$$\sigma_{01}(u,v) := \frac{1}{v}\int_0^v s(u,z)\,dz = \int_0^u\int_0^v \left(1-\frac{y}{v}\right) f(x,y)\,dx\,dy \quad \text{as } u,v\to\infty.$$

Each of Theorems 3, 4 and Corollaries 5–8 has a symmetric counterpart when summability $(C,0,1)$ is considered in place of summability $(C,1,0)$ of the integral (1.1).

REMARK 2. Analogous results were proved in [3] for double numerical series with rectangular partial sums s_{jk}, $j,k=0,1,2,\ldots$ Making use of the method of this paper, we are now able to improve some of those results. For example, [3, Corollary 1] remains valid if we drop the condition of slow decrease in $(1,1)$ sense. Likewise, condition (2.3) in [3, Corollary 2] and condition (5.1) in [3, Corollary 5] are superfluous.

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Vector series whose lacunary subseries converge

by

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Abstract. The area of research of this paper goes back to a 1930 result of H. Auerbach showing that a scalar series is (absolutely) convergent if all its zero-density subseries converge. A series $\sum_n x_n$ in a topological vector space X is called $\mathcal{L}$-convergent if each of its lacunary subseries $\sum_k x_{n_k}$ (i.e. those with $n_{k+1}-n_k\to\infty$) converges. The space X is said to have the Lacunary Convergence Property, or LCP, if every $\mathcal{L}$-convergent series in X is convergent; in fact, it is then subseries convergent. The Zero-Density Convergence Property, or ZCP, is defined similarly though of lesser importance here. It is shown that for every $\mathcal{L}$-convergent series the set of all its finite sums is metrically bounded; however, it need not be topologically bounded. Next, a space with the LCP contains no copy of the space c_0. The converse holds for Banach spaces and, more generally, sequentially complete locally pseudoconvex spaces. However, an F-lattice of measurable functions is constructed that has both the Lebesgue and Levi properties, and thus contains no copy of c_0, and, nonetheless, lacks the LCP. The main (and most difficult) result of the paper is that if a Banach space E contains no copy of c_0 and λ is a finite measure, then the Bochner space $L_0(\lambda,E)$ has the LCP. From this, with the help of some Orlicz–Pettis type theorems proved earlier by the authors, the LCP is deduced for a vast class of spaces of (scalar and vector) measurable functions that have the Lebesgue type property and are "metrically-boundedly sequentially closed" in the containing L_0 space. Analogous results about the convergence of $\mathcal{L}$-convergent positive series in topological Riesz spaces are also obtained. Finally, while the LCP implies the ZCP trivially, an example is given that the converse is false, in general.

1. Introduction. We first recall a few more or less standard definitions and facts. As usual, a series in a topological vector space X is said to be *con-*

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vergent (resp. *bounded*) if the sequence of its partial sums is convergent (resp. bounded) in X. It is called *subseries convergent* (resp. *subseries bounded*) in X if each of its subseries is convergent (resp. bounded). A strictly increasing sequence (n_k) of positive integers, or the set $\{n_1, n_2, \ldots\}$, is called *lacunary* if $\lim_k (n_{k+1} - n_k) = \infty$, and *of density zero* if $\lim_k (k/n_k) = 0$. The *lacunary subseries* (resp. *zero-density subseries*) of a given series are those corresponding to lacunary sequences of indices (resp. sequences of indices of density zero).

As is well known, a subseries convergent series is unconditionally convergent, and the converse holds in sequentially complete spaces. Next, the range of a subseries convergent series, that is, the set of sums of all its (finite and infinite) subseries, is compact. Also, a series is subseries bounded iff it is *perfectly bounded*, i.e., the set of all its finite sums is bounded.

Our point of departure is a 1930 result of Auerbach [Au, Hilfsatz], obtained already in 1923 (*loc.cit.*, Footnote 1), originally stated for series with positive terms:

(A) *A scalar series is subseries* (*or unconditionally, or absolutely*) *convergent provided each of its zero-density subseries converges.*

Without mentioning Auerbach, this result reappeared in a 1947 paper by Agnew [Ag, Thm. 1], a 1986 paper by Estrada and Kanwal [EK, Thm. 1], a 1989 paper by Noll and Stadler [NS, Lemma on p. 116] and, in a stronger form, in the already mentioned paper by Agnew [Ag, Thm. 2], and a 1981 paper by Sember and Freedman [SF, Prop. 2]:

(B) *A scalar series is subseries convergent provided each of its lacunary subseries converges.*

Some other extensions of (A) can be found in Paštéka [P] and Drewnowski, Florencio and Paúl [DFP]. We note that the proofs given by the authors mentioned so far are essentially variations of Auerbach's; in fact, Auerbach's original proof is also a proof of (B).

On the other hand, also without any reference to Auerbach, in 1955 Orlicz studied the questions of boundedness or convergence of those series in Banach spaces whose zero-density subseries were bounded or convergent. In particular, he proved the following two results [O, Thm. 2′ and Thm. 3.B]:

(C) *A series in a normed space is subseries bounded provided it has a "large" set of bounded zero-density subseries.*

(D) *A series in a weakly sequentially complete Banach space is subseries convergent provided it has a "large" set of convergent zero-density subseries.*

Orlicz used Baire category methods, and the "large" sets of subseries in the statements above correspond to the second Baire category sets in a suitable metric space of zero-density sequences; see the remark at the end of Section 3 below. He also considered similar questions for the convergence and boundedness in measure of series of measurable functions, but did not arrive at conclusive results.

It seemed natural to examine to what extent the results of the above types were valid for series (or some special series, e.g. those with positive terms) in more general spaces. Working in this direction, we have recently proved in [DL1] the following.

(E) *For a finite measure λ, a series in the F-space $L_0(\lambda)$ is subseries convergent provided each of its lacunary subseries converges.*

In this paper we not only generalize all the previous results mentioned so far, but also give a more systematic, comprehensive account of our investigations arising in connection with the lacunary convergence of series in abstract spaces.

Throughout, we use the abbreviations TVS and TRS for Hausdorff *topological vector space* and Hausdorff *locally solid topological Riesz space*, respectively. As a rule, we follow [J] and [AB] in the terminology concerning such spaces.

We introduce the following definitions.

Let $\mathfrak{L}$ denote the family of all lacunary subsets of $\mathbb{N}$. We shall say that a series is *$\mathfrak{L}$-convergent* (resp. *$\mathfrak{L}$-bounded*) if each of its lacunary subseries, or *$\mathfrak{L}$-subseries* for short, is convergent (resp. bounded). We shall say that a TVS X has

- the *Lacunary Convergence Property*, LCP, if every $\mathfrak{L}$-convergent series in X is convergent;
- the *Lacunary Boundedness Property*, LBP, if every $\mathfrak{L}$-bounded series in X is bounded.

If X is a TRS and the above requirements concern positive series only, we obtain

- the *Positive* LCP, and the *Positive* LBP.

The following two facts are easily verified and will be used without explicit reference:

1) If a series is $\mathfrak{L}$-convergent (resp. $\mathfrak{L}$-bounded), then each of its $\mathfrak{L}$-subseries is subseries convergent (resp. subseries bounded), and each of its subseries is $\mathfrak{L}$-convergent (resp. $\mathfrak{L}$-bounded).

2) If a TVS X has the LCP (resp. LBP), then every $\mathfrak{L}$-convergent (resp. $\mathfrak{L}$-bounded) series in X is subseries convergent (resp. subseries bounded). Likewise for the Positive LCP and the Positive LBP when X is a TRS.

Similar definitions apply, and similar facts hold, if $\mathcal{L}$ is replaced by the family $\mathcal{Z}$ of all subsets of density zero in $\mathbb{N}$. In this case we speak about the *Zero-Density Convergence* (resp. *Boundedness*) *Property*. Since $\mathcal{L}$ is a proper subclass of $\mathcal{Z}$, the lacunary properties are stronger than their zero-density counterparts. Although in many cases these two properties are equivalent, it is not so in general (see Example 11.4).

We now describe briefly the content of the paper. We start with two basic though relatively easy results. First, we prove that

(1) *Every $\mathcal{L}$-bounded (in particular, $\mathcal{L}$-convergent) series in a* TVS *is subseries metrically bounded.*

We recall that a sequence or set A in a TVS X is said to be *metrically bounded* (or *additively bounded*) if every continuous F-seminorm in X is bounded on A, or equivalently, if for every neighborhood U of zero in X there is n such that $A \subset U + \ldots + U$ (n summands). Note that if X is a TRS, then it is enough to use continuous Riesz (or monotone) F-seminorms (and solid neighborhoods of zero).

Our second result shows that there is a natural limitation on the class of spaces enjoying the Lacunary Convergence Properties:

(2) *If a* TVS *has the* LCP, *then it contains no copy of c_0. If a* TRS *has the Positive* LCP, *then it contains no positive copy of c_0.*

It is seeking various converses to the latter statement that will take most of our effort in this paper. In view of the known results characterizing the spaces without copies of c_0, the case of locally pseudoconvex spaces is easy:

(3) *If a sequentially complete locally pseudoconvex* TVS *contains no copy of c_0, then it has the* LCP. *If a sequentially complete locally pseudoconvex* TRS *contains no positive (or lattice) copy of c_0, then it has the Positive* LCP.

In our approach to the non-locally pseudoconvex case, the following two results about L_0 spaces of measurable functions, with the topology of convergence in (sub)measure, are of primary importance. They may be viewed as the main results of the paper.

(4) *For each order continuous submeasure μ, the space $L_0(\mu)$ has the Positive* LCP.

