COLLOQUIUM MATHEMATICUM

VOL. 83 2000 NO. 2

INTERPOLATION SETS FOR FRÉCHET MEASURES

BY

J. CAGGIANO (STATE UNIVERSITY, AR)

Abstract. We introduce various classes of interpolation sets for Fréchet measures—the measure-theoretic analogues of bounded multilinear forms on products of $C(K)$ spaces.

1. Introduction. The classical theory of interpolation sets in a harmonic-analytic context can be roughly described as the study of norm properties of "one-dimensional" objects (bounded linear forms) in relation to some underlying spectral set. The study of interpolation sets for naturally multi-dimensional structures has developed only in the last twenty years; see [GMc], [GS2]. In this work, it is our aim to examine certain harmonic-analytic interpolation properties of Fourier transforms of Fréchet measures—the measure-theoretic counterparts of multi-linear forms on products of $C_0(K)$ spaces. There are some interesting departures from the one-dimensional theory.

DEFINITION 1 ([B5, Def. 1.1]). Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be locally compact spaces with respective Borel fields $\mathcal{A}_1, \ldots, \mathcal{A}_n$. A set function $\mu : \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \to \mathbb{C}$ is an $\mathcal{F}_n$-*measure* if, when $n-1$ coordinates are fixed, μ is a measure in the remaining coordinate. When the measure spaces are arbitrary or understood, we denote the space of $\mathcal{F}_n$-measures by $\mathcal{F}_n = \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)$.

For our purposes, each space $\mathcal{X}_i$ will be the circle group $\mathbb{T}$. There is a natural identification between the space of $\mathcal{F}_n$-measures on $\mathbb{T} \times \ldots \times \mathbb{T}$ and the space of bounded n-linear forms on $C(\mathbb{T}) \times \ldots \times C(\mathbb{T})$ [B4, Thm. 4.12]. Denoting this identification by

$$\beta_\eta \leftrightarrow \eta,$$

we define the Fourier transform of an $\mathcal{F}_n$-measure η on $\mathbb{T}^n$ to be the function

2000 *Mathematics Subject Classification*: 43A25, 43A46, 46A32.

This is a portion of my doctoral thesis. I thank my advisor, Ron Blei, for his patience and encouragement. I also thank the Department of Mathematics at the University of Connecticut for its generous support during my graduate studies.

on $\mathbb{Z}^n$ given by

$$\begin{aligned}\widehat{\eta}(m_1,\ldots,m_n) &= \beta_\eta(e^{-im_1t_1},\ldots,e^{-im_nt_n})\\ &= \int_{\mathbb{T}^n} e^{-im_1t_1}\otimes\ldots\otimes e^{-im_nt_n}\,\eta(dt_1,\ldots,dt_n)\\ &= \int_{\mathbb{T}^{n-1}} e^{-im_1t_1}\otimes\ldots\otimes e^{-im_{n-1}t_{n-1}}\int_{\mathbb{T}} e^{-im_nt_n}\,\eta(dt_1,\ldots,dt_n).\end{aligned}$$

The integral above is defined iteratively, i.e.,

$$\int_{\mathbb{T}} e^{-im_nt_n}\,\eta(dt_1,\ldots,dt_n)\in\mathcal{F}_{n-1}(\mathbb{T},\ldots,\mathbb{T});$$

see [B4, Lemma 4.9] for details.

The space of $\mathcal{F}_2$-measures on $\mathcal{X}\times\mathcal{Y}$ (referred to as the space of *bimeasures* on $\mathcal{X}\times\mathcal{Y}$ in the literature) is a convolution Banach *-algebra [GS1] whose structure extends that of the space of measures on $\mathcal{X}\times\mathcal{Y}$. Convolution of $\mathcal{F}_n$-measures is not well defined in general when $n>2$ [GS3], essentially because there is no general Grothendieck-type inequality for $n>2$. If we restrict our attention to the so-called *projectively bounded* Fréchet measures, we have a well defined convolution, as well as suitable extensions of the Grothendieck inequality. The class of *completely bounded* multi-linear forms has also been considered as a natural class of $\mathcal{F}_n$-measures which satisfies a Grothendieck-type inequality; see [CS], [ZS], [Y].

DEFINITION 2 ([B5]). Let $\mu\in\mathcal{F}_n(\mathbb{T},\ldots,\mathbb{T})$, and let $E_1,\ldots,E_n$ be finite subsets of the unit ball of $\mathcal{L}^\infty(\mathbb{T})$. For $(f_1,\ldots,f_n)\in E_1\times\ldots\times E_n$ define

$$\phi_\mu(f_1,\ldots,f_n)=\int_{\mathbb{T}^n} f_1\otimes\ldots\otimes f_n\,\mu(dt_1,\ldots,dt_n). \tag{1}$$

Let

$$\|\mu\|_{\mathrm{pb}_n}=\sup\{\|\phi_\mu\|_{\mathcal{V}_n(E_1,\ldots,E_n)}: E_j\subset\mathrm{Ball}(\mathcal{L}^\infty(\mathbb{T})),\ |E_j|<\infty,\ j=1,\ldots,n\}. \tag{2}$$

Then μ is *projectively bounded* if $\|\mu\|_{\mathrm{pb}_n}<\infty$. The space of projectively bounded $\mathcal{F}_n$-measures on $\mathbb{T}\times\ldots\times\mathbb{T}$ is denoted by $\mathcal{PBF}_n=\mathcal{PBF}_n(\mathbb{T},\ldots,\mathbb{T})$.

