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Filters and sequences

by

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Abstract. We consider two situations which relate properties of filters with properties of the limit operators with respect to these filters. In the first one, we show that the space of sequences having limits with respect to a Π_3^0 filter is itself Π_3^0 and therefore, by a result of Dobrowolski and Marciszewski, such spaces are topologically indistinguishable. This answers a question of Dobrowolski and Marciszewski. In the second one, we characterize universally measurable filters which fulfill Fatou's lemma.

A family F of subsets of ω is a *filter* if $x \cap y \in F$ for $x, y \in F$ and $x \in F$, $x \subseteq y$ implies $y \in F$. We assume that $\emptyset \notin F$ and $\omega \setminus \{n\} \in F$ for $n \in \omega$. A family I of subsets of ω is an *ideal* if $x, y \in I$ implies $x \cup y \in I$ and $x \subseteq y \in I$ implies $x \in I$. We always assume that $\omega \notin I$ and that $\{n\} \in I$ for any $n \in \omega$. For an ideal I we denote by I^* the *dual filter* of I, that is, $I^* = \{\omega \setminus x : x \in I\}$. Similarly if F is a filter, F^* denotes its dual ideal.

Terminology and notation concerning Polish spaces and Borel sets follow [K].

1. Separating ideals from their dual filters. In this section, we investigate the space of all real sequences which have limits with respect to a filter. For a filter F on ω, define C_F to be the space of all functions $f : \omega \to \mathbb{R}$ which have limit with respect to F, that is,

$$C_F = \{f \in \mathbb{R}^\omega : \exists r \in \mathbb{R}\ \forall \varepsilon > 0\ \{n : |f(n) - r| < \varepsilon\} \in F\}.$$

These and similar function spaces have been studied intensively in recent years from descriptive set theoretic and topological points of view (see, for example, the references in [DM]). It was proved in [DM, Propositions 3.4 and 3.3] that if F is Π_α^0, then C_F is the difference of two Π_α^0 sets, and that if F is Π_α^0-hard, then C_F is Π_α^0-hard, for $\alpha < \omega_1$. The authors asked if $F \in \Pi_\alpha^0$

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implies $C_F \in \Pi^0_\alpha$ [DM, Question 3.5]. For $\alpha = 3$ an affirmative answer to this descriptive set theoretic question is of topological importance since it was showed in [DM, Theorem 5.10] that all C_F with $C_F \in \Pi^0_3$ are homeomorphic to each other and to σ^ω where $\sigma = \{x \in \mathbb{R}^\omega : \exists n\ \forall m > n\ x_m = 0\}$, that is, the Borel complexity of C_F determines it up to homeomorphism in this case. We answer the Dobrowolski–Marciszewski question affirmatively for this case $\alpha = 3$. (I would like to point out a certain peculiarity of the situation here. Normally in estimating Borel complexity of sets it is establishing the lower bounds that causes most problems. This is reversed in the case of C_F: the lower bound (C_F Π^0_3-hard for F Π^0_3-hard) is relatively simple to establish, see [DM, Proposition 3.3], while the upper bound ($C_F \in \Pi^0_3$ for $F \in \Pi^0_3$) is much trickier.)

THEOREM 1.1. *For a Π^0_3 filter F, C_F is Π^0_3.*

Theorem 1.1 is an immediate consequence of Corollary 1.5 and Lemma 1.3 which will be proved later. Corollary 1.2 below follows from Theorem 1.1 and the already mentioned result of Dobrowolski and Marciszewski [DM, Theorem 5.10].

COROLLARY 1.2. *For a Π^0_3 filter F, C_F is homeomorphic to σ^ω where $\sigma = \{x \in \mathbb{R}^\omega : \exists n\ \forall m > n\ x_m = 0\}$.*

Now, we prove a result which reveals that the descriptive set theoretic complexity of C_F depends on the possibility of separating the ideal dual to F from F by a set which is simpler than it might be expected. For $A, B \subseteq 2^\omega$, we say that A is *separated from B by C* if $A \subseteq C$ and $C \cap B = \emptyset$. It is well known that if A and B are disjoint and both are Π^0_α, $\alpha < \omega_1$, then A can be separated from B by a Δ^0_α ($= \Sigma^0_\alpha \cap \Pi^0_\alpha$) set [K, 22.16]. If F is a Π^0_α filter, then F^* is Π^0_α as well, and F and F^* are disjoint, so they can be separated by a Δ^0_α set. Lemma 1.3 shows that if for this particular pair of disjoint Π^0_α sets one can improve the general result and separate F^* from F by a Σ^0_β set for some $\beta < \alpha$, then the complexity of C_F can be sharply determined.

LEMMA 1.3. *Let F be a Π^0_α filter. If the dual ideal of F can be separated from F by a Σ^0_β set for some $\beta < \alpha$, then $C_F \in \Pi^0_\alpha$.*

Proof. Let G be a Σ^0_β set, for some $\beta < \alpha$, separating F^* from F. Pick perfect, closed, zero-dimensional sets $C_n \subseteq [0,1]$ and continuous surjections $h_n : C_n \to [0,1]$ so that

- $C_n \subseteq C_{n+1}$;
- $\bigcup_n C_n$ dense in $[0,1]$;
- $\mathbb{Q} \cap \bigcup_n C_n = \{0,1\}$;
- $h_n^{-1}(0) = \{0\}$, $h_n^{-1}(1) = \{1\}$;

- h_n increasing;
- $h_{n+1}|C_n = \mathrm{Id}|C_n$.

