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The existence of solution for boundary value problems for differential equations with deviating arguments and p-Laplacian

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Abstract. We consider a boundary value problem for a differential equation with deviating arguments and p-Laplacian: $-(\phi_p(x'))' + \frac{d}{dt}\operatorname{grad} F(x) + g(t, x(t), x(\delta(t)), x'(t), x'(\tau(t))) = 0$, $t \in [0,1]$; $x(t) = \underline{\varphi}(t)$, $t \le 0$; $x(t) = \overline{\varphi}(t)$, $t \ge 1$. An existence result is obtained with the help of the Leray–Schauder degree theory, with no restriction on the damping forces $\frac{d}{dt}\operatorname{grad} F(x)$.

1. Introduction. The main purpose of the present paper is to get the solvability of the following boundary value problem (BVP for short) for a differential equation with deviating arguments and p-Laplacian:

$$\begin{aligned}(1)\qquad & -(\phi_p(x'))' + \frac{d}{dt}\operatorname{grad} F(x)\\ &\qquad + g(t, x(t), x(\delta(t)), x'(t), x'(\tau(t))) = 0, \qquad t \in [0,1],\end{aligned}$$

$$(2)\qquad \begin{aligned} x(t) &= \underline{\varphi}(t), \qquad t \le 0,\\ x(t) &= \overline{\varphi}(t), \qquad t \ge 1,\end{aligned}$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable function, $g : [0,1] \times (\mathbb{R}^n)^4 \to \mathbb{R}^n$ is a Carathéodory function, $\delta, \tau : [0,1] \to \mathbb{R}$ are differentiable functions such that $\{t \in [0,1] : \delta(t) = 0 \text{ or } \tau(t) = 1\}$ is finite and $\phi_p : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\phi_p(x) = \phi_p(x_1, \ldots, x_n) = (|x_1|^{p-2}x_1, \ldots, |x_n|^{p-2}x_n)$$

where $1 < p < \infty$. Note that ϕ_p is a homeomorphism of $\mathbb{R}^n$ with inverse ϕ_q $(1/q + 1/p = 1)$. Moreover, we suppose that

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$$-\infty < -r = \min_{t\in[0,1]}\{\delta(t),\tau(t)\} < 0 \quad \text{and} \quad 1 < \max_{t\in[0,1]}\{\delta(t),\tau(t)\} = d < \infty,$$

and $\underline{\varphi} : [-r,0] \to \mathbb{R}^n$ and $\overline{\varphi} : [1,d] \to \mathbb{R}^n$ are continuously differentiable functions.

By a *solution* x of the BVP (1), (2) we mean that $x \in C^1([-r,d],\mathbb{R}^n)$ and $\phi_p(x')$ is absolutely continuous on $[0,1]$, $x|_{[0,1]}$ satisfies the equation (1) and $x|_{[-r,0]} = \underline{\varphi}$, $x|_{[1,d]} = \overline{\varphi}$.

When $p = 2$ or $\phi_p(x) = x$, the above BVP was recently studied by Tsamatos and Ntouyas [5] by using the Topological Transversality Method. However, the existence results in [5] mainly depend upon a strict damping force condition, i.e., there exists a nonnegative constant Q such that

$$\langle A(u)v, v\rangle \le Q|v|^2 \qquad \text{for all } u,v \text{ in } \mathbb{R}^n$$

where A is the Hessian matrix of F, and $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the Euclidean norm and Euclidean inner product on $\mathbb{R}^n$ respectively. When no damping is present in (1), i.e., $F(x) \equiv 0$ and $p = 2$, the above BVP (1), (2) is also considered by Tsamatos and Ntouyas [6]. It is therefore natural to ask whether one can obtain an existence result with no restriction on the damping forces $\frac{d}{dt}\operatorname{grad} F(x)$. In this paper, we establish an existence result which can be applied to any damping forces without imposing more conditions on g. Moreover, the general exponent p is allowed, and our results seem to be new even if $p = 2$.

