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ON UNRESTRICTED PRODUCTS OF (W) CONTRACTIONS

BY

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Abstract. Given a family of (W) contractions $T_1, \ldots, T_N$ on a reflexive Banach space X we discuss unrestricted sequences $T_{r_n} \circ \ldots \circ T_{r_1}(x)$. We show that they converge weakly to a common fixed point, which depends only on x and not on the order of the operators T_{r_n} if and only if the weak operator closed semigroups generated by $T_1, \ldots, T_N$ are right amenable.

Let $(X, \|\cdot\|)$ be a reflexive Banach space. Its dual space is denoted by $(X^*, \|\cdot\|)$. The dual operation, where $x \in X$ and $\lambda \in X^*$, is denoted by $\lambda(x)$ or $\langle x, \lambda\rangle$. We say that a linear contraction $T : X \to X$ satisfies the (W) *condition* if for every sequence $x_n \in X$ we have $\text{w-lim}_{n\to\infty}(x_n - T(x_n)) \to 0$ whenever x_n is bounded and satisfies $\|x_n\| - \|T(x_n)\| \to 0$ (we write w-lim for weak limits). If for every $x \in X$ we have $\|T(x)\| = \|x\|$ if and only if $T(x) = x$ (i.e. when x is a T-fixed point) then we say that T satisfies the (W′) *condition.* Clearly (W)⇒(W′).

Given a finite collection $T_1, \ldots, T_N$ of linear operators on X we study the asymptotic behaviour of $T_{r_n} \circ T_{r_{n-1}} \circ \ldots \circ T_{r_1}$, where $1 \le r_j \le N$. If $F \subseteq \{1, \ldots, N\}$ we define $\boldsymbol{S}_F = \{T_{r_n} \circ T_{r_{n-1}} \circ \ldots \circ T_{r_1} : r_j \in F\}$ to be the semigroup of linear operators generated by T_j, where $j \in F$. Elements of $\boldsymbol{S}_F$ are called *F-words*. We say that $\boldsymbol{S}_F$ has property (W) if for every bounded sequence of vectors $x_n \in X$ and F-words W_n, if $\lim_{n\to\infty}(\|x_n\| - \|W_n(x_n)\|) = 0$, then $\text{w-lim}_{n\to\infty}(x_n - W_n(x_n)) = 0$.

The closure of $\boldsymbol{S}_F$ in the weak operator topology (w.o.t.) is denoted by $\mathfrak{S}_F$. Obviously all $\mathfrak{S}_F$ as well as their adjoints $\mathfrak{S}_F^* = \{P^* : P \in \mathfrak{S}_F\}$ are w.o.t. compact semitopological semigroups (X is reflexive). An infinite sequence $\underline{r} = (r_j)_{j=1}^\infty$, where all $r_j \in F$, is called *F-unrestricted* if every index

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from F appears in r_j infinitely many times. The set of all F-unrestricted sequences is denoted by $\mathcal{R}_F$.

If $x \in X$ is simultaneously a fixed point for all T_j, where $j \in F \subseteq \{1, \ldots, N\}$, then it is called an *$F$-common fixed point*. Clearly all F-common fixed points form a closed linear subspace of X which is denoted by X_F. The same terminology applies to fixed points of the adjoint operators T_j^*, acting on X^*. We say that X_F *separates* X_F^* if for any $\lambda_1 \neq \lambda_2$ in X_F^*, there exists $u \in X_F$ such that $\langle u, \lambda_1 - \lambda_2 \rangle \neq 0$. Similarly X_F^* *separates* X_F if for any $u \neq v$ in X_F there exists $\lambda \in X_F^*$ such that $\langle u - v, \lambda \rangle \neq 0$.