This can be easily accomplished. Define $H : \prod_n C_n \to [0,1]^\omega$ by letting $H(f) = (h_n(f(n)))_n$. Then H is clearly a continuous surjection. By the properties of the C_n's and the h_n's listed above, we get

$$\Big\{f \in \prod_n C_n : \forall n\ f(n) \in (0,1) \text{ and } \lim_F f \text{ exists and belongs to } (0,1)\Big\}$$
$$= H^{-1}(\{f \in [0,1]^\omega : \forall n\ f(n) \in (0,1) \text{ and } \lim_F f \text{ exists and belongs to } (0,1)\}).$$

Note that the set inside $H^{-1}(\cdot)$ on the right hand side of the above equation is homeomorphic to C_F, so, by Saint Raymond's theorem [K, 24.20], it is enough to show that the set on the left hand side is $\varPi^0_\alpha$. But elements of this last set are precisely those $f \in \prod_n C_n$ which fulfill all the conditions listed below:

(i) $\forall n\ 0 < f(n) < 1$;
(ii) $\forall m\, \exists k \le m+1\ \{n : |f(n) - k/(m+1)| < 1/(m+1)\} \in F$;
(iii) $\exists m\ \{n : |f(n) - 0| < 1/(m+1)\} \in G$;
(iv) $\exists m\ \{n : |f(n) - 1| < 1/(m+1)\} \in G$.

Indeed, for any $f \in [0,1]^\omega$, condition (ii) is equivalent to the existence of $\lim_F f$. For any f for which $\lim_F f$ exists (that is, f fulfilling (ii)), (iii) and (iv) are equivalent to saying that $\lim_F f \ne 0$ and $\lim_F f \ne 1$.

Condition (i) is obviously $\varPi^0_2$. Note that for any $f \in \prod_n C_n$ fulfilling (i), the range of f is disjoint from $\mathbb{Q}$. Thus, for any $q, r \in \mathbb{Q}$ the function $f \mapsto \{n : |f(n) - q| < r\}$ is continuous on the set of all $f \in \prod_n C_n$ with (i). Hence, on the $\varPi^0_2$ set described by (i), condition (ii) is $\varPi^0_\alpha$ and conditions (iii) and (iv) are $\varSigma^0_\beta$.

Remark. It should be pointed out that two particular instances of the above lemma were proved, also using Saint Raymond's theorem, in [DM]: Proposition 3.4(a) ($F \in \varSigma^0_\beta$ implies $C_F \in \varPi^0_{\beta+1}$) and Corollary 3.7 (if $F_n \in \varSigma^0_{\beta(n)}$, $n \in \omega$, $\beta(n) < \alpha$, then $C_F \in \varPi^0_\alpha$ for the filter $F = \prod_n F_n$ on $\omega \times \omega$).

Now, passing to duals for convenience, we address the question of the possibility of separating an ideal I from I^*, $I \in \varPi^0_\alpha$, by a $\varSigma^0_\beta$ set for $\beta < \alpha$. This question seems to be of independent interest. We answer it for $\alpha = 3$. The following theorem gives a bit more than we actually need in applications.

Theorem 1.4. *Let I be an ideal. Assume I cannot be separated from I^* by a $\varSigma^0_2$ set. Then for any $\varPi^0_3$ sets $A \supseteq I$, $B \supseteq I^*$ there is a partition of ω, $\{x_n : n \in \omega\}$, into elements of I such that $\bigcup_{n\in a} x_n \in A \cap B$ for any infinite and coinfinite $a \subseteq \omega$.*

P r o o f. It will be enough to show that for any Π^0_3 set $A \supseteq I$ there exists a partition $\{x_n : n \in \omega\} \subseteq I$ of ω such that $\bigcup_{n\in a} x_n \in A$ for any $a \subseteq \omega$ infinite and coinfinite. Indeed, granting this, if $A \supseteq I$, $B \supseteq I^*$ are Π^0_3, let $A' = A \cap \{\omega \setminus x : x \in B\}$. Now find a partition $\{x_n : n \in \omega\}$ for A'. Then if $a \subseteq \omega$ is infinite and coinfinite, so is $\omega \setminus a$. Thus, $\bigcup_{n\in a} x_n, \bigcup_{n\in \omega\setminus a} x_n \in A'$. Now it follows immediately that $\bigcup_{n\in a} x_n \in A \cap B$.

Let F_n be Σ^0_2 and such that $A = \bigcap_n F_n$. We can assume that $F_{n+1} \subseteq F_n$.

Claim. *Let $y_0 \subseteq x_0 \in I$ and let $n \in \omega$. Then for some finite k_0 with $k_0 \cap x_0 = \emptyset$ and some $z_0 \in I^*$ with $z_0 \cap (k_0 \cup x_0) = \emptyset$ we have*

$$\forall z \subseteq z_0 \; y_0 \cup k_0 \cup z \in F_n.$$

P r o o f. Let

$$F'_n = \{x \in F_n : x \cap x_0 = y_0 \text{ and } \{z : z \subseteq x,\ z \cap x_0 = y_0\} \cap F_n \\ \text{is not meager in } \{z : z \subseteq x,\ z \cap x_0 = y_0\}\}.$$

Note the following three facts: $F'_n \subseteq F_n$, $\{x \in I : x \cap x_0 = y_0\} \subseteq F'_n$, and $F'_n \in \Sigma^0_2$. I check this last assertion more carefully. It is clearly enough to show that, for any Σ^0_2 set $F \subseteq 2^\omega$,

$$F' = \{x : \{z \subseteq x : z \in F\} \text{ is not meager in } \{z : z \subseteq x\}\}$$

is Σ^0_2. To see this define $f : 2^\omega \times 2^\omega \to 2^\omega$ by

$$n \in f(x,y) \Leftrightarrow n \text{ is the } k\text{th element of } x \text{ and } k-1 \in y.$$

