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TAME TRIANGULAR MATRIX ALGEBRAS

BY

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Abstract. We describe all finite-dimensional algebras A over an algebraically closed field for which the algebra $T_2(A)$ of 2×2 upper triangular matrices over A is of tame representation type. Moreover, the algebras A for which $T_2(A)$ is of polynomial growth (respectively, domestic, of finite representation type) are also characterized.

Introduction. The class of finite-dimensional algebras (associative, with an identity) over an algebraically closed field K may be divided into two disjoint classes [19] (see also [13]). One class consists of tame algebras for which the indecomposable modules occur, in each dimension, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite-dimensional vector spaces together with two noncommuting endomorphisms, for which the classification is a well known difficult problem. Hence we can realistically hope to describe modules only over tame algebras.

For a finite-dimensional algebra A over an algebraically closed field K we denote by $T_2(A) = \left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$ the algebra of 2×2 upper triangular matrices over A. It is well known that the category $\operatorname{mod} T_2(A)$ of finite-dimensional (over K) modules over $T_2(A)$ is equivalent to the category whose objects are A-homomorphisms $f : X \to Y$ between finite-dimensional A-modules X and Y, and morphisms are pairs of homomorphisms making the obvious squares commutative. We are concerned with the problem of deciding when $T_2(A)$ is tame. Certain classes of tame triangular matrix algebras $T_2(A)$ have been investigated in [3], [10], [11], [23], [28], [29], [31], [33], [40], [41]. In particular, it has been proved in [41] that if $T_2(A)$ is tame then A is of finite representation type and admits a simply connected Galois covering, and consequently, $T_2(A)$ also admits a simply connected Galois covering. Moreover, it follows from [3] that, for A of finite representation type, the tameness of $T_2(A)$ is equivalent to the tameness of the Auslander algebra

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$\mathcal{A}(A)$ of A (see [4]), that is, the algebra of the form $\mathrm{End}_A(M_1 \oplus \ldots \oplus M_n)$ for a fixed set $M_1, \ldots, M_n$ of representatives of isoclasses of indecomposable A-modules. We also mention that the representation theory of triangular matrix algebras is related to the representation theory of tensor products of algebras (see [30]).

The main aim of this paper is to give a complete description of all finite-dimensional algebras A over an algebraically closed field for which the algebra $T_2(A)$ is tame. Moreover, criteria for the tameness of $T_2(A)$ in terms of its simply connected Galois covering are also established. As a consequence we also obtain complete characterizations of triangular matrix algebras $T_2(A)$ which are of polynomial growth (respectively, domestic, of finite representation type). Therefore our main results solve completely the representation type problem of algebras $T_2(A)$, raised almost 30 years ago in [10], [11]. Some results presented in this paper have been announced in [32].

The paper is organized as follows. In Section 1 we present our main results and recall the related background. In Sections 2–5 we define some families of simply connected algebras A for which the triangular matrix algebras $T_2(A)$ are respectively wild, of nonpolynomial growth, nondomestic, of infinite representation type, playing a crucial role in the proofs of our main results. In Section 6 we introduce a family of algebras A of finite representation type and show that their triangular matrix algebras $T_2(A)$ are tame. In Section 7 we show that all simply connected algebras A with tame weakly sincere algebras $T_2(A)$ are factor algebras of algebras introduced in Section 6. The final Section 8 is devoted to the proofs of our main results.

For basic background from the representation theory of algebras we refer to [4], [20], and [38]. Moreover, we refer to [9], [15], [17], [21], [47] for basic results on Galois covering techniques in the representation theory of algebras, and to [37], [38] and [39] for the vector space category methods.

1. The main results and related background. Throughout the paper K will denote a fixed algebraically closed field. By an *algebra* is meant an associative finite-dimensional K-algebra with identity, which we shall assume (without loss of generality) to be basic and connected. For such an algebra A, there exists an isomorphism $A \cong KQ/I$, where KQ is the path algebra of the Gabriel quiver $Q = Q_A$ of A and I is an admissible ideal of KQ, generated by a (finite) system of forms $\sum_{1\le j\le t} \lambda_j \alpha_{m_j,j} \ldots \alpha_{1,j}$ (called *K-linear relations*), where $\lambda_1, \ldots, \lambda_t$ are elements of K and $\alpha_{m_j,j}, \ldots, \alpha_{1,j}$, $1 \le j \le t$, are paths of length ≥ 2 in Q with a common source and common end. Denote by Q_0 the set of vertices of Q, by Q_1 the set of arrows of Q, and by $s, e : Q_1 \to Q_0$ the maps which assign to each arrow $\alpha \in Q_1$ its source $s(\alpha)$ and its end $e(\alpha)$. The category $\mathrm{mod}\, A$ of all finite-dimensional (over K) left A-modules is equivalent to the category $\mathrm{rep}_K(Q, I)$ of all finite-dimensional K-linear representations $V = (V_i, \varphi_\alpha)_{i\in Q_0,\,\alpha\in Q_1}$ of Q, where V_i, $i \in Q_0$, are finite-dimensional K-vector spaces and $\varphi_\alpha : V_{s(\alpha)} \to V_{e(\alpha)}$, $\alpha \in Q_1$, are K-linear maps satisfying the equalities $\sum_{1\le j\le t} \lambda_j \varphi_{\alpha_{m_j,j}} \dots \varphi_{\alpha_{1,j}} = 0$ for all K-linear relations $\sum_{1\le j\le t} \lambda_j \varphi_{\alpha_{m_j,j}} \dots \varphi_{\alpha_{1,j}} \in I$ (see [20, Section 4]). We shall identify $\operatorname{mod} A$ with $\operatorname{rep}_K(Q, I)$ and call finite-dimensional left A-modules briefly A-*modules*.

Let $A = KQ/I$ be an algebra. Following [41] (see also [30]) the triangular matrix algebra $T_2(A)$ has the presentation $T_2(A) = KQ^{(2)}/I^{(2)}$, where the set $Q_0^{(2)}$ of vertices of $Q^{(2)}$ consists of x and x^* for $x \in Q_0$, the set $Q_1^{(2)}$ of arrows of $Q^{(2)}$ consists of α, α^* for $\alpha \in Q_1$, and additional arrows $\gamma_x : x^* \to x$ for $x \in Q_0$, and the ideal $I^{(2)}$ of $KQ^{(2)}$ is generated by the K-linear relations $\varrho = \sum \lambda_j \alpha_{m_j,j} \dots \alpha_{1,j}$ and $\varrho^* = \sum \lambda_j \alpha^*_{m_j,j} \dots \alpha^*_{1,j}$ for all K-linear relations $\varrho = \sum \lambda_j \alpha_{m_j,j} \dots \alpha_{1,j}$ generating the ideal I, and the differences $\gamma_{e(\alpha)}\alpha^* - \alpha\gamma_{s(\alpha)}$ for all $\alpha \in Q_1$.

An algebra $A = KQ/I$ may be equivalently considered as a K-category whose objects are the vertices of Q, and the set of morphisms $A(x, y)$ from x to y is the quotient of the K-space $KQ(x, y)$, formed by the K-linear combinations of paths in Q from x to y, by the subspace $I(x, y) = KQ(x, y) \cap I$. An algebra A with Q_A having no oriented cycle is called *triangular*. A full subcategory C of A is said to be *convex* if any path in Q_A with source and target in Q_C lies entirely in Q_C. Finally, a triangular algebra (respectively, triangular locally bounded category [9]) is called *simply connected* [1] if, for any presentation $A \cong KA/I$ of A as a bound quiver algebra (respectively, bound quiver category), the fundamental group $\pi_1(Q, I)$ of (Q, I) is trivial.

Let A be an algebra and $K[x]$ the polynomial algebra in one variable. Recall that following Drozd [19] an algebra A is called of *tame representation type* (briefly, *tame*) if, for any dimension d, there exist a finite number of A-$K[x]$-bimodules M_i, $1 \le i \le n_d$, which are finitely generated and free as right $K[x]$-modules, and all but finitely many isoclasses of indecomposable A-modules of dimension d are of the form $M_i \otimes_{K[x]} K[x]/(x - \lambda)$ for some $\lambda \in K$ and some i. Let $\mu_A(d)$ be the least number of A-$K[x]$-bimodules satisfying the above condition for d. Then A is said to be of *polynomial growth* [42] (respectively, *domestic* [37], [43], [14]) if there is a positive integer m such that $\mu_A(d) \le d^m$ (respectively, $\mu_A(d) \le m$) for all $d \ge 1$. Finally, A is said to be of *finite representation* type if there are only finitely many isoclasses of indecomposable A-modules. From the validity of the second Brauer–Trall conjecture (see [5]) we know that A is of finite representation type if and only if $\mu_A(d) = 0$ for all $d \ge 1$. We also refer to [14] and [16] for equivalent definitions of tameness.

Let $A = KQ/I$ be a triangular algebra. The *Tits quadratic form* q_A of A is the integral quadratic form on the Grothendieck group $K_0(A) = \mathbb{Z}^{Q_0}$

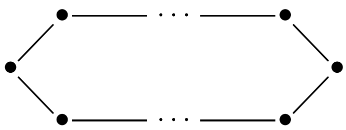
of A, defined for $\mathbf{x} = (x_i)_{i \in Q_0} \in K_0(A)$ as follows:

$$q_A(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{e(\alpha)} + \sum_{i,j \in Q_0} r_{ij} x_i x_j$$

where r_{ij} is the cardinality of $\mathcal{R} \cap I(i,j)$ for a minimal (finite) set $\mathcal{R} \subset \bigcup_{i,j \in Q_0} I(i,j)$ of K-linear relations generating the ideal I (see [6]). It is well known (see [36]) that if A is tame then q_A is *weakly nonnegative*, that is, $q_A(\mathbf{x}) \geq 0$ for any $\mathbf{x}$ in $K_0(A)$ with nonnegative coordinates.

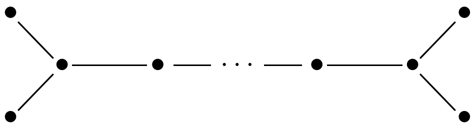
Consider the Euclidean graphs

$\widetilde{\mathbb{A}}_m$:



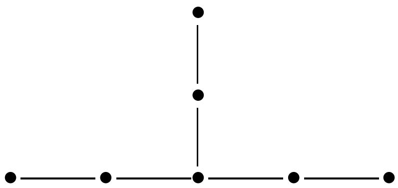
($m+1$ vertices, $m \geq 1$)

$\widetilde{\mathbb{D}}_n$:

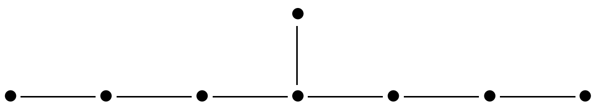


($n+1$ vertices, $n \geq 4$)

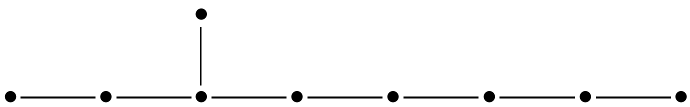
$\widetilde{\mathbb{E}}_6$:



$\widetilde{\mathbb{E}}_7$:

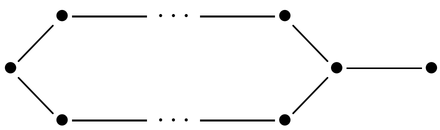


$\widetilde{\mathbb{E}}_8$:



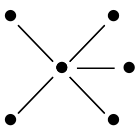
and the extended Euclidean graphs

$\widetilde{\widetilde{\mathbb{A}}}_m$:

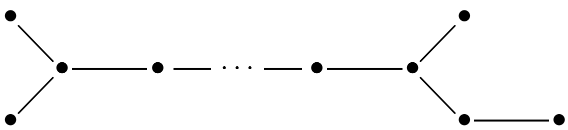


($m+2$ vertices, $m \geq 1$)

T_5 :

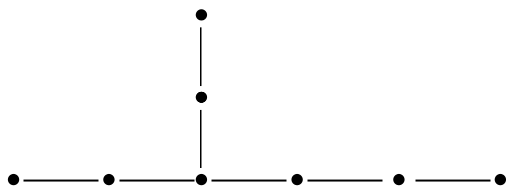


$\widetilde{\widetilde{\mathbb{D}}}_n$:

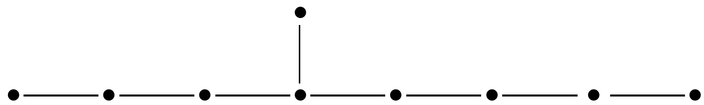


($n+2$ vertices, $n \geq 4$)

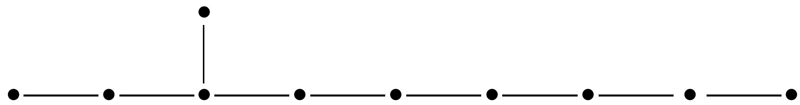
$\widetilde{\widetilde{\mathbb{E}}}_6$:



$\widetilde{\widetilde{\mathbb{E}}}_7$:

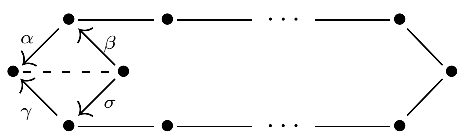


$\widetilde{\widetilde{\mathbb{E}}}_8$:



Let $H = K\Delta$ be the path algebra of a quiver Δ (without oriented cycles) whose underlying graph $\overline{\Delta}$ is one of the above Euclidean or extended Euclidean graphs, and T be a preprojective tilting H-module, that is, $\mathrm{Ext}^1_H(T,T) = 0$ and T is a direct sum of $|\Delta_0|$ pairwise nonisomorphic H-modules lying in different TrD-orbits of indecomposable projective H-modules. Then $C = \mathrm{End}_H(T)$ is said to be a *concealed algebra of type* $\overline{\Delta}$. It is known that $\mathrm{gl.dim}\, C \le 2$, the opposite algebra C^{op} of C is also a concealed algebra of type $\overline{\Delta}$, and C has the same representation type as H. In particular (see [25], [35]), the Tits form q_C of C is weakly nonnegative if and only if C is of Euclidean type. Moreover, concealed algebras of Euclidean type (respectively, extended Euclidean type) are of infinite representation type (respectively, wild).