(5) *For each finite measure λ and every Banach space E containing no copy of c_0, the space $L_0(\lambda, E)$ of E-valued λ-measurable functions has the* LCP.

As in [DL1], where the scalar case (E) of (5) has been established, we prove (4) and (5) by intertwining, roughly speaking, combinatorial arguments of Auerbach's type with standard measure-theoretic arguments involving Egoroff's theorem. However, in contrast to the self-contained proof of (4), in proving (5) we crucially depend on a result essentially due to Kwapień and Hoffmann-Jørgensen which is a vector-valued form of the Orlicz Theorem used in [DL1].

Actually, the two results just stated, in particular (E), hold true for the larger classes of locally order continuous submeasures μ, and locally finite measures λ of type (SC), i.e. those whose L_0 space is sequentially complete.

We then apply (4) and (5) to those TRS's and suitable TVS's X that can be continuously embedded in an L_0 space of one of the above types. Our idea is the following: Consider a positive (resp. arbitrary) $\mathcal{L}$-convergent series in X. Then it is also $\mathcal{L}$-convergent in L_0 whence, by (4) or (5), subseries convergent in L_0. Now, suppose we know that $(*)$ the L_0-sums of all the subseries are in X. Then our series is subseries convergent in X provided that X is a σ-Lebesgue TRS (resp. X has the Orlicz–Pettis property relative to L_0), and we are done. As we see it, the most essential point in this approach is $(*)$. To ensure $(*)$, we introduce some conditions on X that are strongly motivated by (1) and, therefore, are metric in character.

In the case of positive series in TRS's, we use a strengthening of the familiar disjoint Levi property. We say that a TRS X has the *disjoint metric Levi property* if every disjoint positive sequence in X with metrically bounded partial sums has a supremum in X. Combining (1) and (4) with a representation theorem (Theorem 2.3) yields our most general result about positive series:

(6) *If a* TRS *X is Dedekind complete, Lebesgue, and has the disjoint metric Levi property, then it has the Positive* LCP.

The other type of spaces considered is that of *topological vector spaces X of Bochner measurable functions* with values in a Banach space E, a class recently introduced and studied in [DL4]. In this case we impose on X the requirement that $X \subset L_0(\lambda, E)$ continuously and is *metrically-boundedly sequentially closed* there. That is, the sequential closure of a metrically bounded subset of X, taken in $L_0(\lambda, E)$, is again a set in X. This is a metric version of a similar topological condition used in [DL4]. Then, by combining (1) and (5) with an Orlicz–Pettis type theorem (Theorem 2.5), we obtain the following.

(7) *Let the Banach space E contain no isomorphic copy of c_0, the measure λ be of type (SC), and assume that a* TVS *X of E-valued λ-measurable functions is continuously included in $L_0(\lambda, E)$. If X is λ-continuous and metrically-boundedly sequentially closed in $L_0(\lambda, E)$, then it has the* LCP.

Towards the end of the paper, we construct in Example 11.1 a Lebesgue Levi F-lattice (of measurable functions) failing both the Positive Lacunary Convergence Property and the Positive Lacunary Boundedness Property. In fact, it contains a $\mathcal{Z}$-convergent positive series all of whose subseries of nonzero density are unbounded.

2. A few basic notions and results. We collect here some notions and results that will be of primary importance in what follows, or at least are worth noting for the sake of clarity.

Following Orlicz (see e.g. [MO]), a TVS X is said to have

- *Property* (O) if every subseries (or perfectly) bounded series in X is subseries convergent.

X is said to *contain a copy* of c_0 if there exists a linear homeomorphism from the Banach space c_0 onto a subspace of X. Evidently, if X has Property (O), then it contains no copy of c_0. Also the following is obvious.

2.1. PROPOSITION. *In a* TVS *with Property* (O), *the Lacunary Convergence Property and the Lacunary Boundedness Property are equivalent.*

Since a TRS is σ-Lebesgue σ-Levi iff every bounded positive series converges, we also have a "positive" analogue of the above proposition.

2.2. PROPOSITION. *In a σ-Lebesgue σ-Levi* TRS, *the Positive Lacunary Convergence Property and the Positive Lacunary Boundedness Property are equivalent.*

We now briefly recall some terminology from [DL4] concerning submeasures and spaces of Bochner measurable functions; the reader is referred to [DL4] for more details. Thus a triple (S, Σ, μ) is called a *submeasure space* if S is a set, Σ is a σ-algebra of subsets of S, and $\mu : \Sigma \to \overline{\mathbb{R}}_+$ is a submeasure (i.e., μ is nondecreasing, subadditive, and $\mu(\emptyset) = 0$) that is assumed throughout to be null-complete and locally order continuous. To explain the latter condition, denote by $\Sigma^+_{\text{oc}}(\mu)$ the class of all sets $B \in \Sigma$ such that $\mu(B) > 0$ and μ is order continuous on B; that is, $\mu(B_n) \to 0$ whenever $B_n \in \Sigma$, $B_n \subset B$ and $B_n \downarrow \emptyset$. Now, the assumption that μ is *locally order continuous* means that each $A \in \Sigma$ with $\mu(A) > 0$ contains a $B \in \Sigma^+_{\text{oc}}(\mu)$.

Let (S, Σ, μ) be a submeasure space. We denote by $L_0(\mu) = L_0(S, \Sigma, \mu)$ the vector lattice (Riesz space) of all μ-equivalence classes of measurable scalar functions on S. It is equipped with the locally solid vector topology of convergence in submeasure μ on all sets in $\Sigma^+_{\text{oc}}(\mu)$. The submeasure μ is said to be of

- *type* (C) (resp. (SC)) if the TVS $L_0(\mu)$ is complete (resp. sequentially complete).

By a TRS *of μ-measurable functions* we mean a solid subspace X of $L_0(\mu)$ equipped with a Hausdorff locally solid topology.

Let now E be a Banach space. As in [DL4], a function $f : S \to E$ is called (*Bochner*) *μ-measurable* if it is essentially separably valued and Borel measurable. (For another possible definition, see Remark 3.3 in [DL4].) We denote by $L_0(\mu, E) = L_0(S, \Sigma, \mu; E)$ the space of all μ-equivalence classes of E-valued μ-measurable functions on S, equipped with the Hausdorff vector topology of convergence in submeasure μ on sets in $\Sigma^+_{\text{oc}}(\mu)$. We say that a subset V of $L_0(\mu, E)$ is *Σ-solid* if

$$V_A := \{1_A f : f \in V\} \subset V \quad \text{for all } A \in \Sigma.$$

By a TVS *of E-valued μ-measurable functions* we mean a Σ-solid subspace X of $L_0(\mu, E)$ equipped with a Hausdorff vector topology τ such that

1) τ has a base of Σ-solid neighborhoods of zero;

2) each $A \in \Sigma^+_{\text{oc}}(\mu)$ contains a $B \in \Sigma^+_{\text{oc}}(\mu)$ such that $\tau|X_B$ is weaker than the topology of μ-a.e. uniform convergence on B.

Such a space X is said to be *μ-continuous* (resp. *sequentially μ-continuous*) if $f = \tau\text{-}\lim_i 1_{A_i} f$ for every $f \in X$ and every net (resp. sequence) (A_i) in Σ with $A_i \uparrow S$.

All of the above applies in particular when in place of a submeasure space we are given a *measure space* (S, Σ, λ), with a positive countably additive measure λ that is assumed throughout to be null-complete and *locally finite* (i.e., each $A \in \Sigma$ with $\lambda(A) > 0$ contains a $B \in \Sigma$ with $0 < \lambda(B) < \infty$) instead of being locally order continuous. Note that in this case $\Sigma^+_{\text{oc}}(\lambda)$ is simply the class of sets of finite positive λ measure.

The following is a somewhat specialized and modified form of the Representation Theorem 2.7 in [L2]; for the universal completions, see [AB].

2.3. THEOREM. *If a* TRS *X is Lebesgue, then there exists a type (C) submeasure space (S, Σ, μ) such that X is continuously included as an order dense sublattice in $L_0(\mu)$. In consequence, the vector lattice $L_0(\mu)$ is a universal completion of X.*

The next result is an extension of the Orlicz Theorem used in [DL1]. It is implicit in the works of Kwapień [Kw] and Hoffmann-Jørgensen [HJ]; for an explicit formulation and proof see [L3, Thm. 2.11]; see also Theorems 6.1 and 9.1 in [DL4].

2.4. THEOREM. *If (S, Σ, λ) is a measure space of type (SC) and E is a Banach space containing no copy of c_0, then $L_0(\lambda, E)$ has Property (O).*

The following is an Orlicz–Pettis type theorem from [DL4].

2.5. THEOREM. *Let (S, Σ, μ) be a submeasure space, E a Banach space, and X a μ-continuous* TVS *of μ-measurable E-valued functions. If a series* *in X is subseries convergent in the topology induced from $L_0(\mu, E)$, then it is also subseries convergent in the original topology of X.*

Throughout, if A is a set, then $\mathcal{F}(A)$ stands for the family of all finite subsets of A, and $\mathcal{P}(A)$ for the power set of A.

3. The lacunary and zero-density subsets of $\mathbb{N}$. We begin by modifying the definitions of the families $\mathcal{Z}$ and $\mathcal{L}$ so that they include the finite subsets of $\mathbb{N}$. Here and in the sequel, we write $|F|$ for the number of elements of a finite set F. A set $A \subset \mathbb{N}$ is said to be

- *of density zero* ($A \in \mathcal{Z}$) if
$$\lim_n d_n(A) = 0, \quad \text{where} \quad d_n(A) = \frac{|A \cap \{1, \ldots, n\}|}{n};$$
- *r-rare*, where $r \in \mathbb{N}$, if $|m - m'| \geq r$ for all distinct $m, m' \in A$;
- *lacunary* ($A \in \mathcal{L}$) if for every $r \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that the "tail" set
$$A(n) := A \cap \{n, n+1, \ldots\}$$
is r-rare.

