive law and the damped response condition are chosen as in [14]. We model the friction assuming that the tangential shear on the contact surface is given as a function of the tangential velocity. Considering friction laws of this form leads to a new mathematical model involving a family of nonlinear evolution equations parametrized with time. We establish the existence of a unique solution of the model using again fixed point theorems, but for different operators, since the contact conditions, and therefore the settings, in [14] and here are different. Then we prove the stability of the problem with respect to perturbations of the contact conditions, which is important from the point of view of applications.

The paper is organized as follows. In Section 2 we introduce some notation and preliminary material. In Section 3 we present the mechanical problem, derive its variational formulation and state our main existence and uniqueness result, Theorem 3.1. The proof of Theorem 3.1 is given in Section 4. Finally, in Section 5 we study the dependence of the solution on the contact boundary conditions and we establish a convergence result, Theorem 5.1.

2. Notation and preliminaries. In this short section, we present the notation we shall use and some preliminary material. For further details, we refer the reader to [5, 6, 10].

We denote by S_N the space of second order symmetric tensors on \mathbb{R}^N ($N = 2, 3$), while “ \cdot ” and $|\cdot|$ will represent the inner product and the Euclidean norm on S_N and \mathbb{R}^N , respectively. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ . In what follows the indices i and j run between 1 and N , the summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. We denote by $D(\Omega)$ the space of infinitely differentiable real-valued functions with compact support in Ω and by $L^2(\Omega)$ the space of measurable and square summable real-valued functions on Ω . We use the notation $H^{1/2}(\Gamma)$ for the Sobolev space of order $s = 1/2$ on Γ (see, e.g., [5], p. 42) while \mathcal{D} and H_Γ denote the spaces

$$\mathcal{D} = \{\varphi = (\varphi_i) \mid \varphi_i \in D(\Omega)\}, \quad H_\Gamma = \{\theta = (\theta_i) \mid \theta_i \in H^{1/2}(\Gamma)\}.$$

Throughout the paper we also use the following notation:

$$\begin{aligned} H &= \{u = (u_i) \mid u_i \in L^2(\Omega)\}, & \mathcal{H} &= \{\sigma = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \\ H_1 &= \{u = (u_i) \mid \varepsilon(u) \in \mathcal{H}\}, & \mathcal{H}_1 &= \{\sigma \in \mathcal{H} \mid \text{Div } \sigma \in H\}. \end{aligned}$$

Here ε and Div denote the deformation and the divergence operators, respectively, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div } \sigma = (\sigma_{ij,j}).$$

The spaces H , \mathcal{H} , H_1 and \mathcal{H}_1 are real Hilbert spaces endowed with their canonical inner products given by

$$\begin{aligned} \langle u, v \rangle_H &= \int_\Omega u_i v_i \, dx, & \langle \sigma, \tau \rangle_{\mathcal{H}} &= \int_\Omega \sigma_{ij} \tau_{ij} \, dx, \\ \langle u, v \rangle_{H_1} &= \langle u, v \rangle_H + \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}}, & \langle \sigma, \tau \rangle_{\mathcal{H}_1} &= \langle \sigma, \tau \rangle_{\mathcal{H}} + \langle \text{Div } \sigma, \text{Div } \tau \rangle_H. \end{aligned}$$

The associated norms are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively.

Let $\gamma : H_1 \rightarrow H_\Gamma$ be the trace map. For every $u \in H_1$, we still write u for the trace γu of u on Γ and we denote by u_ν and u_τ the normal and tangential components of u on Γ given by

$$(2.1) \quad u_\nu = u \cdot \nu, \quad u_\tau = u - u_\nu \nu.$$

Let H'_Γ be the dual of H_Γ and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between H'_Γ and H_Γ . For every $\sigma \in \mathcal{H}_1$, $\sigma\nu$ can be defined as the element in H'_Γ which satisfies

$$(2.2) \quad \langle \sigma\nu, \gamma u \rangle = \langle \sigma, \varepsilon(u) \rangle_{\mathcal{H}} + \langle \text{Div } \sigma, u \rangle_H \quad \forall u \in H_1.$$

Let also σ_ν and σ_τ represent the normal and tangential traces of σ , respectively. If σ is a smooth function, e.g. $\sigma \in C^1$, then

$$(2.3) \quad \langle \sigma\nu, \gamma u \rangle = \int_\Gamma \sigma\nu \cdot u \, da \quad \forall u \in H_1,$$

where da is the surface measure element, and

$$(2.4) \quad \sigma_\nu = (\sigma\nu) \cdot \nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

Finally, let $(X, |\cdot|_X)$ be a real normed space. Then $C(0, T; X)$ and $C^1(0, T; X)$ denote the spaces of continuous and continuously differentiable functions from $[0, T]$ to X , with the norms

$$|x|_{C(0,T;X)} = \max_{t \in [0,T]} |x(t)|_X, \quad |x|_{C^1(0,T;X)} = \max_{t \in [0,T]} (|x(t)|_X + |\dot{x}(t)|_X),$$

respectively. Here and everywhere below, the dot above a symbol represents the time derivative.