Since f is continuous, $f^{-1}(F) \in \Sigma^0_2$. By a straightforward argument we check that

$$F' = \{x : \{y : (x,y) \in f^{-1}(F)\} \text{ is not meager}\}$$

so F' is Σ^0_2 by [K, 22.22]. (Here is another, perhaps more direct, way of seeing that F' is Σ^0_2 suggested by the referee. Let $F = \bigcup_n K_n$ with K_n closed. Then

$$\begin{aligned} x \in F' \Leftrightarrow\ & \exists n\ \exists k_0 \subseteq k_1 \subseteq \omega \text{ finite} \\ & \quad (k_1 \subseteq x \text{ and } \{z \subseteq x : z \cap k_1 = k_0\} \subseteq K_n) \\ \Leftrightarrow\ & \exists n\ \exists k_0 \subseteq k_1 \text{ finite } \forall z \text{ finite} \\ & \quad (k_1 \subseteq x \text{ and } ((z \cap k_1 = k_0 \text{ and } z \subseteq x) \Rightarrow z \in K_n)). \end{aligned}$$

Since the condition following the quantifiers in the last line is closed, F' is Σ^0_2.)

We now show that the supposition that $F'_n \cap I^* = \emptyset$ leads to the conclusion that I can be separated from I^* by a Σ^0_2 set, which contradicts our assumption. Note that the mapping $x \mapsto x \setminus x_0$ is a homeomorphism on F'_n; thus, $\{x \setminus x_0 : x \in F'_n\} \in \Sigma^0_2$ whence

$$F''_n = \{z \cup (x \setminus x_0) : z \subseteq x_0 \text{ and } x \in F'_n\} \in \Sigma^0_2.$$

We check that F''_n separates I from I^*. If $x \in I$, then $(x \setminus x_0) \cup y_0 \in I \subseteq F'_n$, whence

$$x = (x \cap x_0) \cup (((x \setminus x_0) \cup y_0) \setminus x_0) \in F''_n,$$

so $I \subseteq F''_n$. Assume towards a contradiction that $F''_n \cap I^* \neq \emptyset$, and let $y \in F''_n \cap I^*$. Let $z \subseteq x_0$, $x \in F'_n$ be such that $y = z \cup (x \setminus x_0)$. Then clearly $x \in I^*$, so $F'_n \cap I^* \neq \emptyset$, contradiction.

Suppose therefore that $F'_n \cap I^* \neq \emptyset$. Let $z_1 \in F'_n \cap I^*$. Since $\{z : z \subseteq z_1,\ z \cap x_0 = y_0,\ z \in F_n\}$ is nonmeager in $\{z : z \subseteq z_1 \text{ and } z \cap x_0 = y_0\}$ and is Σ^0_2, it must have nonempty interior in $\{z : z \subseteq z_1 \text{ and } z \cap x_0 = y_0\}$. So there exist finite sets k_0, k_1 such that $k_1 \cap x_0 = \emptyset$, $k_0 \subseteq k_1 \cap z_1$, and for any z with $z \subseteq z_1$, $z \cap x_0 = y_0$, and $z \cap k_1 = k_0$, we have $z \in F_n$. Let $z_0 = z_1 \setminus (x_0 \cup k_1)$. Now z_0 and k_0 are as required, which proves the claim.

Now, we recursively construct, for each $n \in \omega$, $y^n \in I$ and finite sets k^n such that for each n,

(o) $k^n \cap \bigcup_{i \le n} y^i = \emptyset$;

(i) $y^n \cap \bigcup_{i<n} y^i = \emptyset$;

(ii) $n \in \bigcup_{i \le n} y^i$;

(iii) $k^{n-1} \subseteq y^n$ for $n \ge 1$;

(iv) $\forall x \subseteq \omega$ if $y^n \cap x = \emptyset$, $k^n \subseteq x$, and $\forall i < n\ y^i \subseteq x$ or $y^i \cap x = \emptyset$, then $x \in F_n$.

Assume the construction has been carried out. Let $a \subseteq \omega$ be infinite and coinfinite. We show that $x = \bigcup_{n \in a} y^n$ is in A. Let n_i, $i \in \omega$, be such that $n_i \notin a$, $n_i + 1 \in a$, $n_i < n_{i+1}$. Note that by (i) and (iii), $k^n \subseteq y^{n+1} \setminus \bigcup_{i \le n} y^i$ for each n. Thus, $k^{n_i} \subseteq x$. Since all other assumptions from (iv) are clearly fulfilled for $n = n_i$, we get $x \in F_{n_i}$. Since this happens for each n_i, $x \in \bigcap_n F_n = A$. Note also that (i) and (ii) guarantee that $\{y^n : n \in \omega\}$ is a partition of ω.