We remark that a number of studies are concerned with boundary value problems for differential equations with deviating argument by means of the Leray–Schauder Alternative Theorem (see for example [1–4]). The key tool in our approach is the Leray–Schauder degree theory. This method reduces the problems of existence of a solution for the BVP (1), (2) to establishing suitable a priori bounds for the solutions.

Throughout this paper, we assume that

$$\underline{\varphi}(0) = \overline{\varphi}(1) = 0,$$

but this restriction is no loss of generality, since an appropriate change of variables reduces the problem with $\underline{\varphi}(0)\overline{\varphi}(1) \ne 0$ to this case.

Furthermore, the function $g : [0,1] \times (\mathbb{R}^n)^4 \to \mathbb{R}^n$ is a Carathéodory function, which means:

(i) for almost every $t \in [0,1]$ the function $g(t,\cdot,\cdot,\cdot,\cdot)$ is continuous;

(ii) for every $(x,y,u,v) \in (\mathbb{R}^n)^4$ the function $f(\cdot,x,y,u,v)$ is measurable on [0,1];

(iii) for each $\varrho > 0$ there is $\overline{g}_\varrho \in L^1([0,1],\mathbb{R})$ such that, for almost every $t \in [0,1]$ and $[x,y,u,v] \in (\mathbb{R}^n)^4$ with $|x| \le \varrho$, $|y| \le \varrho$, $|u| \le \varrho$, $|v| \le \varrho$, one has

$$|g(t,x,y,u,v)| \le \overline{g}_\varrho(t).$$

2. Main results. In what follows, we denote the Euclidean inner product in $\mathbb{R}^n$ by $\langle\cdot,\cdot\rangle$, and the l^p-norm in $\mathbb{R}^n$ by $|\cdot|$, i.e.

$$|x| = |(x_1,\ldots,x_n)| = \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p}.$$

The corresponding L^p-norm in $L^p([0,1],\mathbb{R}^n)$ is defined by

$$\|x\|_p = \Big(\sum_{i=1}^{n}\int_0^1 |x_i(t)|^p\,dt\Big)^{1/p}.$$

The L^∞-norm in $L^\infty([0,1],\mathbb{R}^n)$ is

$$\|x\|_\infty = \max_{1\le i\le n}\|x_i\|_\infty = \max_{1\le i\le n}\sup_{t\in[0,1]}|x_i(t)|.$$

Now, we introduce the space

$$X = C([-r,d],\mathbb{R}^n)\cap C^1([-r,0]\cup[1,d],\mathbb{R}^n)\cap C_0^1([0,1],\mathbb{R}^n)$$

with the norm

$$\|x\|_* = \max\{\|x\|_\infty,\|x\|_-,\|x\|_+,\|x'\|_-,\|x'\|_+,\|x'\|_\infty\}$$

where

$$C_0^1([0,1],\mathbb{R}^n) = \{x\in C^1([0,1],\mathbb{R}^n) : x(0)=x(1)=0\},$$
$$\|x\|_- = \max_{1\le i\le n}\|x_i\|_- = \max_{1\le i\le n}\sup_{t\in[-r,0]}|x_i(t)|,$$
$$\|x\|_+ = \max_{1\le i\le n}\|x_i\|_+ = \max_{1\le i\le n}\sup_{t\in[1,d]}|x_i(t)|.$$

Moreover

$$Z = L^1([0,1],\mathbb{R}^n).$$

Define the *p-Laplacian* $\Delta_p : \operatorname{dom}\Delta_p \subset X \to Z$ by

$$(\Delta_p x)(t) = (\phi_p(x'(t)))'$$

where $\operatorname{dom}\Delta_p = \{x\in X : \phi_p(x') \text{ is absolutely continuous on } [0,1]\}$.

