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A converse of the Arsenin–Kunugui theorem on Borel sets with σ-compact sections

by

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Abstract. Let f be a Borel measurable mapping of a Luzin (i.e. absolute Borel metric) space L onto a metric space M such that $f(F)$ is a Borel subset of M if F is closed in L. We show that then $f^{-1}(y)$ is a $\boldsymbol{K}_\sigma$ set for all except countably many $y \in M$, that M is also Luzin, and that the Borel classes of the sets $f(F)$, F closed in L, are bounded by a fixed countable ordinal. This gives a converse of the classical theorem of Arsenin and Kunugui. As a particular case we get Taĭmanov's theorem saying that the image of a Luzin space under a closed continuous mapping is a Luzin space. The method is based on a parametrized version of a Hurewicz type theorem and on the use of the Jankov–von Neumann selection theorem.

1. Introduction. Luzin's classical result says that a Borel measurable mapping $f : L \to Y$ of a Borel subset L of a Polish space X to a Polish space Y is *Borel bimeasurable* (i.e. $f(B)$ is Borel for every Borel subset B of L) if $f^{-1}(y)$ is at most countable for each $y \in Y$. Let us remark that moreover the graph of f is the countable union of graphs of one-to-one Borel measurable mappings [2, Theorem 18.10].

R. Purves [7] showed that a Borel measurable $f : L \to Y$, with L and Y as above, is Borel bimeasurable if and only if the set $\{y \in Y : f^{-1}(y) \text{ is uncountable}\}$ is at most countable. In the final remark below we give a slightly stronger formulation of this theorem.

The classical Arsenin–Kunugui theorem says that the projection to Y of a Borel subset B of the product $X \times Y$ of two Polish spaces is a Borel subset of Y if the *sections* of B defined by $B^y = \{x \in X : (x, y) \in B\}$ are $\boldsymbol{K}_\sigma$ for all $y \in Y$. Considering the projection of the graph, it follows that, if

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$f : L \to Y$ is Borel measurable with L and Y as above, then the set $f(L)$ is Borel in Y whenever $f^{-1}(y)$ is $\boldsymbol{K}_\sigma$ for each $y \in Y$. Let us remark that also this theorem has a strengthening analogous to the result mentioned above. J. Saint-Raymond [8] showed that the graph of such an f is the union of countably many graphs of Borel measurable mappings with compact fibers. Then the result of Arsenin and Kunugui follows from a well known result of Novikov about projections of Borel sets with compact sections.

In this paper we prove the following converse of the Arsenin–Kunugui theorem that is similar to the converse of Luzin's theorem given by Purves.

Main Theorem. *Let $f : L \to Y$ be Borel measurable, L be a Luzin space, and Y be a metric space. Then the following statements are equivalent.*

(a) *For every closed set $F \subset L$ the set $f(F)$ is Borel in $f(L)$.*
(b) *The set $S = \{y \in Y : f^{-1}(y)$ is not $\boldsymbol{K}_\sigma\}$ is at most countable.*

The proof is based on Lemma 1, where we give a construction of a parametrized family of closed homeomorphic copies of the space of irrationals in sections of an analytic set A with non-$\boldsymbol{K}_\sigma$ sections. The formulation of Lemma 1 was inspired by proofs of Purves' theorem ([7], [5], [9], [6]). We would like to point out that Roman Pol showed us another possibility to prove Lemma 1 by an elegant reduction to a Hurewicz type theorem (cf. [2, Theorem 21.22]). We guess that his proof is of comparable difficulty with that using infinite games below.

2. Some needed notions and facts. For standard notions and results of classical descriptive set theory, we refer the reader to [1], [2], [3], [9], etc. However let us recall the following crucial notions.

A metric space L is called a *Luzin space* if there is a continuous (or equivalently Borel measurable) bijection $f : F \to L$, where F is a closed subset of the space $\mathbb{N}^{\mathbb{N}}$. It is well known that L is Luzin if and only if it is a Borel subset of some (or all) Polish spaces in which it is embedded.

A metric space A is called *analytic* if it is empty or if there is a continuous surjection of $\mathbb{N}^{\mathbb{N}}$ onto A. An analytic subspace of a metric space is called an analytic set.

We introduce some more notation. We use the symbol $M^{<\mathbb{N}}$ for the set of finite sequences (including the empty one) of elements of a set M. Further, $|s|$ denotes the *length* of s which is k if $s \in M^k$ or zero if $s = \emptyset$. We recall that $s \subset t$ denotes that $s \in M^{<\mathbb{N}}$ is an initial segment of $t \in M^{<\mathbb{N}}$. If $s \in M^{<\mathbb{N}}$, $k \in \mathbb{N}$ and $|s| \geq k$, then we sometimes write $s|k$ instead of $(s_1, \ldots, s_k)$. We use $s^\wedge t$ to denote the concatenation of s and t, where $s, t \in M^{<\mathbb{N}}$. We also use $\mathcal{N}(s)$ for the set $\{\nu \in M^{\mathbb{N}} : (\nu_1, \ldots, \nu_{|s|}) = s\}$ if $s \in M^{<\mathbb{N}}$, when speaking about $M^{\mathbb{N}}$.