3.1. Proposition. (a) *For every $A \subset \mathbb{N}$ and $r \in \mathbb{N}$ the sets*
$$A_j = A \cap \{j + (k-1)r : k \in \mathbb{N}\}, \quad j = 1, \ldots, r,$$
form a partition of A into r-rare subsets.

(b) *An infinite set $A \subset \mathbb{N}$ is lacunary iff it is the union of a sequence (A_r) of finite subsets of $\mathbb{N}$ such that each A_r is r-rare and* $\max A_r + r \leq \min A_{r+1}$.

3.2. Proposition. *A set $A \subset \mathbb{N}$ is of density zero if and only if*
$$\lim_k |A \cap \Delta_k|/2^k = 0,$$
where $\Delta_k = \{n \in \mathbb{N} : 2^k \leq n < 2^{k+1}\}$ for $k = 0, 1, 2, \ldots$ (Note that $|\Delta_k| = 2^k$.)

Proof. Define $I_k = \{j \in \mathbb{N} : 1 \leq j < 2^{k+1}\}$ for $k = 0, 1, \ldots$ If $A \in \mathcal{Z}$, then
$$2^{-k}|A \cap \Delta_k| \leq 2^{-k}|A \cap I_k| = (2 - 2^{-k})|I_k|^{-1}|A \cap I_k| \to 0 \quad \text{as } k \to \infty.$$

Conversely, assume that the condition stated in the proposition is satisfied. Given $n \in \mathbb{N}$, let m be the greatest integer with $2^m \leq n$. Then
$$d_n(A) \leq 2^{-m}|A \cap I_m| = \sum_{k=0}^{m} 2^{k-m} 2^{-k} |A \cap \Delta_k|$$
from which (by using a simple direct argument or the Silverman–Toeplitz summability theorem) it follows that $A \in \mathcal{Z}$. ∎

Identifying $\mathcal{P} = \mathcal{P}(\mathbb{N})$ with the Cantor cube $\{0,1\}^{\mathbb{N}}$ via the map $A \mapsto 1_A$ makes $\mathcal{P}$ a compact metric space. Indeed, $\mathcal{P}$ is a commutative metric group (with the symmetric difference Δ as addition) whose topology can be defined by the group norm ϱ given by the formula
$$\varrho(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} 1_A(n).$$
Clearly, $A \mapsto \varrho(A)$ is also a submeasure (in fact, a σ-additive measure) on $\mathcal{P}$.

Consider also two other submeasures on $\mathcal{P}$, d and δ, defined by
$$d(A) = \limsup_n d_n(A), \quad \delta(A) = \sup_n d_n(A).$$
Clearly, $\mathcal{Z}$ is precisely the null ideal of d. Furthermore, we have

3.3. Proposition. *$\mathcal{Z}$ is a first Baire category subset of the space $(\mathcal{P}, \varrho)$.*

Proof. In view of the definition of d, it is not difficult to see that
$$\mathcal{Z} = \{A \in \mathcal{P} : d(A) = 0\} = \bigcap_{r=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{n=k}^{\infty} \{A \in \mathcal{P} : d_n(A) \leq (1 - q^{-1})r^{-1}\}.$$
Since the functions d_n are continuous on $(\mathcal{P}, \varrho)$, it follows that $\mathcal{Z}$ is an $F_{\sigma\delta}$ subset of $(\mathcal{P}, \varrho)$. On the other hand, $\mathcal{Z}$ is a proper subgroup of $\mathcal{P}$, so it has to be of the first Baire category by the Banach theorem [B, Ch. I, Th. 2]. ∎

A similar relationship can be displayed between $\mathcal{L}$ and $\mathcal{Z}$. But now, following Orlicz [O], the submeasure δ is used to convert $\mathcal{P}$ into a *complete* metric group $(\mathcal{P}, \delta)$ in which $\mathcal{Z}$ is a closed subgroup. As is easily seen, the δ-topology is stronger than the ϱ-topology.

3.4. Proposition. *The ideal $\widetilde{\mathcal{L}}$ in $\mathcal{P}$ generated by $\mathcal{L}$ is a first Baire category subset of the space $(\mathcal{Z}, \delta)$.*

Proof. Note that $\widetilde{\mathcal{L}} = \bigcup_k \mathcal{L}_k$, where $\mathcal{L}_k$ is the class of sets that can be written as the union of at most k lacunary sets. From a result in [DL5] it follows that each $\mathcal{L}_k$ is an $F_{\sigma\delta}$ subset of $(\mathcal{P}, \varrho)$. In consequence, $\widetilde{\mathcal{L}}$ is an $F_{\sigma\delta\sigma}$ subset of $(\mathcal{P}, \varrho)$ and, a fortiori, of $(\mathcal{Z}, \delta)$. Moreover, $\widetilde{\mathcal{L}}$ is a proper subset of $\mathcal{Z}$. The theorem of Banach applies again. ∎

Remark. As mentioned in the introduction, Orlicz uses Baire category methods in [O] which are connected to the techniques of so-called Saks spaces developed by him in several papers. (Orlicz works with the metric spaces H and H^* of zero-one sequences which correspond to $(\mathcal{P}, \varrho)$ and $(\mathcal{Z}, \delta)$, respectively.) The two propositions above, besides being of interest in themselves, show that no results about the convergence or boundedness of $\mathcal{Z}$-convergent or $\mathcal{Z}$-bounded series can be reduced to the "second category results" of Orlicz in $\mathcal{P}$ and, likewise, no results about the convergence or boundedness of

$\mathcal{L}$-convergent or $\mathcal{L}$-bounded series can be reduced to the "second category results" of Orlicz in $\mathbb{Z}$ (compare [O] for the exact statements).

4. $\mathcal{L}$-convergence and boundedness in terms of finite r-rare sets. As is well known, a series $\sum_n x_n$ in a sequentially complete TVS X is subseries (or unconditionally) convergent iff it satisfies the following Cauchy type condition:

- For every neighborhood U of zero in X there exists $m \in \mathbb{N}$ such that $\sum_{n\in F} x_n \in U$ for all $F \in \mathcal{F}(\mathbb{N})$ with $\min F \geq m$.

We will see below that $\mathcal{L}$-convergence and $\mathcal{L}$-boundedness admit similar characterizations. For $m, r \in \mathbb{N}$, let

$\mathcal{F}(m,r)$ = the family of all finite r-rare sets $F \subset \mathbb{N}$ with $\min F \geq m$.

4.1. PROPOSITION. *A series $\sum_n x_n$ in a sequentially complete* TVS *X is $\mathcal{L}$-convergent if and only if*

(LC) *for every neighborhood U of zero in X there exist $m, r \in \mathbb{N}$ such that*

$$\sum_{n\in F} x_n \in U \quad \text{for all } F \in \mathcal{F}(m,r).$$

Proof. "Only if": Suppose (LC) is false for some U. Then we can construct a sequence (A_r) in $\mathcal{F}(\mathbb{N})$ such that each A_r is r-rare, $\max A_r + r < \min A_{r+1}$ and $\sum_{n\in A_r} x_n \notin U$. Then the union A of the A_r's is lacunary and the series $\sum_{n\in A} x_n$ is not convergent.

"If": Let $A \in \mathcal{L}$. Take any neighborhood U of zero in X and next choose m and r according to (LC). Since A is lacunary, there is $k \geq m$ such that the tail set $A(k)$ is r-rare. In view of (LC), for every finite subset F of $A(k)$ one has $\sum_{n\in F} x_n \in U$. Thus the subseries $\sum_{n\in A} x_n$ is (unconditionally) Cauchy. ∎

The next two propositions are established by a similar argument.

4.2. PROPOSITION. *A series $\sum_n x_n$ in a* TVS *X is $\mathcal{L}$-bounded if and only if*

(LB) *for every neighborhood U of zero in X there exist $m, r, s \in \mathbb{N}$ such that*

$$\sum_{n\in F} x_n \in sU \quad \text{for all } F \in \mathcal{F}(m,r).$$

For a series in a TVS, the properties of being *metrically bounded*, *subseries metrically bounded*, and *$\mathcal{L}$-metrically bounded* are defined in an obvious way. As is easily seen, a series is subseries metrically bounded iff it is *perfectly metrically bounded*, i.e., the set of all its finite sums is metrically bounded. (This will be generalized in Corollary 4.6 below.)

4.3. PROPOSITION. *A series $\sum_n x_n$ in a* TVS *X is $\mathcal{L}$-metrically bounded if and only if*

(LM) *for every neighborhood U of zero in X there exist $m, r, s \in \mathbb{N}$ such that*

$$\sum_{n\in F} x_n \in U + \ldots + U \ (s \text{ summands}) \quad \text{for all } F \in \mathcal{F}(m,r).$$

Also perfect boundedness can be expressed in similar terms; note, however, the difference between (LB) and condition (PB) below.

4.4. PROPOSITION. *A series $\sum_n x_n$ in a* TVS *X is perfectly bounded if and only if*

(PB) *there exists $r \in \mathbb{N}$ such that for every neighborhood U of zero in X there exist $m, s \in \mathbb{N}$ such that*

$$\sum_{n\in F} x_n \in sU \quad \text{for all } F \in \mathcal{F}(m,r).$$

Proof. "Only if" holds trivially with $r = 1$.