Let $N : X\to Z$ be the Nemytskiĭ operator associated with g:

$$(Nx)(t) = -\frac{d}{dt}\operatorname{grad}F(x) - g(t,x(t),x(\delta(t)),x'(t),x'(\tau(t))).$$

Since the operator $\Delta_p : \operatorname{dom}\Delta_p \to Z$ is invertible [7], we can define $A : X\to X$ as follows:

$$(Ax)(t) = \begin{cases} \underline{\varphi}(t), & t\in[-r,0],\\ (-\Delta_p)^{-1}(Nx)(t), & t\in[0,1],\\ \overline{\varphi}(t), & t\in[1,d].\end{cases}$$

Thus, the BVP (1), (2) is equivalent to solving the fixed point problem

$$x = Ax, \qquad x\in X. \tag{3}$$

Now, by using the same methods as in the proof of Lemmas 1 and 2 of [7], we can show

Lemma 1. *The mapping $A : X \to X$ is completely continuous, i.e. A is continuous and maps bounded sets to relatively compact sets.*

Next, let $W^{1,p}([0,1],\mathbb{R}^n)$ be the Sobolev space.

Lemma 2 (see [7]). *If $x \in W^{1,p}([0,1],\mathbb{R}^n)$ and $x(0) = x(1) = 0$, then*

$$\|x\|_p \le \pi_p^{-1}\|x'\|_p \quad \text{and} \quad \|x\|_\infty \le 2^{-1/q}\|x'\|_p$$

where $1/p + 1/q = 1$ and

$$\pi_p = 2\int_0^{(p-1)^{1/p}} \frac{ds}{(1-s^p/(p-1))^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}. \tag{4}$$

Theorem 1. *Let $p > 1$ be an integer. Assume that there exist constants δ_0, τ_0 such that*

$$|\delta'(t)| \ge \delta_0 > 0 \quad \text{and} \quad |\tau'(t)| \ge \tau_0 > 0 \quad \text{for all } t \in [0,1].$$

Furthermore, suppose that:

(H_1) There exist nonnegative integers m_1 $(< p)$, m_3 $(< p)$, nonnegative constants m_2 $(< p)$, θ $(< p)$, $\overline{a}$, $\overline{b}_i$ $(i = 1,2,3)$, and real functions b_i $(i = 1,2,3)$, c defined on $[0,1]$ with

$$|a(t)| \le \overline{a}, \qquad |b_i(t)| \le \overline{b}_i \qquad (i = 1,2,3)$$

for all $t \in [0,1]$, $c \in L^1([0,1],\mathbb{R})$ and such that

$$\langle x, g(t,x,u_1,u_2,u_3)\rangle \ge a(t)|x|^p + \sum_{i=1}^{3} b_i(t)|x|^{p-m_i}|u_i|^{m_i} + c(t)|x|^\theta$$

for all $x, u_1, u_2, u_3 \in \mathbb{R}^n$ and almost $t \in [0,1]$.

(H_2) There exist constants $\alpha \ge 0$, $\beta \ge 0$, a nonnegative integer n_1 $(< p)$, $h \in L^1([0,1],\mathbb{R}_+)$, and a Carathéodory function $G : [0,1]\times(\mathbb{R}^n)^2 \to \mathbb{R}^n$ such that

$$|g(t,x,u,v,w)| \le |G(t,x,u)| + \alpha|v|^p + \beta|v|^{p-n_1}|w|^{n_1} + h(t)$$

for all $x,u,v,w \in \mathbb{R}^n$ and almost all $t \in [0,1]$.

Then the BVP (1), (2) has at least one solution provided that

$$\overline{a} + \overline{b}_1\delta_0^{-m_1/p} + \overline{b}_2\pi_p^{m_2} + \overline{b}_3\tau_0^{-m_3/p}\pi_p^{m_3} < \pi_p^p$$

where π_p is defined by (4).