We write $X \cong Y$ if the metric spaces X and Y are homeomorphic.

We say that a set $S \subset Z$ can be $\boldsymbol{K}_\sigma$ *separated from a set* $T \subset Z$ in a metric space Z if there is a $\boldsymbol{K}_\sigma$ subset K of Z such that $S \subset K$ and $K \cap T = \emptyset$.

We need

Jankov–von Neumann Selection Theorem (see e.g. [9, Theorem 5.5.2]). *Let $A \subset X \times Y$ be an analytic subset of the product of two Polish spaces, X and Y. Let $p_Y(A)$ be the image of A under the projection p_Y of $X \times Y$ to Y. Then there is a mapping $f : p_Y(A) \to X$ such that $(f(y), y) \in A$ for $y \in p_Y(A)$ and the preimage of every open subset of X belongs to the smallest σ-algebra generated by analytic sets in Y. In particular, f has the Baire property* (see [3, Chapter 2, Section 32, I] for the definition).

The key results of Arsenin and Kunugui that we are investigating in this note are summarized in the following theorem.

Arsenin–Kunugui Theorem (cf. [1, Chapter III, 32, Remark (b)]). *Let X and Y be Polish spaces and $B \subset X \times Y$ be a Borel set. Then*

(a) $\{y \in Y : B^y$ *is not* $\boldsymbol{K}_\sigma\}$ *is analytic and*
(b) $\{y \in Y : B^y$ *is not* $\boldsymbol{K}_\sigma$ *or* B^y *is empty*$\}$ *is analytic.*

In particular, the projection of B to Y is Borel in Y if B^y is $\boldsymbol{K}_\sigma$ for every $y \in Y$.

Considering the Borel graph of f and its projection to Y, we may restate the very last claim as follows. Let $f : L \to Y$ be a Borel measurable mapping of a Luzin space L such that $f^{-1}(y)$ is $\boldsymbol{K}_\sigma$ for every $y \in Y$. Then $f(L)$ is a Borel subset of Y.

3. Proof of Main Theorem. We use the notation $P_{\mathrm{f}} = \{x \in \{0,1\}^{\mathbb{N}} : x^{-1}(1)$ is finite$\}$, and $P_\infty = \{0,1\}^{\mathbb{N}} \setminus P_{\mathrm{f}}$.

We are going to use a result of Louveau and Saint-Raymond [4] on infinite games.

Louveau–Saint-Raymond game $\mathrm{LSR}(E_0, E_1)$. Let E_0 and E_1 be closed subsets of $\{0,1\}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ whose projections, A and B respectively, along $\mathbb{N}^{\mathbb{N}}$ are disjoint analytic subsets of $\{0,1\}^{\mathbb{N}}$.

Player I plays $\varepsilon_n \in \{0,1\}$ and player II plays $(\alpha_n, \beta_n) \in \{0,1\} \times \mathbb{N}$ in their nth move. The payoff set $V(E_0, E_1)$ for player II is defined by

$$\begin{aligned} V(E_0, E_1) = \{((\varepsilon_n, (\alpha_n, \beta_n)))_{n=1}^{\infty} \in (\{0,1\} \times (\{0,1\} \times \mathbb{N}))^{\mathbb{N}} :& \\ (\forall k \in \mathbb{N})\ \mathcal{N}(r((\varepsilon_1, \ldots, \varepsilon_k), ((\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)))) \cap E_{i(\varepsilon_1, \ldots, \varepsilon_k)} \neq \emptyset\}&, \end{aligned}$$

where r maps each $((\varepsilon_1, \ldots, \varepsilon_k), ((\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)))$, $k \in \mathbb{N}$, to a finite sequence of elements from $\{0,1\} \times \mathbb{N}$, and i maps $\{0,1\}^{<\mathbb{N}}$ to $\{0,1\}$. The mappings r and i do not depend on the choice of E_0, E_1 (and thus also on the choice of A, B). We shall not need the concrete way how r and i are defined.

Now we are going to state a special case of [4, Section 1, Theorem 1] that is proved in Section 2.1 of [4]. We do not use exactly the same formulations as Louveau and Saint-Raymond.

We say that the strategy σ for player II in that game *induces a mapping* $\tau : \{0,1\}^{\mathbb{N}} \to (\{0,1\} \times \mathbb{N})^{\mathbb{N}}$ if $\tau(\varepsilon) = ((\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots)$ whenever $(\varepsilon_1, (\alpha_1, \beta_1), \varepsilon_2, (\alpha_2, \beta_2), \ldots)$ is a run of the game when I played $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ and II followed the strategy σ. This is a continuous (in fact 1-Lipschitz) mapping with respect to the standard product metrics.