The concealed algebras of type $\overline{\Delta} = \widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ (respectively, $\overline{\Delta} = T_5$, $\widetilde{\widetilde{\mathbb{D}}}_n$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$, $\widetilde{\widetilde{\mathbb{E}}}_8$) are (strongly) simply connected and have been classified completely in [7], [22] (respectively, [27], [48], [49]). Moreover, every concealed algebra of type $\widetilde{\mathbb{A}}_m$ is the path algebra of a quiver of type $\widetilde{\mathbb{A}}_m$ (see [22]). Finally, it has been noted in [48] that every concealed algebra of type $\widetilde{\widetilde{\mathbb{A}}}_m$ is either the path algebra of a quiver of type $\widetilde{\widetilde{\mathbb{A}}}_m$ or isomorphic to the bound quiver algebra given by a quiver of the form



$(m+2$ vertices, $m \ge 1)$

and the ideal generated by $\alpha\beta - \gamma\sigma$, where $\bullet$ —— $\bullet$ means $\bullet \longrightarrow \bullet$ or $\bullet \longleftarrow \bullet$.

Following Ringel [38], by a *tubular algebra* we mean a tubular extension of a concealed algebra of Euclidean type (tame concealed algebra) of tubular type $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$. It is known that if A is a tubular algebra then:

(1) A is nondomestic of polynomial growth,
(2) $\operatorname{gl.dim} A = 2$,
(3) A is simply connected,
(4) the opposite algebra A^{op} is also tubular

(see [38, (5.2)] and [43, (3.6)]).

In the representation theory of tame simply connected algebras an important role is played by polynomial growth critical algebras introduced and investigated by R. Nörenberg and A. Skowroński in [34]. Recall that by a *polynomial growth critical algebra* (briefly *pg-critical algebra*) is meant an algebra satisfying the following conditions:

(i) A is one of the matrix algebras

$$B[X] = \begin{bmatrix} B & X \\ 0 & K \end{bmatrix}, \qquad B[Y,t] = \begin{bmatrix} B & 0 & 0 & 0 & \dots & 0 & Y \\ & K & 0 & K & \dots & K & K \\ & & K & K & \dots & K & K \\ & & & & \ddots & \vdots & \vdots \\ & & & & & K & K \\ & 0 & & & & & K \end{bmatrix}$$

where B is a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{D}}_n$, $n \geq 4$, with a complete slice in the preinjective component of its Auslander–Reiten quiver, X (respectively, Y) is an indecomposable regular B-module of regular length 2 (respectively, regular length 1) lying in a tube with $n-2$ rays, and $t+1$ ($t \geq 2$) is the number of isoclasses of simple $B[Y,t]$-modules which are not B-modules.

(ii) Every proper convex subcategory of A is of polynomial growth.

The *pg*-critical algebras have been classified by quivers and relations in [34]. There are 31 frames of such algebras. In particular, if A is a *pg*-critical algebra then:

(1) A is tame minimal of nonpolynomial growth,
(2) $\operatorname{gl.dim} A = 2$,
(3) A is simply connected,
(4) the opposite algebra A^{op} is also *pg*-critical.

Assume $A = KQ/I$ is an algebra such that the triangular matrix algebra $T_2(A)$ is tame. Then, by [41], A is of finite representation type and standard [9]. In particular, A admits a Galois covering $F : \widetilde{A} \to \widetilde{A}/G = A$, where $\widetilde{A} = K\widetilde{Q}/\widetilde{I}$ is a simply connected locally bounded K-category and

G is the fundamental group $\pi_1(Q,I)$, which is moreover a finitely generated free group. Clearly, $\widetilde{A}=A$ if A is simply connected. Since A is standard, applying [12] we may assume that I is generated by paths $\alpha_m\ldots\alpha_1$ (*zero-relations*) and differences $\beta_r\ldots\beta_1-\gamma_s\ldots\gamma_1$ of paths with a common source and common end (*commutativity relations*). Therefore, in our considerations we may restrict to the algebras A of finite representation type having such a nice bound quiver presentation. Then in the bound quiver presentation $T_2(A)=KQ^{(2)}/I^{(2)}$ of $T_2(A)$ described before, the ideal $I^{(2)}$ is also generated by paths and differences of paths. Moreover, the fundamental groups $\pi_1(Q^{(2)},I^{(2)})$ and $\pi_1(Q,I)$ are isomorphic, and the Galois covering $F:\widetilde{A}\to\widetilde{A}/G=A$ with $G=\pi_1(Q,I)$ induces a Galois covering $F^{(2)}:\widetilde{T_2(A)}\to\widetilde{T_2(A)}/G=T_2(A)$, where $\widetilde{T_2(A)}=T_2(\widetilde{A})=K\widetilde{Q}^{(2)}/\widetilde{I}^{(2)}$ is simply connected. Finally, we note that nonstandard algebras of finite representation type can only occur in characteristic 2 (see [5]).

Below we shall present the families (W), (NPG), (ND), (IT) of standard algebras Λ of finite representation type and show later that the corresponding triangular matrix algebras $T_2(\Lambda)$ are wild, not of polynomial growth, nondomestic, of infinite representation type, respectively.

Our main results are the following five theorems.

THEOREM 1. *Let A be a standard algebra of finite representation type. The following conditions are equivalent*:

(i) $T_2(A)$ *is tame.*

(ii) *The Tits form q_B of any finite convex subcategory B of $T_2(\widetilde{A})$ is weakly nonnegative.*

(iii) $T_2(\widetilde{A})$ *does not contain a finite convex subcategory which is concealed of type $\widetilde{\widetilde{\mathbb{A}}}_m$, $m\ge 1$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$, $n\ge 4$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$ or $\widetilde{\widetilde{\mathbb{E}}}_8$.*

(iv) $\widetilde{A}$ *does not contain a finite convex subcategory Λ such that one of the algebras from the family* (W) *is a factor algebra of Λ or Λ^{op}.*

THEOREM 2. *Let A be a standard algebra of finite representation type. The following conditions are equivalent*:

(i) $T_2(A)$ *is of polynomial growth.*

(ii) $T_2(\widetilde{A})$ *does not contain a finite convex subcategory which is pg-critical or concealed of type $\widetilde{\widetilde{\mathbb{A}}}_m$, $m\ge 1$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$, $n\ge 4$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$ or $\widetilde{\widetilde{\mathbb{E}}}_8$.*

(iii) $\widetilde{A}$ *does not contain a finite convex subcategory Λ such that one of the algebras from the families* (W) *and* (NPG) *is a factor algebra of Λ or Λ^{op}.*

THEOREM 3. *Let A be a standard algebra of finite representation type. The following conditions are equivalent*:

(i) $T_2(A)$ *is domestic.*

(ii) $T_2(\widetilde{A})$ *does not contain a finite convex subcategory which is tubular, pg-critical or concealed of type* $\widetilde{\widetilde{\mathbb{A}}}_m$, $m \geq 1$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$, $n \geq 4$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$ *or* $\widetilde{\widetilde{\mathbb{E}}}_8$.

(iii) $\widetilde{A}$ *does not contain a finite convex subcategory* Λ *such that one of the algebras from the families* (W) *and* (ND) *is a factor algebra of* Λ *or* Λ^{op}.

THEOREM 4. *Let* A *be a standard algebra of finite representation type. The following conditions are equivalent*:

(i) $T_2(A)$ *is of finite representation type.*

(ii) $T_2(\widetilde{A})$ *does not contain a finite convex subcategory which is concealed of type* $\widetilde{\mathbb{A}}_m$, $m \geq 1$, $\widetilde{\mathbb{D}}_n$, $n \geq 4$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ *or* $\widetilde{\mathbb{E}}_8$.

(iii) $\widetilde{A}$ *does not contain a finite convex subcategory* Λ *such that one of the algebras from the family* (IT) *is a factor algebra of* Λ *or* Λ^{op}.

In the course of our proofs we also establish the following fact.

THEOREM 5. *Let* A *be an algebra such that* $T_2(A)$ *is of polynomial growth. Then the push-down functor*

$$F_\lambda^{(2)} : \operatorname{mod} T_2(\widetilde{A}) \to \operatorname{mod} T_2(A),$$

associated with the Galois covering $F^{(2)} : T_2(\widetilde{A}) \to T_2(A)$, *is a Galois covering of module categories* (*in the sense of* [9]). *In particular*, *the Auslander–Reiten quiver* $\Gamma_{T_2(A)}$ *of* $T_2(A)$ *is the orbit quiver* $\Gamma_{T_2(\widetilde{A})}/G$ *of the Auslander–Reiten quiver* $\Gamma_{T_2(\widetilde{A})}$ *with respect to the action of the fundamental group* $G = \Pi_1(Q, I) = \pi_1(Q^{(2)}, I^{(2)})$.

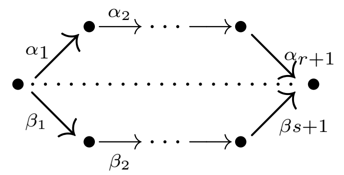
In a forthcoming paper we shall prove that for an algebra A, the algebra $T_2(A)$ is of polynomial growth (respectively, domestic) if and only if the infinite radical $\operatorname{rad}^\infty(\operatorname{mod} T_2(A))$ of $\operatorname{mod} T_2(A)$ is locally nilpotent (respectively, nilpotent). We refer to [26], [45] and [46] for basic definitions and results in this direction.

As we have already pointed out, if A is an algebra of finite representation type, then the algebra $T_2(A)$ has the same representation type as the Auslander algebra $\mathcal{A}(A)$ of A (by a discussion in [3]). Therefore, the above theorems also give complete characterizations of the Auslander algebras of tame representation type, polynomial growth, domestic, of finite representation type, respectively (see [32]). We mention that the Auslander algebras of finite representation type have already been characterized (in different terms) by Igusa–Platzeck–Todorov–Zacharia [24].

In the present paper we shall use the following notation. For a bound quiver (Q, I):

(i) an unoriented edge $\bullet\!\!-\!\!\!-\!\!\bullet$ means $\bullet\!\longrightarrow\!\bullet$ or $\bullet\!\longleftarrow\!\bullet$.

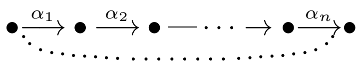
(ii)



$r, s \geq 1$

means that $\alpha_{r+1} \dots \alpha_1 - \beta_{s+1} \dots \beta_1 \in I$ but $\alpha_{r+1} \dots \alpha_1 \notin I$, $\beta_{s+1} \dots \beta_1 \notin I$.

(iii)

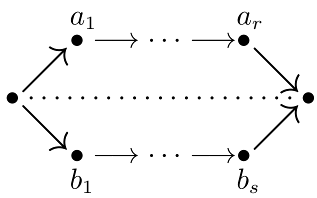


$n \geq 2$

means that $\alpha_n \dots \alpha_1 \in I$ but $\alpha_n \dots \alpha_2 \notin I$, $\alpha_{n-1} \dots \alpha_1 \notin I$.

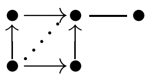
2. Wild triangular algebras. Consider the following family (W) of bound quiver algebras KQ/I given by the bound quivers (Q, I):

(1)

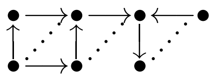


$r, s \geq 1,\ r + s \geq 3$

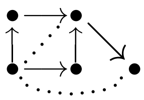
(2)



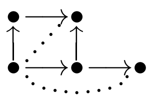
(3)



(4)



(5)



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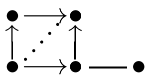

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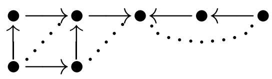

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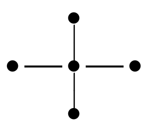

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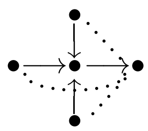

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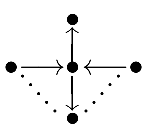

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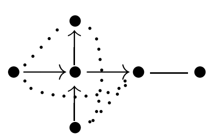
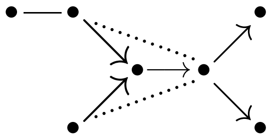
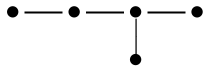
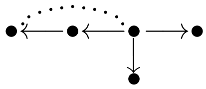
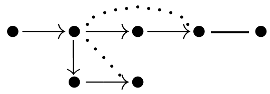
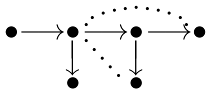


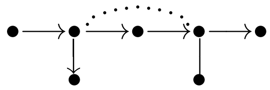
(12) 

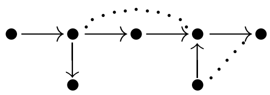
(13) 

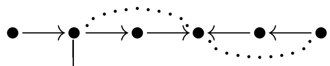
(14) 

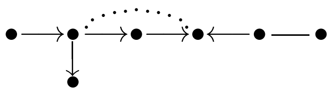
(15) 

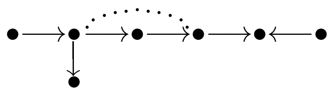
(16) 

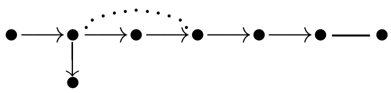
(17) 

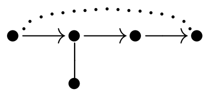
(18) 

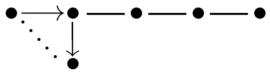
(19) 

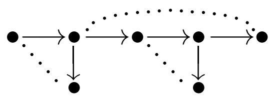
(20) 

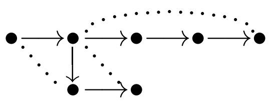
(21) 

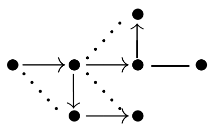
(22) 

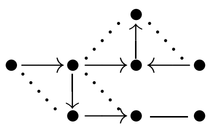
(23) 

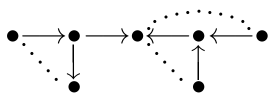
(24) 

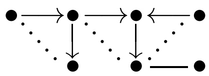
(25) 

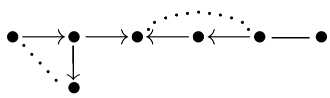
(26) 

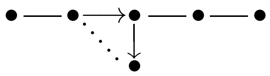
(27) 

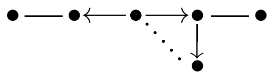
(28) 

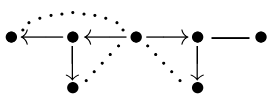
(29) 

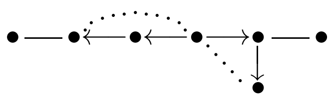
(30) 