Proof. Consider the auxiliary BVP

$$
(5)\qquad \begin{cases} -(\phi_p(x'))' + \lambda \dfrac{d}{dt}\operatorname{grad} F(x) \\ \qquad\qquad + \lambda g(t, x(t), x(\delta(t)), x'(t), x'(\tau(t))) = 0, \quad t \in [0,1], \\ x(t) = \lambda \underline{\varphi}(t), \quad t \in [-r, 0], \\ x(t) = \lambda \overline{\varphi}(t), \quad t \in [1, d], \end{cases}
$$

where $\lambda \in [0,1]$. In view of the reduction from (1), (2) to (3), the BVP (5) is equivalent to the equation

$$
(6)\qquad x = A(x, \lambda), \qquad x \in X,
$$

where

$$
(7)\qquad A(x,\lambda)(t) = \begin{cases} \lambda \underline{\varphi}(t), & t \in [-r, 0], \\ (-\Delta_p)^{-1}(\lambda N x)(t), & t \in [0,1], \\ \lambda \overline{\varphi}(t), & t \in [1, d]. \end{cases}
$$

First, we verify that the set of all possible solutions of the family (5) of BVPs, $\lambda \in [0,1]$, is a priori bounded by a constant independent of λ. In fact, suppose $x \in X$ is a solution of (5) for some $\lambda \in [0,1]$. Note that $x(0) = x(1) = 0$. Then we get

$$
(8)\qquad \|x'\|_p^p = \int_0^1 \langle x, -(\phi_p(x'))' \rangle \, dx
$$

and

$$
(9)\qquad \int_0^1 \left\langle x, \frac{d}{dt}\operatorname{grad} F(x) \right\rangle dt = \int_0^1 \frac{d}{dt} \langle x, \operatorname{grad} F(x) \rangle \, dt - \int_0^1 \frac{d}{dt} F(x) \, dt = 0.
$$

Thus, in view of (H_1), Hölder's inequality, and (8), (9), we have

$$
\begin{aligned}
(10)\qquad 0 &= \int_0^1 \langle x, -(\phi_p(x'))' \rangle \, dx + \lambda \int_0^1 \left\langle x(t), \frac{d}{dt}\operatorname{grad} F(x) \right\rangle dt \\
&\quad + \lambda \int_0^1 \langle x, g(t, x(t), x(\delta(t)), x'(t), x'(\tau(t))) \rangle \, dt \\
&= \|x'\|_p^p + \lambda \int_0^1 \langle x, g(t, x(t), x'(\delta(t)), x'(t), x'(\tau(t))) \rangle \, dt \\
&\ge \|x'\|_p^p + \lambda \int_0^1 a(t)|x(t)|^p \, dt + \lambda \int_0^1 b_1(t)|x(t)|^{p-m_1}|x(\delta(t))|^{m_1} \, dt \\
&\quad + \lambda \int_0^1 b_2(t)|x(t)|^{p-m_2}|x'(t)|^{m_2} \, dt
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda\int_0^1 b_3(t)|x(t)|^{p-m_3}|x'(\tau(t))|^{m_3}\,dt+\lambda\int_0^1 c(t)|x(t)|^{\theta}\,dt\\
\geq{}& \|x'\|_p^p-\int_0^1 |a(t)|\cdot|x(t)|^p\,dt-\int_0^1 |b_1(t)|\cdot|x(t)|^{p-m_1}|x(\delta(t))|^{m_1}\,dt\\
&-\int_0^1 |b_2(t)|\cdot|x(t)|^{p-m_2}|x'(t)|^{m_2}\,dt\\
&-\int_0^1 |b_3(t)|\cdot|x(t)|^{p-m_3}|x'(\tau(t))|^{m_3}\,dt-\|x\|_\infty^{\theta}\int_0^1 |c(t)|\,dt\\
\geq{}& \|x'\|_p^p-\overline{a}\int_0^1 |x(t)|^p\,dt-\overline{b}_1\int_0^1 |x(t)|^{p-m_1}|x(\delta(t))|^{m_1}\,dt\\
&-\overline{b}_2\int_0^1 |x(t)|^{p-m_2}|x'(t)|^{m_2}\,dt\\
&-\overline{b}_3\int_0^1 |x(t)|^{p-m_3}|x'(\tau(t))|^{m_3}\,dt-\|x\|_\infty^{\theta}\|c\|_1\\
\geq{}& \|x'\|_p^p-\overline{a}\|x\|_p^p-\overline{b}_1\|x\|_p^{p-m_1}\Big(\int_0^1 |x(\delta(t))|^p\,dt\Big)^{m_1/p}\\
&-\overline{b}_2\|x\|_p^{p-m_2}\|x'\|_p^{m_2}\\
&-\overline{b}_3\|x\|_p^{p-m_3}\Big(\int_0^1 |x'(\tau(t))|^p\,dt\Big)^{m_3/p}-\|x\|_\infty^{\theta}\|c\|_1.
\end{aligned}
$$