The class of projectively bounded $\mathcal{F}_n$-measures is a non-empty proper subspace of $\mathcal{F}_n$ for $n>2$, and $\mathcal{PBF}_n=\mathcal{F}_n$ for $n<3$ (see [B5]). Projectively bounded $\mathcal{F}_n$-measures obey a Grothendieck-type inequality in the sense that $\widehat{\mu}\in\widetilde{\mathcal{V}}_n(\mathbb{Z},\ldots,\mathbb{Z})$ for all $\mu\in\mathcal{PBF}_n$. To see this, let $E_N=\{e^{-iNt},\ldots,1,\ldots,e^{iNt}\}$, and let $m_1,\ldots,m_n\in[N]=\{-N,\ldots,-1,0,1,\ldots,N\}$. Then

$$(3) \qquad \|\widehat{\mu}1_{[N]^n}\|_{\mathcal{V}_n([N],\dots,[N])} = \|\phi_\mu\|_{\mathcal{V}_n(E_N,\dots,E_N)} \le \|\mu\|_{\mathcal{PBF}_n}.$$

Since $\widehat{\mu}1_{[N]^n} \to \widehat{\mu}$ pointwise, we see immediately that $\widehat{\mu} \in \widetilde{\mathcal{V}}_n(\mathbb{Z},\dots,\mathbb{Z})$.

Given $E \subset \mathbb{Z}^n$ and $m \le n$, we define

$$B_m(E) = \{\phi \in \ell^\infty(E) : \exists \mu \in \mathcal{F}_m,\ \widehat{\mu}(j_1,\dots,j_n) = \phi(j_1,\dots,j_n) \text{ on } E\},$$

with

$$\|\phi\|_{B_m(E)} = \inf\{\|\mu\|_{\mathcal{F}_m} : \widehat{\mu} = \phi \text{ on } E\},$$

and

$$PB_m(E) = \{\phi \in \ell^\infty(E) : \exists \mu \in \mathcal{PBF}_m,\ \widehat{\mu}(j_1,\dots,j_n) = \phi(j_1,\dots,j_n) \text{ on } E\},$$

with

$$\|\phi\|_{PB_m(E)} = \inf\{\|\mu\|_{\mathcal{PBF}_m} : \widehat{\mu} = \phi \text{ on } E\}.$$

A word about the condition $m \le n$: there are certain canonical containments in $\mathcal{F}_n(\mathbb{T},\dots,\mathbb{T})$, which yield corresponding containments in the restriction algebras defined above. Consider the case $n=3$. We have $\mathcal{F}_1(\mathbb{T}^3) \subsetneq \mathcal{F}_2(\mathbb{T}^2,\mathbb{T}) \subsetneq \mathcal{F}_3(\mathbb{T},\mathbb{T},\mathbb{T})$, so $B_1(\mathbb{Z}^3) \subsetneq B_2(\mathbb{Z}^3) \subsetneq B_3(\mathbb{Z}^3)$. For certain subsets of $\mathbb{Z}^n$ we may have equality of restriction algebras; see Def. 11.

For a given Banach space A of functions on $\mathbb{Z}^n$ and $S \subset \mathbb{Z}^n$, we use the notation $[A]|_S$ to denote the quotient space A/J_S, where

$$J_S = \{f \in A : f = 0 \text{ on } S\}.$$

Similarly, for a given Banach space B of functions on $\mathbb{T}^n$ and $S \subset \mathbb{Z}^n$, we use the notation $[B]_S$ to denote $\{f \in B : \widehat{f} = 0 \text{ on } S^{\mathrm{c}}\}$. We define $\mathcal{V}_n = \mathcal{V}_n(\mathbb{T},\dots,\mathbb{T}) \equiv \widehat{\bigotimes}_{k=1}^n C(\mathbb{T})$. The Banach space dual of $\mathcal{V}_n$ is $\mathcal{F}_n = \mathcal{F}_n(\mathbb{T},\dots,\mathbb{T})$ [B4, Thm. 4.12]. The Banach space dual of $\mathcal{F}_n(\mathbb{Z},\dots,\mathbb{Z})$ is the space $\widetilde{\mathcal{V}}_n(\mathbb{Z},\dots,\mathbb{Z})$, given by

$$\widetilde{\mathcal{V}}_n(\mathbb{Z},\dots,\mathbb{Z}) = \{\phi \in \ell^\infty(\mathbb{Z}^n) : \phi = \lim_k \phi_k \text{ pointwise},\ \sup_k \|\phi_k\|_{\widehat{\otimes}} < \infty\},$$

where $\|\ \|_{\widehat{\otimes}}$ denotes the norm in $\mathcal{V}_n(\mathbb{Z},\dots,\mathbb{Z})$. We will use $\|\cdot\|_{\check{\otimes}}$ to denote the injective tensor norm, and we note that it is straightforward to show that $\mathcal{F}_n(\mathbb{Z},\dots,\mathbb{Z})$ is canonically isomorphic to $\check{\bigotimes}_{j=1}^n \ell^1(\mathbb{Z})$.

2. Interpolation sets

2.1. $\mathcal{PBF}_n$-Sidon sets

Proposition 3 ([GS2, Thm. 1]). *$\widehat{\eta} \in \widetilde{\mathcal{V}}_2(\mathbb{Z},\mathbb{Z})$ for all $\eta \in \mathcal{F}_2(\mathbb{T},\mathbb{T})$.*

Proof. Choose Grothendieck probability measures ν_1,ν_2 ([G, Corollaire 2, p. 61], [GS1, Thm. 1.2]), so that η extends to $L^2(\mathbb{T},\nu_1) \times L^2(\mathbb{T},\nu_2)$. Still denoting this extension by η, we have

$$\|\eta\| \le K_G \|\eta\|_{\mathcal{F}_2}.$$

Let $S : L^2(\mathbb{T},\nu_1) \to L^2(\mathbb{T},\nu_2)$ satisfy $\eta(f,g) = \langle Sf, g\rangle$ and let $\{a_{jk}\} \in \mathcal{F}_2(\mathbb{Z},\mathbb{Z})$. Finally, choose finite subsets A and B of $\mathbb{Z}$. Then

$$\Big|\sum_{j\in A,\,k\in B} a_{jk}\widehat{\eta}(j,k)\Big| = K_G\|\eta\|_{\mathcal{F}_2}\Big|\sum_{j\in A,\,k\in B} a_{jk}\Big\langle \frac{Se^{-ijs}}{K_G\|\eta\|_{\mathcal{F}_2}}, e^{-ikt}\Big\rangle\Big| \le 4K_G^2\|\eta\|_{\mathcal{F}_2}\|\{a_{jk}\}\|_{\mathcal{F}_2(\mathbb{Z},\mathbb{Z})}.$$