Given a sequence $\underline{r}$ of numbers $1 \leq r_j \leq N$ we set

$$S_n = T_{r_n} \circ T_{r_{n-1}} \circ \ldots \circ T_{r_1},$$

which is a sequence of contractions on X. The purpose of this paper is to study asymptotic properties of such products, mainly when $\underline{r} \in \mathcal{R}_F$ and F goes through the subsets of $\{1, \ldots, N\}$. This is motivated by applications in various mathematical fields, or even in computer tomography (see [DKR] for more details in this regard). It was John von Neumann (see [N]) who proved that if T_1 and T_2 are orthogonal projections on a Hilbert space, then for every x the sequence $(T_1 \circ T_2)^n(x)$ converges strongly to a common fixed point. This has been generalized in several directions (see [AA], [B], [BA], [D], [DKLR], [DR], [DKR], [R], and [RZ]). In particular, [DKR] shows that any unrestricted $S_n(x)$ converges weakly to a common fixed point $Q(x)$ of $T_1, \ldots, T_N$ as long as the space X is reflexive and smooth. We emphasise here that $Q(x)$ does not depend on a specific $\underline{r}$ as long as all $T_1, \ldots, T_N$ appear in S_n infinitely many times. This has recently been extended in [L], where the Banach space X remains reflexive but the smoothness condition is replaced by the weaker assumption that for every $X \ni x \neq 0$ the set

$$\{x^* \in X^* : \|x^*\| = 1 \text{ and } x^*(x) = \|x\|\}$$

is norm compact. In [L] it is proved that unrestricted sequences $S_n(x)$ converge weakly to a limit $Q(x, \underline{r})$, which is a common fixed point depending on $\underline{r}$ however.

Given $x \in X$ and $F \subseteq \{1, \ldots, N\}$ we denote by $\mathcal{O}_F(x)$ the weak orbit (i.e. $\mathcal{O}_F(x) = \{T(x) : T \in \mathfrak{S}_F\}$). A vector $x \in X$ is called *F-reversible* if for every $y \in \mathcal{O}_F(x)$ we also have $x \in \mathcal{O}_F(y)$. The set of all F-reversible vectors is denoted by $X_{\mathrm{r},F}$. We start with the following:

LEMMA 1. *For every finite collection of* (W′) *linear contractions* $T_1, \ldots, T_N$ *on a reflexive Banach space* X *and any set* $F \subseteq \{1, \ldots, n\}$ *we have* $X_{\mathrm{r},F} = X_F$.

Proof. The inclusion $X_F \subseteq X_{\mathrm{r},F}$ is obvious. Now suppose that $x \in X_{\mathrm{r},F}$. Choose $y \in \mathcal{O}_F(x)$ which is the weak limit of $S_n(x) = T_{r_n} \circ \ldots \circ T_{r_1}(x)$ for some $\underline{r} \in \mathcal{R}_F$. Because x is reversible, it can be recovered from y. Namely,

$x = \text{w-lim}_{n\to\infty} T_n(x) = \text{w-lim}_{n\to\infty} T_{p^n_n} \circ \ldots \circ T_{p^n_1}(y)$ for some sequences $(p^n_j)^n_{j=1}$. Hence $\|x\| \le \lim_{n\to\infty} \|T_n(y)\| \le \|y\| \le \lim_{n\to\infty} \|S_n(x)\| \le \|x\|$. It follows from property (W′) that $T_{r_j}(x) = x$ for all r_j. Since $\underline{r} \in \mathcal{R}_F$ it follows that $x \in X_F$. ■

It follows directly from the above lemma and Theorem 4.10 of [DLG] that if $T_1, \ldots, T_N$ are (W′) contractions on a reflexive Banach space X then for every $F \subseteq \{1, \ldots, N\}$ the Banach space $C(\mathfrak{S}_F)$ of all continuous functions on the w.o.t. compact $\mathfrak{S}_F$ has a left invariant probability (mean), or in other words $\mathfrak{S}_F$ is left amenable. Clearly left amenability of $\mathfrak{S}_F$ is equivalent to right amenability of the adjoint semigroup $\mathfrak{S}^*_F$.

Remark 1. The existence of right invariant means does not follow from property (W). For instance if $X = \mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$, then the operators $T_j((x_1, x_2)) = \big(x_1 + \frac{1}{j+1} x_2, 0\big)$, where $j = 1, 2$, are contractive projections. It is easy to verify that T_1 and T_2 satisfy condition (W). On the other hand $T_1 \circ T_2 = T_2 \ne T_1$. Therefore $C(\mathfrak{S})$ has no right invariant mean (see [DLG], Theorem 4.9).