Again

$$
\begin{aligned}
(11)\qquad &\Big(\int_0^1 |x(\delta(t))|^p\,dt\Big)^{m_1/p}\\
&=\Big[\int_0^1 |x(\delta(t))|^p\cdot\frac{1}{\delta'(t)}d(\delta(t))\Big]^{m_1/p}\leq\delta_0^{-m_1/p}\Big[\int_{\delta([0,1])}|x(s)|^p ds\Big]^{m_1/p}\\
&=\delta_0^{-m_1/p}\Big[\int_0^1|x(s)|^p\,dt+\int_{-r}^0|x(s)|^p\,ds+\int_1^d|x(s)|^p\,ds\Big]^{m_1/p}\\
&=\delta_0^{-m_1/p}\Big[\|x\|_p^p+\int_{-r}^0|\underline{\varphi}(t)|^p\,ds+\int_1^d|\overline{\varphi}(s)|^p\,ds\Big]^{m_1/p}\\
&=\delta_0^{-m_1/p}[\|x\|_p^p+\Delta_1^p]^{m_1/p}\leq\delta_0^{-m_1/p}[\|x\|_p+\Delta_1]^{m_1}\\
&=\delta_0^{-m_1/p}\Big[\|x\|_p^{m_1}+\sum_{k=1}^{m_1}\binom{m_1}{k}\|x\|_p^{m_1-k}\Delta_1^k\Big]
\end{aligned}
$$

where $\varDelta_1 = (\int_{-r}^0 |\underline{\varphi}(s)|^p\,ds + \int_1^d |\overline{\varphi}(s)|^p\,ds)^{1/p}$. Similarly

$$\Big(\int_0^1 |x'(\tau(t))|^p\,dt\Big)^{m_3/p} \le \tau_0^{-m_3/p}\Big[\|x'\|_p^{m_3} + \sum_{k=1}^{m_3}\binom{m_3}{k}\|x'\|_p^{m_3-k}\varDelta_2^k\Big] \tag{12}$$

where $\varDelta_2 = (\int_{-r}^0 |\underline{\varphi}'(s)|^p\,ds + \int_1^d |\overline{\varphi}'(s)|^p\,ds)^{1/p}$. From (10)–(12) and Lemma 2, we obtain

$$\begin{aligned}
0 &\ge \|x'\|_p^p - \overline{a}\|x\|_p^p - \overline{b}_1\delta_0^{-m_1/p}\Big[\|x\|_p^p + \sum_{k=1}^{m_1}\binom{m_1}{k}\|x\|_p^{p-k}\varDelta_1^k\Big]\\
&\quad - \overline{b}_2\|x\|_p^{p-m_2}\|x'\|_p^{m_2}\\
&\quad - \overline{b}_3\tau_0^{-m_3/p}\|x\|_p^{p-m_3}\Big[\|x'\|_p^{m_3} + \sum_{k=1}^{k}\|x'\|_p^{m_3-k}\varDelta_2^k\Big] - \|x\|_\infty^\theta\|c\|_1\\
&\ge \|x'\|_p^p - \overline{a}\pi_p^{-p}\|x'\|_p^p - \overline{b}_1\delta_0^{-m_1/p}\pi_p^{-p}\|x'\|_p^p - \overline{b}_2\pi_p^{m_2-p}\|x'\|_p^p\\
&\quad - \overline{b}_3\tau_0^{-m_3/p}\pi_p^{m_3-p}\|x'\|_p^p - \overline{b}_1\delta_0^{-m_1/p}\sum_{k=1}^{m_1}\binom{m_1}{k}\varDelta_1^k\pi_p^{k-p}\|x'\|_p^{p-k}\\
&\quad - \overline{b}_3\tau_0^{-m_3/p}\pi_p^{m_3-p}\sum_{k=1}^{m_3}\binom{m_3}{k}\varDelta_2^k\|x'\|_p^{p-k} - 2^{-1/q}\|c\|_1\|x'\|_p^\theta,
\end{aligned}$$