The last inequality follows immediately from the Grothendieck inequality [LP, Thm. 2.1]. ■

We note that Proposition 3 is equivalent to $\ell^1(\mathbb{Z})\check{\otimes}\ell^1(\mathbb{Z}) \subset C(\mathbb{T})\hat{\otimes}C(\mathbb{T})$ under the correspondence

$$\{a_{mn}\} \leftrightarrow \sum_{m,n} a_{mn}e^{ims}e^{int}.$$

Definition 4. A set $S \subset \mathbb{Z}\times\mathbb{Z}$ is called *$\mathcal{PBF}_n$-Sidon* if $PB_n(S) = [\widetilde{\mathcal{V}}_n(\mathbb{Z},\ldots,\mathbb{Z})]|_S$. The *$\mathcal{PBF}_n$-Sidon constant* of S is

$$\gamma_S = \sup\{\|\phi\|_{PB_n(S)} : \|\phi\|_{[\widetilde{\mathcal{V}}_n(Z,\ldots,Z)]|_S} = 1\}.$$

In [GS1] (resp. [GS2]), the authors define *BM-Sidon* (resp. *BM-interpolation*) sets to be those subsets E of $\widehat{G}\times\widehat{H}$ for which $PB_2(E) = C(E)$, where G and H are LCA groups. The case $n = 2$ in Definition 4 is different, and we see that BM-Sidon sets are necessarily $\mathcal{PBF}_2$-Sidon.

The sections of $\mathcal{PBF}_n$-Sidon sets behave as expected; let E be $\mathcal{PBF}_n$-Sidon, and let $S \subset \{1,\ldots,n\}$ be an ordered subset with $|S| = m$. Define the projection $\pi_S : \mathbb{Z}^n \to \mathbb{Z}^m$ in the obvious way. Then the $(n-m)$-section

$$E\cap\pi_S^{-1}(j_1,\ldots,j_m)$$

is $\mathcal{PBF}_{n-m}$-Sidon. To show this, we need only interpolate elementary tensors in $\widetilde{\mathcal{V}}_{n-m}(\mathbb{Z},\ldots,\mathbb{Z})$. Any such tensor ψ is extendible to a tensor $\overline{\psi} \in \widetilde{\mathcal{V}}_n(\mathbb{Z},\ldots,\mathbb{Z})$ in the obvious way. Since E is $\mathcal{PBF}_n$-Sidon, we can find a projectively bounded Fréchet measure $\mu_{\overline{\psi}}$ which interpolates $\overline{\psi}$ on E. Viewing $\mu_{\overline{\psi}}$ as an n-linear form, we see that we obtain a bounded $(n-m)$-linear form by simply fixing the coordinates of $\mu_{\overline{\psi}}$ corresponding to S. This restriction is projectively bounded, and interpolates the original tensor ψ on $E\cap\pi_S^{-1}$.

For $S \subset \mathbb{Z}\times\mathbb{Z}$ we let $I_S(\mathbb{T},\mathbb{T}) = \{f \in [\mathcal{V}_2(\mathbb{T},\mathbb{T})]_S : \widehat{f} \in [\ell^1\check{\otimes}\ell^1]_S\}$. The proof of the following theorem is straightforward.

Theorem 5. *Let $S \subset \mathbb{Z}\times\mathbb{Z}$. The following are equivalent*:

(i) $I_S(\mathbb{T},\mathbb{T}) = [\mathcal{V}_2(\mathbb{T},\mathbb{T})]_S$ (*i.e.*, *S is $\mathcal{PBF}_2$-Sidon*).
(ii) $\exists C > 0$ *with* $\|\widehat{f}\|_{\check{\otimes}} \le C\|f\|_{\mathcal{V}_2(T,T)}$, $\forall f \in I_S(\mathbb{T},\mathbb{T})$.
(iii) $I_S(\mathbb{T},\mathbb{T}) = [L^\infty(\mathbb{T})\hat{\otimes}L^\infty(\mathbb{T})]_S$.

PROPOSITION 6. (i) *Let $E, F \subset \mathbb{Z}$ be Sidon. Then $E\times F$ is $\mathcal{PBF}_2$-Sidon.*

(ii) *Let $f : \mathbb{Z} \to \mathbb{Z}$ be strictly monotone. Then the graph of f is $\mathcal{PBF}_2$-Sidon.*

Proof. (i) Clearly $[\widetilde{\mathcal{V}}_2(\mathbb{Z},\mathbb{Z})]|_{E\times F} = \widetilde{\mathcal{V}}_2(E,F)$. We show $\widetilde{\mathcal{V}}_2(E,F) \subset PB_2(E\times F)$. By a standard compactness argument, we need only interpolate restrictions of elementary tensors to $E\times F$. Let α_E, α_F be the Sidon constants of E and F, and let $\phi\otimes\psi$ be an elementary tensor of norm 1 in $\mathcal{V}_2(E,F)$. Then we can find measures μ and ν such that $\widehat{\mu}(m)\widehat{\nu}(n) = \phi(m)\psi(n)$ for any $(m,n)\in E\times F$, with $\|\mu\otimes\nu\|_{\mathcal{F}_2} \le \alpha_E\alpha_F$.

(ii) As shown in [GS1, Thm. 6.3], *any* bounded sequence on graph f can be interpolated by the transform of an $\mathcal{F}_2$-measure. ■

The existence of other examples of $\mathcal{PBF}_2$-Sidon sets is not known. In one dimension, the use of Riesz products as interpolating measures suggests a number of arithmetic criteria on subsets as sufficient conditions for satisfaction of the Sidon property. There is no clear connection between arithmetic properties of a given subset of $\mathbb{Z}^2$ and the $\mathcal{PBF}_2$-Sidon property, making the question of sufficiency somewhat more delicate.