Louveau–Saint-Raymond Theorem ([4, Section 2.1]). *Let E_0 and E_1 be closed subsets of $\{0,1\}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ whose projections, A and B respectively, along $\mathbb{N}^{\mathbb{N}}$ are disjoint analytic subsets of $\{0,1\}^{\mathbb{N}}$. If A cannot be $\boldsymbol{K}_\sigma$ separated from B in $\{0,1\}^{\mathbb{N}}$, then player II has a winning strategy σ that induces a mapping $\tau : \varepsilon \in \{0,1\}^{\mathbb{N}} \mapsto ((\alpha_n, \beta_n))_{n=1}^{\infty} \in (\{0,1\} \times \mathbb{N})^{\mathbb{N}}$ such that projecting $\tau(\varepsilon)$ to $\{0,1\}^{\mathbb{N}}$ along the β-coordinates, we get a continuous mapping $\tau^* : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ with $\tau^*(P_\infty) \subset A$ and $\tau^*(P_{\mathrm{f}}) \subset B$.*

Let $\mathcal{T}_{\mathrm{II}}$ denote the set of all mappings induced by all (even not winning) strategies for II as above. The elements τ of $\mathcal{T}_{\mathrm{II}}$ are in one-to-one correspondence with the mappings $\overline{\tau} \in ((\{0,1\} \times \mathbb{N})^{<\mathbb{N}})^{\{0,1\}^{<\mathbb{N}}}$ defined by $\overline{\tau}(e_1, \ldots, e_n) = ((a_1, b_1), \ldots, (a_n, b_n))$ if $\tau(\varepsilon) = ((\alpha_n, \beta_n))_{n=1}^{\infty}$, where $\varepsilon \in \{0,1\}^{\mathbb{N}}$, $\alpha = (\alpha_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ and $\beta = (\beta_n)_{n=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ are arbitrary such that $\varepsilon|n = (e_1, \ldots, e_n)$, $\alpha|n = (a_1, \ldots, a_n)$, and $\beta|n = (b_1, \ldots, b_n)$. We use $\overline{\mathcal{T}}_{\mathrm{II}}$ to denote the set of all such mappings $\overline{\tau}$.

We consider the topology of uniform convergence on $\mathcal{T}_{\mathrm{II}}$ and the topology induced by the product topology from $((\{0,1\} \times \mathbb{N})^{<\mathbb{N}})^{\{0,1\}^{<\mathbb{N}}}$ on $\overline{\mathcal{T}}_{\mathrm{II}}$. It is easy to notice that, due to the compactness of $\{0,1\}^{\mathbb{N}}$, the above correspondence is even a homeomorphism and so we may and shall identify $\mathcal{T}_{\mathrm{II}}$ and $\overline{\mathcal{T}}_{\mathrm{II}}$. Since $\overline{\mathcal{T}}_{\mathrm{II}}$ is obviously closed in $((\{0,1\} \times \mathbb{N})^{<\mathbb{N}})^{\{0,1\}^{<\mathbb{N}}}$, $\mathcal{T}_{\mathrm{II}}$ is a Polish space.

The following lemma is the crucial part of the proof of the Main Theorem ((a)$\Rightarrow$(b)) and it can be viewed as a "parametrized version" of a Hurewicz type theorem (cf. [2, Theorem 21.22]).

Lemma 1. *Let $G, H \subset K \times Y_0$ be disjoint analytic subspaces of the product of a compact space (K, ϱ) and $Y_0 \cong \{0,1\}^{\mathbb{N}}$. Suppose the set G^y cannot be $\boldsymbol{K}_\sigma$ separated from H^y in K for every $y \in Y_0$. Then there are $C \subset Y_0$ with $C \cong \{0,1\}^{\mathbb{N}}$, $Q \cong \{0,1\}^{\mathbb{N}}$, a countable dense subset Q_{f} of Q, and a homeomorphic embedding Φ of $Q \times C$ into $K \times C$ such that $\Phi((Q \setminus Q_{\mathrm{f}}) \times C) \subset G$, $\Phi(Q_{\mathrm{f}} \times C) \subset H$, and $\Phi(Q \times \{y\}) \subset K \times \{y\}$ for every $y \in C$.*

Proof. We are going to use infinite games to construct the mapping Φ. To this end we consider a continuous surjection $\psi : \{0,1\}^{\mathbb{N}} \to K$ and the identity mapping id on Y_0. Set $A = (\psi \times \mathrm{id})^{-1}(G)$ and $B = (\psi \times \mathrm{id})^{-1}(H)$. These sets are disjoint analytic and A^y cannot be $\boldsymbol{K}_\sigma$ separated from B^y in $\{0,1\}^{\mathbb{N}}$ for every $y \in Y_0$ by our assumptions on G and H.

We find closed sets F_0 and F_1 in $\{0,1\}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times Y_0$ whose projections along $\mathbb{N}^{\mathbb{N}}$ are A and B respectively.

Let $\mathcal{W}_{\mathrm{II}}(y)$, $y \in Y_0$, stand for the subset of those elements τ of $\mathcal{T}_{\mathrm{II}}$ that are induced by a winning strategy for II in the game $\mathrm{LSR}(F_0^y, F_1^y)$.