which yields

$$\begin{aligned}
\|x'\|_p^p \le \frac{1}{\varLambda}\Big[&\overline{b}_1\delta_1^{-m_1/p}\sum_{k=1}^{m_1}\binom{m_1}{k}\varDelta_1^k\pi_p^{k-p}\|x'\|_p^{p-k}\\
&+ \overline{b}_3\tau_0^{-m_3/p}\pi_p^{m_3-p}\sum_{k=1}^{m_3}\binom{m_3}{k}\varDelta_2^k\|x'\|_p^{p-k}\\
&+ 2^{-1/q}\|c\|_1\|x'\|_p^\theta\Big]
\end{aligned} \tag{13}$$

where

$$\varLambda = 1 - [\overline{a} + \overline{b}_1\delta_0^{-m_1/p} + \overline{b}_2\pi_p^{m_2} + \overline{b}_3\tau_0^{-m_3/p}\pi_p^{m_3}]\pi_p^{-p} > 0.$$

Since $m_1 < p$, $m_3 < p$, $\theta < p$, from (13) we see that there exists a constant $M > 0$ such that

$$\|x'\|_p \le M. \tag{14}$$

Hence by Lemma 2, there exists a constant $M_1 = 2^{-1/q}M$ such that

$$\|x\|_\infty \le M_1. \tag{15}$$

By (15), $|x(t)| = (\sum_{i=1}^n |x_i(t)|^p)^{1/p}$ is bounded, thus since $F \in C^2(\mathbb{R}^n,\mathbb{R})$, there exists a constant $M_2 > 0$ such that $\big|\frac{\partial^2 F(x)}{\partial x^2}\big| \le M_2$. Therefore, from

(H_2) and (15), we have

$$\tag{16} \int_0^1 |(\phi_p(x'))'|\,dt = \lambda\int_0^1 \left|\frac{d}{dt}\operatorname{grad}F(x) + g(t,x(\delta(t)),x'(t),x'(\tau(t)))\right| dt$$
$$\le \int_0^1 \left|\frac{\partial^2 F(x)}{\partial x^2}\right| |x'|\,dt + \int_0^1 |G(t,x(t)),x(\delta(t))|\,dt$$
$$+ \alpha\int_0^1 |x'(t)|^p\,dt$$
$$+ \beta\int_0^1 |x'(t)|^{p-n_1}|x'(\tau(t))|^{n_1}\,dt + \int_0^1 h(t)\,dt$$
$$\le \int_0^1 \overline{G}_\varrho(t)\,dt + M_2\|x'\|_p + \alpha\|x'\|_p^p$$
$$+ \beta\|x'\|_p^{p-n_1}\Big(\int_0^1 |x'(\tau(t))|^p\,dt\Big)^{n_1/p} + \|h\|_1$$

where $\varrho = \max\{M_1, \|\underline{\varphi}\|_-, \|\varphi\|_+\}$, and $\overline{G}_\varrho \in L^1([0,1],\mathbb{R})$ is such that

$$|G(t,x,y)| \le \overline{G}_\varrho(t)$$

when $|x| \le \varrho$, $|y| \le \varrho$. The existence of $\overline{G}_\varrho$ is guaranteed by the fact that G is of Carathéodory type.