"If": Take any neighborhood V of zero and next a balanced neighborhood U of zero such that $U + \ldots + U \subset V$ ($r+1$ summands). Choose $m, s \in \mathbb{N}$ according to (PB). Let $t \geq s$ be such that $\sum_{n\in F} x_n \in tU$ for all $F \subset \{1,\ldots,m\}$. Now, if $F \in \mathcal{F}(\mathbb{N})$, define $F_0 = F \cap \{1,\ldots,m\}$ and choose a partition $F_1,\ldots,F_r$ of $F \setminus F_0$ into r-rare subsets (see Proposition 2.6(a)). Then

$$\sum_{n\in F} x_n = \sum_{i=0}^{r} \sum_{n\in F_i} x_n \in tU + (sU + \ldots + sU) \subset t(U + \ldots + U) \subset tV,$$

where the last two sums have $r+1$ summands. This completes the proof. ∎

4.5. PROPOSITION. *Let $\varphi : \mathcal{F}(\mathbb{N}) \to \mathbb{R}_+$ be a subadditive set function. Suppose that*

$$(*) \qquad \sup_n \varphi(A \cap \{1,\ldots,n\}) < \infty \quad \text{for every } A \in \mathcal{L}.$$

Then φ is bounded, that is, $\sup\{\varphi(F) : F \in \mathcal{F}(\mathbb{N})\} < \infty$.

Proof. We first show that

$$(**) \qquad \text{there exist } m, r, s \in \mathbb{N} \text{ such that } \varphi(F) \leq s \text{ for all } F \in \mathcal{F}(m,r).$$

Suppose $(**)$ is false. Then we can construct a sequence (A_r) in $\mathcal{F}(\mathbb{N})$ such that each A_r is r-rare, $\max A_r + r < \min A_{r+1}$ and $\varphi(A_r) > r$. Then the union A of the A_r's is lacunary and condition $(*)$ is not satisfied for A.

Now, take any $F \in \mathcal{F}(\mathbb{N})$. Let $F_0 = F \cap \{1, \ldots, m\}$, and choose a partition $F_1, \ldots, F_r$ of $F \setminus F_0$ into r-rare subsets. Define $t = \sup\{\varphi(E) : \max E < m\}$. Then, by (**), $\varphi(F) \le \sum_{j=0}^{r} \varphi(F_j) \le t + rs$, and the assertion follows. ∎

4.6. COROLLARY. *If a series in a* TVS *is $\mathcal{L}$-metrically bounded (in particular, $\mathcal{L}$-convergent), then it is perfectly metrically bounded.*

Proof. This is an easy consequence of Proposition 4.3 or 4.5. If the series in question is $\mathcal{L}$-convergent, Proposition 4.1 can be applied as well. ∎

REMARK. For an example of an $\mathcal{L}$-convergent series that is not perfectly (topologically) bounded, see Example 11.1 below.

A TVS is said to be *locally pseudoconvex* if its topology can be defined by a family of F-seminorms each of which is an r-seminorm for some $0 < r \le 1$ (see [J, §6.5]).

4.7. COROLLARY. *A* TVS *in which metrically bounded sets are bounded has the Lacunary Boundedness Property. In particular, every locally pseudoconvex* TVS *has the Lacunary Boundedness Property.*

REMARK. Proposition 4.5, applied to $\varphi(F) := \sum_{n\in F} |a_n|$, gives the result (B) quoted in the Introduction.

5. Lacunary Convergence Property and copies of c_0. We start with the following.

5.1. PROPOSITION. *If a* TVS *X has the Zero-Density Convergence Property, then it contains no copy of c_0.*

Proof. It suffices to construct a $\mathcal{Z}$-convergent series in c_0 which is nonconvergent. Let (e_n) be the sequence of unit vectors in c_0. Define a sequence (x_n) in c_0 as follows:

$$x_n = 2^{-k} e_{k+1} \quad \text{for } n \in \Delta_k,\ k = 0, 1, 2, \ldots,$$

where $\Delta_k = \{n \in \mathbb{N} : 2^k \le n < 2^{k+1}\}$. Then for any $A \subset \mathbb{N}$,

$$\Big\| \sum_{n\in A\cap\Delta_k} x_n \Big\| = 2^{-k} |A \cap \Delta_k|.$$

From this, and in view of Proposition 3.2, it follows easily that the series $\sum_{n\in A} x_n$ converges iff $A \in \mathcal{Z}$. ∎

Since the proof of the next proposition appeals directly to (B) of the introduction, we decided to state it separately; it will be generalized in Theorem 5.3.

5.2. PROPOSITION. *A Banach space (more generally, a sequentially complete locally convex space) has the Lacunary Convergence Property iff it contains no copy of c_0.*

Proof. In view of Proposition 5.1, we only have to verify the "if" part. Assume a Banach space X contains no copy of c_0, and let $\sum_n x_n$ be an $\mathcal{L}$-convergent series in X. Then applying (B) it is easily seen that $\sum_n |x^*(x_n)| < \infty$ for all $x^* \in X^*$. Hence, by a well-known result of Bessaga and Pełczyński [BP, Thm. 5], the series $\sum_n x_n$ converges in X. ∎

In a locally pseudoconvex space, every perfectly bounded series is convexly bounded (cf. [R, Thm. 3.6.11]). This fact combined with Corollary 4.7, Proposition 5.1, and [DL3, Prop. 1.3] yields the following.

5.3. THEOREM. *For a sequentially complete locally pseudoconvex* TVS *X, the following are equivalent.*

(a) *X contains no copy of c_0.*
(b) *X has the Zero-Density Convergence Property.*
(c) *X has the Lacunary Convergence Property.*

We say that a TRS X *contains a positive copy* of c_0 if there is a positive linear homeomorphism from the Banach lattice c_0 onto a subspace of X, and that it *contains a lattice copy* of c_0 if there is a homeomorphic Riesz isomorphism from c_0 onto a sublattice of X.

Since the series in c_0 constructed in the proof of Proposition 5.1 is positive, that proof also shows the following.

5.4. PROPOSITION. *If a* TRS *X has the Positive Zero-Density Convergence Property, then it contains no positive copy of c_0.*

By [DL3, Thm. 2.4], a sequentially complete TRS X is σ-Lebesgue and σ-Levi iff it contains no positive (or lattice) copy of c_0. Hence also the following "positive" version of Theorem 5.3 is true:

5.5. THEOREM. *For a sequentially complete locally pseudoconvex* TRS *X the following are equivalent.*

(a) *X contains no positive (or lattice) copy of c_0.*
(b) *X has the Positive Zero-Density Convergence Property.*
(c) *X has the Positive Lacunary Convergence Property.*

6. Positive Lacunary Convergence Property in $L_0(\mu)$ spaces. We refer the reader to Section 2 for the exact meaning of the term "submeasure space". The main result of this section is the following.

6.1. THEOREM. *If (S, Σ, μ) is a submeasure space, then $L_0(\mu)$ has the Positive Lacunary Convergence Property. Thus whenever a positive series $\sum_n f_n$ in $L_0(\mu)$ is such that, for all $Z \in \mathcal{L}$, $\sum_{n\in Z} f_n(s) < \infty$ a.e. in S, then $\sum_{n=1}^{\infty} f_n(s) < \infty$ a.e. in S.*

Clearly, in proving this we may (and will) assume that the submeasure μ is order continuous. We first show the following.

6.2. LEMMA. *Let $0 \le f_n \in L_0(\mu)$ be such that $\sum_{n=1}^{\infty} f_n(s) = \infty$ a.e. in S. Then for all $m, r \in \mathbb{N}$ and $\varepsilon > 0$ there exists a finite subset K of $\mathbb{N}$ such that* 1) $m \le \min K$, 2) *K is r-rare,* 3) $\mu(\{s \in S : \sum_{i \in K} f_i(s) < 1\}) < \varepsilon$.

Proof. For $j = 1, \ldots, r$, let

$$M_j = \{j + (k-1)r : k \in \mathbb{N}\}, \qquad D_j = \Big\{s \in S : \sum_{i \in M_j} f_i(s) = \infty\Big\}.$$

Clearly, $S = D_1 \cup \ldots \cup D_r$ (a.e.).