As in the one-dimensional case, there is an approximate interpolation condition for $\mathcal{PBF}_2$-Sidon sets.

PROPOSITION 7. *Let $E\subset\mathbb{Z}\times\mathbb{Z}$. If there are $0<\delta<1$ and $0<C<\infty$ such that for all $f,g\in \mathrm{Ball}_1(\ell^\infty(\mathbb{Z}))$, there is $\mu\in\mathcal{F}_2(\mathbb{T},\mathbb{T})$, $\|\mu\|_{\mathcal{F}_2} < C$, satisfying*

$$\|f\otimes g-\widehat{\mu}\|_{[\widetilde{\mathcal{V}}_2(Z,Z)]|_E} < \delta, \tag{4}$$

then E is $\mathcal{PBF}_2$-Sidon.

Proof. We show that $[C(\mathbb{T})\mathbin{\hat{\otimes}} C(\mathbb{T})]_E \subset [\ell^1 \mathbin{\check{\otimes}} \ell^1]_E$. Choose a polynomial f with support in E, and select $\omega_1,\omega_2\in\{-1,1\}^{\mathbb{Z}}$. The canonical projections $r_j : \{-1,1\}^{\mathbb{Z}} \to \{-1,1\}$ given by $r_j(\omega) = \omega(j)$ are the Rademacher functions. Choose an $\mathcal{F}_2$-measure μ_{ω_1,ω_2} satisfying

$$\|\widehat{\mu}_{\omega_1,\omega_2}(j,k) - r_j(\omega_1)r_k(\omega_2)\|_{[\widetilde{\mathcal{V}}_2(Z,Z)]|_E} < \delta.$$

By (4) and duality,

$$\begin{aligned}\Big|\sum_{(j,k)\in E}\widehat{f}(j,k)r_j(\omega_1)r_k(\omega_2)\Big| &\le \Big|\sum_{(j,k)\in E}\widehat{f}(j,k)(r_j(\omega_1)r_k(\omega_2)-\widehat{\mu}_{\omega_1,\omega_2}(j,k))\Big| \\ &\quad + \Big|\sum_{(j,k)\in E}\widehat{f}(j,k)\widehat{\mu}_{\omega_1,\omega_2}(j,k)\Big| \\ &\le \delta\|\widehat{f}\|_{\check{\otimes}} + \|f\|_{\mathcal{V}_2(\mathbb{T},\mathbb{T})}\|\mu_{\omega_1,\omega_2}\|_{\mathcal{F}_2(\mathbb{T},\mathbb{T})}.\end{aligned}$$

Taking suprema over all choices of ω_1 and ω_2, we obtain

$$\|\widehat{f}\|_{\check{\otimes}} \le \frac{4C}{1-\delta}\|f\|_{\mathcal{V}_2(\mathbb{T},\mathbb{T})}.$$

The factor 4 appears due to the consideration of real and imaginary parts in the calculation of the injective norm of $\widehat{f}$. ■

We note that it is straightforward to show that the union of $\mathcal{PBF}_2$-Sidon sets of the type described in Proposition 6 is again $\mathcal{PBF}_2$-Sidon, whereas the union problem in general remains open. An analogous approximate interpolation condition holds for $\mathcal{PBF}_n$-Sidon sets.

That the diagonal of $\mathbb{Z} \times \mathbb{Z}$ is $\mathcal{PBF}_2$-Sidon allows us to demonstrate a fundamental difference between multiplier properties of $\mathcal{F}_1$-measures and $\mathcal{F}_2$-measures. Define $U : \mathcal{F}_2(\mathbb{T}, \mathbb{T}) \times \mathcal{F}_2(\mathbb{T}, \mathbb{T}) \to \mathcal{F}_2(\mathbb{T}, \mathbb{T})$ by $U(\mu, \nu) = \mu * \nu$. Then U is a bounded bilinear operator [GS1, Thm. 2.6]. Interestingly, U is not bounded on $L^p(\mathbb{T}^2) \times \mathcal{F}_2(\mathbb{T}, \mathbb{T})$, in direct contrast with the situation for ($\mathcal{F}_1$-)measures.

PROPOSITION 8. *Let $2 < p < \infty$. Then U is not a bounded operator on $L^p(\mathbb{T}^2) \times \mathcal{F}_2(\mathbb{T}, \mathbb{T})$.*

Let Δ denote the diagonal in $\mathbb{Z} \times \mathbb{Z}$, and let f be a Δ-polynomial,

$$f(s,t) = \sum a_j e^{ij(s+t)}.$$

For $\omega \in \{-1, 1\}^{\mathbb{Z}}$, let

$$f_\omega(s,t) = \sum a_j e^{ij(s+t)} r_j(\omega).$$

Since Δ is $\mathcal{PBF}_2$-Sidon, we can find $\mu_\omega \in \mathcal{F}_2(\mathbb{T}, \mathbb{T})$ such that

$$\widehat{\mu}(j,j) = r_j(\omega), \quad j \in \mathbb{Z},$$

and $\|\mu_\omega\|_{\mathcal{F}_2} \le \gamma_\Delta$ for all ω. Suppose $\|U\| = C < \infty$. Then

$$f = f_\omega * \mu_\omega = U(f_\omega, \mu_\omega)$$

and $\|f\|_{L^p} \le C\gamma_\Delta \|f_\omega\|_{L^p}$, which implies that

$$\|f\|_{L^p} \le C\gamma_\Delta \mathrm{E}_\omega \|f_\omega\|_{L^p}.$$

Applying Khinchin's inequalites, we obtain

$$\begin{aligned}\|f\|_{L^p}^p &\le C^p \gamma_\Delta^p \mathrm{E}_\omega \int_{\mathbb{T}}\int_{\mathbb{T}} \Big| \sum a_j e^{ij(s+t)} r_j(\omega) \Big|^p \\ &= C^p \gamma_\Delta^p \int_{\mathbb{T}}\int_{\mathbb{T}} \mathrm{E}_\omega \Big| \sum a_j e^{ij(s+t)} r_j(\omega) \Big|^p \le C^p \gamma_\Delta^p p^{p/2} \Big(\sum |a_j|^2 \Big)^{p/2},\end{aligned}$$

which shows that Δ is a $\Lambda(p)$-set, a contradiction. ■

We now consider $\mathcal{PBF}_n$-Sidon sets for $n > 2$. The space of $\mathcal{F}_1$-measures on $\mathbb{T}^n$ is denoted by $\mathcal{M}(\mathbb{T}^n)$.