Similarly to (13), we have

$$\tag{17} \Big(\int_0^1 |x'(\tau(t))|^p\,dt\Big)^{n_1/p} \le \tau_0^{-n_1/p}\Big[\|x'\|_p^{n_1} + \sum_{k=1}^{n_1}\binom{n_1}{k}\|x'\|_p^{n_1-k}\Delta_2^k\Big].$$

Thus from (14), (16), (17) one has

$$\int_0^1 |(\phi_p(x'))'|\,dt \le \int_0^1 \overline{G}_\varrho(t)\,dt + M_2M + \alpha M^p$$
$$+ \beta\tau_0^{-n_1/p}\Big[M^p + \sum_{k=1}^{n_1}\binom{n_1}{k}\Delta_2^k M^{n_1-k}\Big] = M_3.$$

Again for each $i = 1,\ldots,n$, as $x_i(0) = x_i(1) = 0$, we have $x_i'(t_i) = 0$ for some $t_i \in (0,1)$. Thus for any $t \in [0,1]$, we obtain

$$|\phi_p(x_i'(t))| = |\phi_p(x_i'(t)) - \phi_p(x_i'(t_i))| = \Big|\int_{t_i}^t (\phi_p(x_i'(s)))'\,ds\Big| \le M_3.$$

Hence for all $i \in \{1,\ldots,n\}$ and $t \in [0,1]$, one has $|x_i'(t)| \le \phi_q(M_3)$, which yields $\|x'\|_\infty \le \phi_q(M_3)$. Thus, from (15) we have

$$\begin{aligned}\|x\|_* &= \max\{\|x\|_\infty, \|x\|_-, \|x\|_+, \|x'\|_-, \|x'\|_+, \|x'\|_\infty\}\\ &\le \max\{M_1, \|\underline{\varphi}\|_-, \|\overline{\varphi}\|_+, \|\underline{\varphi}'\|_-, \|\overline{\varphi}'\|_+, \phi_q(M_3)\} = M_4,\end{aligned}$$

which implies $\|x\|_*$ is bounded.

Next, taking $r > M_4$, set

$$\Omega = \{x \in X : \|x\|_* < r\}.$$

From the above argument, (6) does not have a solution for $(x,\lambda) \in \partial\Omega \times [0,1]$. Thus for each $\lambda \in [0,1]$, the Leray–Schauder degree $\deg_{\mathrm{LS}}[I - A(\cdot,\lambda), \Omega, 0]$ is well defined and by the properties of that degree,

$$\deg_{\mathrm{LS}}[I - A(\cdot,1), \Omega, 0] = \deg_{\mathrm{LS}}[I - A(\cdot,0), \Omega, 0]. \tag{18}$$

Now it is clear that the problem

$$x = A(x,1) \tag{19}$$

is equivalent to (3). Since $A(x,0) = 0$ for all $x \in \Omega$, from (18) we have

$$\deg_{\mathrm{LS}}[I - A(\cdot,1), \Omega, 0] = \deg_{LS}[I, \Omega, 0] \ne 0,$$

which yields that the problem (19), and hence (3), has a solution, so that the BVP (1), (2) has at least one solution. This completes the proof.

Remark 1. Similar results can be obtained for the BVP

$$\begin{cases} -(\phi_p(x'))' + \dfrac{d}{dt}\operatorname{grad} F(x) + g(t, x(t), x(\delta_1(t)), \ldots, x(\delta_k(t)), \\ \qquad\qquad x'(t), x'(\tau_1(t)), \ldots, x'(\tau_l(t))) = 0, \quad t \in [0,1], \\ x(t) = \underline{\varphi}(t), \quad t \le 0, \\ x(t) = \overline{\varphi}(t), \quad t \ge 1. \end{cases}$$

Remark 2. The BVP

$$\begin{cases} (\phi_p(x'))' + \dfrac{d}{dt}\operatorname{grad} F(x) \\ \qquad\qquad + g(t, x(t), x(\delta(t)), x'(t), x'(\tau(t))) = 0, \qquad t \in [0,1], \\ x(t) = \underline{\varphi}(t), \qquad t \le 0, \\ x(t) = \overline{\varphi}(t), \qquad t \ge 1, \end{cases}$$

can also be studied by the methods of this paper.

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