The idea of the next result comes from [L] (see also Proposition 1 in [DKR]).

Lemma 2. *Let $T = T_1$ be a single* (W) *contraction on a reflexive Banach space X. Then the semigroup $\boldsymbol{S} = \{T^n : n \ge 1\}$ has property* (W). *As a result, for every $x \in X$ the limit* $\text{w-lim}_{n\to\infty} T^n(x)$ *exists and is a fixed point.*

Proof. Let x_n be a bounded sequence of vectors from X and $k_n \ge 0$ be such that $\|x_n\| - \|T^{k_n}(x_n)\| \to 0$. Suppose that

$$\operatorname*{w-lim}_{n\to\infty} x_n = u \ne v = \operatorname*{w-lim}_{n\to\infty} T^{k_n}(x_n).$$

We have $\|x_n\| - \|T^{k_n}(x_n)\| \ge \|x_n\| - \|T(x_n)\| \to 0$. Since T is a (W) contraction it follows that $\text{w-lim}_{n\to\infty}(x_n - T(x_n)) = 0$. This gives $T(u) = u$. Similarly $\|x_n\| - \|T^{k_n}(x_n)\| \ge \|T^{k_n-1}(x_n)\| - \|T^{k_n}(x_n)\| \to 0$. Therefore

$$\operatorname*{w-lim}_{n\to\infty}(T^{k_n-1}(x_n) - T^{k_n}(x_n)) = 0.$$

Applying T to the last limit we get

$$\operatorname*{w-lim}_{n\to\infty}(T^{k_n}(x_n) - T^{k_n+1}(x_n)) = v - T(v) = 0$$

so that both $u, v \in X_{\{1\}}$. Now let $\lambda \in X^*$ with $\|\lambda\| = 1$ be such that $\lambda(u - v) = \|u - v\|$. We notice that the set $C^*_{u,v} = \{\lambda \in X^* : \|\lambda\| \le 1$ and $\lambda(u - v) = \|u - v\|\}$ is convex, weakly compact and T^*-invariant. By the mean ergodic theorem the Cesàro means

$$A_K(\lambda) = \frac{1}{K} \sum_{k=0}^{K-1} T^{*k}(\lambda) \to \lambda^*$$

(in norm) and the limit functional $\lambda^* \in C^*_{u,v}$ is T^*-invariant. We get

$$\begin{aligned}\lambda^*(u) &= \operatorname*{w-lim}_{n\to\infty} \langle x_n, \lambda^* \rangle = \operatorname*{w-lim}_{n\to\infty} \langle x_n, T^{*k_n}(\lambda^*) \rangle \\ &= \operatorname*{w-lim}_{n\to\infty} \langle T^{k_n}(x_n), \lambda^* \rangle = \lambda^*(v)\end{aligned}$$

contradicting $\lambda^*(u - v) = \|u - v\| \neq 0$. Hence $\boldsymbol{S}$ has property (W).

We have already proved that the sequence $T^n(x)$ has only one cluster point (for the weak topology). We conclude that $\text{w-lim}_{n\to\infty} T^n(x)$ exists and is a fixed point. ■

Now we are in a position to formulate the main result of the paper. Some elements of our proof come from [L] and [DKR].

THEOREM 1. *Let $T_1, \ldots, T_N$ be a finite collection of* (W) *contractions on a reflexive Banach space X. Then the following conditions are equivalent*:

(a) *For every $F \subseteq \{1, \ldots, N\}$ the semigroup $\mathfrak{S}_F$ has an invariant mean.*

(b) *For every $F \subseteq \{1, \ldots, N\}$ the semigroup $\mathfrak{S}_F$ has a right invariant mean.*

(c) *For every $F \subseteq \{1, \ldots, N\}$ the space X^*_F separates X_F.*

(d) *For every $F \subseteq \{1, \ldots, N\}$ the semigroup $\boldsymbol{S}_F$ has property* (W), *unrestricted sequences $S_n(x) = T_{r_n} \circ \ldots \circ T_{r_1}(x)$, where $\underline{r} \in \mathcal{R}_F$, converge weakly to $Q_F(x) \in X_F$, and the limit $Q_F(x)$ does not depend on the sequence $\underline{r} \in \mathcal{R}_F$.*