By Egoroff's theorem, there exists $E_1 \subset D_1$ with $\mu(D_1 \setminus E_1) < \varepsilon/r$ such that if $M_1(n) := \{i \in M_1 : m \le i \le n\}$, then

$$\sum_{i \in M_1(n)} f_i(s) \to \infty \quad \text{as } n \to \infty, \text{ uniformly for } s \in E_1.$$

Hence there is $n_1 \ge m$ so large that if $K_1 := M_1(n_1)$, then

$$\sum_{i \in K_1} f_i(s) \ge 1 \quad \text{for all } s \in E_1.$$

Let $m_1 = n_1 + r$. Since

$$\sum_{m_1 \le i \in M_2} f_i(s) = \infty \quad \text{for } s \in D_2,$$

we can find as above a set $E_2 \subset D_2$ with $\mu(D_2 \setminus E_2) < \varepsilon/r$, and $n_2 \ge m_1$ so that, writing $K_2 = \{i \in M_2 : m_1 \le i \le n_2\}$, we have

$$\sum_{i \in K_2} f_i(s) \ge 1 \quad \text{for all } s \in E_2.$$

We proceed by an obvious induction. Finally, set $K = K_1 \cup \ldots \cup K_r$ and $E = E_1 \cup \ldots \cup E_r$. Clearly, K satisfies conditions 1) and 2). Since $\mu(S \setminus E) < \varepsilon$ and $\sum_{i \in K} f_i(s) \ge 1$ for all $s \in E$, also condition 3) is satisfied. ■

Proof of Theorem 6.1. Suppose $\sum_{n=1}^{\infty} f_n(s) = \infty$ on a set $E \in \Sigma$ with $\mu(E) > 0$. Then, of course, we may assume that $E = S$. Applying Lemma 6.2, we find a sequence (K_r) of finite subsets of $\mathbb{N}$ and a sequence (E_r) in Σ such that for every r,

$$K_r \text{ is } r\text{-rare}, \quad \max K_r + r < \min K_{r+1},$$
$$\mu(S \setminus E_r) < 2^{-r}, \quad \sum_{i \in K_r} f_i(s) \ge 1 \quad \text{for all } s \in E_r.$$

Let

$$Z = \bigcup_{r=1}^{\infty} K_r, \qquad E = \bigcup_{r=1}^{\infty} \bigcap_{k=r}^{\infty} E_k.$$

Then $Z \in \mathcal{L}$ by Proposition 3.1(b), and $\mu(S \setminus E) = 0$. If $s \in E$, then there is r such that for every $k \ge r$ we have $s \in E_k$ and, therefore, $\sum_{i \in K_k} f_i(s) \ge 1$. In consequence, $\sum_{i \in Z} f_i(s) = \infty$, contradicting the hypothesis of the theorem. ■

7. Lacunary Convergence Property and Lacunary Boundedness Property in $L_0(\lambda, E)$ spaces. By Theorem 2.4, for every measure λ of type (SC) the space $L_0(\lambda)$ has Property (O). However, as is well known, if λ is not purely atomic, then $L_0(\lambda)$ is not locally pseudoconvex (see e.g. [R]). Thus Theorem 5.3 is not applicable to $L_0(\lambda)$. Nevertheless, as was shown in [DL1], $L_0(\lambda)$ spaces behave very well as far as the lacunary properties are concerned. The main results of this section generalize those obtained in [DL1] to the case of spaces of vector-valued functions.

7.1. THEOREM. *If (S, Σ, λ) is a measure space and E is a Banach space, then the space $L_0(\lambda, E)$ has the Lacunary Boundedness Property.*

From this we easily deduce a much more interesting result:

7.2. THEOREM. *If (S, Σ, λ) is a measure space of type (SC) and E is a Banach space containing no copy of c_0, then the space $L_0(\lambda, E)$ has the Lacunary Convergence Property.*

Proof. By Theorem 2.4, $L_0(\lambda, E)$ has Property (O). Therefore, in view of Proposition 2.1, the assertion is immediate from Theorem 7.1. ■

We now start proving Theorem 7.1. Below, if $g \in L_0(\lambda, E)$, then $\|g\|$ denotes the function $s \mapsto \|g(s)\|$ in $L_0(\lambda)$.

7.3. LEMMA. *Let (S, Σ, λ) be a measure space and E a Banach space.*

(a) *If a series $\sum_n g_n$ in $L_0(\lambda, E)$ is subseries convergent (or Cauchy), then*

$$\lim_{r \to \infty} \sup_{i \in I_r} \|g_i\| = 0 \quad \text{in } L_0(\lambda)$$

for every sequence (I_r) of finite subsets of $\mathbb{N}$ such that $\min I_r \to \infty$.

(b) *If a series $\sum_n g_n$ in $L_0(\lambda, E)$ is perfectly bounded, then the sequence*

$$\sup_{1 \le i \le n} \|g_i\|, \qquad n = 1, 2, \ldots,$$

is bounded in $L_0(\lambda)$.

Proof. Assume, as we may, that $\lambda(S) = 1$. The topology of $L_0(\lambda, E)$ is defined by the F-norm d given by $d(f) = \inf\{a > 0 : \lambda\{\|f\| > a\} \le a\}$; likewise for $L_0(\lambda)$. According to Sublemma 1(b) in [D1], for every finite family $(h_i)_{i \in I}$ in $L_0(\lambda, E)$ there exists $J \subset I$ such that $d(\sum_{j \in J} h_j) \ge \frac{1}{4} d(\sup_{i \in I} \|h_i\|)$. From this the assertions follow immediately. ■

For the purpose of the rest of the proof we introduce some technical terminology. Let $K \subset \mathbb{N}$. An *interval* in K is a subset $I = \{k \in K : m \leq k \leq n\}$, where $m, n \in K$ and $m \leq n$. An *interval partition* of K is a (finite or infinite) partition of K consisting of intervals in K.

Let $\sum_{n=1}^{\infty} f_n$ be a series in $L_0(\lambda, E)$. Given $r \in \mathbb{N}$ and $M > 0$, we shall say that a set $B \in \Sigma$ has

◇ *Property* (b_r) if there exists an infinite r-rare subset K of $\mathbb{N}$ and an interval partition (I_n) of K such that

$$\limsup_{n\to\infty} \Big\| \sum_{i\in I_n} f_i(s) \Big\| = \infty \quad \text{for all } s \in B;$$

◇ *Property* (b_{rM}) if there exists a finite r-rare subset K of $\mathbb{N}$ and an interval partition $I_1, \ldots, I_q$ of K such that

$$\sup_{1\leq n\leq q} \Big\| \sum_{i\in I_n} f_i(s) \Big\| \geq M \quad \text{for all } s \in B.$$

REMARK. By applying Egoroff's theorem it is easily seen that if a set B of finite λ measure has Property (b_r), then for every M it contains subsets C having Property (b_{rM}) and such that $\lambda(C)$ is arbitrarily close to $\lambda(B)$.

The key technical ingredients of our proof of Theorem 7.1 are the following two lemmas. We shall say that a series $\sum_n f_n$ in $L_0(\lambda, E)$ is *perfectly bounded in measure on a set* $A \in \Sigma$ if the series $\sum_n f_n 1_A$ is perfectly bounded in $L_0(\lambda, E)$.

7.4. LEMMA. *Assume that a series $\sum_n f_n$ in $L_0(\lambda, E)$ is not perfectly bounded in measure on a set A with $0 < \lambda(A) < \infty$. Then for every $r \in \mathbb{N}$ there exists $B \subset A$ with $\lambda(B) > 0$ such that B has Property* (b_r).

Proof. Fix $r \in \mathbb{N}$. By the assumption and Proposition 4.4, condition (PB) is not satisfied for the series $\sum_n f_n 1_A$ in $X = L_0(\lambda, E)$. It follows that for some $\varepsilon_r > 0$ we can find a sequence (I_k) of finite r-rare subsets of $\mathbb{N}$ with $\max I_k + r < \min I_{k+1}$ and

$$\lambda(B_k) > \varepsilon_r, \quad \text{where} \quad B_k := \Big\{ s \in A : \Big\| \sum_{n\in I_k} f_n(s) \Big\| \geq k\varepsilon_r \Big\}.$$

Clearly, the set

$$B := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k$$

is of λ measure $\geq \varepsilon_r$. Also, the set $K := \bigcup_{k=1}^{\infty} I_k$ is r-rare, and (I_k) is an interval partition of K. Moreover, if $s \in B$, then $\|\sum_{n\in I_k} f_n(s)\| \geq k\varepsilon_r$ for infinitely many k. Hence $\limsup_k \|\sum_{n\in I_k} f_n(s)\| = \infty$ for all $s \in B$. Thus B has Property (b_r). ∎

7.5. LEMMA. *Let $C \in \Sigma$, $0 < \lambda(C) < \infty$, and assume that a series $\sum_n f_n$ in $L_0(\lambda, E)$ is perfectly bounded in measure on no subset A of C with $\lambda(A) > 0$. Then for all $r \in \mathbb{N}$, $M > 0$ and $\eta > 0$ there exists $D \subset C$ with $\lambda(C \setminus D) < \eta$ such that D has Property* (b_{rM}).

Proof. Let $\mathcal{B}$ be a maximal disjoint family of subsets B of C such that $\lambda(B) > 0$ and B has Property (b_r). Then $\mathcal{B}$ is countable, say $\mathcal{B} = \{B_1, B_2, \ldots\}$. Moreover, by Lemma 7.4, $\lambda(C \setminus \bigcup_j B_j) = 0$. Fix k such that $\lambda(C \setminus B) < \eta/2$, where $B := \sum_{j=1}^{k} B_j$. Now, each B_j has Property (b_r), hence there exists an infinite r-rare subset K_j of $\mathbb{N}$, and an interval partition $(I_n(j))$ of K_j such that

$$\sup_{n\geq N} \Big\| \sum_{i\in I_n(j)} f_i(s) \Big\| = \infty \quad \text{for all } s \in B_j \text{ and } N = 1, 2, \ldots$$

By applying Egoroff's theorem and proceeding by an easy induction, we can find for $j = 1, \ldots, k$ a set $C_j \subset B_j$ and a block $\{I_n(j) : m_j \leq n \leq n_j\}$ in the sequence $(I_n(j))$, where $m_j \leq n_j$, such that

$$\begin{aligned}
\lambda(B_j \setminus C_j) &< \eta/(2k) && \text{for } j = 1, \ldots, k,\\
\max I_{n_j}(j) + r &\leq \min I_{m_{j+1}}(j+1) && \text{for } j = 1, \ldots, k-1,\\
\sup_{m_j\leq n\leq n_j} \Big\| \sum_{i\in I_n(j)} f_i(s) \Big\| &\geq M && \text{for } s \in C_j,\ j = 1, \ldots, k.
\end{aligned}$$

Define

$$D = \bigcup_{j=1}^{k} C_j, \qquad K = \bigcup_{j=1}^{k} \bigcup_{n=m_j}^{n_j} I_n(j).$$