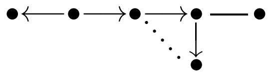
(31) 

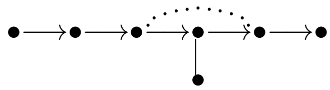
(32) 

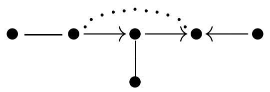
(33) 

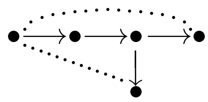
(34) 

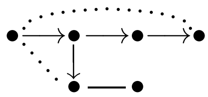
(35) 

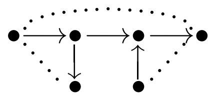
(36) 

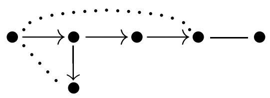
(37) 

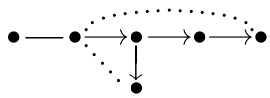
(38) 

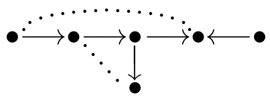
(39) 

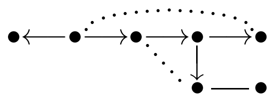
(40) 

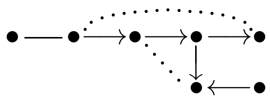
(41) 

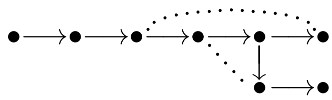
(42) 

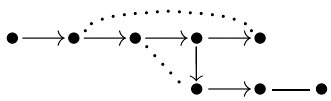
(43) 

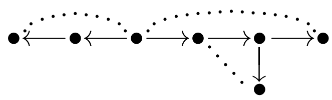
(44) 

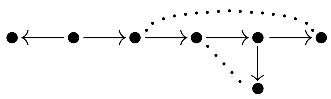
(45) 

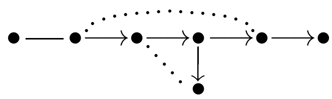
(46) 

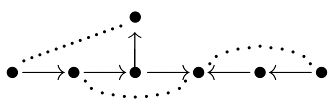
(47) 

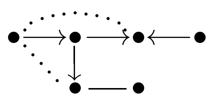
(48) 

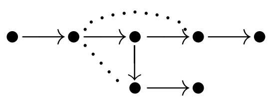
(49) 

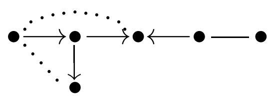
(50) 

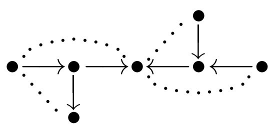
(51) 

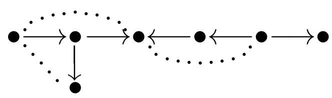
(52) 

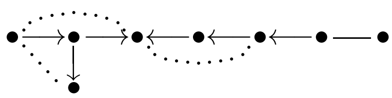
(53) 

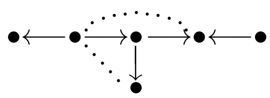
(54) 

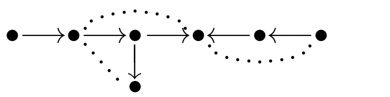
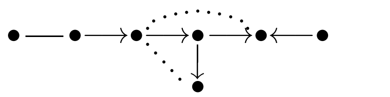
(55) 

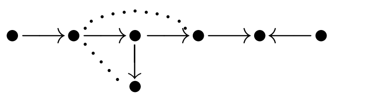
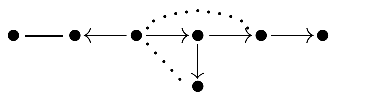
(56) 

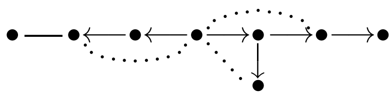
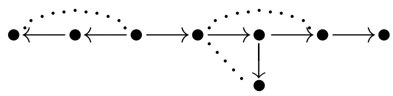
(57) 

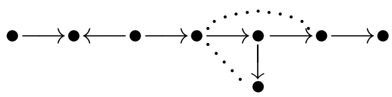
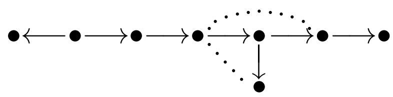
(58) 

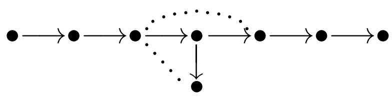
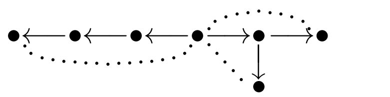
(59) 

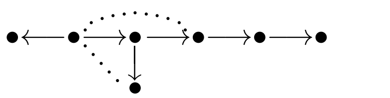
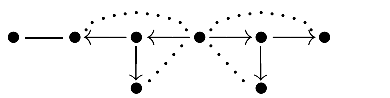
(60)  (61) 

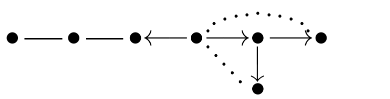
(62)  (63) 

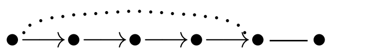
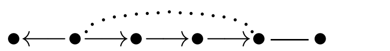
(64)  (65) 

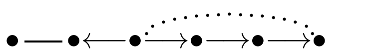
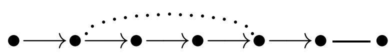
(66)  (67) 

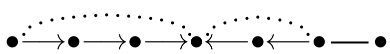
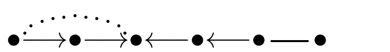
(68)  (69) 

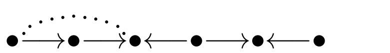
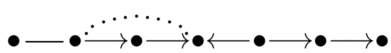
(70)  (71) 

(72)  (73)

(74)  (75) 

(76)  (77) 

(78)  (79) 

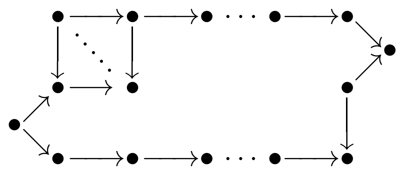
(80)  (81) 

We shall denote by (Wn) the nth quiver from the above family (W).

PROPOSITION 1. *Let A be a simply connected algebra of finite representation type. Assume that A admits a factor algebra B such that B or B^{op} is the bound quiver algebra of one of the bound quivers* (W1)–(W81). *Then $T_2(A)$ contains a convex subcategory which is concealed of type* $\widetilde{\widetilde{\mathbb{A}}}_m$, $m \geq 1$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$, $n \geq 4$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$ *or* $\widetilde{\widetilde{\mathbb{E}}}_8$.

P r o o f. This is a direct but tedious checking. We shall illustrate it by a few examples.

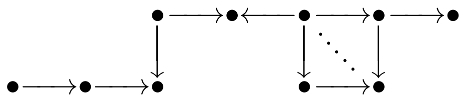
Let $A = B = KQ/I$ where (Q, I) is of type (W1), say with $r \geq 2$, $s \geq 1$. Then invoking the bound quiver presentation $T_2(A) = KQ^{(2)}/I^{(2)}$ of $T_2(A)$ described in Section 1, we easily observe that $T_2(A)$ has a convex subcategory given by the bound quiver



of a concealed algebra of type $\widetilde{\widetilde{\mathbb{A}}}_{r+s+3}$.

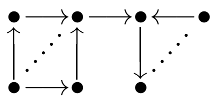
Let $A = B$ be the path algebra of a quiver Q of type $\widetilde{\mathbb{D}}_4$. Then obviously $T_2(A)$ contains a convex subcategory which is the path algebra of the corresponding tree of type T_5.

Let $A = B = KQ/I$ where (Q, I) is of the form (W81). Then $T_2(A)$ contains a convex subcategory given by the bound quiver

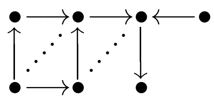


which is a concealed algebra type $\widetilde{\widetilde{\widetilde{\mathbb{E}}}}_8$ (see for example [48]).

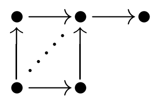
Assume now that A admits a proper factor algebra B given by the bound quiver (W3). We may assume $Q_A = Q_B$. Since A is simply connected and of finite representation type we conclude that A is given by one of the bound quivers



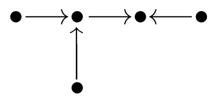
or



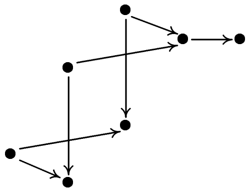
Hence, A contains a convex bound subquiver of one of the forms



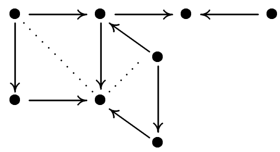
or



of type (W2) or (W13), respectively. Therefore, $T_2(A)$ contains a convex subcategory given by one of the bound quivers



or



and so a concealed algebra of type $\widetilde{\widetilde{\mathbb{A}}}_5$ or $\widetilde{\widetilde{\mathbb{E}}}_6$, respectively.

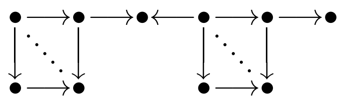
Finally, assume that A admits a proper factor algebra B given by the bound quiver (W81) and again that $Q_A = Q_B$. Then A is the path algebra of one of the quivers

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \quad \text{or} \quad \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

of type (W73), and then A contains a convex subcategory given by the convex subquiver

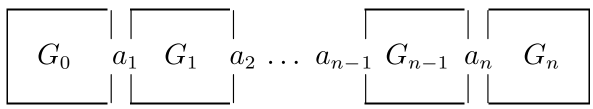
$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

Then $T_2(A)$ contains a convex subcategory given by the bound quiver



which is a concealed algebra of type $\widetilde{\widetilde{\mathbb{D}}}_8$.

3. Nonpolynomial growth triangular matrix algebras. Consider the family (NPG) of bound quiver algebras KQ/I given by the bound quivers (Q, I) of the form

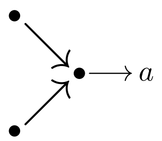


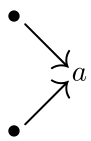
$n \geq 1$

and satisfying the following conditions:

(α) for $i = 0$ and $i = n$, G_i or G_i^{op} is one of the quivers

$$\bullet - \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow a \qquad \bullet - \bullet \longleftarrow \bullet \longrightarrow a$$





with $a = a_1$ and $a = a_n$, respectively,

(β) if $n \geq 2$, then for $1 \leq i \leq n-1$, G_i or G_i^{op} is one of the quivers

$$a_i \longleftarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow a_{i+1} \qquad a_i \longleftarrow \bullet \longrightarrow\!\!\!\!\!\!- \bullet \longrightarrow a_{i+1} \qquad a_i \longleftarrow \bullet \longrightarrow a_{i+1}$$

(γ) for $1 \leq i \leq n$, the vertex a_i is a source (respectively, target) of G_{i-1} if and only if a_i is a target (respectively, source) of G_i,

(δ) the composition of any two arrows in Q having a_i, $1 \leq i \leq n$, as a common vertex belongs to I,

(σ) either at least one of G_0, G_0^{op}, G_n, G_n^{op} has one of the forms

$$\bullet \text{---} \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow a \qquad\qquad \begin{tikzcd}[row sep=small] \bullet \arrow[dr] & & \\ & \bullet \arrow[r] & a \\ \bullet \arrow[ur] & & \end{tikzcd}$$

or $n \geq 2$ and, for some $1 \leq i \leq n-1$, G_i or G_i^{op} has one of the forms

$$a_i \longleftarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow a_{i+1} \qquad\qquad a_i \longleftarrow \bullet \text{---} \bullet \longrightarrow a_{i+1}$$

PROPOSITION 2. *Let A be a simply connected algebra of finite representation type satisfying the following conditions*:

(i) *A admits a factor algebra B such that B or B^{op} is the bound quiver algebra of one of the bound quivers from the family* (NPG).

(ii) *A has no factor algebra given by one of the bound quivers from the family* (W).

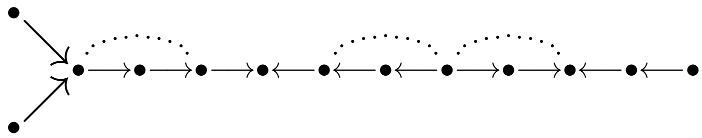
Then $T_2(A)$ contains a convex pg-critical subcategory. In particular, $T_2(A)$ is not of polynomial growth.

Proof. This follows by direct analysis of all possible shapes of bound quiver algebras from the family (NPG) and inspection of the list of all *pg*-critical algebras given in [34, Theorem 3.2]. We illustrate it by one of the typical cases. Let $n = 3$ and A be the bound quiver algebra from the list (NPG) given by the quivers

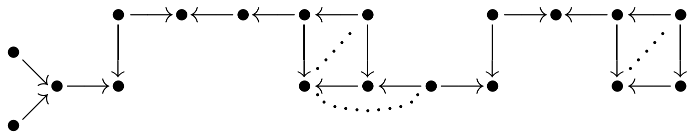
$$G_0\colon \begin{tikzcd}[row sep=small] \bullet \arrow[dr] & & \\ & \bullet \arrow[r] & a_1 \\ \bullet \arrow[ur] & & \end{tikzcd} \qquad\qquad G_1\colon\ a_1 \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longleftarrow a_2$$

$$G_2\colon\ a_2 \longleftarrow \bullet \longrightarrow \bullet \quad a_3 \qquad\qquad G_3\colon\ a_3 \longrightarrow \bullet \longleftarrow \bullet \longleftarrow \bullet$$

Then A is given by

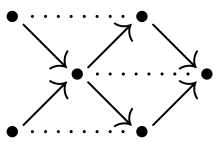


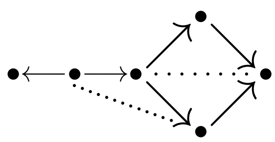
and $T_2(A)$ contains a convex subcategory given by the bound quiver

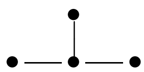


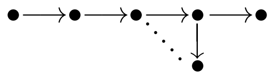
which is pg-critical (see the frame (3) in [34, Theorem 3.2]).