Now, it remains to construct the y^n's and the k^n's. To find y^0 and k^0 apply the claim to $y_0 = \emptyset$, $x_0 = \{0\}$. We get k_0 and z_0. Let $y^0 = (\omega \setminus z_0) \setminus k_0$ and $k^0 = k_0$. Assume y^i, $i < n$, and k^{n-1} have been constructed. Let s_j, $j < 2^n$, enumerate all subsets of $n = \{0, 1, \ldots, n-1\}$. We now produce $z_j \in I^*$, $k^0_j \subseteq \omega$ finite, for $j < 2^n$, as follows: applying the claim to

$$x_0 = \bigcup_{i<n} y^i \cup k^{n-1} \cup \bigcup_{i<j} k^0_i \cup \{n\} \cup \bigcup_{i<j} \omega \setminus z_i$$

and

$$y_0 = \bigcup_{i \in s_j} y^i \cup \bigcup_{i<j} k^0_i,$$

we get $z_j \in I^*$ and $k_j^0 = k_0$. Now let

$$y^n = \Big(\Big(\omega \setminus \bigcup_{i<n} y^i\Big) \setminus \bigcap_{j<2^n} z_j\Big) \setminus \bigcup_{j<2^n} k_j^0$$

and $k^n = \bigcup_{j<2^n} k_j^0$.

This choice does the job. Conditions (o)–(iii) for n are evident and only the validity of (iv) may pose any doubt. So, let $x \subseteq \omega$ be such that $y^n \cap x = \emptyset$, $k^n \subseteq x$, and $x \cap y^i = \emptyset$ or $y^i \subseteq x$ for $i < n$. Let $s = \{i < n : y^i \subseteq x\}$. Then $s = s_{j_0}$ for some $j_0 < 2^n$ in our enumeration of 2^n. By definition of y^n and k^n, for some $z \subseteq \bigcap_{j<2^n} z_j$,

$$x = \bigcup_{i\in s_{j_0}} y^i \cup \bigcup_{j<2^n} k_j^0 \cup z = \Big(\bigcup_{i\in s_{j_0}} y^i \cup \bigcup_{j<j_0} k_j^0\Big) \cup k_{j_0}^0 \cup \Big(\bigcup_{j_0<j<2^n} k_j^0 \cup z\Big).$$

Since $\bigcup_{j_0<j<2^n} k_j^0 \cup z \subseteq z_{j_0}$, it follows that $x \in F_n$.

The following corollary is an immediate consequence of Theorem 1.4 and, in conjunction with Lemma 1.3, it implies Theorem 1.1. The second sentence of the corollary follows from the first one since two disjoint $\varPi_3^0$ sets can be separated by a $\varDelta_3^0$ set (see [K, 22.16]).

Corollary 1.5. *If I can be separated from I^* by a $\varDelta_3^0$ set, then it can be separated from I^* by a $\varSigma_2^0$ set. In particular, if I is $\varPi_3^0$, then it can be separated from I^* by a $\varSigma_2^0$ set.*

The above corollary may shed some light on a question of Mazur from [M]: is each $\varPi_3^0$ ideal I included in a hereditary $\varSigma_2^0$ set F such that $\{x \cup y : x, y \in F\}$ is meager? As is easy to see, this is equivalent to looking for a $\varSigma_2^0$ hereditary set F such that no set $x \cup y$ with $x, y \in F$ is cofinite. (By a *hereditary set* we mean a set of subsets of ω which is closed under taking subsets.) Proposition 1.6 below shows that separation of an ideal from its dual filter by a $\varSigma_2^0$ set implies something in this direction. Given a $\varPi_3^0$ ideal I, by letting L in this proposition be equal to the family of all subsets of ω whose complement has no more than n elements, we obtain a $\varSigma_2^0$ hereditary set F_n containing I such that $\{x \cup y : x, y \in F_n\}$ does not contain sets whose complement has $\le n$ elements. Note that this is a strengthening of separation: one easily checks that any set F containing I and such that $x \cup y \ne \omega$, for $x, y \in F$, is disjoint from I^*.

Proposition 1.6. *Assume an ideal I can be separated from I^* by a $\varSigma_2^0$ set. (So, by Corollary 1.5, $I \in \varPi_3^0$ is sufficient.) Let $L \subseteq I^*$ be compact. Then there is a $\varSigma_2^0$ hereditary set F such that $I \subseteq F$ and $\{x \cup y : x, y \in F\} \cap L = \emptyset$.*

Proof. Let F' be a Σ_2^0 set separating I from I^*. Then $F' = \bigcup_n L_n$ with L_n compact. Note that the set consisting of all subsets of elements of L_n is also compact, contains L_n and is disjoint from I^*. Thus, from the start we can assume that each L_n is hereditary. Without loss of generality we can also suppose that $L_n \subseteq L_{n+1}$ for each n. We can also assume that L is upwards closed. Note that for any n, $\{(\omega\setminus x)\cap z : x \in L_n, z \in L\}$ is compact being the image of $L_n \times L$ by a continuous function. Let

$$K_n = L_n \setminus \{(\omega\setminus x)\cap z : x \in L_n,\ z \in L\}.$$

Each K_n is a hereditary Σ_2^0 set. Moreover, it follows from $L \subseteq I^*$ that $K_n \cap I = L_n \cap I$. Thus, $I \subseteq \bigcup_n K_n$. Put $F = \bigcup_n K_n$. It remains to check that if $x, y \in F$, then $x \cup y \notin L$. Assume otherwise and fix $x_0, y_0 \in F$ with $x_0 \cup y_0 = z_0 \in L$. Let n_0 be the smallest natural number with $x_0 \in L_{n_0}$ or $y_0 \in L_{n_0}$. Say $x_0 \in L_{n_0}$. Since L_{n_0} is hereditary and L is upwards closed, by making x_0 smaller and z_0 bigger if necessary, we can suppose that

$$y_0 = (\omega\setminus x_0)\cap z_0 \quad \text{for some } z_0 \in L. \tag{1}$$

By definition of n_0, $y_0 \notin L_n$ for $n < n_0$, so $y_0 \notin K_n$ with $n < n_0$. Also $x_0 \in L_n$ for all $n \geq n_0$ whence, by (1), $y_0 \in \{(\omega\setminus x)\cap z : x \in L_n,\ z \in L\}$ for $n \geq n_0$. Therefore, $y_0 \notin K_n$ for $n \geq n_0$. It follows that $y \notin \bigcup_n K_n = F$, contradiction.