(d′) *For every $F \subseteq \{1, \ldots, N\}$ and any $\underline{r} \in \mathcal{R}_F$ unrestricted sequences $S_n(x) = T_{r_n} \circ \ldots \circ T_{r_1}(x)$ converge weakly to $Q_F(x) \in X_F$ and the limit $Q_F(x)$ does not depend on the sequence $\underline{r} \in \mathcal{R}_F$.*

(e) *For every $F \subseteq \{1, \ldots, N\}$ the Banach space X can be represented as a direct sum $X = X_{0,F} \oplus X_F$, where $X_{0,F}$ consists of those $x \in X$ such that* $\text{w-lim}_{n\to\infty} S_n(x) = 0$ *for every $\underline{r} \in \mathcal{R}_F$.*

(f) *For every $F \subseteq \{1, \ldots, N\}$ the convex hull $\overline{\operatorname{conv} \mathcal{O}_F(x)}$ contains exactly one $\mathfrak{S}_F$-fixed point.*

P r o o f. (a)⇒(b) is obvious.

(b)⇒(c). Let $u \neq v$ be arbitrary vectors in X_F. We choose a normalized $\lambda_0 \in X^*$ such that $\langle u - v, \lambda_0 \rangle = \|u - v\| \neq 0$. Clearly

$$\langle u - v, \lambda_0 \rangle = \langle T(u - v), \lambda_0 \rangle = \langle u - v, T^*(\lambda_0) \rangle = \text{const} \neq 0$$

for all $T^* \in \mathfrak{S}^*_F$. We show that the set

$$\{\lambda \in X^* : \|\lambda\| = 1,\ \langle u - v, \lambda \rangle = \|u - v\|\} = C^*_{u,v}$$

is $\mathfrak{S}_F^*$-invariant and contains a $\mathfrak{S}_F^*$-invariant vector. In fact, if μ is a left invariant probability measure on $\mathfrak{S}_F^*$ then define

$$\lambda^*(x) = \int T^* \lambda_0(x)\, d\mu(T^*), \tag{1}$$

where $x \in X$. Notice that $\mathfrak{S}_F^* \ni T^* \mapsto f_x(T^*) = T^*\lambda_0(x) = \langle x, T^*(\lambda_0)\rangle$ is w.o.t. continuous. The linear functional defined by (1) is continuous. By invariance of u and v and convexity of $C^*_{u,v}$ we also have $\lambda^* \in C^*_{u,v}$. It remains to verify that λ^* is $\mathfrak{S}_F^*$-invariant. For this choose $S^* \in \mathfrak{S}_F^*$ and $x \in X$. We have

$$\begin{aligned} S^*(\lambda^*)(x) &= \int S^* \circ T^*(\lambda_0)(x)\, d\mu(T^*) = \int f_x(S^* \circ T^*)\, d\mu(T^*) \\ &= \int f_x(T^*) d\mu(T^*) = \lambda^*(x). \end{aligned}$$

Hence λ^* is $\mathfrak{S}_F^*$-invariant.

(c)$\Rightarrow$(d). We proceed by induction. By Lemma 2 all semigroups $\boldsymbol{S}_F$, where $F = \{m\}$ is a singleton, have property (W) and unrestricted sequences $S_n(x) = T_m^n(x)$ converge to a unique fixed point which is contained in $\mathcal{O}_{\{m\}}(x)$. Now assume that (d) holds for all $F \subseteq \{1, \ldots, N\}$ with $\#F \le j$. Consider an arbitrary F with $\#F = j+1$. Let $\|x_n\| - \|W_n(x_n)\| \to 0$, where W_n are F-words. Suppose that

$$\operatorname*{w-lim}_{n\to\infty} x_n = u \neq v = \operatorname*{w-lim}_{n\to\infty} W_n(x_n). \tag{2}$$