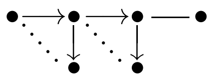
4. Nondomestic triangular matrix algebras. Consider the family (ND) of bound quiver algebras KQ/I given by the following quivers:

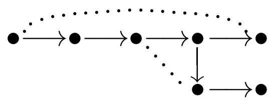
(1) 

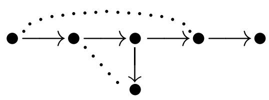
(2) 

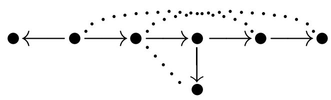
(3) 

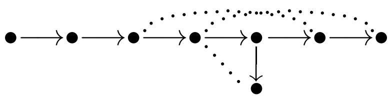
(4) 

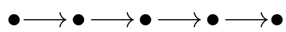
(5) 

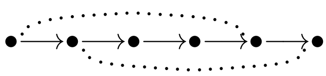
(6) 

(7) 

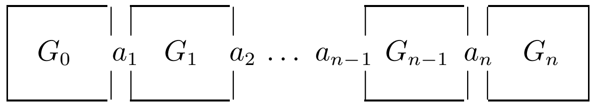
(8) 

(9) 

(10) but different from 

(11) 

(12) Q is of the form



$n \geq 1$

and the following conditions are satisfied:

(α) for $i = 0$ and $i = n$, G_i or G_i^{op} is one of the quivers

$$\bullet \;\text{---}\; \bullet \longleftarrow \bullet \longrightarrow a \qquad\qquad \begin{tikzcd}[row sep=small] \bullet \arrow[dr] & \\ & a \\ \bullet \arrow[ur] & \end{tikzcd}$$

with $a = a_1$ and $a = a_n$, respectively,

(β) if $n \geq 2$, then for $1 \leq i \leq n-1$, G_i or G_i^{op} is one of the quivers

$$a_i \longleftarrow \bullet \;\text{---}\; \bullet \longrightarrow a_{i+1} \qquad\qquad a_i \longleftarrow \bullet \longrightarrow a_{i+1}$$

(γ) for $1 \leq i \leq n$, the vertex a_i is a source (respectively, target) of G_{i-1} if and only if a_i is a target (respectively, source) of G_i,

(σ) the composition of any two arrows in Q having a_i, $1 \leq i \leq n$, as a common vertex belongs to I.

Note that bound quiver algebras of type (12) are special cases of algebras from the list (NPG).

Proposition 3. *Let A be a simply connected algebra of finite representation type satisfying the following conditions*:

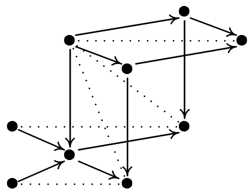
(i) *A admits a factor algebra B such that B or B^{op} is the bound quiver algebra of one of the bound quivers* (1)–(12) *in* (ND).

(ii) *A has no factor algebra given by one of the bound quivers from the families* (W) *and* (NPG).

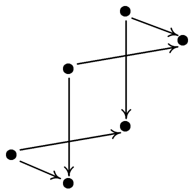
Then $T_2(A)$ contains a convex tubular subcategory. In particular, *$T_2(A)$ is nondomestic.*

P r o o f. We shall prove the claim in two typical cases.

Let A be of type (1). Then $T_2(A)$ contains a convex subcategory B given by the bound quiver

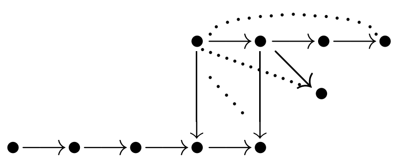


Then B is a tubular extension of tubular type $(2,4,4)$ of the path algebra of the Euclidean quiver

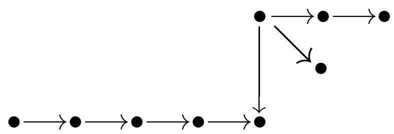


of type $\widetilde{\mathbb{A}}_6$, and hence is a tubular algebra.

Let A be of type (9). Then $T_2(A)$ contains a convex subcategory D given by the bound quiver

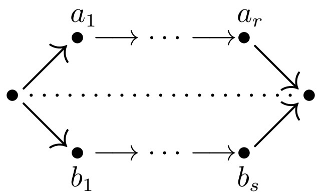


Then D is the one-point extension of the path algebra H of the Euclidean quiver

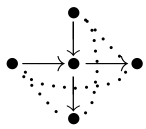
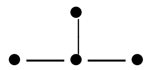


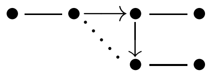
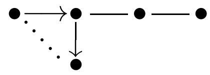
of type $\widetilde{\widetilde{\mathbb{E}}}_8$ by a simple regular module lying in the stable tube of rank 5 of the Auslander–Reiten quiver of A, and consequently D is a tubular algebra of tubular type $(2,3,6)$.

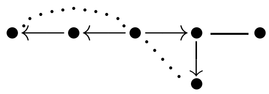
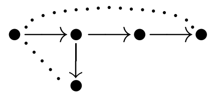
5. Triangular matrix algebras of infinite representation type. Consider the family (IT) of bound quiver algebras KQ/I given by the following bound quivers (Q,I):

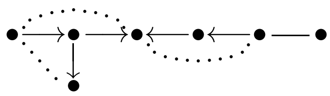
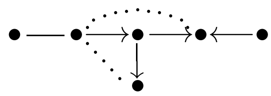
(1) 

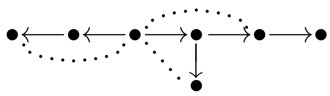
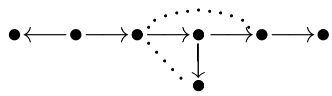
$r, s \geq 1$

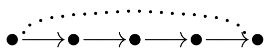
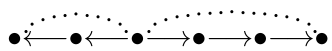
(2)  (3) 

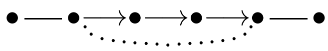
(4)  (5) 

(6)  (7) 

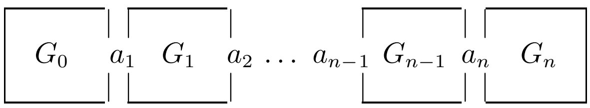
(8)  (9) 

(10)  (11) 

(12)  (13) 

(14)  (15)

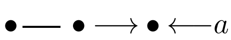
(16) Q is of the form

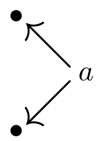


$n \geq 1$

and the following conditions are satisfied:

(α) for $i = 0$ and $i = n$, G_i or G_i^{op} is one of the the quivers





with $a = a_1$ and $a = a_n$, respectively,

(β) if $n \geq 2$, then for $1 \leq i \leq n-1$, G_i or G_i^{op} has the form

$$a_i \longleftarrow \bullet \longrightarrow a_{i+1}$$

(γ) for $1 \leq i \leq n$, the vertex a_i is a source (respectively, target) of G_{i-1} if and only if a_i is a target (respectively, source) of G_i,

(σ) the composition of any two arrows in Q having a_i, $1 \leq i \leq n$, as a common vertex belongs to I.

Proposition 4. *Let A be a simply connected algebra of finite representation type satisfying the following conditions*:

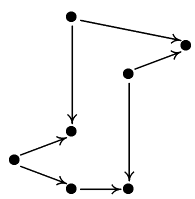
(i) *A admits a factor algebra B such that B or B^{op} is the bound quiver algebra of one of the bound quivers from the family* (IT).

(ii) *A has no factor algebra given by one of the bound quivers from the families* (W), (NPG) *or* (ND).

Then $T_2(A)$ contains a convex subcategory which is concealed of type $\widetilde{\mathbb{A}}_m$, $m \geq 1$, $\widetilde{\mathbb{D}}_n$, $n \geq 4$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$. In particular, $T_2(A)$ is of infinite representation type.

Proof. We shall prove the claim in three typical cases.

Assume A is of type (1) with $r = 2$, $s = 3$. Then $T_2(A)$ contains a convex subcategory which is the path algebra of the quiver



of Euclidean type $\widetilde{\mathbb{A}}_6$.

Let A be of type (12). Then $T_2(A)$ contains a convex subcategory given by the bound quiver

$$\begin{tikzcd}
& \bullet \arrow[r] \arrow[d] \arrow[dr, dash, dotted] & \bullet \arrow[r] \arrow[d] \arrow[dr, dash, dotted] & \bullet \arrow[r] \arrow[d] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet &
\end{tikzcd}$$

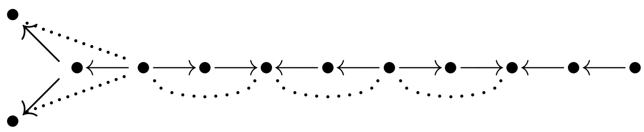
which is concealed of type $\widetilde{\mathbb{E}}_7$ (see [7], [22]).

Finally, let A be of type (16) with $n = 4$ and G_0, G_1, G_2, G_3, G_4 as follows:

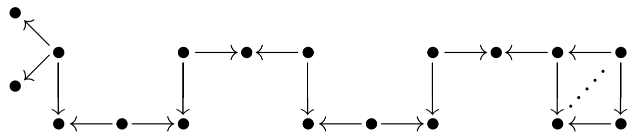
$$G_0: \begin{tikzcd}[row sep=small] \bullet & \\ & a_1 \arrow[ul] \arrow[dl] \\ \bullet & \end{tikzcd} \qquad G_1: \quad a_1 \longleftarrow \bullet \longrightarrow a_2 \qquad G_2: \quad a_2 \longrightarrow \bullet \longleftarrow a_3$$

G_3: $a_2 \longleftarrow \bullet \longrightarrow \bullet\ a_3$ $\qquad$ G_4: $a_3 \longrightarrow \bullet \longleftarrow \bullet \longleftarrow \bullet$

Then A is given by the quiver

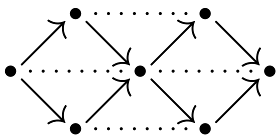


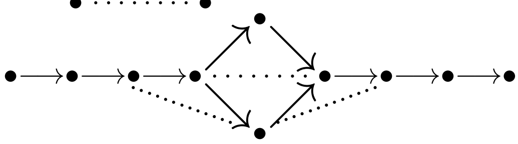
and $T_2(A)$ contains a convex subcategory of the form

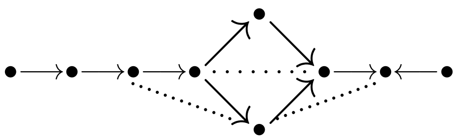


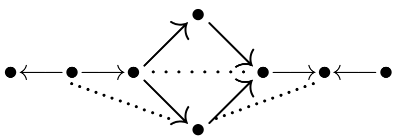
which is concealed of type $\widetilde{\mathbb{D}}_{17}$.

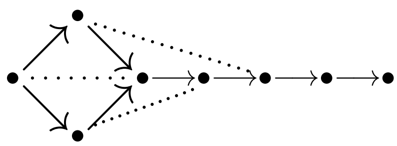
6. Tame triangular matrix algebras. Consider the family (T) of bound quiver algebras KQ/I given by the following quivers:

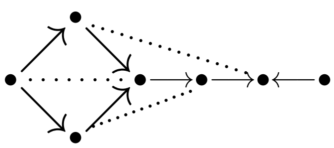
(1) 

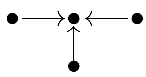
(2) 

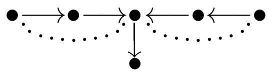
(3) 

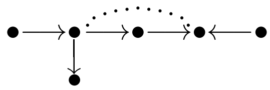
(4) 

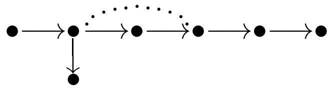
(5) 

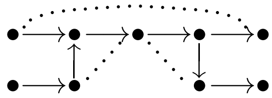
(6) 

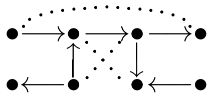
(7) 

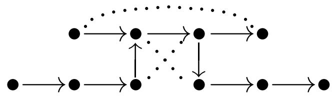
(8) 

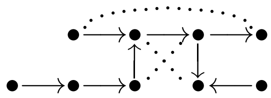
(9) 

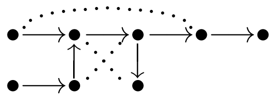
(10) 

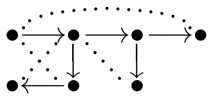
(11) 

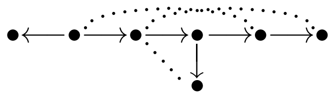
(12) 

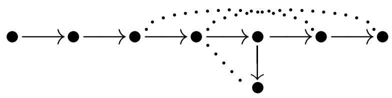
(13) 

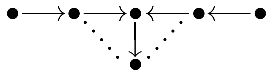
(14) 

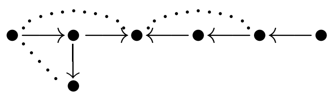
(15) 

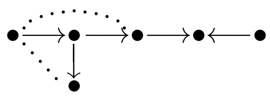
(16) 

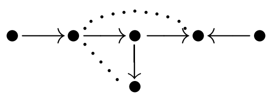
(17) 

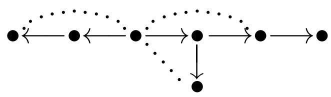
(18) 

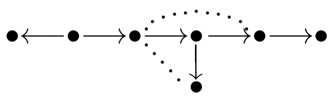
(19) 

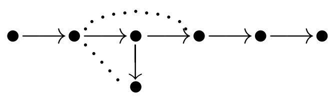
(20) 

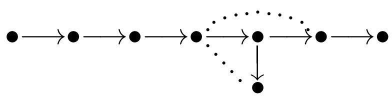
(21) 

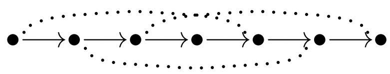
(22) 

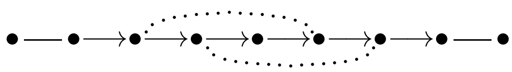
(23) 

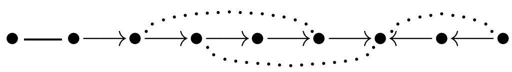
(24) 

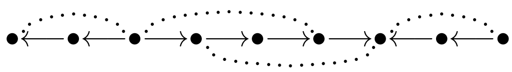
(25) 

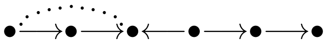
(26) 

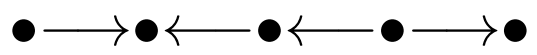
(27) 

(28) 

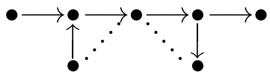
(29) 

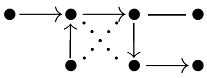
(30) 