Example 1.7. It is not true that any Borel ideal can be separated from its dual filter by a Σ_2^0 set. Let $I = \{x \subseteq \omega\times\omega : \exists n\ \forall m \geq n\ \exists k\ \{i : (m,i) \in x\} \subseteq k\}$. I is a Borel (actually Σ_4^0) ideal. It is generated by vertical "lines" in $\omega\times\omega$ and the subgraphs of functions from ω to ω. Assume a Σ_2^0 set F separates I from I^*. By the Baire category theorem, there exists $s \in \omega^{m_0}$, for some $m_0 \in \omega$, such that $\{\tilde f : s \subseteq f \in \omega^\omega\}$ is included in a closed subset of F, where $\tilde f = \{(i,j) \in \omega\times\omega : j \leq f(i)\}$ is the subgraph of f. Thus, $s \cup (\omega\setminus m_0)\times\omega \in F$ but clearly $s \cup (\omega\setminus m_0)\times\omega \in I^*$, contradiction.

2. Filters fulfilling Fatou's lemma. Investigating certain Borel equivalence relations, Kechris defined a class of filters which led to considering the interesting family of filters fulfilling Fatou's lemma. The question of whether a concrete filter has this property arose already in a much earlier paper by Louveau [L]. Below we give a characterization of such universally measurable filters and show that this class of filters is determined by one filter. Namely, there exists a filter F_0 with the property that a universally measurable filter fulfills Fatou's lemma precisely when it does not "locally contain" F_0.

We denote by λ the product measure on $2^\omega = \{0,1\}^\omega$ obtained from measures assigning equal weight $1/2$ to both 0 and 1 on each coordinate.

λ is sometimes called *Lebesgue measure* on 2^ω. A subset A of a Polish space X is *universally measurable* if it is measurable with respect to any probability Borel measure on X. Note that if $A \subseteq X$ is universally measurable, then for any Borel $f : 2^\omega \to X$, $f^{-1}(A)$ is λ-measurable. (This holds since λ-measurability of $f^{-1}(A)$ is equivalent to μ-measurability of A where μ is the measure obtained by transferring λ to X via f.)

Let Ω consist of all clopen subsets U of 2^ω with $\lambda(U) = 1/2$. Let I_0 be the ideal of subsets of Ω generated by all $x \subseteq \Omega$ with $\bigcap x \neq \emptyset$. One easily checks that I_0 is a proper ideal. Let F_0 be the filter dual to I_0. That is, as is easy to see, F_0 is the family of all $x \subseteq \Omega$ such that for some finite $f \subseteq 2^\omega$, $\{U \in \Omega : f \cap U = \emptyset\} \subseteq x$. ($I_0$ and F_0 are not literally families of sets of natural numbers, but since Ω is countable, we may identify it with ω and then transfer I_0 and F_0 to ω.)

Recall that for a sequence (a_n) of real numbers and a filter G on ω, $\liminf_G a_n = \sup\{r \in \mathbb{R} : \{n \in \omega : a_n < r\} \in G^*\}$ with the understanding that $\sup \emptyset = -\infty$.

Let (X, μ) be a σ-finite measure space with μ defined on some σ-algebra of subsets of X. Let $f_n : X \to [0, \infty)$ be μ-measurable and let G be a filter. We say that *Fatou's lemma holds* on this sequence with respect to G if

$$\underline{\int} \liminf_G f_n \, d\mu \leq \liminf_G \int f_n \, d\mu.$$

(By $\underline{\int}$ we understand the lower integral, so $\underline{\int} g \, d\mu$, for $g \geq 0$, stands for $\sup\{\int f \, d\mu : f \leq g \text{ and } f \ \mu\text{-measurable}\}$.) Given a filter G on ω, we say that Fatou's lemma holds for G if it holds with respect to G for any sequence (f_n) on any σ-finite measure space as above.

Note that many filters fulfill Fatou's lemma. For instance, the Fréchet filter consisting of all cofinite sets (this is simply the classical Fatou's lemma) or the density filter consisting of all subsets of ω with density 1. A most interesting example of such a filter was found by Louveau in [L]. He defines there a coanalytic filter which is in a sense the ultimate extension of the Fréchet filter: it has the property that each Borel function on a Polish space can be obtained as a limit of a sequence of continuous functions with respect to this filter and, on the other hand, only Borel functions are obtainable as such limits. By [L, Lemme 4], this filter fulfills Fatou's lemma. (Actually, what is proved there is that it fulfills Fatou's lemma on sequences of bounded functions defined on a compact metric space and with respect to a Borel probability measure. However, it follows from the arguments below that this is enough: by the statement of Theorem 2.1 and the first part of its proof, if a filter fails to fulfill Fatou's lemma, it fails to fulfill it with respect to a bounded sequence of continuous functions on 2^ω with respect to Lebesgue measure.)