We are going to apply the uniformization theorem of Jankov and von Neumann to the set $\mathcal{W}_{\mathrm{II}} = \{(\tau, y) \in \mathcal{T}_{\mathrm{II}} \times Y_0 : \ \tau \in \mathcal{W}_{\mathrm{II}}(y)\}$.

According to the definition of the payoff set for II we have

$$\tau \in \mathcal{W}_{\mathrm{II}}(y) \Leftrightarrow (\forall s \in \{0,1\}^{<\mathbb{N}})\ \mathcal{N}(r(s,\tau(s))) \cap F_{i(s)}^y \neq \emptyset,$$

for every $y \in Y_0$. The last formula gives that

$$\begin{aligned}\mathcal{W}_{\mathrm{II}} &= \bigcap_{s\in\{0,1\}^{<\mathbb{N}}} \{(\tau,y) \in \mathcal{T}_{\mathrm{II}} \times Y_0 : (\mathcal{N}(r(s,\tau(s))) \times \{y\}) \cap F_{i(s)} \neq \emptyset\} \\ &= \bigcap_{s\in\{0,1\}^{<\mathbb{N}}} \bigcup_{\substack{i\in\{0,1\}\\ i(s)=i}} \bigcup_{t\in(\{0,1\}\times\mathbb{N})^{|s|}} \{\tau \in \mathcal{T}_{\mathrm{II}} : \tau(s) = t\} \\ &\qquad\qquad \times \{y \in Y_0 : (\mathcal{N}(r(s,t)) \times \{y\}) \cap F_i \neq \emptyset\}\end{aligned}$$

and so the set $\mathcal{W}_{\mathrm{II}}$ is analytic. By the Louveau–Saint-Raymond theorem each section $\mathcal{W}_{\mathrm{II}}^y$ $(= \mathcal{W}_{\mathrm{II}}(y))$, $y \in Y_0$, is non-empty, and we may use the Jankov–von Neumann theorem as stated above.

We obtain a mapping $\boldsymbol{\tau} : Y_0 \to \mathcal{T}_{\mathrm{II}}$ with the Baire property such that $\boldsymbol{\tau}(y) \in \mathcal{W}_{\mathrm{II}}(y)$ for every $y \in Y_0$. We can find a dense $\boldsymbol{G}_\delta$ subset T of Y_0 such that $\boldsymbol{\tau}$ is continuous on T (see e.g. [3, Chapter 2, Section 32, II]). Now we find $Y_1 \subset T$ with $Y_1 \cong \{0,1\}^{\mathbb{N}}$. We define a mapping $\widetilde{\Phi} : \{0,1\}^{\mathbb{N}} \times Y_1 \to K \times Y_0$ by

$$\widetilde{\Phi}(x,y) = (\psi[(\boldsymbol{\tau}(y))^*(x)], y).$$

Let us recall that $(\boldsymbol{\tau}(z))^*$, $z \in Y_0$, is obtained from $\boldsymbol{\tau}(z)$ by deleting suitable β-coordinates. Thus $(\boldsymbol{\tau}(z))^*$ is obtained by a projection of $\boldsymbol{\tau}(z)$. According to the properties of the games $\mathrm{LSR}(F_0^y, F_1^y)$, we have $\widetilde{\Phi}(P_\infty \times Y_1) \subset G$, $\widetilde{\Phi}(P_{\mathrm{f}} \times Y_1) \subset H$. By the definition of $\widetilde{\Phi}$ we have $\widetilde{\Phi}(\{0,1\}^{\mathbb{N}} \times \{y\}) \subset K \times \{y\}$ for every $y \in Y_1$. The mapping $(x,y) \mapsto (\boldsymbol{\tau}(y))^*(x)$ is continuous since $\boldsymbol{\tau}$ is continuous as a mapping to the subspace $\mathcal{T}_{\mathrm{II}}$ of the space of continuous mappings from $\{0,1\}^{\mathbb{N}}$ to $(\{0,1\}\times\mathbb{N})^{\mathbb{N}}$ endowed with the topology of uniform convergence on $\{0,1\}^{\mathbb{N}}$ as mentioned before Lemma 1. We conclude that $\widetilde{\Phi}$ is continuous.

Moreover, $\widetilde{\Phi}$ is non-constant on every set $V \times \{y\}$, where $y \in Y_1$ and V is a non-empty open subset of $\{0,1\}^{\mathbb{N}}$. Using this, the analogue of the standard Cantor construction, and the continuity of $\widetilde{\Phi}$, we construct by induction points $x_s \in P_{\mathrm{f}}$ and open balls $V_s \subset \{0,1\}^{\mathbb{N}}$, $U_s \subset Y_1$, such that for every $s \in \{0,1\}^{<\mathbb{N}}$ we have

(i) $x_{s^\wedge 0} = x_s$,
(ii) V_s has its center at x_s,
(iii) $\overline{V}_{s^\wedge 0} \cup \overline{V}_{s^\wedge 1} \subset V_s$, $\overline{V}_{s^\wedge 0} \cap \overline{V}_{s^\wedge 1} = \emptyset$, $\operatorname{diam} V_s < 2^{-|s|}$,
(iv) $\overline{U}_{s^\wedge 0} \cup \overline{U}_{s^\wedge 1} \subset U_s$, $\overline{U}_{s^\wedge 0} \cap \overline{U}_{s^\wedge 1} = \emptyset$, $\operatorname{diam} U_s < 2^{-|s|}$,
(v) $\widetilde{\Phi}(V_{s^\wedge 0} \times U_{s^\wedge i}) \cap \widetilde{\Phi}(V_{s^\wedge 1} \times U_{s^\wedge i}) = \emptyset$, $i = 0, 1$.