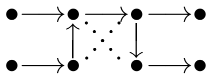
(31) 

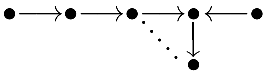
(32) 

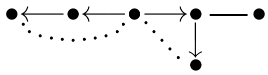
(33) 

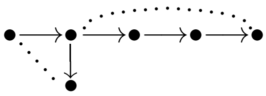
(34) 

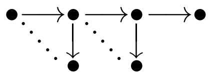
(35) 

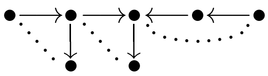
(36) 

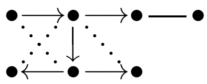
(37) 

(38) 

(39) 

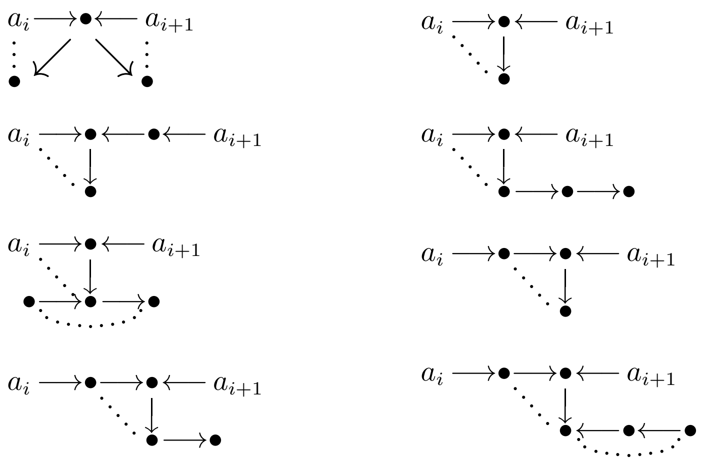
(40) 

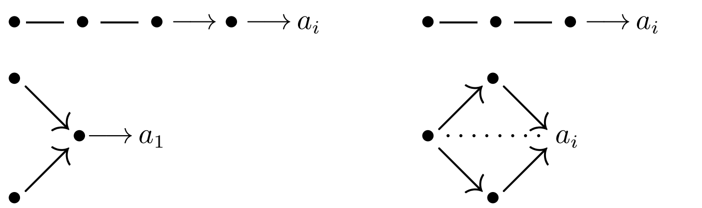
(41) 

(42) 

(43) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be the following families of bound quivers:

$\mathcal{A}$: $a_i \longleftarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow a_{i+1}$ $\quad$ $a_i \longleftarrow \bullet \longrightarrow a_{i+1}$ $\quad$ $a_i \longleftarrow \bullet \;—\; \bullet \longrightarrow a_{i+1}$

$\mathcal{B}$: 

$\mathcal{C}$: 

Then (Q, I) is a bound quiver of the form

$$G_0 \quad a_1 \quad G_1 \quad a_2 \ \ldots \ a_{n-1} \ G_{n-1} \ a_n \quad G_n \qquad\qquad n \geq 1$$

satisfying the following conditions:

(α) for $i = 0$ and $i = n$, G_i or G_i^{op} is one of the bound quivers from $\mathcal{A} \cup \mathcal{C}$,

(β) if $n \geq 2$ then, for $1 \leq i \leq n-1$, G_i or G_i^{op} is one of the bound quivers from $\mathcal{A} \cup \mathcal{B}$,

(γ) for $1 \leq i \leq n$ the vertex a_i is a source (respectively, target) of G_{i-1} if and only if a_i is a target (respectively, source) of G_i,

(δ) the composition of any two arrows in Q having a_i, $1 \leq i \leq n$, as a common vertex belongs to I.

(44) (Q, I) is a bound quiver of the form

$$a_0 \ G_0 \quad a_1 \quad G_1 \quad a_2 \ \ldots \ a_{n-1} \ G_{n-1} \ a_n$$

with $n \geq 1$, $a_n = a_0$, and satisfying the following conditions:

(α) for each $0 \leq i \leq n-1$, G_i or G_i^{op} is one of the bound quivers from $\mathcal{A} \cup \mathcal{B}$,

(β) for $1 \leq i \leq n$, the vertex a_i is a source (respectively, target) of G_{i-1} if and only if a_i is a target (respectively, source) of G_i (where $G_n = G_0$),

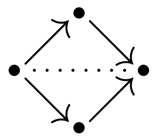
(γ) the composition of any two arrows in Q having a_i, $1 \leq i \leq n$, as a common vertex belongs to I.

We note that (Q, I) contains exactly one (nonoriented) cycle.

We shall write (Tn) for the nth quiver from the above family (T).

Proposition 5. *Let A be a bound quiver algebra from the family* (T1)–(T43). *Then $T_2(A)$ is of polynomial growth*, *provided A is not of type* (T43) *having a factor algebra from the family* (NPG). *Moreover*, *$T_2(A)$ is domestic* (*respectively*, *of finite type*) *if and only if A has no factor algebra Λ such that Λ or Λ^{op} is from the family* (ND) (*respectively*, (IT)).

Proof. Observe that $T_2(A)$ is simply connected, and in fact both $T_2(A)$ and $T_2(A)^{\mathrm{op}}$ satisfy the separation property (see [44], [46]). Moreover, $T_2(A)$ is strongly simply connected if and only if A does not contain a convex subcategory given by a commutative square



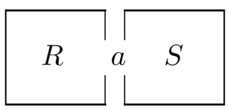
or equivalently, A is the bound quiver algebra of a bound tree. In particular, it is the case for all algebras of types (T7)–(T42). Clearly, if $T_2(A)$ is of polynomial growth then A has no factor algebra Λ from the family (NPG), because otherwise $T_2(\Lambda)$ is a factor algebra of $T_2(A)$, which contradicts Proposition 2. Hence the necessity part follows. Consequently, a direct checking shows that $T_2(A)$ contains a convex *pg*-critical subcategory if and only if A is of type (T43) and admits a factor algebra Λ with Λ or Λ^{op} from the family (NPG). Further, it is easy to check that $T_2(A)$ does not contain a convex subcategory which is concealed of one of the types $\widetilde{\widetilde{\mathbb{A}}}_m$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$ or $\widetilde{\widetilde{\mathbb{E}}}_8$. Applying now [46, Theorem 4.1] (and its proof) we conclude that $T_2(A)$ is a polynomial growth simply connected algebra (even a multicoil algebra with directed component quiver) provided A is not of type (T43) having a factor algebra Λ with Λ or Λ^{op} from the family (NPG). Finally, we easily check that A has no factor algebra Λ with Λ or Λ^{op} from the family (ND) (respectively, (IT)) if and only if $T_2(A)$ does not contain a convex subcategory B which is tubular (respectively, concealed of type $\widetilde{\mathbb{A}}_m$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$), or equivalently $T_2(A)$ is domestic (by [46, Corollary 4.3] and its proof) (respectively, $T_2(A)$ is of finite representation type, by [8]). This finishes the proof.

Our next aim is to prove that, for any algebra A of type (T43) or (T44), the algebra $T_2(A)$ is tame. We need a reduction lemma and the following concept.

For a bound quiver algebra $A = KQ/I$, we say that an object x of A (vertex x of Q) is a *node* of A provided $\beta\alpha \in I$ for any two arrows $\alpha, \beta \in Q$ with $s(\beta) = x$ and $e(\alpha) = x$.

Consider the following two families of bound quiver algebras:

(i) B :



where S or S^{op} is the bound quiver algebra of a bound quiver from the family $\mathcal{A} \cup \mathcal{C}$ in (T43), with $a = a_i$ or $a = a_{i+1}$, a is a source (respectively, target) of S if and only if a is a target (respectively, source) of R, and a is a node of B, and

(ii) $$C: \quad \boxed{R_1 \quad b \quad S \quad c \quad R_2}$$

where possibly $R_1 = R_2$, S or S^{op} is the bound quiver algebra of a bound quiver from the family $\mathcal{A} \cup \mathcal{B}$ in (T43), with $b = a_i$ and $c = a_{i+1}$, b and c are sources (respectively, targets) of S if and only if b and c are targets (respectively, sources) in R_1 and R_2, and b and c are nodes of C.

Let Δ be the quiver $x \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow y$ of type $\mathbb{A}_7$ and $\Lambda = K\Delta$. We now define new families of algebras using B, C, Λ, Λ^{op} as follows. For B, the algebra B' is obtained from B by replacing S by Λ with $x = a$ if a is a source of S, or by replacing S by Λ^{op} with $x = a$ if a is a target of S, and again with a being a node of B'. Similarly, for C, the algebra C' is obtained from C by replacing S by Λ with $x = b$ and $y = c$ if a is a source of S, or by replacing S by Λ^{op} with $x = b$ and $y = c$ if a is a target S, and again with b and c being nodes of C'. Then $T_2(B')$ contains a convex subcategory B'' of the form

(iii)
$$\begin{tikzcd}
\boxed{T_2(R)} & a^* \arrow[d] & \bullet \arrow[l] \arrow[d] & & & & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
 & a & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l]
\end{tikzcd}$$

if a is a target of S, or of the form

(iv)
$$\begin{tikzcd}
\boxed{T_2(R)} & a^* \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] & \bullet & \bullet \arrow[l] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] \\
 & a \arrow[r] & \bullet & & & & \bullet & \bullet \arrow[l]
\end{tikzcd}$$

if a is a source of S. Similarly, $T_2(C')$ contains a convex subcategory C'' of the form

(v)
$$\begin{tikzcd}
\boxed{T_2(R_1)} & b^* \arrow[d] & \bullet \arrow[l] \arrow[d] & & & & \bullet \arrow[r] \arrow[d] & c^* \arrow[d] & \boxed{T_2(R_2)} \\
 & b & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & c &
\end{tikzcd}$$

if b and c are targets of S, or of the form

(vi)
$$\begin{tikzcd}
\boxed{T_2(R_1)} & b^* \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] & \bullet & \bullet \arrow[l] & \bullet \arrow[l] \arrow[d] & c^* \arrow[l] \arrow[d] & \boxed{T_2(R_2)} \\
 & b \arrow[r] & \bullet & & & & \bullet & c \arrow[l] &
\end{tikzcd}$$

if b and c are sources of S.

In the above notation we have the following

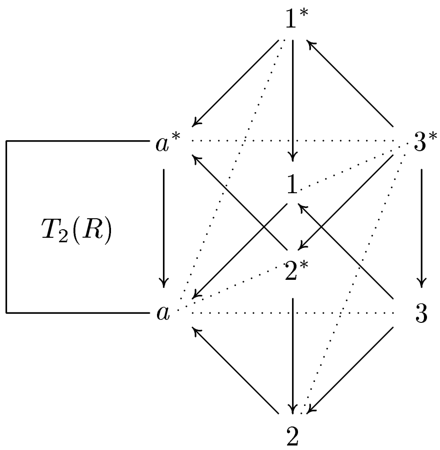
LEMMA 1. *Assume B'' (respectively, C'') is tame. Then $T_2(B)$ (respectively, $T_2(C)$) is tame.*

P r o o f. This is done by case-by-case consideration of all possible shapes of the algebra S. We shall illustrate the procedure in two typical cases.

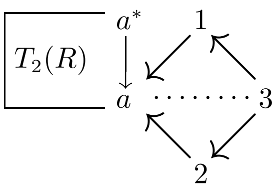
Consider the algebra B with S given by the bound quiver

$$\begin{tikzcd} & 1 \arrow[dl] & \\ a \arrow[rr, dotted, no head] & & 3 \arrow[ul] \arrow[dl] \\ & 2 \arrow[ul] & \end{tikzcd}$$

from the family $\mathcal{C}$. Then $T_2(B)$ is of the form



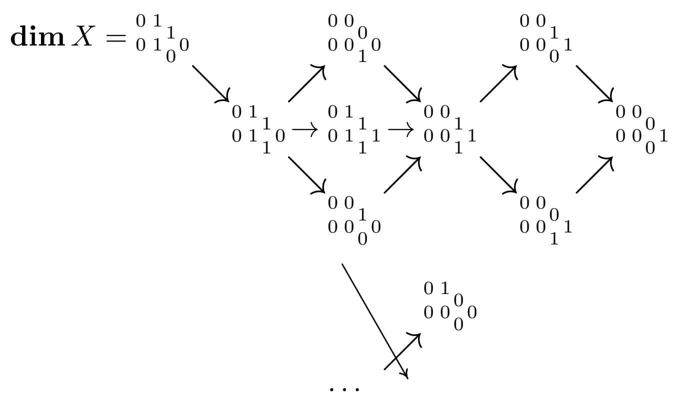
Observe that $T_2(B)$ can be obtained from the algebra D of the form



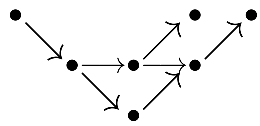
by iterated one-point extensions creating the vertices 1^*, 2^*, 3^*. Consider first the one-point extension $D[X]$ with extension vertex 1^*, where X is the unique indecomposable D-module of dimension vector

$$\mathbf{dim}\, X = \begin{smallmatrix}0&1&&\\ &&1&\\0&1&&0\\&&0&\end{smallmatrix}$$

(having 0 at all the vertices of $T_2(R)$ except a and a^*). Then the Auslander–Reiten quiver of D has a full translation subquiver of the form



Hence the vector space category $\mathrm{Hom}_D(X, \mathrm{mod}\, D)$ is the additive category of the incidence category of the following partially ordered set of finite representation type:



Thus there are only finitely many isoclasses of indecomposable $D[X]$-modules whose dimension vector is nonzero at the vertex 1^*.