3. Problem statement and variational formulation. In this section we describe the model for the process, present its variational formulation and state our main existence and uniqueness result, Theorem 3.1.

The physical setting is as follows. We consider a deformable body which occupies a domain Ω and let $\Gamma_1, \Gamma_2, \Gamma_3$ be a partition of Γ into three disjoint measurable parts such that $\text{meas } \Gamma_1 > 0$. Let $T > 0$ and let $[0, T]$ denote the time interval considered. We assume that the body is fixed on $\Gamma_1 \times (0, T)$ and therefore the displacement field vanishes there. Surface tractions of density f_2 act on $\Gamma_2 \times (0, T)$ and a volume force of density f_0 acts on $\Omega \times (0, T)$. Moreover, the solid is in frictional contact with a foundation on $\Gamma_3 \times (0, T)$.

We denote by u the displacement field, by σ the stress field, and $\varepsilon(u)$ represents the small strain tensor. We assume that the material is viscoelastic and has a general constitutive law of the form

$$(3.1) \quad \sigma = \mathcal{A}(\varepsilon(\dot{u})) + G(\varepsilon(u))$$

where \mathcal{A} and G are given nonlinear constitutive functions. We recall that in linear viscoelasticity, the stress tensor $\sigma = (\sigma_{ij})$ is given by

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(\dot{u}) + g_{ijkl}\varepsilon_{kl}(u)$$

where $\mathcal{A} = (a_{ijkl})$ is the viscosity tensor and $G = (g_{ijkl})$ is the elasticity tensor, for $i, j, k, l = 1, \dots, N$. Kelvin–Voigt viscoelastic materials of the form (3.1) involving nonlinear constitutive functions have been considered recently in [12–14].

We now turn to describing the contact conditions, which are our main object of interest. We assume that the normal stress σ_ν satisfies a general normal damped response condition

$$(3.2) \quad -\sigma_\nu = p_\nu(\dot{u}_\nu)$$

where \dot{u}_ν represents the normal velocity and p_ν is a prescribed function. Equality (3.2) states a general dependence of the normal stress on the normal velocity. In the case when

$$(3.3) \quad p_\nu(r) = kr$$

with $k \geq 0$, the resistance of the foundation to penetration is proportional to the normal velocity. This type of behavior was considered in [15] when modeling the motion of a deformable body on sand or a granular material. We may also consider the case

$$(3.4) \quad p_\nu(r) = \beta(r_+)^m + p_0$$

where $r_+ = \max\{0, r\}$ and $0 < m \leq 1$. For $m = 1$ the boundary condition (3.4) was considered in [13] where the potential contact surface Γ_3 was assumed to be covered with a lubricant that contains solid particles, such as

one of the new smart lubricants, or with worn metallic particles. In (3.4), β represents the damping resistance constant, assumed positive, and p_0 is the oil pressure, which is given and nonnegative. The contact conditions (3.2) and (3.4) model the fact that the oil layer presents resistance or damping only when the surface moves towards the foundation.

The associated friction law is chosen as

$$(3.5) \quad -\sigma_\tau = p_\tau(\dot{u}_\tau).$$

Here p_τ is a prescribed vector-valued function, \dot{u}_τ denotes the tangential velocity and σ_τ represents the tangential force on the contact boundary. As an example we may consider the function

$$(3.6) \quad p_\tau(r) = \mu|r|^{m-1}r$$

where μ represents the coefficient of friction, assumed positive, and $0 < m \leq 1$. This is the case when the contact surface is lubricated with a thin layer of non-Newtonian fluid (see e.g. [15]). In the particular case $m = 1$, (3.5), (3.6) show that the tangential shear is proportional to the tangential velocity.

With these assumptions, denoting by u_0 the initial displacement field, the mechanical problem of frictional contact of the viscoelastic body may be formulated classically as follows.

PROBLEM P. Find a displacement field $u : \Omega \times [0, T] \rightarrow \mathbb{R}^N$ and a stress field $\sigma : \Omega \times [0, T] \rightarrow S_N$ such that

$$(3.7) \quad \sigma = \mathcal{A}(\varepsilon(\dot{u})) + G(\varepsilon(u)) \quad \text{in } \Omega \times (0, T),$$

$$(3.8) \quad \text{Div } \sigma + f_0 = 0 \quad \text{in } \Omega \times (0, T),$$

$$(3.9) \quad u = 0 \quad \text{on } \Gamma_1 \times (0, T),$$

$$(3.10) \quad \sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0, T),$$

$$(3.11) \quad -\sigma_\nu = p_\nu(\dot{u}_\nu), \quad -\sigma_\tau = p_\tau(\dot{u}_\tau) \quad \text{on } \Gamma_3 \times (0, T),$$

$$(3.12) \quad u(0) = u_0 \quad \text{in } \Omega.$$

To provide the variational formulation of the mechanical problem (3.7)–(3.12) we need additional notation. Thus, let V denote the closed subspace of H_1 defined by

$$V = \{v \in H_1 \mid v = 0 \text{ on } \Gamma_1\}.$$

Since $\text{meas } \Gamma_1 > 0$, the following Korn-type inequality holds: there exists $C_K > 0$ which depends only on Ω and Γ_1 such that

$$(3.13) \quad |\varepsilon(v)|_{\mathcal{H}} \geq C_K |v|_{H_1} \quad \forall v \in V.$$