We put

$$Q = \bigcap_{n=0}^{\infty} \bigcup_{s \in \{0,1\}^n} V_s, \qquad C = \bigcap_{n=0}^{\infty} \bigcup_{s \in \{0,1\}^n} U_s,$$

and $Q_{\mathrm{f}} = Q \cap P_{\mathrm{f}}$. The conditions (i) and (ii) give that Q_{f} is dense in Q. The set Q_{f} is countable since $Q_{\mathrm{f}} \subset P_{\mathrm{f}}$. The condition (iii) ((iv), respectively) gives $Q \cong \{0,1\}^{\mathbb{N}}$ ($C \cong \{0,1\}^{\mathbb{N}}$, respectively). Define $\Phi = \widetilde{\Phi}|_{Q \times C}$. The mapping Φ is injective by the condition (v). The other required properties of Φ are clearly satisfied. ∎

To get the proof of (a)⇒(b) of the Main Theorem we need a lemma.

Lemma 2. *Let Y be a separable metric space, $C \subset Y$ be a homeomorphic copy of $\{0,1\}^{\mathbb{N}}$ and $g_n : C \to Y$ be a mapping with the Baire property without fixed points for every $n \in \mathbb{N}$. Then there exists $D \subset C$ such that*

(a) $D \cong \{0,1\}^{\mathbb{N}}$,
(b) *g_n is continuous on D for every $n \in \mathbb{N}$,*
(c) *$g_n(D) \cap D = \emptyset$ for every $n \in \mathbb{N}$.*

Proof. We need the following claim.

Claim. *Let Y be a metric space, $T \cong \{0,1\}^{\mathbb{N}}$, $T \subset Y$, $g : T \to Y$ be a continuous mapping without fixed points, $k \in \mathbb{N}$ and $V_1, \ldots, V_k$ be open sets intersecting T. Then there exists a set $L \subset (V_1 \cup \ldots \cup V_k) \cap T$ such that*

(a) *$L \cong \{0,1\}^{\mathbb{N}}$,*
(b) *L intersects each V_j, $j = 1, \ldots, k$,*
(c) *$g(L) \cap L = \emptyset$.*

Proof. We pick a point $x_j \in T \cap V_j$ for every $j \in \{1, \ldots, k\}$ in such a way that $\{x_1, \ldots, x_k\} \cap \{g(x_1), \ldots, g(x_k)\} = \emptyset$. This can be done as follows.

For $j = 1, \dots, k$ define

$$N_j = \begin{cases} \emptyset & \text{if } g(V_j \cap T) \text{ is infinite,} \\ g(V_j \cap T) & \text{if } g(V_j \cap T) \text{ is finite;} \end{cases} \qquad P = \bigcup_{j=1}^{k} N_j.$$

The set P is clearly finite. Choose $x_1 \in (V_1 \cap T) \setminus P$. Suppose that we have picked points $x_1, \dots, x_s$, $s < k$, such that for all $j = 1, \dots, s$,

$$x_j \in (V_j \cap T) \setminus (P \cup g(\{x_1, \dots, x_{j-1}\}) \cup g^{-1}(\{x_1, \dots, x_{j-1}\})).$$

Observe that we may find

$$x_{s+1} \in X_{s+1} = (V_{s+1} \cap T) \setminus (P \cup g(\{x_1, \dots, x_s\}) \cup g^{-1}(\{x_1, \dots, x_s\})).$$

Indeed, the set X_{s+1} is non-empty since otherwise $g(V_{s+1} \cap T)$ is finite and, as $(V_{s+1} \cap T) \setminus (P \cup g(\{x_1, \dots, x_s\}))$ is infinite, there exists $j_0 \in \{1, \dots, s\}$ such that $x_{j_0} \in g(V_{s+1} \cap T) = N_{s+1} \subset P$, a contradiction. Our construction of $x_1, \dots, x_k$ clearly works.

This construction and continuity of g on each $V_j \cap T$ imply that, for every $j = 1, \dots, k$, we can find in a sufficiently small neighbourhood of x_j a set $L_j \cong \{0,1\}^{\mathbb{N}}$ such that $L_j \subset V_j \cap T$ and $(\bigcup_{j=1}^{k} L_j) \cap g(\bigcup_{j=1}^{k} L_j) = \emptyset$. Put $L = \bigcup_{j=1}^{k} L_j$. It is not difficult to see that $L \cong \{0,1\}^{\mathbb{N}}$ and has all required properties. This finishes the proof of the Claim. ∎

Let $D_0 \subset C$ be a homeomorphic copy of the Cantor set such that all g_n's are continuous on D_0 (using [3, Chapter 2, Section 32, II]). Let $G_\emptyset = C$.