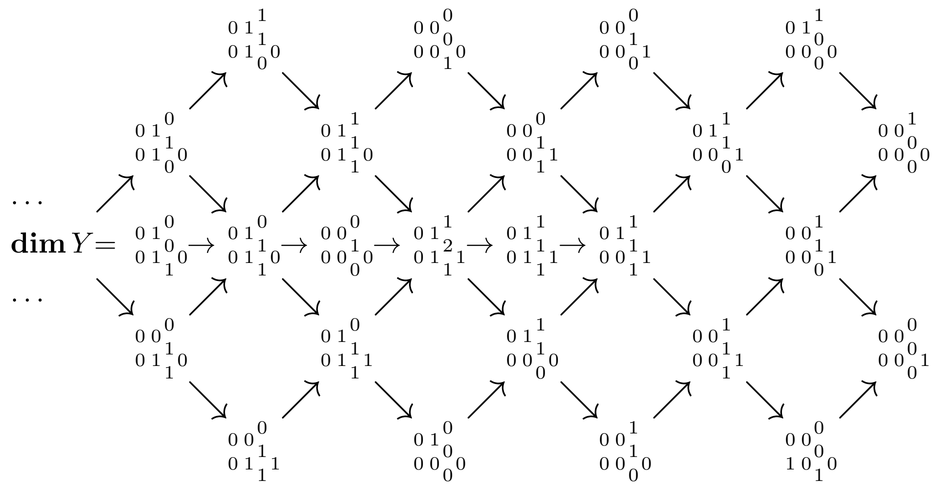
Next, we consider the one-point extension

$$E = (D[X])[Y]$$

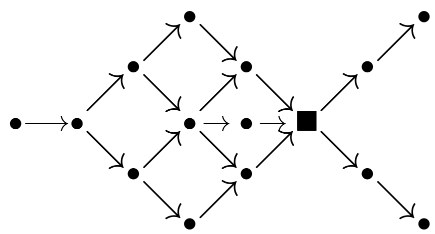
with extension vertex 2^*, where Y is the unique indecomposable $D[X]$-module (in fact even a D-module) of dimension vector

$$\mathbf{dim}\, Y = \begin{smallmatrix}&&0&\\0&1&&\\&&0&\\0&1&&0\\&&1&\end{smallmatrix}$$

The Auslander–Reiten quiver of $D[X]$ contains a full translation subquiver of the form



hence the vector space category $\mathrm{Hom}_{D[X]}(Y, \mathrm{mod}\, D[X])$ is the additive category of the category



(see [37, (2.4)] for the corresponding notation). In particular, $D[X][Y]$ is a domestic (even one-parametric) extension of $D[X]$. Observe that the convex subcategory of $E = D[X][Y]$ given by the vertices a, a^*, 1, 1^*, 2, 2^*, 3 is a tilted algebra F of type $\widetilde{\mathbb{E}}_6$, obtained from the hereditary algebra H of type $\widetilde{\mathbb{A}}_6$, formed by the vertices a^*, 1, 1^*, 2, 2^*, 3, by one-point coextension using a simple regular module lying in a stable tube of rank one of the Auslander–Reiten quiver of H. Moreover, since a is a target in S, we conclude that a is a source of R. Further, a is a node of B. This implies that for any indecomposable injective E-module $I_E(x)$, with x being an object of $T_2(R)$ different from a and a^*, the restriction of $I_E(x)$ to F is projective. In particular, we conclude that the preinjective component and $\mathbb{P}_1(K)$-family of coray tubes of the Auslander–Reiten quiver of F are full components of the Auslander–Reiten quiver of E, and moreover, are closed under successors in $\mathrm{mod}\, E$.

Finally, observe that $T_2(B)$ is the one-point extension $E[Z]$, with extension vertex 3^*, where Z is the unique indecomposable injective module

$I_F(a) = I_E(a)$ in the tubular family of the Auslander–Reiten quiver of F. Therefore, invoking the above remarks, we conclude that the Auslander–Reiten quiver of $T_2(B)$ has a preinjective component of Euclidean type $\widetilde{\mathbb{E}}_6$ and a $\mathbb{P}_1(K)$-family of tubes, one of them containing the projective-injective module $I_{T_2(R)}(a) = I_{T_2(S)}(a) = P_{T_2(S)}(3^*) = P_{T_2(R)}(3^*)$ and the remaining ones being stable tubes of the Auslander–Reiten quiver of H. As a consequence, we deduce that if M is an indecomposable $T_2(B)$-module whose support contains one of the vertices 1^*, 2^*, or 3^*, then the support of M is contained in $T_2(S)$. Therefore, $T_2(B)$ is tame if and only if D is tame.

Further, taking the APR-cotilt of D (in the sense of [2]) with respect to the simple injective nonprojective module $S_D(3)$, we get an algebra Γ of the form

$$\begin{tikzcd} & a^* \arrow[d] & & 1 \arrow[dl] \\ \boxed{T_2(R)} & a & 3 \arrow[l] & \\ & & & 2 \arrow[ul] \end{tikzcd}$$

and D is tame provided Γ is tame. Taking now the APR-cotilt of Γ with respect to the simple injective nonprojective module $S_\Gamma(2)$, we obtain an algebra Ω of the form

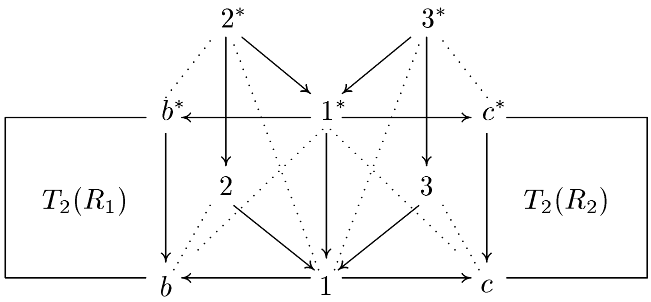
$$\begin{tikzcd} & a^* \arrow[d] & & 1 \arrow[dl] \\ \boxed{T_2(R)} & a & 3 \arrow[l] \arrow[dr] & \\ & & & 2 \end{tikzcd}$$

and such that Ω tame implies Γ tame. We now observe that Ω can be obtained from a full subcategory of the category B'' of type (iii) by shrinking some arrows to identity (see [37, (1.2)]), and consequently, Ω is tame if B'' is tame (see also [16, Lemma 6] for the fact that a full subcategory of a tame algebra is also tame). Summing up the considerations above, we infer that if B'' is tame then $T_2(B)$ is tame.

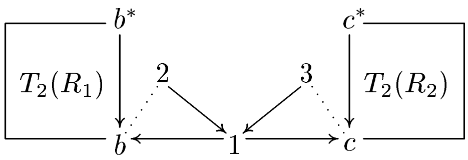
Consider now the algebra C with S given by the bound quiver

$$\begin{tikzcd} 2 \arrow[dr] \arrow[d, dotted, no head] & & 3 \arrow[dl] \arrow[d, dotted, no head] \\ b & 1 \arrow[l] \arrow[r] & c \end{tikzcd}$$

from the family $\mathcal{B}^{\mathrm{op}}$. Then $T_2(C)$ is of the form

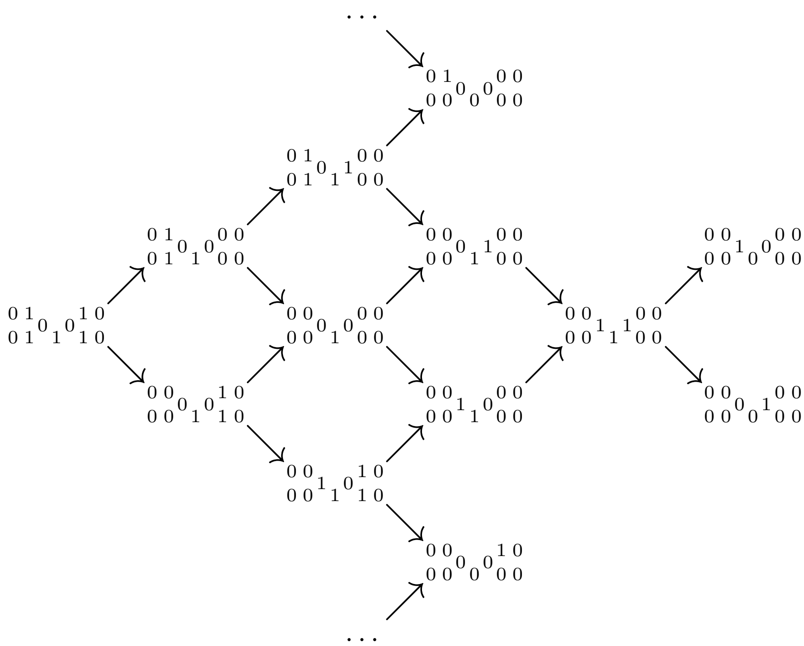


Hence $T_2(C)$ can be obtained from the algebra D of the form

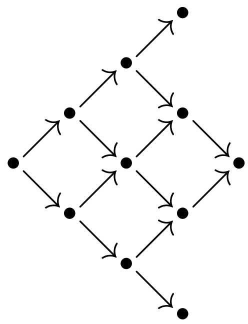


by iterated one-point extensions creating the vertices 1^*, 2^*, 3^*.

Consider first the one-point extension $D[X]$ with extension vertex 1^*, where X is the unique indecomposable D-module of dimension vector $\mathbf{dim}\, X = {}^{0\,1}_{0\,1}{}^{0}_{1}{}^{0}_{1}{}^{1\,0}_{1\,0}$ (having 0 at all vertices of $T_2(R_1)$ and $T_1(R_2)$ except b, b^*, c, c^*). The Auslander–Reiten quiver of D has a full translation subquiver of the form



Hence the vector space category $\operatorname{Hom}_D(X, \operatorname{mod} D)$ is the additive category of the incidence category of the following partially ordered set:



Consequently, $E = D[x]$ is a domestic (even one-parametric) extension of D, and the new one-parametric families of indecomposable E-modules are those from the $\mathbb{P}_1(K)$-family $\mathcal{T}$ of stable tubes of the Auslander–Reiten quiver of the hereditary algebra H given by the quiver

$$\begin{tikzcd} b^* & 1^* \arrow[l] \arrow[r] \arrow[dd] & c^* \\ 2 \arrow[dr] & & 3 \arrow[dl] \\ & 1 & \end{tikzcd}$$

of Euclidean type $\widetilde{\mathbb{D}}_5$.

Let F be the convex subcategory of $T_2(S)$ formed by the vertices b, b^*, 1, 1^*, c, c^*, 2 and 3. Then F is a tubular coextension of H of tubular type $(2, 3, 4)$ given by two one-point coextensions of H by two simple regular modules lying in a stable tube of rank 2 of the tubular family $\mathcal{T}$. Since b and c are sources in $R_1 \cup R_2$ and nodes in C, we deduce that for any indecomposable injective E-module $I_E(x)$ with x being an object of $T_2(R_1)$ or $T_2(R_2)$ different from b, b^*, c, c^*, the restriction of $I_E(x)$ to F is a preprojective F-module. This implies that the preinjective component and the $\mathbb{P}_1(K)$-family $\mathcal{T}'$ of coray tubes of the Auslander–Reiten quiver of F are full components of the Auslander–Reiten quiver of E, and moreover, are closed under successors in $\operatorname{mod} E$.

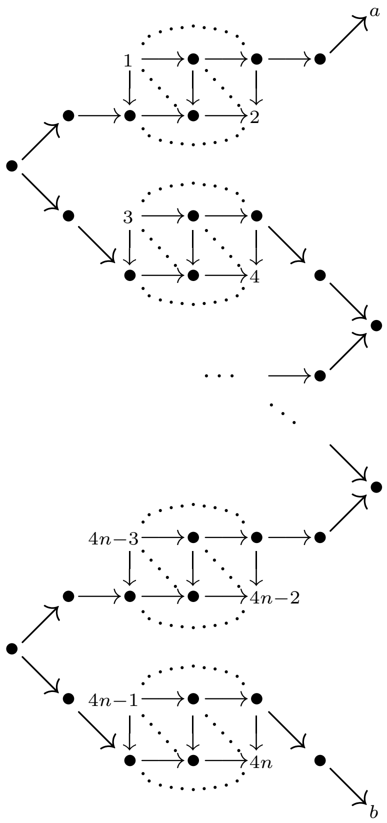
Observe now that $T_2(C)$ is a tubular extension $E[Y][Z]$ of E by the unique two injective E-modules Y and Z lying in the family $\mathcal{T}'$. Therefore, we deduce that the Auslander–Reiten quiver of $T_2(C)$ has a preinjective component of Euclidean type $\widetilde{\mathbb{E}}_7$ and a $\mathbb{P}_1(K)$-family $\mathcal{T}''$ of tubes; one of them contains two projective-injective modules $I_{T_2(C)}(b) = P_{T_2(C)}(3^*)$ and $I_{T_2(C)}(c) = P_{T_2(C)}(2^*)$, and the remaining ones are stable tubes of the Auslander–Reiten quiver of A.

In particular, we deduce that if M is an indecomposable $T_2(C)$-module whose support contains one of the vertices 1^*, 2^*, or 3^*, then the support of M is contained in $T_2(S)$. Therefore, $T_2(C)$ is tame if and only if D is tame. Taking two APR-cotilts of D with respect to the simple injective nonprojective modules $S_D(2)$ and $S_D(3)$, we obtain an algebra Γ of the form

$$\begin{tikzcd}
\boxed{T_2(R_1)} & b^* \arrow[d] & & & & c^* \arrow[d] & \boxed{T_2(R_2)} \\
 & b & 2 \arrow[l] & 1 \arrow[l] \arrow[r] & 3 \arrow[r] & c &
\end{tikzcd}$$

and if Γ is tame then so is D. Finally, we observe that Γ is a full subcategory of the category C'' of type (v). Hence, if C'' is tame then Γ is tame. Summing up our considerations we find that if C'' is tame then $T_2(C)$ is tame. This finishes the proof.

For each positive integer n, we denote by $B[n]$ the algebra given by the bound quiver



with $a = b$.

LEMMA 2. *For each positive integer* n, *the algebra* $B[n]$ *is tame* (*but not of polynomial growth*).

Proof. Fix m. Let $H[n]$ be the convex subcategory (algebra) of $B[n]$ given by all vertices of $B[n]$ except $1, \ldots, 4n$. Then $H[n]$ is the path algebra of a Euclidean quiver of type $\widetilde{\mathbb{A}}_{14n-1} = \widetilde{\mathbb{A}}_{7n,7n}$, and $B[n]$ is the biextension algebra

$$[N_1, \ldots, N_{2n}]H[n][M_1, \ldots, M_{2n}] = \begin{bmatrix} K^{2n} & 0 & 0 \\ M & H[n] & 0 \\ D(N) \otimes_{H[n]} M & D(N) & K^{2n} \end{bmatrix}$$

(in the sense of [18]), where $M = M_1 \oplus \ldots \oplus M_{2n}$, $N = N_1 \oplus \ldots \oplus N_{2n}$, $D(N) = \mathrm{Hom}_K(N, K)$, $M_i = \mathrm{rad}\, P_{H[n]}(2i-1)$, $N_i = I_{H[n]}(2i)/\mathrm{soc}\, I_{H[n]}(2i)$, for $1 \le i \le 2n$. Observe that $M_1, \ldots, M_{2n}, N_1, \ldots, N_{2n}$ are indecomposable regular $H[n]$-modules of regular length 2 lying in two stable tubes of rank $7n$ in the Auslander–Reiten quiver of $H[n]$. Moreover, the modules $M_1, \ldots, M_{2n}$ (respectively, $N_1, \ldots, N_{2n}$) are Hom-orthogonal. Therefore, applying [18, Theorem A], we conclude that $B[n] = [N_1, \ldots, N_{2n}]H[n][m_1, \ldots, M_{2n}]$ is tame. Finally, we note that $B[n]$ contains convex *pg*-critical subcategories, and hence is not of polynomial growth.