LEMMA 9. $\mathcal{M}(\mathbb{T}^n) \subset \mathcal{PBF}_n(\mathbb{T}, \ldots, \mathbb{T})$.

Proof. Let $\mu \in \mathcal{M}(\mathbb{T}^n)$, and choose finite subsets $E_1, \ldots, E_n$ in the unit ball of $\mathcal{L}^\infty(\mathbb{T})$. Then

$$\begin{aligned}
&\|\phi_\mu\|_{\mathcal{V}_n(E_1,\ldots,E_n)} \\
&= \sup_{\|\beta\|_{\mathcal{F}_n} \le 1} \Big| \sum_{f_1 \in E_1, \ldots, f_n \in E_n} \beta(f_1, \ldots, f_n) \int_{\mathbb{T}^n} (f_1 \otimes \ldots \otimes f_n)\, \mu(dt \times \ldots \times dt) \Big| \\
&\le \sup_{\|\beta\|_{\mathcal{F}_n} \le 1} \int_{\mathbb{T}^n} \Big| \sum_{f_1 \in E_1, \ldots, f_n \in E_n} \beta(f_1, \ldots, f_n)(f_1 \otimes \ldots \otimes f_n) \Big|\, |\mu|(dt \times \ldots \times dt) \\
&\le 2^n \|\mu\|_{\mathcal{M}(\mathbb{T}^n)}. \ \blacksquare
\end{aligned}$$

Corollary 10. *If $E_1, \ldots, E_n$ are Sidon, then $E_1 \times \ldots \times E_n$ is $\mathcal{PBF}_n$-Sidon.*

Proof. We need only recall that $B(E_1 \times \ldots \times E_n) = \widetilde{\mathcal{V}}_n(E_1, \ldots, E_n)$. $\blacksquare$

2.2. *$\mathcal{F}_m/\mathcal{F}_n$-sets*

Definition 11. Let $m > n \ge 0$. For $n > 0$, a set $E \subset \mathbb{Z}^m$ is an *$\mathcal{F}_m/\mathcal{F}_n$-set* if $B_m(E) = B_n(E)$; $E \subset \mathbb{Z}^m$ is an *$\mathcal{F}_m/\mathcal{F}_0$-set* if $B_m(E) = \ell^\infty(E)$.

We define $\mathcal{PBF}_m/\mathcal{PBF}_n$, $\mathcal{PBF}_m/\mathcal{F}_n$, and $\mathcal{F}_m/\mathcal{PBF}_n$ sets analogously. In this terminology, Sidon sets are $\mathcal{F}_1/\mathcal{F}_0$-sets and BM-Sidon sets are $\mathcal{F}_2/\mathcal{F}_0$-sets. In [GS2], the authors use the term *BM/B-sets* for those subsets of the dual of an LCA group whose bimeasure restriction algebra coincides with the measure restriction algebra. In our terminology, these are $\mathcal{F}_2/\mathcal{F}_1$-sets. We see immediately that any Sidon set in $\mathbb{Z}^m$ is $\mathcal{F}_m/\mathcal{F}_0$, $\mathcal{F}_m/\mathcal{F}_1$ (and hence $\mathcal{F}_m/\mathcal{F}_n$ for any $n < m$), $\mathcal{PBF}_m/\mathcal{PBF}_n$ and $\mathcal{F}_m/\mathcal{PBF}_m$.

The proof of Proposition 6(i) shows that we need not step outside the space of measures to interpolate all of $\widetilde{\mathcal{V}}_2(E, F)$ when E and F are Sidon. Thus, we have

Corollary 12. *If E and F are Sidon subsets of $\mathbb{Z}$, then $E \times F$ is $\mathcal{F}_2/\mathcal{F}_1$.*

There is a partial converse to the previous corollary: if $A \times A$ is $\mathcal{F}_2/\mathcal{F}_1$ then A is Sidon. To see this, let $\Delta_{A \times A} = \{(a_j, a_j) : a_j \in A\}$. Since the $\mathcal{F}_2/\mathcal{F}_1$ property is inherited by subsets, $\Delta_{A \times A}$ is $\mathcal{F}_2/\mathcal{F}_1$. We claim that $\Delta_{A \times A}$ is Sidon in $\mathbb{Z} \times \mathbb{Z}$. Let $\phi \in \ell^\infty(\Delta_{A \times A})$. For $f, g \in C(\mathbb{T})$, define

$$\eta_\phi(f, g) = \sum_j \phi(j) \widehat{f}(a_j) \widehat{g}(a_j).$$

Then η_ϕ is a bounded linear form on $\mathcal{V}_2(\mathbb{T}, \mathbb{T})$ satisfying

$$\widehat{\eta}_\phi(a_j, a_j) = \phi(j).$$

But $\Delta_{A\times A}$ is $\mathcal{F}_2/\mathcal{F}_1$, and so we can find a measure μ_ϕ satisfying

$$\widehat{\mu}_\phi(a_j, a_j) = \phi(j).$$

So $\Delta_{A\times A}$ is Sidon. Now, let f be an A-polynomial, $f(s) = \sum_j c_j e^{ia_j s}$, and consider

$$F(s,t) = \sum_j c_j e^{ia_j(s+t)}.$$

If B denotes the Sidon constant of $\Delta_{A\times A}$, we have

$$\|\widehat{f}\|_{\ell^1} = \sum_j |c_j| \le B \sup_{s,t} \Big| \sum_j c_j e^{ia_j(s+t)} \Big| = B\|f\|_\infty,$$

and A is Sidon. We also note that the result above need not hold when the factors forming the Cartesian product in $\mathbb{Z}\times\mathbb{Z}$ are different. For example, $\mathbb{Z}\times\{n\}$ is $\mathcal{F}_2/\mathcal{F}_1$. To see this, choose $\mu \in \mathcal{F}_2(\mathbb{T},\mathbb{T})$, and let η_μ be the corresponding bilinear form on $C(\mathbb{T})\times C(\mathbb{T})$. Then

$$\widehat{\mu}|_{\mathbb{Z}\times\{n\}} = (\eta_\mu(\cdot, n)\otimes e^{int}dt)|_{\mathbb{Z}\times\{n\}}.$$

This leads to a question. Given two (different) infinite subsets A and B such that $A\times B$ is $\mathcal{F}_2/\mathcal{F}_1$, must A or B be Sidon? We do not know the answer.