The proof of this inequality may be found in [9], p. 79. On V we consider

the inner product given by

$$(3.14) \quad \langle u, v \rangle_V = \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}} \quad \forall u, v \in V,$$

and let $|\cdot|_V$ be the associated norm, i.e.

$$(3.15) \quad |v|_V = |\varepsilon(v)|_{\mathcal{H}} \quad \forall v \in V.$$

It follows that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V and therefore $(V, |\cdot|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.13), we have a constant $C_0 > 0$ depending only on Ω , Γ_1 , and Γ_3 such that

$$(3.16) \quad |v|_{L^2(\Gamma_3)^N} \leq C_0 |v|_V \quad \forall v \in V.$$

In the study of the mechanical problem (3.7)–(3.12), we assume that the viscosity operator $\mathcal{A} : \Omega \times S_N \rightarrow S_N$ satisfies:

$$(3.17) \quad \left\{ \begin{array}{l} \text{(a) there exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad |\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)| \leq L_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2| \\ \quad \forall \varepsilon_1, \varepsilon_2 \in S_N, \text{ a.e. } x \in \Omega; \\ \text{(b) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|^2 \\ \quad \forall \varepsilon_1, \varepsilon_2 \in S_N, \text{ a.e. } x \in \Omega; \\ \text{(c) the mapping } x \mapsto \mathcal{A}(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \quad \forall \varepsilon \in S_N; \\ \text{(d) the mapping } x \mapsto \mathcal{A}(x, 0) \text{ belongs to } \mathcal{H}. \end{array} \right.$$

The elasticity operator $G : \Omega \times S_N \rightarrow S_N$ satisfies:

$$(3.18) \quad \left\{ \begin{array}{l} \text{(a) there exists } L_G > 0 \text{ such that} \\ \quad |G(x, \varepsilon_1) - G(x, \varepsilon_2)| \leq L_G |\varepsilon_1 - \varepsilon_2| \\ \quad \forall \varepsilon_1, \varepsilon_2 \in S_N, \text{ a.e. } x \in \Omega; \\ \text{(b) the mapping } x \mapsto G(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\ \quad \forall \varepsilon \in S_N; \\ \text{(c) the mapping } x \mapsto G(x, 0) \text{ belongs to } \mathcal{H}. \end{array} \right.$$

We also assume that the contact functions $p_\nu : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}$ and $p_\tau : \Gamma_3 \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfy:

$$(3.19) \quad \left\{ \begin{array}{l} \text{(a) there exist } C_1^\nu, C_2^\nu > 0 \text{ such that} \\ \quad |p_\nu(x, r)| \leq C_1^\nu |r| + C_2^\nu \quad \forall r \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\ \text{(b) } (p_\nu(x, r_1) - p_\nu(x, r_2))(r_1 - r_2) \geq 0 \\ \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\ \text{(c) the mapping } x \mapsto p_\nu(x, r) \text{ is Lebesgue measurable on } \Gamma_3, \\ \quad \forall r \in \mathbb{R}; \\ \text{(d) the mapping } r \mapsto p_\nu(x, r) \text{ is continuous on } \mathbb{R}, \text{ a.e. } x \in \Gamma_3. \end{array} \right.$$

$$(3.20) \quad \left\{ \begin{array}{l} \text{(a) there exist } C_1^\tau, C_2^\tau \text{ such that} \\ \quad |p_\tau(x, r)| \leq C_1^\tau |r| + C_2^\tau \quad \forall r \in \mathbb{R}^N, \text{ a.e. } x \in \Gamma_3; \\ \text{(b) } (p_\tau(x, r_1) - p_\tau(x, r_2)) \cdot (r_1 - r_2) \geq 0 \\ \quad \forall r_1, r_2 \in \mathbb{R}^N, \text{ a.e. } x \in \Gamma_3; \\ \text{(c) the mapping } x \mapsto p_\nu(x, r) \text{ is Lebesgue measurable on } \Gamma_3, \\ \quad \forall r \in \mathbb{R}^N; \\ \text{(d) the mapping } r \mapsto p_\tau(x, r) \text{ is continuous on } \mathbb{R}^N, \text{ a.e. } x \in \Gamma_3; \\ \text{(e) } p_\tau(x, r) \cdot \nu(x) = 0 \quad \forall r \in \mathbb{R}^N \text{ such that } r \cdot \nu(x) = 0, \\ \quad \text{a.e. } x \in \Gamma_3. \end{array} \right.$$

We observe that the assumption (3.19) is satisfied by the functions p_ν defined in (3.3), (3.4) and the assumption (3.20) is satisfied by the function p_τ defined in (3.6). We conclude that our results below are valid for the boundary value problems related to each of these examples.