We are now able to prove the following fact.

PROPOSITION 6. *Let* A *be an algebra from one of the families* (T43) *or* (T44). *Then* $T_2(A)$ *is tame.*

Proof. We replace each part $S = G_i$ from the families $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{A}^{\mathrm{op}}$, $\mathcal{B}^{\mathrm{op}}$, $\mathcal{C}^{\mathrm{op}}$ in A by $\Lambda = K\Delta$, for Δ of the form $\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$ or its opposite, according to the procedure described before Lemma 1, and obtain an algebra A'. Then $T_2(A')$ contains a convex subcategory A'' obtained from $T_2(A')$ by replacing each $T_2(\Lambda)$ by

$$\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[l] \arrow[d] \arrow[dl, dotted, no head] & & & & \bullet \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & \bullet \arrow[d] \\
\bullet & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l]
\end{tikzcd}$$

and each $T_2(\Lambda^{\mathrm{op}})$ by

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, dotted, no head] & \bullet \arrow[d] & & & & \bullet \arrow[d] & \bullet \arrow[l] \arrow[d] \arrow[dl, dotted, no head] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l] & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}$$

Moreover, applying Lemma 1, we infer that $T_2(A)$ is tame if A'' is tame. Finally, observe that there is a positive integer n such that $A'' = B[n]$ if A is of type (T44), or A'' is a proper convex subcategory of $B[n]$ if A is of type

(T43). Applying Lemma 2, we conclude that A'' is tame. Therefore, $T_2(A)$ is also tame.

7. Weakly sincere triangular matrix algebras. For an algebra A, we say that the algebra $T_2(A)$ is *weakly sincere* if there exists an indecomposable $T_2(A)$-module M such that for every proper convex subcategory B of A the support of M is not contained in the convex subcategory $T_2(B)$. Clearly, if $T_2(A)$ is sincere then $T_2(A)$ is weakly sincere.

The main aim of this section is to prove the following fact.

PROPOSITION 7. *Let A be a simply connected algebra of finite representation type with $T_2(A)$ weakly sincere, and assume that neither A nor A^{op} has a factor algebra from the family* (W). *Then A or A^{op} is a factor algebra of one of the algebras from the family* (T1)–(T43).

In order to prove the proposition we need some concepts and lemmas. Throughout this section we assume that $A = KQ/I$ is a bound quiver algebra satisfying the conditions of the above proposition. Since A is simply connected of finite representation type (hence standard), we may also assume that I is generated by paths or differences of paths with common sources and common ends. Moreover, Q has no oriented cycles. We start with the following two lemmas.

LEMMA 3. *Assume there is a K-linear relation $u - w \in I$, where u, w are two paths in Q with a common source a and a common end b. Then $u = \beta\alpha$ and $w = \sigma\gamma$ for a convex bound subquiver of* (Q, I)

$$\begin{tikzcd} & \bullet \arrow[dr, "\beta"] & \\ a \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] \arrow[rr, dotted, no head] & & b \\ & \bullet \arrow[ur, "\sigma"'] & \end{tikzcd}$$

P r o o f. This follows from the fact that the bound quiver algebras (W1) and (W14) are not factor algebras of $A = KQ/I$.

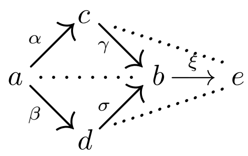
LEMMA 4. *Assume the bound quiver algebra of the bound quiver*

$$\begin{tikzcd} & c \arrow[dr, "\gamma"] & \\ a \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] \arrow[rr, dotted, no head] & & b \\ & d \arrow[ur, "\sigma"'] & \end{tikzcd}$$

is a convex subcategory of A. Then one of the following cases holds:

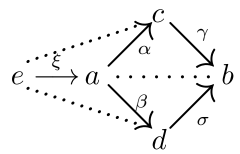
(i) *A or A^{op} is a factor algebra of one of the algebras from the family* (T1)–(T6).

(ii) *A admits a convex subcategory given by the bound quiver*



and α, β, γ, σ, ξ are the unique arrows starting or ending at the vertices a, b, c, d.

(iii)



and α, β, γ, σ are the unique arrows starting or ending at the vertices a, b, c, d.

Proof. This follows by a simple analysis of the neighbourhood of the commutative square formed by $\beta\alpha$ and $\sigma\gamma$ in (Q, I), invoking the facts that $T_2(A)$ is weakly sincere and neither A nor A^{op} has a factor algebra from the family (W2)–(W11).

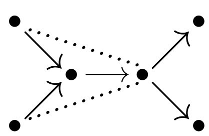
It will follow from our further analysis of (Q, I) that if A contains a convex subcategory of one of the forms (ii) or (iii), then A or A^{op} is a factor algebra of an algebra of type (T43). As a consequence we will find that if Q is not a tree then A or A^{op} is a factor algebra of one of the algebras (T1)–(T6) or (T43).

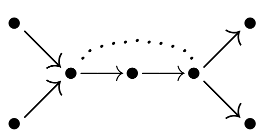
For two vertices x and y of Q, we set $x \leq y$ if there exists a path from x to y (including the trivial one for $x = y$) which does not belong to I. Then we may assign to each vertex x of Q two partially ordered sets

$$x^- = \{y \in Q_0 \mid y \leq x\} \quad \text{and} \quad x^+ = \{y \in Q_0 \mid x \leq y\}.$$

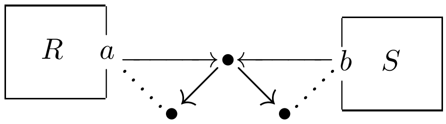
Recall that, for a finite partially ordered set S, its *width* $w(S)$ is the maximal number of pairwise incomparable vertices of S. For each vertex x of Q, consider also the full bound subquiver $N(x)$ of (Q, I) given by all vertices of x^- and x^+. Moreover, we put $w(x) = w(x^-) + w(x^+)$.

LEMMA 5. *Let x be a vertex of Q. Then $w(x) \leq 4$. Moreover, if $w(x) = 4$ then A or A^{op} is a factor algebra of* (T1) *or is given by the bound quiver of one of the forms*





or

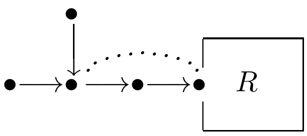


where a and b are nodes.

Proof. Assume that $w(x) \geq 4$. We claim that then $w(x^-) = 2$ and $w(x^+) = 2$, and consequently $w(x) = 4$. Indeed, suppose that $w(x^-) \geq 3$ or $w(x^+) \geq 3$. Then, invoking Lemmas 3 and 4, we easily conclude that A or A^{op} has a factor algebra of one of the forms (W9)–(W11), a contradiction. Hence $w(x^-) = 2 = w(x^+)$. Assume now that x is the source and the end of two arrows. Since A and A^{op} have no factor algebras of types (W8), (W10) and (W11), applying Lemma 4, we then infer that A is a factor algebra of (T1). Finally, assume (by symmetry) that there is only one arrow in Q starting at the vertex x. Using now the fact that algebras of types (W12), (W13) and (W23) are not factor algebras of A and A^{op}, we easily verify that A or A^{op} must be one of the algebras given by the bound quivers presented in the lemma.

From now on we assume that $w(x) \leq 3$ for any vertex x of (Q, I).

LEMMA 6. *Assume there exists a vertex x of Q such that $w(x) = 3$ and $N(x)$ is a quiver* (*without relations*). *Then A or A^{op} is a factor algebra of one of the algebras* (T7)–(T10) *or of an algebra of the form*



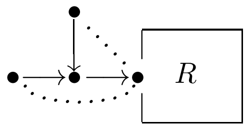
Proof. This follows from Lemmas 3, 4 and that Q is a tree and A contains a convex subcategory which is the path category of a Dynkin quiver of type $\mathbb{D}_4$. Then, since A and A^{op} have no factor algebras from the family (W13)–(W23), a direct analysis shows that A or A^{op} is a factor algebra of one of the algebras presented in the lemma.

LEMMA 7. *Assume there exists a vertex x in Q such that $w(x) = 3$ and the bound quiver $N(x)$ is bound by a zero-relation of length at least* 3. *Then A or A^{op} is a factor algebra of one of the algebras* (T11)–(T18).

Proof. This follows from Lemmas 3, 4 and 6, and the fact that A and A^{op} have no factor algebras from the family (W39)–(W52).

We note that if $A = KQ/I$ satisfies the conditions of the above lemma then for each vertex y of Q with $w(y) = 3$, the quiver $N(y)$ is bound by at least two zero-relations, one of them of length at least 3 and another one of length 2. Observe also that, if $w(x) = 3$ and $N(x)$ is bound only by zero-relations of length 2, then $N(x)$ is bound by at most two zero-relations.

LEMMA 8. *Assume there exists a vertex x in Q with $w(x) = 3$, and, for each vertex y in Q with $w(y) = 3$, the quiver $N(y)$ is bound only by two zero-relations of length* 2. *Then A or A^{op} is a factor algebra of one of the algebras* (T19)–(T26), *or is an algebra of the form*



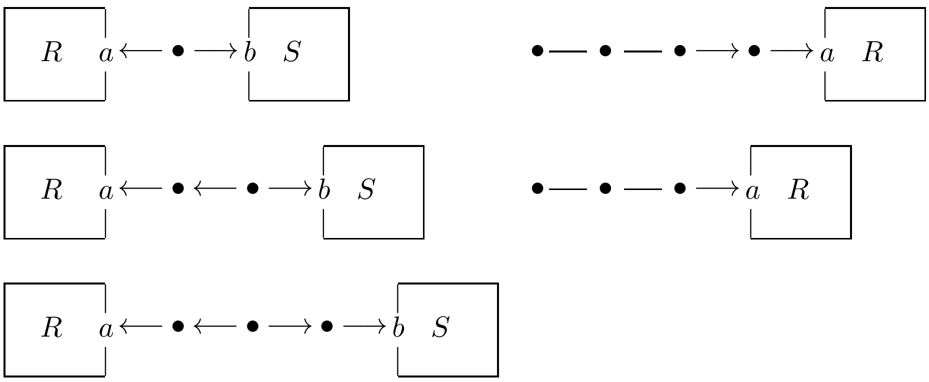
which is a factor algebra of the algebra presented in Lemma 4(ii).

Proof. This follows from Lemmas 3, 4, 6 and 7, and the fact that A and A^{op} have no factor algebras from the family (W53)–(W72).

LEMMA 9. *Assume $w(x) \le 2$ for any vertex x of Q, and there is a vertex y in Q with $N(y)$ bound by a zero-relation of length at least* 3. *Then A or A^{op} is a factor algebra of one of the algebras* (T27)–(T30).

Proof. This follows from the fact that A and A^{op} have no factor algebras of the forms (W73)–(W78) and (W81).

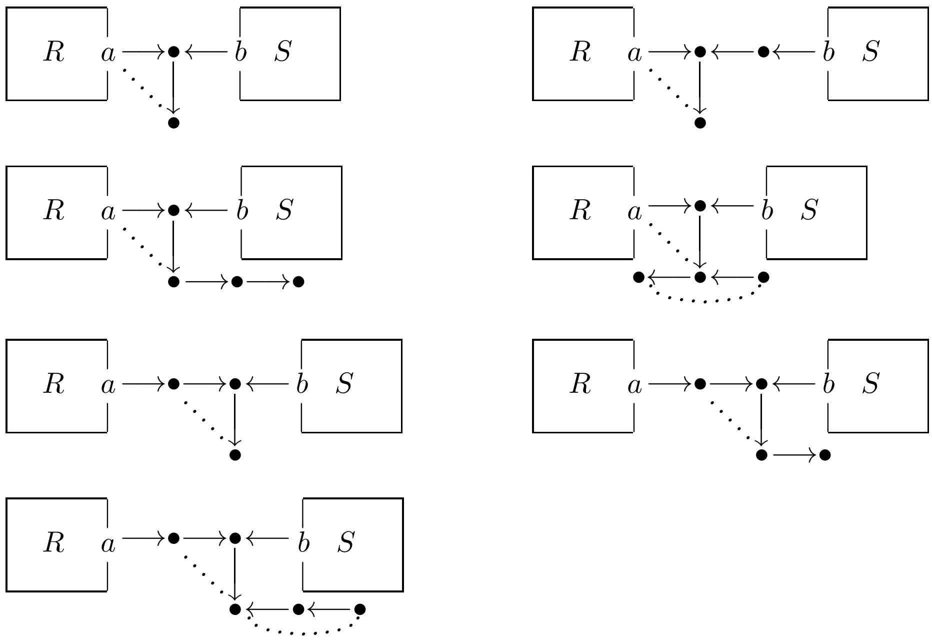
LEMMA 10. *Assume $w(x) \le 2$ for any vertex x of Q and each $N(x)$ is bound by at most one zero-relation of length* 2. *Then A or A^{op} has one of the following forms*: (T31)–(T33) *or*



where a and b are nodes.

Proof. This follows from the weak sincerity of $T_2(A)$ and the fact that A and A^{op} have no factor algebras of the forms (W73), (W79)–(W81).

LEMMA 11. *Assume there exists a vertex x in Q with $w(x) = 3$, and $N(x)$ is bound only by one zero-relation of length* 2, *and for each vertex y in Q with $w(y) = 3$, $N(y)$ is bound by at least one zero-relation. Then A or A^{op} is a factor algebra of one of the algebras* (T34)–(T42), *or has one of the forms*



where a and b are nodes.

Proof. This follows by a tedious analysis invoking the above lemmas and the fact that A and A^{op} have no factor algebras among (W24)–(W38).