We inductively construct compact sets $D_n \subset D_0$, $D_n \cong \{0,1\}^{\mathbb{N}}$, and open sets $G_s \subset C$, $s \in \{0,1\}^n$ for $n \in \mathbb{N}$, such that

(i) $\overline{G}_{s^\wedge 0} \cap \overline{G}_{s^\wedge 1} = \emptyset$, $\overline{G}_{s^\wedge 0} \cup \overline{G}_{s^\wedge 1} \subset G_s$ for $s \in \{0,1\}^{n-1}$,
(ii) $\operatorname{diam} G_s < 2^{-n}$ for $s \in \{0,1\}^n$,
(iii) $D_n \cap G_s \neq \emptyset$ for $s \in \{0,1\}^n$ and $D_n \subset D_{n-1} \cap \bigcup_{t \in \{0,1\}^n} G_t$, and
(iv) $g_n(D_n) \cap D_n = \emptyset$.

Suppose that we have constructed $D_0, \dots, D_{n-1}$ and G_s, $s \in \{0,1\}^{<\mathbb{N}}$, $|s| \le n-1$. Choose G_s intersecting D_{n-1} for $s \in \{0,1\}^n$ such that (i) and (ii) are satisfied. Use the Claim for $T = D_{n-1}$, $g = g_n$ and $\{V_1, \dots, V_{2^n}\} = \{G_s : s \in \{0,1\}^n\}$ to get L and put $D_n = L$.

Define $D = \bigcap_{n=0}^{\infty} D_n$. It is not difficult to see that D has all required properties. ∎

Proof of Main Theorem. The set $f(L)$ is separable by [9, 4.3.8]. Hence, replacing Y by the metric completion of $f(L)$, without any loss of generality, we assume that Y is a Polish space.

Proof of (b)⇒(a). We use the Arsenin–Kunugui theorem as restated for Borel measurable mappings above. If $F \subset L$ is closed, then the restriction of

f to $F\setminus f^{-1}(S)$ fulfils the assumptions and $f(F)=f(F\setminus f^{-1}(S))\cup(f(F)\cap S)$ is Borel in the Polish space Y (and so it is a Luzin space).

Proof of (a)⇒(b). Let K be a metric compactification of L. We extend f to a Borel mapping $\widetilde{f}$ of K to Y by a constant on $K\setminus L$. The graph of f is a Borel subset of $K\times Y$ as L is Borel in K and $f:L\to Y$ is Borel measurable.

Suppose that (b) does not hold, i.e. $S=\{y\in Y:(\operatorname{graph} f)^y \text{ is not } \boldsymbol{K}_\sigma\}$ is uncountable. By (a) of the Arsenin–Kunugui theorem, S is analytic and so there is $Y_0\cong\{0,1\}^{\mathbb{N}}$ in S. Moreover we may and shall assume that $\widetilde{f}(K\setminus L)\cap Y_0=\emptyset$.

Let $G=(\operatorname{graph} f)\cap(L\times Y_0)$ and $H=(K\times Y_0)\setminus G$. Using Lemma 1 we get Φ, C, Q, and Q_{f} satisfying the claim of that lemma. Let $E=\Phi((Q\setminus Q_{\mathrm{f}})\times C)$, $\overline{E}\setminus E=\Phi(Q_{\mathrm{f}}\times C)$ and $Q_{\mathrm{f}}=\{q_n:n\in\mathbb{N}\}$. So $\overline{E}\setminus E$ is covered by the graphs of continuous mappings $\gamma_n:C\to K$, $n\in\mathbb{N}$, where $(\gamma_n(y),y)=\Phi(q_n,y)$, $y\in C$. We put $g_n=\widetilde{f}\circ\gamma_n:C\to Y$ and use Lemma 2 for them. Thus we find $D\cong\{0,1\}^{\mathbb{N}}$ in C such that each $\widetilde{f}\circ\gamma_n:D\to Y$ is continuous and $\widetilde{f}\circ\gamma_n(D)\cap D=\emptyset$. As Φ from Lemma 1 restricted to $(Q\setminus Q_{\mathrm{f}})\times D$ is a homeomorphism onto the set $E_D=(K\times D)\cap E$ with the property that $\Phi(x,y)\in E_D$ for every $(x,y)\in(Q\setminus Q_{\mathrm{f}})\times D$, there is a relatively closed subset U of E_D whose projection N to D is not Borel in D. (It suffices to choose a closed subset of $(Q\setminus Q_{\mathrm{f}})\times D\cong\mathbb{N}^{\mathbb{N}}\times D$ with non-Borel projection N to D and take its image under Φ.) Finally, we consider the relatively closed subset $F=\overline{p_L(U)}^{L}$ in L, where p_L is the projection to L. Since U is a subset of the graph of f and of E_D, we see that $f(p_L(U))=N\subset D$.