Which sets are both $\mathcal{F}_2/\mathcal{F}_1$ and $\mathcal{PBF}_2$-Sidon? It is obvious that any Sidon subset of $\mathbb{Z}\times\mathbb{Z}$ is necessarily $\mathcal{F}_2/\mathcal{F}_1$ and $\mathcal{PBF}_2$-Sidon. We can glean a bit more. It is straightforward to show

PROPOSITION 13. *If S is $\mathcal{F}_2/\mathcal{F}_1$ and $\mathcal{PBF}_2$-Sidon then S is $\Lambda(p)$ for all $p<\infty$.*

We can separate the various interpolation sets described thus far. Let us consider the two-dimensional case. A product of two Sidon sets is $\mathcal{PBF}_2$-Sidon and $\mathcal{F}_2/\mathcal{F}_1$, but not $\mathcal{F}_2/\mathcal{F}_0$. $\mathbb{Z}\times\{n\}$ is $\mathcal{F}_2/\mathcal{F}_1$ but not $\mathcal{PBF}_2$-Sidon or $\mathcal{F}_2/\mathcal{F}_0$, while the diagonal $\Delta = \{(n,n) : n\in\mathbb{Z}\}$ is $\mathcal{F}_2/\mathcal{F}_0$ (hence $\mathcal{PBF}_2$-Sidon) but not $\mathcal{F}_2/\mathcal{F}_1$.

In [GS2], the authors ask: if E, F, and G are infinite subsets of $\mathbb{Z}$ such that $E\cup F$ and G are lacunary ($E\cap F = \emptyset$), must $(E+F)\times G$ be an $\mathcal{F}_2/\mathcal{F}_1$-set? This question turns out to be a "cusp" case, as Theorem 15 and the next lemma demonstrate.

LEMMA 14. *Let E, F, G and H be infinite subsets of $\mathbb{Z}$ with $E\cap F = G\cap H = \emptyset$, and $E\cup F$, $G\cup H$ lacunary. Choose a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{N}^3$, and enumerate E, F, G and H according to this correspondence*:

$$E = \{\lambda_{abc}\},\quad F = \{\nu_{abc}\},\quad G = \{\varrho_{abc}\},\quad H = \{\kappa_{abc}\},\qquad a,b,c\in\mathbb{N}. \tag{8}$$

Then $U \subset (E+F)\times(G+H)$ given by

$$U = \{(\lambda_{abc}+\nu_{bcd})\times(\varrho_{cda}+\kappa_{dab})\}$$

is not Sidon.

Proof. As described in [B3], the scheme above gives rise to a 4/3-product, with Sidon index 8/7.

THEOREM 15. *If E, F, G, and H are as above, then $(E+F)\times(G+H)$ is not $\mathcal{F}_2/\mathcal{F}_1$.*

Proof. We can find a bounded function ϕ on U which is not the transform of a measure on $\mathbb{T}^2$. Then ϕ is a function of twelve variables, but by the linkages described in the lemma we consider ϕ as a function of a, b, c, and d. Let β_ϕ be the bilinear form on $C(\mathbb{T})\times C(\mathbb{T})$ given by

$$\beta_\phi(f,g)=\sum_{a,b,c,d}\phi(a,b,c,d)\widehat{f}(\lambda_{abc}+\nu_{bcd})\widehat{g}(\varrho_{cda}+\kappa_{dab}).$$

An application of the Cauchy–Schwarz inequality gives boundedness of β_ϕ, and we easily verify that

$$\widehat{\beta}_\phi(\lambda_{abc}+\nu_{bcd},\varrho_{cda}+\kappa_{dab})=\phi(a,b,c,d).$$

Thus U is not $\mathcal{F}_2/\mathcal{F}_1$, and any set containing U cannot be $\mathcal{F}_2/\mathcal{F}_1$. ■

Let $A,B\subset\mathbb{Z}$, $A\cap B=\emptyset$, $\mathrm{card}(A)=\mathrm{card}(B)=\infty$. For $m,n\geq 2$, let $E_1,\ldots,E_m$ be pairwise disjoint infinite subsets of A and let $F_1,\ldots,F_n$ be pairwise disjoint infinite subsets of B. By considering translates of a set of the form $(E+F)\times(G+H)$, we see that $(E_1+\ldots+E_m)\times(F_1+\ldots+F_n)$ is not an $\mathcal{F}_2/\mathcal{F}_1$-set.