By the same argument in Lemma 2 of [L] we conclude that u and v belong to X_F. Now we choose $\lambda \in X_F^*$ such that $\langle u - v, \lambda\rangle \neq 0$. But

$$\begin{aligned} \langle u, \lambda\rangle &= \lim_{n\to\infty} \langle x_n, \lambda\rangle = \lim_{n\to\infty} \langle x_n, W_n^*(\lambda)\rangle \\ &= \langle W_n(x_n), \lambda\rangle = \langle v, \lambda\rangle. \end{aligned}$$

In particular (2) fails and $\boldsymbol{S}_F$ has the (W) property. By induction all semigroups $\boldsymbol{S}_F$ have property (W).

Now let $x \in X$, $\underline{r} \in \mathcal{R}_F$, and suppose $x_{n_j} = S_{n_j}(x)$ converges weakly to u and $x_{m_j} = S_{m_j}(x)$ converges weakly to v. We may assume that $W_j = T_{r_{m_j}} \circ \ldots \circ T_{r_{n_j+1}}$ is F-complete (i.e. all indices from F appear in the interval $r_{n_j+1}, r_{n_j+2}, \ldots, r_{m_j}$). Clearly $\lim_{j\to\infty}(\|x_{n_j}\| - \|W_j(x_{n_j})\|) = 0$. By property (W) we get $u - v = 0$. Hence $S_n(x)$ converges weakly. Clearly $\operatorname{w-lim}_{n\to\infty} S_n(x) \in X_F$.

Suppose that for different $\underline{r}^1, \underline{r}^2 \in \mathcal{R}_F$ we have

$$\operatorname*{w-lim}_{n\to\infty} T_{r^1_n} \circ \ldots \circ T_{r^1_1}(x) = u \neq v = \operatorname*{w-lim}_{n\to\infty} T_{r^2_n} \circ \ldots \circ T_{r^2_1}(x).$$

It follows from (c) that u and v can be separated by a $\mathfrak{S}_F^*$-invariant functional, contradicting the fact that u and v are weak limits of sequences coming from the same vector x. We conclude that

$$\operatorname*{w-lim}_{n\to\infty} T_{r_n} \circ \ldots \circ T_{r_1}(x) = Q_F(x)$$

does not depend on the particular sequence $\underline{r} \in \mathcal{R}_F$.

(d)⇒(d′) is obvious.

(d′)⇒(a). We have $T_j \circ Q_F = Q_F$ as $\mathrm{range}(Q_F) \subseteq X_F$. On the other hand, the weak limit of $S_n(x)$ does not depend on the starting operator T_{r_1}, hence $Q_F \circ T_j = Q_F$ for all $j \in F$. By a continuity argument the identities $T_j \circ Q_F = Q_F \circ T_j = Q_F$, where $j \in F$, easily extend to the whole semigroup $\mathfrak{S}_F$, that is, $S \circ Q_F = Q_F \circ S = Q_F$ for all $S \in \mathfrak{S}_F$. Clearly the mapping $C(\mathfrak{S}_F) \ni f \mapsto f(Q_F)$ defines an invariant mean.

(d′)⇒(e). Given F and $x \in X$ consider $x_0 = x - Q_F(x)$. If $\underline{r} \in \mathcal{R}_F$ then

$$\begin{aligned} S_n(x - Q_F(x)) &= S_n(x) - S_n \circ Q_F(x) \\ &= S_n(x) - Q_F(x) \to 0 \quad \text{weakly.} \end{aligned}$$

Hence $x = (x - Q_F(x)) + Q_F(x) = x_0 + x_1$, where $x_0 \in X_{0,F}$ and $x_1 \in X_F$. Obviously $X_{0,F} \cap X_F = \{0\}$.

(e)⇒(d′). Every $x \in X$ may be decomposed as $x = x_{0,F} + x_{1,F}$, where $x_{0,F} \in X_{0,F}$ and $x_{1,F} \in X_F$. Regardless of the order in $\underline{r} \in \mathcal{R}_F$ the limit $\text{w-lim}_{n\to\infty} S_n(x) = x_{1,F}$ exists and belongs to X_F.