Let F and G be filters on ω. Define

$$F \preceq G \quad \text{if } \{\phi^{-1}(x) : x \in F\} \subseteq G \text{ for some } \phi : \omega \to \omega.$$

Define also

$$F \sqsubseteq G \quad \text{if } \{\phi^{-1}(x) : x \in F\} \subseteq G \text{ for some bijection } \phi : \omega \to \omega.$$

For $A \subseteq \omega$, let $G|A$ stand for the filter $\{y \cap A : y \in G\}$ on A. We write $F \preceq_{\mathrm{loc}} G$ if $F \preceq G|A$ for some $A \notin G^*$, and $F \sqsubseteq_{\mathrm{loc}} G$ if $F \sqsubseteq G|A$ for some $A \notin G^*$. Obviously, $F \sqsubseteq G$ implies $F \preceq G$ and so $F \sqsubseteq_{\mathrm{loc}} G$ implies $F \preceq_{\mathrm{loc}} G$.

Recall also for later discussion that for two filters F, G, F is below G in the *Rudin–Keisler order*, $F \leq_{\mathrm{RK}} G$, if for some $\phi : \omega \to \omega$, $x \in F$ if and only if $\phi^{-1}(x) \in G$. So $\preceq$ above is, in a sense, one half of $\leq_{\mathrm{RK}}$.

THEOREM 2.1. *Let G be a universally measurable filter. Then the following are equivalent*:

(i) *G fails to fulfill Fatou's lemma*;
(ii) *$F_0 \preceq_{\mathrm{loc}} G$*;
(iii) *$F_0 \sqsubseteq_{\mathrm{loc}} G$.*

Proof. Since obviously (iii)⇒(ii), only (ii)⇒(i) and (i)⇒(iii) need proving. Assume first that $F_0 \preceq_{\mathrm{loc}} G$. We need to see that Fatou's lemma fails for G.

CLAIM. *If for some (X,μ) and some sequence $f_n : X \to [0,\infty)$, $n \in \omega$, of μ-measurable functions on a σ-finite measure space we have $\underline{\int} \liminf_F f_n \, d\mu > \limsup_F \int f_n \, d\mu$, and $F \preceq_{\mathrm{loc}} G$, then G does not fulfill Fatou's lemma.*

Proof. Fix $A \notin G^*$ and $\phi : A \to \omega$ witnessing $F \preceq_{\mathrm{loc}} G$. Define f_n' to be the constant function equal to n if $n \notin A$ and $f_{\phi(n)}$ if $n \in A$. Then one checks that

$$\liminf_G f_n' \geq \liminf_F f_n \quad \text{and} \quad \limsup_F \int f_n \, d\mu \geq \liminf_G \int f_n' \, d\mu,$$

which implies $\underline{\int} \liminf_G f_n' \, d\mu > \liminf_G \int f_n' \, d\mu$, proving the claim.

It now suffices to show that F_0 satisfies the assumption of the claim for $(2^\omega, \lambda)$. For $U \in \Omega$ let $f_U = \chi_{2^\omega \setminus U}$. Then, since $\lambda(U) = 1/2$ for each relevant U, $\limsup_{F_0} \int f_U \, d\lambda = 1/2$. On the other hand, $\liminf_{F_0} f_U = 1$ since for any $x \in 2^\omega$,

$$\{U \in \Omega : f_U(x) = 1\} = \{U \in \Omega : x \notin U\} \in F_0.$$

Thus, $\underline{\int} \liminf_{F_0} f_U \, d\lambda = 1$.

Now, we show that if G violates Fatou's lemma, then $F_0 \sqsubseteq_{\mathrm{loc}} G$. So assume that for some sequence (f_n) of μ-measurable functions with $f_n \geq 0$ on a σ-finite measure space (X,μ), we have

$$\underline{\int} \liminf_G f_n \, d\mu > \liminf_G \int f_n \, d\mu.$$

Let f be a μ-measurable function with $f \geq 0$, $f \leq \liminf_G f_n$, and

$$\infty > \int f\,d\mu > \liminf_G \int f_n\,d\mu.$$

Consider the set

$$\{(x,t) : 0 \leq t < f(x)\} \subseteq X \times [0,\infty)$$

with the measure ν which is the restriction to this set of the product of μ and Lebesgue measure on $[0,\infty)$. Let $B_n = \{(x,t) : f_n(x) \leq t \leq f(x)\}$. By Fubini's theorem,

$$\limsup_G \nu(B_n) \geq \int f\,d\mu - \liminf_G \int f_n\,d\mu = \delta > 0.$$

Moreover, since $\liminf_G f_n \geq f$, we see that $\limsup_G \chi_{B_n} = 0$. Note that ν is atomless and finite. Now combining the theorems of Carathéodory and Sikorski (see [R, p. 399, Theorem 4, and p. 397, Proposition 3]), and taking into account that 2^ω with λ is isomorphic to $[0,1]$ with Lebesgue measure (see e.g. [R, p. 409, Theorem 16]), we get the following fact: Let (Z,ν) be a probability atomless measure space and let $A_n \subseteq Z$, $n \in \omega$, be ν-measurable. Then there exists $\phi : Z \to 2^\omega$ ν-measurable and such that for any $B \subseteq 2^\omega$ Borel, $\nu(\phi^{-1}(B)) = \lambda(B)$ and for some $A'_n \subseteq 2^\omega$ Borel, $\nu(\phi^{-1}(A'_n) \triangle A_n) = 0$. ($A \triangle B$ stands for the symmetric difference $(A \setminus B)$ $\cup$ $(B \setminus A)$.) Applying this fact to our sets B_n and the measure ν, we can assume that we have Borel sets $B_n \subseteq 2^\omega$ with $\limsup_G \lambda(B_n) \geq \delta' > 0$ and $\lambda^*(\{x \in 2^\omega : \limsup_G \chi_{B_n}(x) = 0\}) = 1$. ($\lambda^*$ is the outer Lebesgue measure.) Since G is universally measurable and the mapping $f : 2^\omega \to 2^\omega$ defined by $f(x) = \chi_{\{n : x \notin B_n\}}$ is Borel, we see that

$$\{x \in 2^\omega : \limsup_G \chi_{B_n}(x) = 0\} = f^{-1}(G)$$

is λ-measurable, so actually $\limsup_G \chi_{B_n} = 0$ λ-almost everywhere.