LEMMA 12. *Assume that Q is not a tree, and neither A nor A^{op} is a factor algebra of one of the algebras* (T1)–(T6). *Then A or A^{op} is a factor algebra of an algebra of type* (T43) *whose quiver is not a tree.*

Proof. This follows from the lemmas proved above.

The final lemma below completes our proof of Proposition 7.

LEMMA 13. *Assume Q is a tree, and neither A nor A^{op} is a factor algebra of one of the algebras* (T7)–(T42). *Then A or A^{op} is a factor algebra of an algebra of type* (T43) *whose quiver is a tree.*

Proof. This is a direct consequence of the lemmas proved above.

8. Proofs of the main results. Let $A = KQ/I$ be a standard algebra of finite representation type and $\widetilde{A} \to A = \widetilde{A}/G$ be its universal

Galois covering, with $\widetilde{A}$ a simply connected locally bounded K-category and G a free group (see [9], [12]). Then the algebra $T_2(A)$ admits a universal Galois covering $F^{(2)} : T_2(\widetilde{A}) \to T_2(A) = T_2(\widetilde{A})/G$ with $T_2(\widetilde{A})$ a simply connected locally bounded K-category, described in Section 1. Denote by $F_\lambda^{(2)} : \operatorname{mod} T_1(\widetilde{A}) \to \operatorname{mod} T_2(A)$ the associated push-down functor. Since G is a free group, the induced action of G on the isoclasses of finite-dimensional indecomposable $T_2(\widetilde{A})$-modules is free, and consequently $F_\lambda^{(2)}$ preserves the indecomposable modules and Auslander–Reiten sequences (see [21]). If $F_\lambda^{(2)}$ is dense then we obtain a Galois covering $F_\lambda^{(2)} : \operatorname{mod} T_2(\widetilde{A}) \to \operatorname{mod} T_2(A)$ of module categories (in the sense of [9], [21]). In particular, in this case, the Auslander–Reiten quiver $\Gamma_{T_2(A)}$ of $T_2(A)$ is the orbit quiver $\Gamma_{T_2(\widetilde{A})}/G$ of the Auslander–Reiten quiver $\Gamma_{T_2(\widetilde{A})}$ with respect to the induced action of G.

We say that an indecomposable locally finite-dimensional $T_2(\widetilde{A})$-module M is *weakly G-periodic* if its support $\operatorname{supp} M$ is infinite and the quotient category $(\operatorname{supp} M)/G_M$ is finite, where $G_M = \{g \in G \mid gM \cong M\}$. Note that then G_M is infinite. Since G is a free group, by [17, Proposition 2.4], we see that the push-down functor $F_\lambda^{(2)} : \operatorname{mod} T_2(\widetilde{A}) \to \operatorname{mod} T_2(A)$ is dense if and only if there is no weakly G-periodic module over $T_2(\widetilde{A})$.

Proof of Theorem 1. Assume $T_2(A)$ is tame. Then it follows from [15, Proposition 2] that $T_2(\widetilde{A})$ is tame. Hence every finite convex subcategory B of $T_2(\widetilde{A})$ is tame and consequently the Tits form q_B of B is weakly nonnegative (see [36]). Therefore (i) implies (ii). The implication (ii)⇒(iii) is a direct consequence of the fact that the Tits form of any concealed algebra of wild type is not weakly nonnegative (see [25, (6.2)]). Further, the implication (iii)⇒(iv) follows from Proposition 1. Therefore, it remains to show that (iv) implies (i).

Assume $\widetilde{A}$ does not contain a finite convex subcategory Λ such that one of the algebras from the family (W) is a factor algebra of Λ or Λ^{op}. Take an indecomposable module M in $\operatorname{mod} T_2(\widetilde{A})$. Since the category $T_2(\widetilde{A})$ is interval-finite in the sense of [12], the convex hull Λ of the support of M is also finite. But then there exists a finite convex subcategory B of $\widetilde{A}$ such that $T_2(B)$ is weakly sincere and Λ is a convex subcategory of $T_2(B)$. In particular, M is an indecomposable $T_2(B)$-module. Clearly, neither B nor B^{op} has a factor algebra from the family (W). Then, applying Proposition 7, we conclude that B or B^{op} is a factor algebra of one of the algebras from the family (T1)–(T43). Therefore, by Propositions 5 and 6, $T_2(B)$ is tame, and so also is Λ. Hence $T_2(\widetilde{A})$ is tame.

If the push-down functor $F_\lambda^{(2)} : \operatorname{mod} T_2(\widetilde{A}) \to \operatorname{mod} T_2(A)$ is dense then $T_2(A)$ is tame (see [15, Lemma 3]). Therefore, assume $F_\lambda^{(2)}$ is not dense.

Then there exists a weakly G-periodic $T_2(\widetilde{A})$-module Y. We need a technique developed in [17, Section 4]. Let $R = T_2(\widetilde{A})$. For a full subcategory C of R we denote by $\widehat{C}$ the full subcategory of R formed by all objects y such that $R(x, y) \neq 0$ or $R(y, x) \neq 0$ for some object x from C. Clearly, if C is finite, then C is also finite because the category R is locally bounded. For an R-module M we denote by $M|C$ the restriction of M to C. For $X, Y \in \operatorname{Mod} R$ we write $X \in\!\!\!\mid Y$ whenever X is isomorphic to a direct summand of Y.

Fix a family C_n, $n \in \mathbb{N}$, of finite convex subcategories of R such that

(1) For each $n \in \mathbb{N}$, C_{n+1} is the convex hull of $\widehat{C}_n$ in R.

(2) $R = \bigcup_{n\in\mathbb{N}} C_n$.

Since R is connected, locally bounded and interval-finite, such a family exists. We shall identify a C_n-module Z with an R-module, by setting $M(x) = 0$ for all objects x of R which are not in C_n. Let $m \in \mathbb{N}$ be the least number such that $Y|C_m \neq 0$. We define a family of modules $Y_n \in \operatorname{ind} C_n$, $n \in \mathbb{N}$, as follows. Put $Y_n = 0$ for $n < m$ and let Y_m be an arbitrary indecomposable direct summand of $Y|C_m$. Then there exist $Y_{m+1} \in \operatorname{ind} C_{m+1}$ and a splittable monomorphism $\varphi_m : Y_m \to Y_{m+1}|C_n$ such that $Y_{m+1} \in\!\!\!\mid (Y|C_{m+1})$. Repeating this procedure we can find, for all $n \geq m$, $Y_n \in \operatorname{ind} C_n$ and splittable monomorphisms $\varphi_n : Y_n \to Y_{n+1}|C_n$ such that $Y_n \in\!\!\!\mid (Y|C_n)$. Thus we obtain a sequence $(Y, \varphi_n)_{n\in\mathbb{N}}$, called in [17] a fundamental R-sequence produced by Y. Since in our case C_n are convex subcategories of R, it is in fact a sequence of finite-dimensional indecomposable R-modules. The following facts are direct consequences of [17, (4.3)–(4.5)]:

(a) $Y = \varinjlim Y_n$.

(b) For each $n \in \mathbb{N}$, there exists $p \geq n$ such that $Y_p|C_n \cong Y|C_n$.

(c) For each $g \in G_Y$ and $n \in \mathbb{N}$, there exists $q \geq n$ such that $gC_n \subset C_q$ and ${}^gY_n \in\!\!\!\mid (Y_q \mid gC_n)$.

For $n \geq m$, denote by D_n the support of Y_n. Clearly, D_n is contained in C_n. Moreover, since Y is indecomposable, infinite-dimensional, locally finite-dimensional, and C_{n+1} contains $\widehat{C}_n$, for each $n \in \mathbb{N}$, we deduce from [15, Lemma 2] that, for any $n \geq m$, D_n is not contained in C_{n-1}. Let $s = 2(m+11)$. Then each of the categories D_n, $n \geq s$, has at least 22 objects. Moreover, we know from the first part of our proof that, for each $n \geq s$, there exists a convex subcategory B_n of $\widetilde{A}$ such that $T_2(B_n)$ is a weakly sincere convex subcategory of $R = T_2(\widetilde{A})$ and Y_n is an indecomposable $T_2(B_n)$-module. Further, it follows from Proposition 7 that B_n or B_n^{op} is a factor algebra of an algebra from the family (T1)–(T43). Since $T_2(B_n)$, for $n \geq s$, has at least 22 objects, we conclude that B_n or B_n^{op} is a factor algebra of an algebra from the family (T43).

Fix now an element $1 \neq g \in G_Y$. We know from (b) and (c) that for any $n \geq s$ there exists $r \geq n$ such that $gC_n \subset C_r$, ${}^gY_n \Subset (Y_r|gC_n)$, and $Y_r|C_n \cong Y|C_n$. Moreover, $Y = \varprojlim Y_n$. Then we conclude that there is a factor algebra B of an algebra of type (T44), whose quiver contains one (unoriented) cycle, such that the universal (simply connected) Galois cover $\widetilde{B}$ of B is a convex subcategory $\widetilde{A}$ and Y is an indecomposable $T_2(\widetilde{B})$-module. We also note that all but finitely many categories B_n have a factor algebra D with D or D^{op} from the family (NPG), and consequently $T_2(\widetilde{A})$ is not of polynomial growth.

Observe now that in our proof that $T_2(A)$ is tame we may assume that $T_2(A)$ is weakly sincere. Under this assumption, applying Proposition 7 and invoking the shape of the algebras of type (T44) and the properties of the convex subcategories C_n, $n \in \mathbb{N}$, we conclude that $T_2(\widetilde{A}) = R = \bigcup_{n\in\mathbb{N}} C_n = T_2(\widetilde{B})$, and consequently $T_2(A) = T_2(B)$ is tame, by Proposition 6. Therefore (iv) implies (i), and this finishes the proof.

Proof of Theorem 2. It follows again from [15, Proposition 2] that $T_2(A)$ of polynomial growth implies $T_2(\widetilde{A})$ of polynomial growth. Then the implication (i)⇒(ii) follows from the fact that all *pg*-critical algebras are not of polynomial growth and all concealed algebras of types $\widetilde{\widetilde{\mathbb{A}}}_m$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$, $\widetilde{\widetilde{\mathbb{E}}}_8$ are wild (see Section 1).

The implication (ii)⇒(iii) is a direct consequence of Propositions 1 and 2.

Assume (iii) holds. Then Theorem 1 yields that $T_2(\widetilde{A})$ does not contain a convex subcategory which is concealed of type $\widetilde{\widetilde{\mathbb{A}}}_m$, T_5, $\widetilde{\widetilde{\mathbb{D}}}_n$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$ or $\widetilde{\widetilde{\mathbb{E}}}_8$. We also know that the support of any indecomposable $T_2(\widetilde{A})$-module is contained in a weakly sincere convex subcategory $T_2(B)$ of $T_2(\widetilde{A})$ for a finite convex subcategory B of $\widetilde{A}$. But then, by our assumption (iii) and Proposition 5, we conclude that $T_2(B)$ is of polynomial growth. Therefore $T_2(\widetilde{A})$ is of polynomial growth. Finally, it follows from the proof of Theorem 1 and the assumption that $\widetilde{A}$ has no factor algebra Λ with Λ or Λ^{op} from the family (NPG) that the push-down functor $F_\lambda^{(2)} : \operatorname{mod} T_2(\widetilde{A}) \to \operatorname{mod} T_2(A)$ is dense. Hence, invoking again [15, Lemma 3], we infer that $T_2(A)$ is of polynomial growth, and (iii) implies (ii).

Proof of Theorem 5. This is a direct consequence of the above proof and the properties of the push-down functor $F_\lambda^{(2)}$ described at the beginning of this section.

Proof of Theorem 3. It follows from [15, Proposition 2] that if $T_2(A)$ is domestic then $T_2(\widetilde{A})$ is domestic, and consequently (i) implies (ii).

The implication (ii)⇒(iii) is a direct consequence of Propositions 1–3.

Assume that (iii) holds. Let Λ be a finite convex subcategory of $\widetilde{A}$. Since Λ and Λ^{op} have no factor algebra from the families (W) and (ND), we easily deduce that Λ and Λ^{op} have no factor algebra from the family (NPG). In particular, by Theorems 2 and 5, $T_2(\widetilde{A})$ is of polynomial growth and the push-down functor $F_\lambda^{(2)} : \operatorname{mod} T_2(\widetilde{A}) \to \operatorname{mod} T_2(A)$ is dense. Further, it follows from Propositions 5 and 7 that every weakly sincere finite convex subcategory of the form $T_2(B)$ in $T_2(\widetilde{A})$ is domestic. Therefore $T_2(\widetilde{A})$ and finally $T_2(A)$ are also domestic. Hence (iii) implies (i) and this finishes the proof.

Proof of Theorem 4. It is well known (see [21, Lemma 3.3]) that if $T_2(A)$ is of finite representation type, then $T_2(\widetilde{A})$ is locally representation-finite, that is, every object of $T_2(\widetilde{A})$ belongs to the supports of finitely many isoclasses of indecomposable $T_2(\widetilde{A})$-modules. Thus, clearly, (i) implies (ii).

The implication (ii)⇒(iii) follows from Proposition 4.

Assume (iii) holds. Then Propositions 5 and 7 imply that every weakly sincere finite convex subcategory of the form $T_2(B)$ in $T_2(\widetilde{A})$ is of finite representation type and consequently every finite convex subcategory of $T_2(\widetilde{A})$ is of finite representation type. In particular, $T_2(\widetilde{A})$ is of polynomial growth, and so the push-down functor $F_\lambda^{(2)} : \operatorname{mod} T_2(\widetilde{A}) \to \operatorname{mod} T_2(A)$ is dense. Since $T_2(\widetilde{A})$ is strongly simply connected, we deduce from [17, Corollary 2.5] that $T_2(\widetilde{A})$ is locally support-finite [15], that is, for each object x of $T_2(\widetilde{A})$ the full subcategory of $T_2(\widetilde{A})$ formed by the supports of all indecomposable finite-dimensional $T_2(\widetilde{A})$-modules having x in the support is finite. But then we conclude that each object x of $T_2(\widetilde{A})$ lies in the support of at most finitely many (up to isomorphism) indecomposable finite-dimensional $T_2(\widetilde{B})$-modules, that is, $T_2(\widetilde{A})$ is locally representation-finite in the sense of [9], [21]. Therefore, by [21, Theorem 3.6], $T_2(A)$ is of finite representation type. Thus (iii) implies (i).

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