Then $\lambda(C \setminus D) < \eta$, the set K is finite and r-rare, the family

$$\{I_n(j) : 1 \leq j \leq k,\ m_j \leq n \leq n_j\} = \{I_1, \ldots, I_q\}$$

(properly arranged) is an interval partition of K, and

$$\sup_{1\leq n\leq q} \Big\| \sum_{i\in I_n} f_i(s) \Big\| \geq M \quad \text{for all } s \in D. \ ∎$$

Proof of Theorem 7.1. We may and will assume that the measure λ is finite. Let $\sum_n f_n$ be an $\mathcal{L}$-bounded series in $L_0(\lambda, E)$. Suppose it is not perfectly bounded in $L_0(\lambda, E)$. Let $\mathcal{A}$ be a maximal disjoint family consisting of sets A with $\lambda(A) > 0$ such that our series is perfectly bounded in measure on A. Then $\mathcal{A}$ is countable, and it is clear that the series is perfectly bounded in measure on the union A_0 of $\mathcal{A}$. From this and the maximality of $\mathcal{A}$ it follows that the set $C := S \setminus A_0$ is of positive λ measure and satisfies the assumptions of Lemma 7.5. Now, applying Lemma 7.5, first with $r =$

$M=1$ to the whole series $\sum_{n=1}^{\infty} f_n$ and then with $r=M=2,3,\ldots$ to its remainders $\sum_{n=N_r}^{\infty} f_n$ with N_r increasing sufficiently fast, we construct

- sequences (K_r) and (I_n) of finite subsets of $\mathbb{N}$,
- a sequence $1=p_1<p_2<\ldots$ in $\mathbb{N}$, and
- a sequence (D_r) of measurable subsets of C

such that for every r:

- K_r is r-rare and $\max K_r+r+1\le \min K_{r+1}$,
- the family $\{I_n : p_r\le n<p_{r+1}\}$ is an interval partition of K_r,
- $\lambda(D_r)>(1-2^{-r})\lambda(C)$

and

$$\gamma_r(s):=\sup_{p_r\le n<p_{r+1}}\|g_n(s)\|\ge r \quad \text{for all } s\in D_r, \text{ where } \quad g_n:=\sum_{i\in I_n} f_i.$$

Set $D=\bigcap_{r=1}^{\infty} D_r$. Then $\lambda(D)>0$ and

$$(*)\qquad \gamma_r(s)\ge r \quad \text{for all } s\in D \text{ and } r=1,2,\ldots$$

Moreover, $Z:=\bigcup_{r=1}^{\infty} K_r\in\mathfrak{L}$ (by 3.1(b)) and the sequence (I_n) is an interval partition of Z. By assumption, the series $\sum_{i\in Z} f_i$ is subseries (or perfectly) bounded in $L_0(\lambda,E)$. Hence also the series $\sum_n g_n$ is subseries bounded in $L_0(\lambda,E)$. Therefore, by Lemma 7.3(b), the sequence (γ_r) is bounded in $L_0(\lambda)$, contradicting $(*)$. ∎

Remark. A direct proof of Theorem 7.2 would be an almost verbatim repetition of the above proof, with some slight simplifications possible: We would need Proposition 4.4 and Lemmas 7.3(a), 7.4, and 7.5 with $M=1$. Then, in the proof of Theorem 7.2, we would have to show that every $\mathfrak{L}$-convergent series $\sum_n f_n$ in $L_0(\lambda,E)$ is perfectly bounded. To this end, we would proceed exactly as in the present proof of Theorem 7.1, though with $M=1$ for each r, arriving at a contradiction with Lemma 7.3(a).

In the scalar case, that is, for the space $L_0(\lambda)$, we could reach a contradiction in a different way: From condition $(*)$, modified as indicated above, it follows that

$$\sum_{n=p_r}^{p_{r+1}-1}|g_n(s)|^2\ge 1 \quad \text{for all } s\in D \text{ and } r=1,2,\ldots$$

On the other hand, since the series $\sum_n g_n$ is unconditionally convergent, we have $\sum_n |g_n(s)|^2<\infty$ for a.e. $s\in S$, by Orlicz's Theorem. This is, more or less, how we proved the scalar version of Theorem 7.2 in [DL1].

In the following two sections we shall assume that (S,Σ,μ) is a submeasure space, in the sense of Section 2.

8. Positive Lacunary Convergence Property and Positive Lacunary Boundedness Property. Let X be a Lebesgue Levi TRS or, equivalently, a complete TRS containing no positive (or lattice) copy of c_0 (see [DL3, Thm. 2.5]). Ideally, we would like X to have the Positive LCP. Unfortunately, as shown in Example 11.1 below, there exists a Lebesgue Levi F-lattice (of measurable functions) without the Positive LCP. Thus some additional conditions on X in order to get this property are needed.

We shall say that a TRS X of μ-measurable functions has

- the *metric Levi* (resp. *σ-Levi*) *property* (in $L_0(\mu)$) if, whenever a net (resp. sequence) (f_i) in X is metrically bounded and $0\le f_i\uparrow f$, where $f\in L_0(\mu)$, then $f\in X$;

and that a general TRS X has

- the *disjoint metric Levi property* if every metrically bounded positive disjoint series in X has an order sum in X.

Remark. Here and in what follows, by a *disjoint series* we mean one with pairwise disjoint terms. Note that a disjoint series in a TRS of μ-measurable functions always has an order (= pointwise) sum in $L_0(\mu)$, while such a series in a general TRS always has an order sum in the universal completion of the space.

8.1. Proposition. *A TRS X of μ-measurable functions has the metric σ-Levi property if (and only if) it has the disjoint metric Levi property.*

Proof. Assume a sequence $(f_n)\subset X$ is metrically bounded and $0\le f_n\uparrow f\in L_0(\mu)$. Set $E_0=\emptyset$ and $E_n=\{s\in S: 2f_n(s)\ge f(s)\}$ for $n\ge 1$. Since $f1_{E_n\setminus E_{n-1}}\le 2f_n$, it follows that $\sum_{n=1}^{\infty} f1_{E_n\setminus E_{n-1}}$ is a metrically bounded positive disjoint series in X converging to f in $L_0(\mu)$. Hence $f\in X$ by the disjoint metric Levi property. ∎

By Theorem 2.3, if X is a Lebesgue TRS, then its universal completion can be identified with $L_0(\mu)$ for a submeasure μ of type (C). Thus, for such X, if the word "metrically" is omitted (or replaced by "topologically") in the above definitions, one obtains the familiar (and weaker) concepts of the Levi, σ-Levi, and disjoint Levi property, respectively. This motivates our terminology.

Of course, for a TRS of measurable functions, the usual Levi properties coincide with their metric counterparts provided the metrically bounded sets are topologically bounded; e.g., it is so in locally pseudoconvex spaces. It is less obvious that also the Musielak–Orlicz spaces $L_\varphi(\lambda)$ (see e.g. [Mu], [T], [W]), in which metrically and topologically bounded sets do not coincide in general, all have the metric σ-Levi property. This follows from the fact that

$L_\varphi(\lambda)$, considered with its usual F-norm

$$\|f\|_\varphi = \inf\Big\{r>0 : \int_S \varphi(r^{-1}|f(s)|, s)\, d\lambda(s) \le r\Big\},$$

has a fundamental system of F-norm-bounded sets that are closed in $L_0(\lambda)$.

8.2. THEOREM. *If a* TRS X *of* μ*-measurable functions is* σ*-Lebesgue and has the metric* σ*-Levi property, then it has the Positive Lacunary Convergence Property.*

Proof. Consider a positive $\mathcal{L}$-convergent series $\sum_n f_n$ in X. Clearly, it is $\mathcal{L}$-convergent in $L_0(\mu)$ and so, by Theorem 6.1, convergent in $L_0(\mu)$ to some f. Combining this with Corollary 4.6 and the metric σ-Levi property of X, we infer that $f \in X$. Since X is σ-Lebesgue, the series converges to f in X. ∎

In view of Theorem 2.3, the preceding result has the following "abstract" variant.

8.3. THEOREM. *If a* TRS X *is Dedekind complete, Lebesgue, and has the disjoint metric Levi property, then it has the Positive Lacunary Convergence Property.*

REMARK. One would like to assume in Theorem 8.3 that X is merely σ-Lebesgue. However, it is the Lebesgue property rather than the σ-Lebesgue property which was used in the proof of Theorem 2.7 in [L2].

If a TVS X is continuously included in a TVS Y, then we shall say that

- X is *polar* (resp. *sequentially polar*) *in* Y if X has a base of neighborhoods of zero that are closed in the topology induced from Y.

8.4. THEOREM. *If a* TRS X *of* μ*-measurable functions is sequentially polar in* $L_0(\mu)$ *and has the metric* σ*-Levi property, then it has the Positive Lacunary Boundedness Property. In fact, if* $\sum_n f_n$ *is a positive* $\mathcal{L}$*-bounded series in* X*, then each of its subseries is order convergent in* X*, and the set* $\mathcal{S} = \{\sum_{n\in N} f_n : N \in \mathcal{P}(\mathbb{N})\}$ *of all sums of the series is bounded in* X.