We claim that

$$(\star)\qquad f(F\setminus p_L(U))=\bigcup_{n\in\mathbb{N}} f(\gamma_n(D)\cap F).$$

Indeed, if $y=f(x)$ with $x\in\gamma_n(D)\cap F$, then $x\notin p_L(U)$. Otherwise y would be an element of $f\circ\gamma_n(D)\cap D$ $(=\emptyset)$. The other inclusion can be proved as follows. Let $z=f(x)$ with $x\in F\setminus p_L(U)$. There is a sequence $(x_n,y_n)\in U$ such that $x_n\to x$ by the definition of F and, as $\overline{E}_D$ is compact, we may suppose that (x_n,y_n) tends to $(x,y)\in\overline{E}_D$ for some $y\in D$. As U is relatively closed in E_D and $x\notin p_L(U)$, we have $(x,y)\in\overline{E}_D\setminus E_D\subset\overline{E}\setminus E$. Hence, there is an $n\in\mathbb{N}$ such that $\gamma_n(y)=x$ and so $z=f(\gamma_n(y))\in f(\gamma_n(D)\cap F)$ and $(\star)$ is proved.

It follows from $(\star)$ that $f(F\setminus p_L(U))\subset\bigcup_{n\in\mathbb{N}}\widetilde{f}\circ\gamma_n(D)$ and so $f(F\setminus p_L(U))\cap D=\emptyset$. Therefore $f(F)\cap D=N$ and, as N is not Borel in the compact set D, the set $f(F)$ is not Borel in Y and the theorem is proved. ∎

4. Some corollaries and remarks. Let us point out that in the proof of (b)⇒(a) of the Main Theorem we proved that (b) implies that $f(F)$ is a Luzin space whenever F is closed in L. Hence using the Main Theorem and this observation we get the following statement.

COROLLARY 3. *Let $f : L \to Y$ be a Borel measurable mapping of a Luzin space L to a metric space Y. Then the following statements are equivalent.*

(a) *For every closed set $F \subset L$ the set $f(F)$ is Borel in $f(L)$.*

(b) *For every closed set $F \subset L$ the image $f(F)$ is a Luzin space.*

REMARK. Our Corollary 3 (a)⇒(b) improves Taĭmanov's result [10, Theorem 3] saying that the image of a Luzin space under a closed continuous mapping is a Luzin space.

COROLLARY 4. *Let $f : L \to Y$ be a Borel measurable mapping of a Luzin space L to a metric space Y. Then exactly one of the following possibilities holds.*

(a) *There is an ordinal $\alpha < \omega_1$ such that $f(F)$ is of Borel class less than α for each closed subset F of L.*

(b) *For every ordinal α, $2 \le \alpha < \omega_1$, there is a closed subset F of L such that $f(F)$ is a Borel subset of exact additive class $\mathbf{\Sigma}^0_\alpha$. If $3 \le \alpha < \omega_1$, then there is a closed set F such that $f(F)$ is of exact multiplicative class $\mathbf{\Pi}^0_\alpha$. Also there is a closed subset F of L such that $f(F)$ is analytic but not Borel.*

In particular, *if f satisfies* (a), *or equivalently* (b), *of the Main Theorem*, *then the Borel classes of the sets $f(F)$*, *with F closed in L*, *are bounded by a fixed countable ordinal.*

P r o o f. As in the proof of the Main Theorem we may assume that Y is Polish. We shall show that (a) holds if the set $S = \{y \in Y : f^{-1}(y)$ is not $\boldsymbol{K}_\sigma\}$ is countable and (b) holds if the set S is uncountable.

Let S be countable. Put

$$G = (\operatorname{graph} f) \cap (L \times (Y \setminus S)).$$

The set G is a Borel subset of $L \times Y$ such that G^y is $\boldsymbol{K}_\sigma$ for every $y \in Y$. According to Saint-Raymond's theorem [8] there exist Borel sets G_n, $n \in \mathbb{N}$, such that $G = \bigcup_{n=1}^\infty G_n$ and G_n^y is compact whenever $y \in Y$ and $n \in \mathbb{N}$. Let F be a closed subset of L. Then

$$f(F) = \Big(\bigcup_{n=1}^{\infty} p_Y((F \times Y) \cap G_n) \Big) \cup (f(F) \cap S),$$

where p_Y is the projection of $L \times Y$ onto Y. It is sufficient to show that if $B \subset L \times Y$ is a Borel set with B^y compact for every $y \in Y$, then there exists an $\alpha < \omega_1$ such that $p_Y((F \times Y) \cap B)$ is of Borel class less than α whenever $F \subset L$ is closed.

Observe that, if $\{F_j\}_{j=1}^\infty$ is a decreasing sequence of closed subsets of L such that $\bigcap_{j=1}^\infty F_j = F$, then

$$p_Y\Big(\bigcap_{j=1}^\infty (F_j \times Y) \cap B\Big) = \bigcap_{j=1}^\infty p_Y((F_j \times Y) \cap B)$$

since B^y's are compact (cf. e.g. [1, II, 19, Theorem]).