Now pick $k \in \omega$ with $(1-\delta')^k < 1/2$. Consider $(2^\omega)^k$ with the measure λ^k and Borel sets

$$B'_n = \bigcup_{i=1}^{k} 2^\omega \times \ldots \times B_n \times \ldots \times 2^\omega$$

with B_n standing at the ith place in the product. Then $\lambda^k(B'_n) = 1 - (1 - \lambda(B_n))^k$ whence

$$\limsup_G \lambda^k(B'_n) \geq 1 - (1-\delta')^k > 1/2. \tag{1}$$

Note also that if $x \in (2^\omega)^k$, $x = (x_i)_{i=1}^k$, is such that $\limsup_G \chi_{B'_n}(x) \neq 0$, then $\{n : x \in B'_n\} \notin G^*$. Since $\{n : x \in B'_n\} = \bigcup_{i=1}^k \{n : x_i \in B_n\}$, we find that for some i, $\{n : x_i \in B_n\} \notin G^*$, so $\limsup_G \chi_{B_n}(x_i) \neq 0$. It follows that

$$\{x \in (2^\omega)^k : \limsup_G \chi_{B'_n}(x) \neq 0\}$$
$$\subseteq \bigcup_{i=1}^{k} 2^\omega \times \ldots \times \{x \in 2^\omega : \limsup_G \chi_{B_n}(x) \neq 0\} \times \ldots \times 2^\omega$$

where $\{x \in 2^\omega : \limsup_G \chi_{B_n}(x) \neq 0\}$ is the ith term in the product. Now from Fubini's theorem we get

$$\lambda^k(\{x \in (2^\omega)^k : \limsup_G \chi_{B'_n}(x) \neq 0\}) = 0. \tag{2}$$

Note, however, that the measure spaces $(2^\omega, \lambda)$ and $((2^\omega)^k, \lambda^k)$ are isomorphic via a homeomorphism between 2^ω and $(2^\omega)^k$, so we can assume by (1) and (2) that we have a sequence of Borel sets $B_n \subseteq 2^\omega$ such that

$$\limsup_G \lambda(B_n) > \frac{1}{2} \quad \text{and} \quad \limsup_G \chi_{B_n} = 0 \ \lambda\text{-almost everywhere.}$$

The same will hold for a sequence of clopen sets $V_n \subseteq 2^\omega$, $n \in \omega$, if we only make sure that $\lambda(B_n \triangle V_n) < 2^{-n}$, which can be done easily. This allows us to find clopen sets $W_n \subseteq 2^\omega$ with $W_n \subseteq W_{n+1}$ and

$$\lambda(W_n) < \frac{1}{2}\Big(\limsup_G \lambda(V_n) - \frac{1}{2}\Big) \quad \text{and} \quad \{x : \limsup_G \chi_{V_n}(x) \neq 0\} \subseteq \bigcup_n W_n.$$

Letting $Z_n = V_n \setminus W_n$, we get

$$\limsup_G \lambda(Z_n) > \frac{1}{2} \quad \text{and} \quad \limsup_G \chi_{Z_n}(x) = 0 \text{ for each } x.$$

Now, let $A = \{n : \lambda(Z_n) > 1/2\}$. Note that $A \notin G^*$. For each $n \in A$, Z_n contains infinitely many distinct clopen sets U with $\lambda(U) = 1/2$, that is, infinitely many distinct members of $\varOmega$. This allows us to pick $U_n \subseteq Z_n$ clopen with $U_n \in \varOmega$ and $U_n \neq U_m$ for $n \neq m$. Define $\phi : A \to \varOmega$ by putting $\phi(n) = U_n$. This ϕ is 1-to-1 and witnesses that $F_0 \preceq_{\mathrm{loc}} G$. We only need to show that if $x \subseteq \varOmega$ and $\bigcap x \neq \emptyset$, then $\phi^{-1}(x) \in G^*$. To see this, pick $\alpha \in \bigcap x$. Then $\limsup_G \chi_{U_n}(\alpha) = 0$ whence $G^* \ni \{n : \alpha \in U_n\} \supseteq \phi^{-1}(x)$ and we are done.

To make the function ϕ onto, note first that each infinite subset of $\varOmega$ contains an infinite subset from F_0^*. Indeed, if $x = \{V_n : n \in \omega\} \subseteq \varOmega$ is infinite, say $V_n \neq V_m$ if $n \neq m$, then

$$\lambda\Big(\bigcap_m \bigcup_{n>m} V_n\Big) = \lim_m \lambda\Big(\bigcup_{n>m} V_n\Big) \geq \frac{1}{2}.$$

In particular, for some α_0, $\{V \in x : \alpha_0 \in V\}$ is infinite, and obviously

$\{V \in x : \alpha_0 \in V\} \in F_0^*$. Pick $x \subset \phi[A]$ infinite and in F_0^*. Then $y = \phi^{-1}(x)$ is infinite and in G^*. Let ϕ' be equal to ϕ on $A \setminus y$ and let it map y bijectively onto $x \cup (\Omega \setminus \phi[\omega])$. This ϕ' still witnesses $F_0 \preceq_{\text{loc}} G$ and is 1-to-1 and onto, so $F_0 \sqsubseteq_{\text{loc}} G$.