Proof. It is clear that the series $\sum_n f_n$ is $\mathcal{L}$-convergent in $L_0(\mu)$. Hence, by Theorem 6.1, it is subseries convergent in $L_0(\mu)$. Combining this with Corollary 4.6 and the metric σ-Levi property yields $\mathcal{S} \subset X$. In consequence, the vector measure $m : \mathcal{P}(\mathbb{N}) \to X$ associated with our series is countably additive in the topology of $L_0(\mu)$. As X is sequentially polar in Y, we are in a position to apply [L1, Thm. 3] and conclude that $\mathcal{S} = m(\mathcal{P}(\mathbb{N}))$ is a bounded set in X. ∎

REMARK. Let X be a TRS of μ-measurable functions. Then: X is polar in $L_0(\mu)$ iff X has the Fatou property; and if X is sequentially polar in $L_0(\mu)$, then X has the σ-Fatou property. All these properties coincide when the submeasure μ is countably o.c.-decomposable, i.e., $S = \bigcup_n S_n$ for a sequence $(S_n) \subset \Sigma^+_{\rm oc}(\mu)$.

9. Metric bounded closedness. Below, $X = (X, \tau)$ is a TVS of E-valued μ-measurable functions, or a TRS of scalar μ-measurable functions if it is so stated explicitly, continuously included in a suitable L_0 space.

In this section, we briefly investigate a metric version of a property used in [DL4], viz., the property of X being "metrically-boundedly closed" in $L_0(\mu, E)$. It will be applied in the next section in much the same way the metric Levi property was in the preceding one.

We shall say that the space X is

- *m-boundedly closed* (resp. *m-boundedly sequentially closed*) *in* $L_0(\mu, E)$ if, for every metrically bounded subset of X, its closure (resp. sequential closure) in $L_0(\mu, E)$ is a subset of X;
- *disjointly m-boundedly closed* if, for every metrically bounded disjoint series in X, its sum in $L_0(\mu, E)$ belongs to X.

Note that if X is a TRS of μ-measurable functions, then X has the disjoint metric Levi property iff it is disjointly m-boundedly closed.

Before proceeding further, let us recall some notions and conditions from [DL4] that will be used below. Let Σ' stand for the family of countable unions of sets in Σ on which μ is order continuous. A *fundamental subspace* of X is one of the form

$$X_A = \{f \in X : f = 1_A f\}, \qquad \text{where } A \in \Sigma'.$$

A *projective net* in X is any net $(f_A : A \in \Sigma')$ in X such that $f_A = 1_A f_B$ whenever $A, B \in \Sigma'$ and $A \subset B$. The space X is said to be

- *projectively complete* if every projective Cauchy net in X has a limit in X;
- *projectively closed in* $L_0(\mu, E)$ if whenever a projective net in X has a limit in $L_0(\mu, E)$, the limit belongs to X;
- *piecewise uniformly closed* in $L_0(\mu, E)$ if each set $A \in \Sigma$ with $\mu(A) > 0$ contains a set $B \in \Sigma$ with $\mu(B) > 0$ and such that whenever (f_n) is a sequence in X_B, $f \in L_0(\mu, E)$, and $f_n \to f$ uniformly, then $f \in X_B$.

Note that if X is quasi-complete, then it is projectively complete and piecewise uniformly closed in $L_0(\mu, E)$ (see [DL4, Prop. 5.4]). Also note that every TRS of μ-measurable functions is piecewise uniformly closed in $L_0(\mu)$.

Trivially, if a TRS of μ-measurable functions is m-boundedly closed (resp. sequentially closed) in $L_0(\mu)$, then it has the metric Levi (resp. σ-Levi) property. As for the converse, see the first proposition below. We omit the proofs of Propositions 9.1 and 9.3 below; they are similar to those of [DL3, Prop. 3.6] and [DL4, Prop. 5.6], respectively. Proposition 9.2 is easy.

Note that a fundamental subspace of X is m-boundedly closed iff it is m-boundedly sequentially closed.

9.1. PROPOSITION. *Let X be a* TRS *of μ-measurable functions.*

(a) *If X has the metric σ-Levi property in $L_0(\mu)$, then every fundamental subspace of X is m-boundedly closed in $L_0(\mu)$.*

(b) *If X has the metric Levi property in $L_0(\mu)$, then X is m-boundedly closed in $L_0(\mu)$.*

9.2. PROPOSITION. *The space X is m-boundedly closed in $L_0(\mu, E)$ iff every fundamental subspace of X is m-boundedly closed in $L_0(\mu, E)$, and X is projectively closed in $L_0(\mu, E)$.*

9.3. PROPOSITION. *A fundamental subspace X_A of the space X is m-boundedly closed in $L_0(\mu, E)$ iff X_A is piecewise uniformly closed in $L_0(\mu, E)$ and disjointly m-boundedly closed.*

Combining Propositions 9.2 and 9.3, we derive the following.

9.4. THEOREM. *The space X is m-boundedly closed in $L_0(\mu, E)$ iff X is piecewise uniformly closed in $L_0(\mu, E)$, each of its fundamental subspaces is disjointly m-boundedly closed, and X is projectively closed in $L_0(\mu, E)$.*

9.5. COROLLARY. *A* TRS *of μ-measurable functions is m-boundedly closed in $L_0(\mu)$ iff it is disjointly m-boundedly closed and projectively closed in $L_0(\mu)$.*

9.6. COROLLARY. *Let the space X be piecewise uniformly closed in $L_0(\mu, E)$, projectively complete, and sequentially μ-continuous. If each fundamental subspace of X is disjointly m-boundedly closed in $L_0(\mu, E)$, then X is m-boundedly closed in $L_0(\mu, E)$.*

Proof. As X is sequentially μ-continuous and projectively complete, it is projectively closed in $L_0(\mu, E)$. Apply Theorem 9.4. ∎

10. Lacunary Convergence Property and Lacunary Boundedness Property in spaces of Bochner measurable functions. The first two results below reveal the "abstract structure" of our main results stated in the second part of this section.

If a TVS X is continuously included in a TVS Y, then we shall say that

- X has the *Orlicz–Pettis Property relative to Y* if a series in X is subseries convergent provided it is subseries convergent in Y and the Y-sums of all the subseries are in X;
- X is *m-boundedly closed* (resp. *m-boundedly sequentially closed*) *in Y* if, for every metrically bounded subset of X, its closure (resp. sequential closure) in Y is a subset of X.

10.1. PROPOSITION. *Let a* TVS *X be continuously included in a* TVS *Y. Assume that X is m-boundedly sequentially closed in Y and has the Orlicz–Pettis Property relative to Y. Then, if Y has the Lacunary Convergence Property, so does X.*

Proof. Let $\sum_n x_n$ be an $\mathfrak{L}$-convergent series in X. Then it is also $\mathfrak{L}$-convergent in Y, hence subseries convergent in Y, by the LCP of Y. In view of Corollary 4.6, and as X is m-boundedly sequentially closed in Y, the Y-sums of all the subseries are actually in X. Apply the Orlicz–Pettis Property of X relative to Y. ∎

10.2. PROPOSITION. *Let a* TVS *X be continuously included in a* TVS *Y. Assume that X is m-boundedly sequentially closed and sequentially polar in Y, and that Y has Property (O). Then, if Y has the Lacunary Boundedness Property, so does X. In fact, every $\mathfrak{L}$-bounded series in X is subseries convergent in Y, and the set of all its Y-sums is a bounded subset of X.*

Proof. Let $\sum_n x_n$ be an $\mathfrak{L}$-bounded series in X. Then it is also $\mathfrak{L}$-bounded in Y, hence subseries convergent in Y, by the LBP and Property (O) of Y. In view of Corollary 4.6, and as X is m-boundedly sequentially closed in Y, the Y-sums of all the subseries are actually in X. Therefore, the associated vector measure $m : \mathcal{P}(\mathbb{N}) \to X$, defined by $m(N) = Y\text{-}\sum_{n\in N} x_n$, is countably additive in the topology of Y. By [L1, Thm. 3], its range is a bounded subset of X. ∎

Below, (S, Σ, λ) is a (locally finite) measure space, E is a Banach space, and X is a TVS of E-valued λ-measurable functions.

10.3. THEOREM. *Let the Banach space E contain no isomorphic copy of c_0, the measure λ be of type (SC), and assume that $X \subset L_0(\lambda, E)$ continuously. If X is λ-continuous and m-boundedly sequentially closed in $L_0(\lambda, E)$, then it has the Lacunary Convergence Property.*

Proof. In view of Theorems 2.5 and 7.2, X has the Orlicz–Pettis Property relative to $L_0(\lambda, E)$, and $L_0(\lambda, E)$ has the LCP. Apply Proposition 10.1. ∎

10.4. THEOREM. *Let the Banach space E contain no isomorphic copy of c_0, and assume that $X \subset L_0(\lambda, E)$ continuously. If X is polar in $L_0(\lambda, E)$, and each fundamental subspace of X is m-boundedly closed in $L_0(\lambda, E)$, then X has the Lacunary Boundedness Property.*

Proof. Let $\sum_n f_n$ be an $\mathfrak{L}$-bounded series in X. First consider the case when λ is σ-finite. Then, by Theorems 2.4 and 7.2, $L_0(\lambda, E)$ has both Property (O) and the LCP, whence also the LBP. Since X is also sequentially polar in $L_0(\lambda, E)$, applying Proposition 10.2 we see that the series $\sum_n f_n$ is perfectly bounded.