We can illustrate something of the "tightness" of the original question in [GS2] as it relates to a generalization of an inequality of Littlewood. One avenue of attack on the problem is as follows. Let

$$E=\{\lambda_i\},\quad F=\{\nu_j\},\quad G=\{\varrho_k\}.$$

Any element of $\ell^2(\mathbb{N}^2)\check{\otimes}\ell^2(\mathbb{N})$ naturally induces a bounded bilinear form on $C(\mathbb{T})\times C(\mathbb{T})$. For $a=\{a_{(j,k),l}\}\in\ell^2(\mathbb{N}^2)\check{\otimes}\ell^2(\mathbb{N})$, define such a form β_a by

$$\beta_a(f,g)=\sum_{j,k,l}a_{(j,k),l}\widehat{f}(\lambda_j+\nu_k)\widehat{g}(\varrho_l).$$

The problem is solved if we can produce a tensor as above which simultaneously is not the transform of a measure restricted to $(E+F)\times G$. But this cannot be done. Littlewood's mixed-norm inequality in three dimensions [D] states that if $\{a_{(j,k),l}\}$ is any finitely supported tensor, then

$$\|a_{(j,k),l}\|_{\widetilde{\mathcal{V}}_3(\mathbb{N},\mathbb{N},\mathbb{N})}\leq 2\sqrt{2}\sup_l\sqrt{\sum_{(j,k)}|a_{(j,k),l}|^2},$$

which implies that $\ell^2(\mathbb{N}^2)\check{\otimes}\ell^2(\mathbb{N})\subset\widetilde{\mathcal{V}}_3(\mathbb{N},\mathbb{N},\mathbb{N})$. Since $B((E+F)\times G)$ contains all elements of $\widetilde{\mathcal{V}}_3(\mathbb{N},\mathbb{N},\mathbb{N})$, we see that it is impossible to find a tensor with the desired properties. As a final comment along this line we remark that in [GS2] the authors prove the following:

PROPOSITION 16. *Let H be an infinite subgroup of the discrete group Γ, and let K be any infinite subset of Γ. Then $H \times K$ is not $\mathcal{F}_2/\mathcal{F}_1$.*

Notice that this is a "limiting case" of $(E_1 + \ldots + E_n) \times K$.

Certain of the "fractional Cartesian products" [B3], [B4] provide examples of $\mathcal{PBF}_n$-Sidon sets, $\mathcal{PBF}_n/\mathcal{F}_1$-sets, and $\mathcal{F}_n/\mathcal{F}_0$-sets. For completeness, we include some of the ideas of [B3] and [B4]. Let E be a lacunary subset of $\mathbb{Z}$. Let $[m] = \{1, \ldots, m\}$. Given $S \subset [m]$, π_S denotes the projection from E^m to $E^{|S|}$ ($|S| = \mathrm{card}(S)$) given by

$$\pi_S(e_1, \ldots, e_m) = (e_j : j \in S),$$

with the $|S|$-tuple on the right of the equality above ordered canonically. Let $\mathcal{S} = \{S_k : k = 1, \ldots, n\}$ be a collection of subsets of $[m]$ whose union is $[m]$. Further, we require that each element of $[m]$ appears in at least two elements of $\mathcal{S}$. For each $k = 1, \ldots, n$, consider $\ell^2(\mathbb{Z}^{|S_k|})$. Let $\phi \in \ell^\infty(\mathbb{Z}^m)$, and for $(x_1, \ldots, x_n) \in \ell^2(\mathbb{Z}^{|S_1|}) \times \ldots \times \ell^2(\mathbb{Z}^{|S_n|})$ define

$$\begin{aligned}(6)\qquad &\eta_{\phi,\mathcal{S}}(x_1, \ldots, x_n) \\ &\quad = \sum_{\vec{a} \in \mathbb{Z}^m} \phi(\vec{a}) x_1(\pi_{S_1}(\vec{a})) \ldots x_n(\pi_{S_n}(\vec{a})), \qquad x_j \in \ell^2(\mathbb{Z}^{|S_j|}).\end{aligned}$$

In [B1], Blei shows that for all bounded arrays ϕ, $\eta_{\phi,\mathcal{S}}$ is a well defined n-linear form whose norm is bounded by $\|\phi\|_\infty$. As such, $\eta_{\phi,\mathcal{S}}$ can be regarded as an n-linear form on $C(\mathbb{T}^{|S_1|}) \times \ldots \times C(\mathbb{T}^{|S_n|})$, or (equivalently) as an $\mathcal{F}_n$-measure on the product of the respective Borel fields of the given products of $\mathbb{T}$. Let

$$\mathcal{V}_{\mathcal{S}}(\mathbb{Z}^m) = \Big\{ \phi(\vec{a}) = \sum_{j=1}^{\infty} \alpha_j \psi_{j1}(\pi_{S_1}(\vec{a})) \ldots \psi_{jn}(\pi_{S_n}(\vec{a})), \qquad \psi_{ji} \in c_0(\mathbb{Z}^{|S_i|}),\ \sum |\alpha_j| < \infty \Big\}.$$

Identifying arrays which are the same pointwise on $\mathbb{Z}^m$, we obtain a quotient space, with norm

$$\|\phi\|_{\mathcal{V}_{\mathcal{S}}} = \inf \Big\{ \sum |\beta_j| : \phi(\vec{a}) = \sum_{j=1}^{\infty} \beta_j \psi_{j1}(\pi_{S_1}(\vec{a})) \ldots \psi_{jn}(\pi_{S_n}(\vec{a})) \quad \text{pointwise on } \mathbb{Z}^m \Big\}.$$

$\widetilde{\mathcal{V}}_{\mathcal{S}}(\mathbb{Z}^m)$ is the space of arrays on $\mathbb{Z}^m$ obtained by taking pointwise limits of uniformly bounded sequences of elements in $\mathcal{V}_{\mathcal{S}}(\mathbb{Z}^m)$.