We suppose that the forces and the tractions have the regularity

$$(3.21) \quad f_0 \in C(0, T; H), \quad f_2 \in C(0, T; L^2(\Gamma_2)^N),$$

and, finally, the initial displacement satisfies

$$(3.22) \quad u_0 \in V.$$

Next, we denote by $f(t)$ the element of V given by

$$(3.23) \quad \langle f(t), v \rangle_V = \langle f_0(t), v \rangle_H + \langle f_2(t), \gamma v \rangle_{L^2(\Gamma_2)^N},$$

for all $v \in V$ and $t \in [0, T]$, and we note that conditions (3.21) imply

$$(3.24) \quad f \in C(0, T; V).$$

Let $j : V \times V \rightarrow \mathbb{R}$ be the friction functional

$$(3.25) \quad j(u, v) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu \, da + \int_{\Gamma_3} p_\tau(u_\tau) \cdot v_\tau \, da \quad \forall u, v \in V.$$

Keeping in mind (3.19) and (3.20) we observe that the integrals in (3.25) are well defined.

Next, we assume that $\{u, \sigma\}$ are regular functions satisfying (3.8)–(3.11) and let $v \in V, t \in [0, T]$. Using (2.2), (2.3) and (3.8) we have

$$\langle \sigma(t), \varepsilon(v) \rangle_{\mathcal{H}} = \langle f_0(t), v \rangle_H + \int_{\Gamma} \sigma(t) \nu \cdot v \, da$$

and, by (3.10) and (3.23), we find

$$(3.26) \quad \langle \sigma(t), \varepsilon(v) \rangle_{\mathcal{H}} = \langle f(t), v \rangle_V + \int_{\Gamma_3} \sigma(t) \nu \cdot v \, da.$$

It now follows from (2.1), (2.4) and (3.11) that

$$\sigma(t) \nu \cdot v = -p_\nu(\dot{u}_\nu(t)) v_\nu - p_\tau(\dot{u}_\tau(t)) \cdot v_\tau \quad \text{on } \Gamma_3$$

and therefore from (3.26) and (3.25) we find

$$(3.27) \quad \langle \sigma(t), \varepsilon(v) \rangle_{\mathcal{H}} + j(\dot{u}(t), v) = \langle f(t), v \rangle_V.$$

To conclude, from (3.7), (3.12) and (3.27) we obtain the following variational formulation of the mechanical problem P .

PROBLEM P_V . Find a displacement field $u : [0, T] \rightarrow V$ and a stress field $\sigma : [0, T] \rightarrow \mathcal{H}_1$ such that, for $t \in [0, T]$,

$$(3.28) \quad \sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + G(\varepsilon(u(t))),$$

$$(3.29) \quad \langle \sigma(t), \varepsilon(v) \rangle_{\mathcal{H}} + j(\dot{u}(t), v) = \langle f(t), v \rangle_V \quad \forall v \in V,$$

$$(3.30) \quad u(0) = u_0.$$

The well posedness of the problem P_V results from the following existence and uniqueness result, which we establish in the next section.

THEOREM 3.1. *Assume that (3.17)–(3.22) hold. Then there is a unique solution $\{u, \sigma\}$ to the problem (3.28)–(3.30). Moreover, the solution satisfies*

$$(3.31) \quad u \in C^1(0, T; V), \quad \sigma \in C(0, T; \mathcal{H}_1).$$

Let $\{u, \sigma\}$ be the solution specified in Theorem 3.1. Then, using standard arguments (see, e.g., [5, 10]) it follows that

$$(3.32) \quad \text{Div } \sigma(t) + f_0(t) = 0 \quad \text{a.e. in } \Omega,$$

for all $t \in [0, T]$. Moreover, the following equalities hold, in the sense of traces, for all $t \in [0, T]$:

$$(3.33) \quad u(t) = 0 \quad \text{a.e. on } \Gamma_1,$$

$$(3.34) \quad \sigma(t)\nu = f_2(t) \quad \text{a.e. on } \Gamma_2,$$

$$(3.35) \quad -\sigma_\nu(t) = p_\nu(\dot{u}_\nu(t)), \quad -\sigma_\tau(t) = p_\tau(\dot{u}_\tau(t)) \quad \text{a.e. on } \Gamma_3.$$

Keeping in mind (3.28), (3.30), (3.32)–(3.35), we may consider $\{u, \sigma\}$ as the weak solution of the mechanical problem P . Therefore, by Theorem 3.1 we conclude that under the assumptions (3.17)–(3.22), problem P has a unique weak solution with the regularity (3.31).

4. Proof of Theorem 3.1. The proof of Theorem 3.1 will be carried out in several steps. It is based on fixed point arguments, similar to those used in [12–14] but with a different choice of the operators. We assume in what follows that (3.17)–(3.22) hold. We start with the following existence and uniqueness result.

LEMMA 4.1. *For every $\eta \in C(0, T; \mathcal{H})$, there exists a unique couple $\{v_\eta, \sigma_\eta\}$ of functions such that, for $t \in [0, T]$,*

$$(4.1) \quad \sigma_\eta(t) = \mathcal{A}(\varepsilon(v_\eta(t))) + \eta(t),$$

$$(4.2) \quad \langle \sigma_\eta(t), \varepsilon(v) \rangle_{\mathcal{H}} + j(v_\eta(t), v) = \langle f(t), v \rangle_V \quad \forall v \in V.$$

Moreover, $v_\eta \in C(0, T; V)$ and $\sigma_\eta \in C(0, T; \mathcal{H}_1)$.