(d)+(c)⇒(f). For every $x \in X$, $Q_F(x) \in \overline{\mathrm{conv}\, \mathcal{O}_F(x)}$ is a $\mathfrak{S}_F$-fixed point. Suppose that $u \in \overline{\mathrm{conv}\, \mathcal{O}_F(x)}$ is another $\mathfrak{S}_F$-fixed point. By (c) we choose $\lambda \in X_F^*$ such that $\langle u - Q_F(x), \lambda\rangle \neq 0$. Let $W_n \in \mathrm{conv}(\boldsymbol{S}_F)$ be a sequence such that $\text{w-lim}_{n\to\infty} W_n(x) = u$. Then

$$\begin{aligned} \langle u - Q_F(x), \lambda\rangle &= \lim_{n\to\infty} \langle W_n(x) - Q_F(x), \lambda\rangle \\ &= \langle x, (W_n^* - Q_F^*)(\lambda)\rangle = \langle x, 0\rangle = 0 \end{aligned}$$

contradicting the assumption that u and $Q_F(x)$ are different.

(f)⇒(c). Let $u \neq v$ be two different $\mathfrak{S}_F$-fixed points. We choose $\lambda \in X^*$ with $\|\lambda\| = 1$ such that $\langle u - v, \lambda\rangle = \|u - v\| > 0$ and let $C_{u,v}^*$ be as before. Clearly the set $C_{u,v}^*$ is $\overline{\mathrm{conv}\, \mathfrak{S}_F^*}$-invariant. Combining Theorems 4.9, 7.2 and 7.4 from [DLG] we deduce that $\overline{\mathrm{conv}\, \mathfrak{S}_F}$ contains a unique projection E. We infer that E^* is a unique projection in $\overline{\mathrm{conv}\, \mathfrak{S}_F^*}$. By the same results of [DLG] we find that the orbit $\overline{\mathrm{conv}\, \mathfrak{S}_F^*}(\lambda)$ contains exactly one $\overline{\mathrm{conv}\, \mathfrak{S}_F^*}$-fixed point λ^*, which obviously belongs to $C_{u,v}^*$. It follows that X_F^* separates X_F. ■

Remark 2. Let X, T_1, T_2 be as in Remark 1. We introduce a third contraction $T_3 = \frac{1}{2}\,\mathrm{Id}$. Clearly $\|S_n\| \to 0$ as long as T_3 appears in S_n infinitely many times. In particular (d′) of Theorem 1 (hence all (a)–(f)) holds if $F = \{1, 2, 3\}$. On the other hand it follows from Remark 1 that conditions (a)–(f) fail if $F = \{1, 2\}$.

The next two corollaries should be compared with the corresponding results in [DKR] (Theorem 1) and [L] (Theorem 6).

COROLLARY 1. *Let* $T_1, \ldots, T_N$ *be* (W) *contractions on a smooth reflexive Banach space* X. *Then* (a)–(f) *of Theorem* 1 *hold.*

Proof. Let $F \subseteq \{1, \ldots, N\}$ be arbitrary. By Lemma 1 the Banach space $C(\mathfrak{S}_F)$ has a left invariant mean. It follows from the smoothness of X (apply Corollary 4.13 and Theorem 4.9 of [DLG]) that $C(\mathfrak{S}_F)$ has a right invariant mean. Applying Corollary 2.9 of [DLG], we conclude that $C(\mathfrak{S}_F)$ has an invariant mean. ∎

COROLLARY 2. *Let* $T_1, \ldots, T_N$ *be* (W) *contractions on a reflexive Banach space* X. *If* $T_1^*, \ldots, T_N^*$ *satisfy condition* (W′) *then* (a)–(f) *of Theorem* 1 *hold.*

Proof. By Lemma 1 both $C(\mathfrak{S}_F)$ and $C(\mathfrak{S}_F^*)$ have a left invariant mean, for every $F \subseteq \{1, \ldots, N\}$. In particular, $C(\mathfrak{S}_F)$ also has a right invariant mean. By Corollary 2.9 of [DLG] the Banach space $C(\mathfrak{S}_F)$ has an invariant mean. ∎

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