We now transfer the constructions above to $\mathcal{F}_n(\mathbb{T},\ldots,\mathbb{T})$. Let $E \subset \mathbb{Z}$ be lacunary, and let $\mathcal{S} = \{S_k : k = 1,\ldots,n\}$ be a cover of $[m]$ with the properties described above. Consider an m-fold enumeration of E : $E = \{e_{a_1\ldots a_m} : a_j \in \mathbb{N}\}$ along with $|S_j|$-fold enumerations of E : $E_j = \{e_{a_1\ldots a_{|S_j|}}\}$. Then we define a subset $E^{\mathcal{S}}$ of E^n by

$$E^{\mathcal{S}} = \{(e^{(1)}_{\pi_{S_1}(j_1,\ldots,j_m)}, e^{(2)}_{\pi_{S_2}(j_1,\ldots,j_m)}, \ldots, e^{(n)}_{\pi_{S_n}(j_1,\ldots,j_m)}) : e^{(i)}_{\pi_{S_i}(j_1,\ldots,j_m)} \in E_i \ \forall i\}.$$

We view $\eta_{\phi,\mathcal{S}}$ as an $\mathcal{F}_n$-measure in the natural way. It is known [B4] that $\widetilde{\mathcal{V}}_{\mathcal{S}}(\mathbb{Z}^m)$ can be realized as a restriction algebra of Fourier–Stieltjes transforms of measures on $\mathbb{T}^n$, namely,

$$\widetilde{\mathcal{V}}_{\mathcal{S}}(\mathbb{Z}^m) = B(E^{\mathcal{S}}) = \mathcal{M}(\mathbb{T}^m)/\{\mu \in \mathcal{M}(\mathbb{T}^m) : \widehat{\mu} = 0 \text{ on } (E^{\mathcal{S}})^{\mathrm{c}}\}.$$

THEOREM 17 ([B1]). *The n-linear form $\eta_{\phi,\mathcal{S}}$ defined by* (6) *is projectively bounded if and only if $\phi \in \widetilde{\mathcal{V}}_{\mathcal{S}}(\mathbb{Z}^m)$.*

Let $e_{\mathcal{S}}$ be the combinatorial dimension of $E^{\mathcal{S}}$ ([BS]). By [B5, Cor. 7.4] we see that if $e_{\mathcal{S}} = 1$, then $E_{\mathcal{S}}$ is $\mathcal{PBF}_m/\mathcal{F}_0$. This is a generalization of the "monotone graphs" of Proposition 6.

THEOREM 18. *Let E be lacunary, and let $\mathcal{S}$ be a cover of $[m]$ so that every element of $[m]$ appears in at least two elements of $\mathcal{S}$. If $e_{\mathcal{S}} > 1$, then $E^{\mathcal{S}} \subset \mathbb{Z}^m$ is $\mathcal{PBF}_m$-Sidon, $\mathcal{PBF}_m/\mathcal{F}_1$, and $\mathcal{F}_m/\mathcal{F}_0$, but not $\mathcal{F}_m/\mathcal{PBF}_m$.*

Proof. $E^{\mathcal{S}}$ is $\mathcal{PBF}_m$-Sidon and $\mathcal{PBF}_m/\mathcal{F}_1$ since

$$\widetilde{\mathcal{V}}_{\mathcal{S}} = B(E^{\mathcal{S}}) \subset PB_m(E^{\mathcal{S}}) \subset \widetilde{\mathcal{V}}_m|_{E^{\mathcal{S}}} = \widetilde{\mathcal{V}}_{\mathcal{S}}.$$

The last equality follows from the fact that $1_{E^{\mathcal{S}}} \in \widetilde{\mathcal{V}}_{\mathcal{S}}$. Next, $E^{\mathcal{S}}$ is $\mathcal{F}_m/\mathcal{F}_0$ since (6) is bounded for all arrays $\vec{a}$. Finally, because (6) can be projectively unbounded for some choice of ϕ ([B5, Cor. 7.4]) we see that $E^{\mathcal{S}}$ is not $\mathcal{F}_m/\mathcal{PBF}_m$. ∎

I am grateful to the referee of this paper for a number of useful comments and corrections, and for the suggested definition of $\mathcal{F}_m/\mathcal{F}_0$-sets.

REFERENCES

[B1] R. C. Blei, *Multi-dimensional extensions of the Grothendieck inequality and applications*, Ark. Mat. 17 (1979), 51–68.

[B2] —, *Rosenthal sets that cannot be sup-norm partitioned and an application to tensor products*, Colloq. Math. 37 (1977), 295–298.

[B3] —, *Fractional Cartesian products of sets*, Ann. Inst. Fourier (Grenoble) 29 (1979), no. 2, 79–105.

[B4] —, *Fractional dimensions and bounded fractional forms*, Mem. Amer. Math. Soc. 331 (1985).

[B5] R. C. Blei, *Projectively bounded Fréchet measures*, Trans. Amer. Math. Soc. 348 (1996), 4409–4432.

[BS] R. C. Blei and J. Schmerl, *Combinatorial dimension and fractional Cartesian products*, Proc. Amer. Math. Soc. 120 (1994), 73–77.

[CS] E. Christensen and A. M. Sinclair, *Representations of completely bounded multilinear operators*, J. Funct. Anal. 72 (1987), 151–181.

[D] A. M. Davie, *Quotient algebras of uniform algebras*, J. London Math. Soc. 7 (1973), 31–40.

[GMc] C. C. Graham and O. C. McGehee, *Essays in Commutative Harmonic Analysis*, Grundlehren Math. Wiss. 238, Springer, New York, 1979.

[GS1] C. C. Graham and B. M. Schreiber, *Bimeasure algebras on LCA groups*, Pacific J. Math. 115 (1984), 91–127.

[GS2] —, —, *Sets of interpolation for Fourier transforms of bimeasures*, Colloq. Math. 51 (1987), 149–154.

[GS3] —, —, *Projections in spaces of bimeasures*, Canad. Math. Bull. 31 (1988), 19–25.

[G] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. Sao Paulo 8 (1956), 1–79.

[LP] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications*, Studia Math. 29 (1968), 275–326.

[R] W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. 9 (1960), 203–227.

[V] N. Th. Varopoulos, *On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory*, J. Funct. Anal. 16 (1974), 83–100.

[Y] K. Ylinen, *Noncommutative Fourier transforms of bounded bilinear forms and completely bounded multilinear operators*, ibid. 79 (1988), 144–165.

[ZS] G. Zhao and B. M. Schreiber, *Algebras of multilinear forms on groups*, in: Contemp. Math. 189, Amer. Math. Soc., 1995, 497–511.

Department of Computer Science and Mathematics
Box 70, Arkansas State University
State University, AR 72467, U.S.A.

Received 6 November 1998;
revised 6 April 1999 (3653)