Proof. Using Riesz's representation theorem we define the operator $B : V \rightarrow V$ and the element $f_\eta(t) \in V$ by

$$(4.3) \quad \langle Bu, v \rangle_V = \langle \mathcal{A}(\varepsilon(u)), \varepsilon(v) \rangle_{\mathcal{H}} + j(u, v),$$

$$(4.4) \quad \langle f_\eta(t), v \rangle_V = \langle f(t), v \rangle_V - \langle \eta(t), \varepsilon(v) \rangle_{\mathcal{H}}$$

for all $u, v \in V$, $t \in [0, T]$. Let $u_1, u_2 \in V$. Using (4.3) and (3.25), we find

$$\begin{aligned} \langle Bu_1 - Bu_2, u_1 - u_2 \rangle_V &= \langle \mathcal{A}(\varepsilon(u_1)) - \mathcal{A}(\varepsilon(u_2)), \varepsilon(u_1) - \varepsilon(u_2) \rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} (p_\nu(u_{1\nu}) - p_\nu(u_{2\nu}))(u_{1\nu} - u_{2\nu}) \, da \\ &\quad + \int_{\Gamma_3} (p_\tau(u_{1\tau}) - p_\tau(u_{2\tau})) \cdot (u_{1\tau} - u_{2\tau}) \, da \end{aligned}$$

and, keeping in mind (3.17), (3.19) and (3.20), we obtain

$$(4.5) \quad \langle Bu_1 - Bu_2, u_1 - u_2 \rangle_V \geq m_{\mathcal{A}} |u_1 - u_2|_V^2.$$

Using again (4.3) and (3.25) it follows that

$$\begin{aligned} \langle Bu_1 - Bu_2, v \rangle_V &= \langle \mathcal{A}(\varepsilon(u_1)) - \mathcal{A}(\varepsilon(u_2)), \varepsilon(v) \rangle_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} (p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})) v_\nu \, da \\ &\quad + \int_{\Gamma_3} (p_\tau(u_{1\tau}) - p_\tau(u_{2\tau})) \cdot v_\tau \, da \end{aligned}$$

for all $v \in V$ and, by (3.15), (3.16), we deduce that

$$(4.6) \quad |Bu_1 - Bu_2|_V \leq L_{\mathcal{A}} |u_1 - u_2|_V + C_0 |p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})|_{L^2(\Gamma_3)} \\ + C_0 |p_\tau(u_{1\tau}) - p_\tau(u_{2\tau})|_{L^2(\Gamma_3)^N}.$$

Inequality (4.5) shows that B is a strongly monotone operator on V . Moreover, inequality (4.6) and assumptions (3.19), (3.20) imply that $B : V \rightarrow V$ is continuous. Therefore, using a standard result for nonlinear equations (see, e.g., [3], Corollary 15), there exists a unique element $v_\eta(t) \in V$ such that

$$(4.7) \quad Bv_\eta(t) = f_\eta(t) \quad \forall t \in [0, T].$$

Let $\sigma_\eta : [0, T] \rightarrow \mathcal{H}$ be defined by (4.1). Using (4.7), (4.3) and (4.4) it follows that the pair $\{v_\eta, \sigma_\eta\}$ solves (4.1) and (4.2) for all $t \in [0, T]$. Choosing $v = \pm\varphi$ where $\varphi \in \mathcal{D}$ in (4.2) yields

$$\langle \sigma_\eta, \varepsilon(\varphi) \rangle_{\mathcal{H}} = \langle f(t), \varphi \rangle_V \quad \forall t \in [0, T]$$

and, using (3.23), we find

$$(4.8) \quad \operatorname{Div} \sigma_\eta(t) + f_0(t) = 0 \quad \text{in } \Omega, \quad \forall t \in [0, T].$$

The assumption (3.21) and (4.8) imply that $\sigma_\eta(t) \in \mathcal{H}_1$ for all $t \in [0, T]$.

Now, let $t_1, t_2 \in [0, T]$. Using (4.7), (4.5) and algebraic manipulations we obtain

$$(4.9) \quad m_{\mathcal{A}}|v_{\eta}(t_1) - v_{\eta}(t_2)|_V \leq |f(t_1) - f(t_2)|_V + |\eta(t_1) - \eta(t_2)|_{\mathcal{H}}.$$

Moreover, (4.1), (3.17) and (3.15) yield

$$(4.10) \quad |\sigma_{\eta}(t_1) - \sigma_{\eta}(t_2)|_{\mathcal{H}} \leq L_{\mathcal{A}}|v_{\eta}(t_1) - v_{\eta}(t_2)|_V + |\eta(t_1) - \eta(t_2)|_{\mathcal{H}}.$$

Now, since $f : [0, T] \rightarrow V$ and $\eta : [0, T] \rightarrow \mathcal{H}$ are continuous functions, from (4.9) and (4.10) we deduce that $v_{\eta} \in C(0, T; V)$ and $\sigma_{\eta} \in C(0, T; \mathcal{H})$. Moreover, it follows from (4.8) and (3.21) that $\sigma_{\eta} \in C(0, T; \mathcal{H}_1)$.