Fix a countable open basis $\mathcal{B}$ of L. Consider a countable family $\mathcal{F}$ of all closed sets of the form $L \setminus (U_1 \cup \ldots \cup U_m)$, where $m \in \mathbb{N}$ and $U_i \in \mathcal{B}$ for every $i = 1, \ldots, m$. Since $\mathcal{F}$ is countable there exists a $\beta < \omega_1$ such that for every $F \in \mathcal{F}$, $p_Y((F \times Y) \cap B)$ is of Borel class less than β. We see that each closed subset of L can be written as an intersection of a decreasing sequence of sets from $\mathcal{F}$. Thus we conclude that $p_Y((F \times Y) \cap B)$ is of Borel class less than $\alpha = \beta + 1$ whenever F is closed in L.

Now let S be uncountable. Let K be a metric compactification of L, $E \subset K \times Y$, D, E_D be sets from the proof of (a)$\Rightarrow$(b) of the Main Theorem.

For every ordinal $2 \le \alpha < \omega_1$ there is a relatively closed subset $H \subset E_D$ whose projection N to D is of exact class $\mathbf{\Sigma}_\alpha^0$ ($\mathbf{\Pi}_\alpha^0$, respectively). Put $F = \overline{p_L(H)}^L$. Using ($\star$) from the proof of (a)$\Rightarrow$(b) of the Main Theorem, we deduce that $f(F)$ is a Borel set of exact class $\mathbf{\Sigma}_\alpha^0$ if $\alpha \ge 2$ (and $\mathbf{\Pi}_\alpha^0$ if $\alpha \ge 3$, respectively) since $f(F)$ is the union of the set $N \subset D$ and the set $f(F \setminus p_L(H))$, which is $\boldsymbol{K}_\sigma$ and disjoint from the compact set D. ∎

Remark. In analogy to our Main Theorem and Corollary 3, we may reformulate the results of Luzin and Purves as follows.

Luzin–Purves theorem. *Let $f : L \to Y$ be Borel measurable, L be a Luzin space, and Y be a metric space. Then the following statements are equivalent.*

(1) *For every $\boldsymbol{G}_\delta$ set $B \subset L$, the set $f(B)$ is Borel in $f(L)$.*

(2) *For every Borel set $B \subset L$, the image $f(B)$ is a Luzin space.*

(3) *The set $\{y \in Y : f^{-1}(y)$ is not countable$\}$ is at most countable.*

We shall show how the proof can be obtained from the following lemma that is an analogue of Lemma 1 above.

Lemma. *Let $G \subset L \times Y$ be analytic and $\{y \in Y : G^y$ is uncountable$\}$ be uncountable, where L is a Luzin space and Y is a metric space. Then* *there is a homeomorphism Φ of $\{0,1\}^{\mathbb{N}} \times C$ onto $E \subset G$, where $C \subset Y$ is a homeomorphic copy of $\{0,1\}^{\mathbb{N}}$, such that $\Phi(\cdot, y)$ is a homeomorphism of $\{0,1\}^{\mathbb{N}}$ onto $E^y \times \{y\}$ for every $y \in C$.*

Proof. A particular case of this lemma with $L = Y = \{0,1\}^{\mathbb{N}}$ is Mauldin's theorem stated in [6, Section 4.1]. We may reduce the general case to that particular one as follows. Since the uncountable set $\{y \in Y : G^y \text{ is uncountable}\}$ is analytic ([3, Chapter 3, Section 39, VII, Theorem 3]), it contains a homeomorphic copy of the Cantor set, and so we may suppose that $Y = \{0,1\}^{\mathbb{N}}$. There is a one-to-one continuous mapping φ of a $\boldsymbol{G}_\delta$ subset H of $\{0,1\}^{\mathbb{N}}$ onto L. We get Φ as the composition of the mapping that takes $(x,y) \in H \times Y$ to $(\varphi(x), y) \in L \times Y$ with the mapping obtained by Mauldin's theorem applied to the set $\{(x,y) \in H \times \{0,1\}^{\mathbb{N}} : (\varphi(x), y) \in G\} \subset \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ that is analytic as a continuous preimage of the analytic set G. ∎

Proof of Luzin–Purves theorem. As in the proof of the Main Theorem we may assume that Y is Polish. The implication (2)⇒(1) is obvious, (3)⇒(2) follows immediately from the Luzin theorem mentioned in the introduction. Thus we need to prove (1)⇒(3) only. Suppose (3) does not hold and let G be the graph of f. Let Φ, C, and E be as in the lemma above. As E is compact, the projection of E into L is a homeomorphism. We choose a $\boldsymbol{G}_\delta$ subset F of $\{0,1\}^{\mathbb{N}} \times C$ whose projection P to C is not Borel. Then the projection B of the set $\Phi(F)$ to L is $\boldsymbol{G}_\delta$ in L and its image under f equals P. ∎

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