Remark. One easily checks that $\preceq$ and $\sqsubseteq$ are transitive. It turns out, however, that the relations $\preceq_{\text{loc}}$ and $\sqsubseteq_{\text{loc}}$ are not. In particular, it does not follow from Theorem 2.1 that if F does not fulfill Fatou's lemma and $F \sqsubseteq_{\text{loc}} G$, then neither does G. In fact, it is false. (However, by Theorem 2.1, this is true for $F = F_0$. The reason is that F_0 fulfills a stronger condition stated in the claim in the proof of Theorem 2.1. See also (iv)–(vi) in the second part of this remark.) The example below illustrates the above statements.

Take two copies of ω and let F restricted to one of them be F_0 and F restricted to the other one be the Fréchet filter of all cofinite sets. Let G contain all subsets of the first copy of ω and only cofinite subsets of the second copy. Now note the following points.

(i) $F|A$, for some $A \notin F^*$, is isomorphic to F_0. In particular, $F_0 \sqsubseteq_{\text{loc}} F$.
(ii) $F \subseteq G$. Thus, $F \sqsubseteq_{\text{loc}} G$.
(iii) $F_0 \not\preceq_{\text{loc}} G$.

Points (i) and (ii) being obvious, only (iii) needs justification. Assume towards a contradiction that $\phi : A \to \Omega$, $A \notin G^*$, witnesses $F_0 \preceq_{\text{loc}} G$. Note that there exists $B \subseteq A$ with $A \setminus B \in G^*$ such that $G|B$ consists of cofinite subsets of B. Then $\phi[B]$ is infinite since otherwise A would be in G^*. By the last paragraph of the proof of Theorem 2.1, there exists $x \subseteq \phi[B]$ infinite with $x \in F_0^*$. Then $\phi^{-1}(x)$ is in G^* and has infinite intersection with B, contradiction.

It follows from (i)–(iii) above that neither $\sqsubseteq_{\text{loc}}$ nor $\preceq_{\text{loc}}$ are transitive. Also, using Theorem 2.1, we deduce that G fulfills Fatou's lemma by (iii) while F fails it by (i) even though $F \sqsubseteq_{\text{loc}} G$ by (ii).

Points (iv)–(vi) in the statement below complement the above example by showing that $\preceq_{\text{loc}}$ has weak forms of transitivity and that if $F \leq_{\text{RK}} G|A$ for some $A \notin G^*$, then G fails Fatou's lemma if F does.

Let F, G, and H be filters.

(iv) If $F \preceq_{\text{loc}} G$, $G \leq_{\text{RK}} H|A$ for some $A \notin H^*$, then $F \preceq_{\text{loc}} H$.
(v) If $F \preceq G$ and $G \preceq_{\text{loc}} H$, then $F \preceq_{\text{loc}} H$.
(vi) If F fails Fatou's lemma and $F \leq_{\text{RK}} G|A$ for some $A \notin G^*$, then G does not fulfill Fatou's lemma either.

We leave verifying (iv) and (v) to the reader. Point (vi) follows immediately from (iv) and Theorem 2.1 if we notice that in the proof of the implication (ii)⇒(i) of Theorem 2.1 we did not use the assumption that G is universally measurable.

EXAMPLE 2.3. The following is a simple and interesting example of a filter not fulfilling Fatou's lemma. Consider the full graph on a countable infinite set. ("Full" means that each pair of vertices is joined by an edge.) The set underlying the filter is the set of all edges. A subset of edges x is in the ideal dual to our filter if the graph spanned by x can be colored by finitely many colors. (This means that each vertex can be assigned a color out of finitely many possible colors in such a way that two vertices joined by an edge are assigned different colors.) For a set x, let $[x]^2$ be the set of all two-element subsets of x. The above filter is isomorphic to the filter F on $[\omega]^2$ generated by the family $\{[x]^2 \cup [\omega \setminus x]^2 : x \subseteq \omega\}$. (This follows from two facts which are not difficult to verify. First, if the graph spanned by a set y of edges can be colored by $\leq 2^n$ colors, then y is the union of $\leq n$ sets each of which spans a graph that can be colored with 2 colors. Second, if we interpret $[\omega]^2$ as the set of all edges, then subsets of $[\omega]^2$ spanning graphs which are 2-colorable are precisely those which can be covered by sets of the form $[\omega]^2 \setminus ([x]^2 \cup [\omega \setminus x]^2)$ for some $x \subseteq \omega$.) To see that F does not fulfill Fatou's lemma, let $\phi : [\omega]^2 \to \Omega$ be defined by

$$\phi(\{n, m\}) = \{\alpha \in 2^\omega : \alpha(n) \neq \alpha(m)\}.$$

This ϕ witnesses that $F_0 \preceq_{\mathrm{loc}} F$. Indeed, if $x \subseteq \Omega$ and $\bigcap x \neq \emptyset$, pick $\alpha_0 \in \bigcap x$. Then

$$\begin{aligned}\phi^{-1}(x) &\subseteq \{\{n, m\} : \alpha_0(n) \neq \alpha_0(m)\} \\ &= [\omega]^2 \setminus ([\{n : \alpha_0(n) = 0\}]^2 \cup [\{n : \alpha_0(n) = 1\}]^2) \in F^*.\end{aligned}$$

The following question is open. Let F be the filter from the above example. Is it true that G fails Fatou's lemma iff $F \preceq_{\mathrm{loc}} G$, for a universally measurable G? (The implication from right to left is true since F fulfills the assumption of the claim stated at the beginning of the proof of Theorem 2.1.)

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