This concludes the existence part of Lemma 4.1. The uniqueness part follows from the unique solvability of the time-dependent nonlinear equation (4.7). ■

Next, for all $\eta \in C(0, T; \mathcal{H})$ let $u_{\eta} : [0, T] \rightarrow V$ denote the function

$$(4.11) \quad u_{\eta}(t) = \int_0^t v_{\eta}(s) ds + u_0 \quad \forall t \in [0, T].$$

We consider the operator $\Lambda : C(0, T; \mathcal{H}) \rightarrow C(0, T; \mathcal{H})$ defined by

$$(4.12) \quad \Lambda\eta(t) = G(\varepsilon(u_{\eta}(t))) \quad \forall \eta \in C(0, T; \mathcal{H}), t \in [0, T].$$

We have the following result.

LEMMA 4.2. *The operator Λ has a unique fixed point $\eta^* \in C(0, T; \mathcal{H})$.*

PROOF. Let $\eta_1, \eta_2 \in C(0, T; \mathcal{H})$ and set $v_i = v_{\eta_i}$, $\sigma_i = \sigma_{\eta_i}$ for $i = 1, 2$. Using (4.1) and (4.2) we obtain

$$(4.13) \quad \sigma_i(t) = \mathcal{A}(\varepsilon(v_i(t))) + \eta_i,$$

$$(4.14) \quad \langle \sigma_i(t), \varepsilon(v) \rangle_{\mathcal{H}} + j(v_i(t), v) = \langle f(t), v \rangle_V \quad \forall v \in V,$$

for all $t \in [0, T]$, $i = 1, 2$. Using now (4.13), (4.14) and algebraic manipulations we find

$$(4.15) \quad \begin{aligned} & \langle \mathcal{A}(\varepsilon(v_1(t))) - \mathcal{A}(\varepsilon(v_2(t))), \varepsilon(v_1(t)) - \varepsilon(v_2(t)) \rangle_{\mathcal{H}} \\ & \quad + j(v_1(t), v_1(t) - v_2(t)) - j(v_2(t), v_1(t) - v_2(t)) \\ & = \langle \eta_2(t) - \eta_1(t), \varepsilon(v_1(t)) - \varepsilon(v_2(t)) \rangle_{\mathcal{H}} \end{aligned}$$

for all $t \in [0, T]$. Keeping in mind (3.17) and (3.15) we deduce that

$$(4.16) \quad \begin{aligned} & \langle \mathcal{A}(\varepsilon(v_1(t))) - \mathcal{A}(\varepsilon(v_2(t))), \varepsilon(v_1(t)) - \varepsilon(v_2(t)) \rangle_{\mathcal{H}} \\ & \quad \geq m_{\mathcal{A}}|v_1(t) - v_2(t)|_V^2 \quad \forall t \in [0, T] \end{aligned}$$

and, from (3.25), (3.19) and (3.20), we find

$$(4.17) \quad j(v_1(t), v_1(t) - v_2(t)) - j(v_2(t), v_1(t) - v_2(t)) \geq 0 \quad \forall t \in [0, T].$$

It follows from (4.15)–(4.16) that

$$(4.18) \quad |v_1(t) - v_2(t)|_V \leq \frac{1}{m_{\mathcal{A}}} |\eta_1(t) - \eta_2(t)|_{\mathcal{H}} \quad \forall t \in [0, T].$$

Therefore, from (4.12), (3.18), (4.11) and (4.18) we find

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{\mathcal{H}} \leq \frac{L_G}{m_{\mathcal{A}}} \int_0^t |\eta_1(t) - \eta_2(t)|_{\mathcal{H}} \quad \forall t \in [0, T].$$

Iterating this inequality n times leads to

$$|\Lambda^n \eta_1 - \Lambda^n \eta_2|_{C(0, T; \mathcal{H})} \leq \left(\frac{L_G}{m_{\mathcal{A}}}\right)^n \frac{T^n}{n!} |\eta_1 - \eta_2|_{C(0, T; \mathcal{H})},$$

which implies that for n sufficiently large a power Λ^n of Λ is a contraction on $C(0, T; \mathcal{H})$. Thus, there exists a unique $\eta^* \in C(0, T; \mathcal{H})$ such that $\Lambda^n \eta^* = \eta^*$ and η^* is also the unique fixed point of Λ . ■

We now have all the ingredients to prove the theorem.

Proof of Theorem 3.1. Existence. Let $\eta^* \in C(0, T; \mathcal{H})$ be the fixed point of Λ and let $\{v_{\eta^*}, \sigma_{\eta^*}\}$ be the solution of the variational problem (4.1), (4.2) for $\eta = \eta^*$. We denote by u_{η^*} the function given by (4.11) for $\eta = \eta^*$. We show that the pair $\{u_{\eta^*}, \sigma_{\eta^*}\}$ is a solution of problem P_V which satisfies (3.31). Indeed, equality (3.28) follows from (4.11), (4.12) and (4.1) since

$$v_{\eta^*}(t) = \dot{u}_{\eta^*}(t), \quad \eta^*(t) = \Lambda\eta^*(t) = G(\varepsilon(u_{\eta^*}(t))) \quad \forall t \in [0, T].$$

Equality (3.29) is a direct consequence of (4.2). Finally, the initial condition (3.30) as well as the regularity (3.31) follow from Lemma 4.1 and (4.11).

Uniqueness. The uniqueness part of Theorem 3.1 results by using arguments similar to those in [12]. It is a consequence of the uniqueness of the fixed point of the operator Λ given by (4.12). ■

5. Continuous dependence result. In this section we study the dependence of the solution to problem P_V on perturbations of the contact conditions. We suppose in what follows that (3.17)–(3.22) hold and, for every $\alpha \geq 0$, let $p_{\nu}^{\alpha}, p_{\tau}^{\alpha}$ be a perturbation of p_{ν}, p_{τ} which satisfies (3.19) and (3.20), respectively. We introduce the functional j^{α} that is obtained from j by replacing p_{ν}, p_{τ} with $p_{\nu}^{\alpha}, p_{\tau}^{\alpha}$. We now consider the following variational problem:

PROBLEM P_V^{α} . For $\alpha \geq 0$, find a displacement field $u^{\alpha} : [0, T] \rightarrow V$ and a stress field $\sigma^{\alpha} : [0, T] \rightarrow \mathcal{H}_1$ such that, for $t \in [0, T]$,

$$(5.1) \quad \sigma^{\alpha}(t) = \mathcal{A}(\varepsilon(\dot{u}^{\alpha}(t))) + G(\varepsilon(u^{\alpha}(t))),$$

$$(5.2) \quad \langle \sigma^{\alpha}(t), \varepsilon(v) \rangle_{\mathcal{H}} + j^{\alpha}(\dot{u}^{\alpha}(t), v) = \langle f(t), v \rangle_V \quad \forall v \in V,$$

$$(5.3) \quad u^{\alpha}(0) = u_0.$$

We deduce from Theorem 3.1 that for each $\alpha \geq 0$ problem P_V^α has a unique solution $\{u^\alpha, \sigma^\alpha\}$ satisfying $u^\alpha \in C^1(0, T; V)$, $\sigma^\alpha \in C(0, T; \mathcal{H}_1)$. Suppose now that the contact functions satisfy the following assumptions:

$$(5.4) \quad \begin{cases} \text{there exist } \beta_\nu \in \mathbb{R} \text{ and } \varphi_\nu : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ such that:} \\ \text{(a) } |p_\nu^\alpha(x, r) - p_\nu(x, r)| \leq \varphi_\nu(\alpha)(|r| + \beta_\nu) \quad \forall r \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\ \text{(b) } \lim_{\alpha \rightarrow 0} \varphi_\nu(\alpha) = 0; \end{cases}$$

$$(5.5) \quad \begin{cases} \text{there exist } \beta_\tau \in \mathbb{R} \text{ and } \varphi_\tau : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ such that:} \\ \text{(a) } |p_\tau^\alpha(x, r) - p_\tau(x, r)| \leq \varphi_\tau(\alpha)(|r| + \beta_\tau) \quad \forall r \in \mathbb{R}^N, \text{ a.e. } x \in \Gamma_3; \\ \text{(b) } \lim_{\alpha \rightarrow 0} \varphi_\tau(\alpha) = 0. \end{cases}$$

Under these assumptions, we have the following convergence result.

THEOREM 5.1. *The solution $\{u^\alpha, \sigma^\alpha\}$ of problem P_V^α converges uniformly to the solution $\{u, \sigma\}$ of problem P_V :*

$$(5.6) \quad \begin{aligned} u^\alpha &\rightarrow u && \text{in } C^1(0, T; V), \\ \sigma^\alpha &\rightarrow \sigma && \text{in } C(0, T; \mathcal{H}_1) \text{ as } \alpha \rightarrow 0. \end{aligned}$$

In addition to the mathematical interest in this result, it is of importance in applications, as it indicates that small inaccuracies in the contact conditions lead to small inaccuracies in the solution.

Proof of Theorem 5.1. Let $\alpha \geq 0$. To simplify the notation we shall not indicate explicitly the dependence on $t \in [0, T]$. Everywhere below C will represent a positive constant which depends on the data but is independent of α and whose value may change from line to line.

Using (3.28), (3.29), (5.1) and (5.2), we obtain

$$(5.7) \quad \begin{aligned} \langle \mathcal{A}(\varepsilon(\dot{u}^\alpha)) - \mathcal{A}(\varepsilon(\dot{u})), \varepsilon(\dot{u}^\alpha) - \varepsilon(\dot{u}) \rangle_{\mathcal{H}} \\ + \langle G(\varepsilon(u^\alpha)) - G(\varepsilon(u)), \varepsilon(\dot{u}^\alpha) - \varepsilon(\dot{u}) \rangle_{\mathcal{H}} \\ + j^\alpha(\dot{u}^\alpha, \dot{u}^\alpha - \dot{u}) - j(\dot{u}, \dot{u}^\alpha - \dot{u}) = 0. \end{aligned}$$

Moreover, from (3.15), (3.17) and (3.18) it follows that

$$(5.8) \quad \langle \mathcal{A}(\varepsilon(\dot{u}^\alpha)) - \mathcal{A}(\varepsilon(\dot{u})), \varepsilon(\dot{u}^\alpha) - \varepsilon(\dot{u}) \rangle_{\mathcal{H}} \geq C|\dot{u}^\alpha - \dot{u}|_V^2,$$

$$(5.9) \quad -\langle G(\varepsilon(u^\alpha)) - G(\varepsilon(u)), \varepsilon(\dot{u}^\alpha) - \varepsilon(\dot{u}) \rangle_{\mathcal{H}} \leq C|u^\alpha - u|_V|\dot{u}^\alpha - \dot{u}|_V.$$

We note that

$$\begin{aligned} j(\dot{u}, \dot{u}^\alpha - \dot{u}) - j^\alpha(\dot{u}^\alpha, \dot{u}^\alpha - \dot{u}) &= j(\dot{u}, \dot{u}^\alpha - \dot{u}) - j^\alpha(\dot{u}, \dot{u}^\alpha - \dot{u}) \\ &\quad + j^\alpha(\dot{u}, \dot{u}^\alpha - \dot{u}) - j^\alpha(\dot{u}^\alpha, \dot{u}^\alpha - \dot{u}) \end{aligned}$$

and, using (3.19), (3.20), we find

$$\begin{aligned}
j(\dot{u}, \dot{u}^\alpha - \dot{u}) - j^\alpha(\dot{u}^\alpha, \dot{u}^\alpha - \dot{u}) &\leq j(\dot{u}, \dot{u}^\alpha - \dot{u}) - j^\alpha(\dot{u}, \dot{u}^\alpha - \dot{u}) \\
&= \int_{\Gamma_3} (p_\nu(\dot{u}_\nu) - p_\nu^\alpha(\dot{u}_\nu))(\dot{u}_\nu^\alpha - \dot{u}_\nu) \, da \\
&\quad + \int_{\Gamma_3} (p_\tau(\dot{u}_\tau) - p_\tau^\alpha(\dot{u}_\tau)) \cdot (\dot{u}_\tau^\alpha - \dot{u}_\tau) \, da.
\end{aligned}$$

Now, in view of (5.4), (5.5) and (3.16), the previous inequality implies

$$(5.10) \quad j(\dot{u}, \dot{u}^\alpha - \dot{u}) - j^\alpha(\dot{u}^\alpha, \dot{u}^\alpha - \dot{u}) \leq C(\varphi_\nu(\alpha) + \varphi_\tau(\alpha))|\dot{u}_\alpha - \dot{u}|_V.$$

Combining (5.7)–(5.10) we deduce that

$$(5.11) \quad |\dot{u}^\alpha - \dot{u}|_V \leq C|u^\alpha - u|_V + C(\varphi_\nu(\alpha) + \varphi_\tau(\alpha)).$$

Now, an integration over $(0, t)$ and (3.30), (5.3) and (5.11) lead to

$$\begin{aligned}
|u^\alpha(t) - u(t)|_V &\leq C \int_0^t |\dot{u}^\alpha(s) - \dot{u}(s)|_V \, ds \\
&\leq C \int_0^t |u^\alpha(s) - u(s)|_V \, ds + C(\varphi_\nu(\alpha) + \varphi_\tau(\alpha))
\end{aligned}$$

for all $t \in [0, T]$, and a Gronwall-type inequality yields

$$(5.12) \quad |u^\alpha - u|_V \leq C(\varphi_\nu(\alpha) + \varphi_\tau(\alpha)).$$

Moreover, from (5.1), (3.28), (3.17) and (3.18), we obtain

$$|\sigma^\alpha - \sigma|_{\mathcal{H}} \leq C(|\dot{u}^\alpha - \dot{u}|_V + |u^\alpha - u|_V)$$

and, since by (4.8), $\text{Div } \sigma = \text{Div } \sigma^\alpha = -f_0$, we find

$$(5.13) \quad |\sigma^\alpha - \sigma|_{\mathcal{H}_1} \leq C(|\dot{u}^\alpha - \dot{u}|_V + |u^\alpha - u|_V).$$

Using now (5.11)–(5.13) yields

$$(5.14) \quad |\sigma^\alpha - \sigma|_{\mathcal{H}_1} \leq C(\varphi_\nu(\alpha) + \varphi_\tau(\alpha)).$$

Finally, it follows from (5.11), (5.12) and (5.14) that

$$(5.15) \quad |u^\alpha - u|_{C^1(0,T;V)} + |\sigma^\alpha - \sigma|_{C(0,T;\mathcal{H}_1)} \leq C(\varphi_\nu(\alpha) + \varphi_\tau(\alpha)).$$

Theorem 5.1 is now a consequence of (5.4), (5.5) and (5.15). ■

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