In the general case, let Σ' denote the family of all sets in Σ of σ-finite λ measure, upward directed by inclusion. By the previous case, for each $A \in \Sigma'$ the set $R_A = \{\sum_{n\in F} 1_A f_n : F \in \mathcal{F}(\mathbb{N})\}$ is a bounded subset of X. From this and the fact that Σ' is a σ-ideal it follows that also the union R of all R_A's is bounded in X. Let $\overline{R}$ denote the closure of R in the space X equipped with the topology induced from $L_0(\lambda, E)$. Using the assumption that X is polar in Y it is readily seen that $\overline{R}$ is bounded in X. Finally, for each $F \in \mathcal{F}(\mathbb{N})$ the net $(\sum_{n\in F} 1_A f_n : A \in \Sigma')$ converges in $L_0(\lambda, E)$ to $\sum_{n\in F} f_n$. In consequence, $\{\sum_{n\in F} f_n : F \in \mathcal{F}(\mathbb{N})\} \subset \overline{R}$. ∎

11. Examples and questions

11.1. EXAMPLE (A Lebesgue Levi F-lattice without the Positive Zero-Density Convergence Property or Positive Boundedness Property). Let L_0 = the L_0 space over the half-line $[0,\infty)$ with Lebesgue measure λ. Define $I_k = [k, k+1)$ for $k = 0,1,2,\ldots$ For each $f \in L_0$ let

$$\|f\|_{m,k} = \int_{I_k} \min(1, (k+1)^m|f|)\, d\lambda \quad \text{for } k,m = 0,1,2,\ldots$$

Note that $\|f\|_{m,k} = \|(k+1)^m f\|_{0,k}$. Next, let

$$\|f\|_m = \sup_k \|f\|_{m,k} \quad \text{for } m = 0,1,2,\ldots$$

Clearly, each $\|\cdot\|_m$ is a monotone FG-norm (in the sense of [D2]) on L_0. Thus $\|\cdot\|_m$ is subadditive, vanishes only at zero, and $\|f\|_m \le \|g\|_m$ whenever $f,g \in L_0$ and $|f| \le |g|$. Moreover, each $\|\cdot\|_m$ is lower semicontinuous on L_0. That is, if $f_n \to f$ in L_0 (i.e. in λ measure on sets of finite measure), then

$$\|f\|_m \le \liminf_n \|f_n\|_m.$$

This follows from the estimates $\|f\|_{m,k} \le \liminf_n \|f_n\|_{m,k} \le \liminf_n \|f_n\|_m$.

Note that $\|(k+1)^{-1}f\|_{m+1,k} = \|f\|_{m,k}$, whence $\|(k+1)^{-1}f\|_{m+1} \ge \|f\|_{m,k}$. Using this it is easy to see that for any $f \in L_0$ the following two conditions are equivalent:

$$(*) \qquad \lim_{t\to 0} \|tf\|_m = 0 \quad \text{for all } m,$$

$$(**) \qquad \lim_{k\to\infty} \|f\|_{m,k} = 0 \quad \text{for all } m.$$

Let X denote the solid subspace of L_0 consisting of all f satisfying the above conditions. By condition $(*)$, (the restriction to X of) each $\|\cdot\|_m$ is a monotone F-norm on X. In what follows we equip X with the sequence of F-norms $\|\cdot\|_m$ $(m = 0,1,2,\ldots)$, thus converting X into an F-lattice. Note that the inclusion $X \subset L_0$ is continuous. Also note that for each k, we have $L_0(I_k) \subset X$, and all the F-norms $\|\cdot\|_m$ are equivalent on $L_0(I_k)$. Hence each of the spaces $L_0(I_k)$ with its usual topology is a topological vector subspace of X.

Using condition $(**)$ it is easy to verify that X has the Lebesgue property. Similarly, using condition $(*)$ along with the lower semicontinuity of $\|\cdot\|_m$ on L_0, one verifies that X has the Levi property. Thus X is a complete, metrizable, Lebesgue, Levi TRS of λ-measurable functions on $[0,\infty)$. Hence, by [DL3, Th. 5.5], X contains no isomorphic copy of c_0.

For $k = 0,1,2,\ldots$, let $\{H_n : n = 2^k,\ldots,2^{k+1}-1\}$, be the partition of the interval I_k into 2^k subintervals of length 2^{-k}. Let f_n denote the characteristic function of H_n. For $A \subset \mathbb{N}$, let f_A stand for the pointwise sum of the series $\sum_{n\in A} f_n$. Then the following statements are equivalent:

(a) $A \in \mathcal{Z}$.
(b) $f_A \in X$.
(c) $\sum_{n\in A} f_n$ converges in X.
(d) $\sum_{n\in A} f_n$ is bounded in X.

To see this, observe that for each m

$$\|f_A\|_{m,k} = 2^{-k}|A \cap \{n \in \mathbb{N} : 2^k \le n < 2^{k+1}\}| \quad \text{for } k = 0,1,2,\ldots$$

Hence $f_A \in X$, i.e. f_A satisfies condition $(**)$, iff $A \in \mathcal{Z}$ (see Proposition 3.2). Conditions (b) and (c) are equivalent by the Lebesgue property of X. Finally, it is easy to see that $f_A \in X$, i.e. f_A satisfies $(*)$, iff condition (d) holds.

Thus X has neither the Positive Zero-Density Convergence Property nor the Positive Zero-Density Boundedness Property. In particular, it fails to have both the Positive LCP and the Positive LBP. Also note that the $\mathcal{Z}$-convergent series $\sum_n f_n$ above, while being perfectly metrically bounded (by Corollary 4.6), is not perfectly (topologically) bounded. Finally, in view of Theorem 8.2, the space X lacks the (disjoint) metric Levi property.

11.2. EXAMPLE (An $\mathcal{L}$-convergent but non-$\mathcal{Z}$-convergent series in c_0). As in Section 3, given $A \subset \mathbb{N}$, let $A(n) := A \cap \{n, n+1, \ldots\}$ for $n = 1,2,\ldots$ Set $E_k = \{n \in \mathbb{N} : 2^{2k} \le n < 2^{2k} + 2^k\}$ for $k = 1,2,\ldots$, and define a sequence (x_n) in c_0 as follows:

$$x_n = \begin{cases} 2^{-k}e_k & \text{if } n \in E_k,\ k = 1,2,\ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then the series $\sum_n x_n$ is not $\mathcal{Z}$-convergent. Indeed, $E := \bigcup_k E_k \in \mathcal{Z}$ and

$$\Big\| \sum_{n\in E_k} x_n \Big\| = \|e_k\| = 1 \quad \text{for all } k.$$

However, $\sum_n x_n$ is $\mathcal{L}$-convergent. Indeed, let $A \in \mathcal{L}$. Take any $\varepsilon > 0$, and next $r \in \mathbb{N}$ with $2/r < \varepsilon$. Then there is m such that the set $A(m)$ is r-rare.

Choose k so that $2^{2k} \geq m$ and $2^k \geq r$. Let B be a finite subset of $A(2^{2k})$ and let l be such that $\max B < 2^{2l} + 2^l$. Then

$$\Big\| \sum_{n\in B} x_n \Big\| = \Big\| \sum_{j=k}^{l} \sum_{n\in B\cap E_j} x_n \Big\| = \max_{k\le j<l} \Big\| \sum_{n\in B\cap E_j} x_n \Big\|$$
$$= \max_{k\le j<l} \frac{|B\cap E_j|}{2^j} \le \max_{j\ge k} \frac{|A\cap E_j|}{2^j},$$

and since there are at most $2^j/r + 1$ elements of A in E_j for $j \geq k$, we have

$$\Big\| \sum_{n\in B} x_n \Big\| \le \max_{j\ge k} \frac{2^j + r}{2^j r} = \frac{1}{r} + \frac{1}{2^k} \le \frac{2}{r} < \varepsilon.$$

Thus the subseries $\sum_{n\in A} x_n$ is unconditionally Cauchy.

11.3. EXAMPLE (Incomplete normed spaces with the Lacunary Convergence Property). Let m_0 denote the dense subspace of the Banach space ℓ_∞ consisting of all elements $x = (\xi_j)$ having finitely many distinct terms. It is known that every subseries convergent series $\sum_n x_n$ in m_0 is finite-dimensional, i.e., $\dim \mathrm{lin}\{x_n : n \in \mathbb{N}\} < \infty$ (see [BDV, Thm. 1] and [DDD, Thm. 4(b)]). From this it follows easily that also every $\mathcal{L}$-convergent series in m_0 is finite-dimensional. In consequence, m_0 has the LCP. Note however that ℓ_∞, which is a completion of m_0, fails the LCP.

It is easy to construct even simpler examples to the same effect: Take for instance any Banach space E with an unconditional basis (e_n) such that E contains no copy of c_0. Then its (incomplete) subspace $E_0 = \mathrm{lin}\,(e_n)$ admits only finite-dimensional subseries convergent series (cf. [DL5]). Hence, as above, E_0 has the LCP.

11.4. EXAMPLE (Spaces with the Zero-Density Convergence Property but without the Lacunary Convergence Property). Denote by $\widetilde{\mathcal{L}}$ the ideal in $\mathcal{P}(\mathbb{N})$ generated by $\mathcal{L}$. Take any solid sequence F-lattice X containing all the unit vectors e_n that has the Zero-Density Convergence Property, e.g., $X = \ell_p$ for $0 < p < \infty$. Then its subspace

$$X(\widetilde{\mathcal{L}}) = \{x \in X : \operatorname{supp} x \in \widetilde{\mathcal{L}}\}$$

has the Zero-Density Convergence Property but lacks the Lacunary Convergence Property, and even the Positive Lacunary Convergence Property. We refer the reader to [DL5] for the (quite technical) proof of this assertion. Unfortunately, all these spaces $X(\widetilde{\mathcal{L}})$ are incomplete.

11.5. QUESTIONS. (a) Does there exist an F-space that has the Zero-Density Convergence Property but lacks the Lacunary Convergence Property?

(b) Does there exist a TVS (preferably an F-lattice) of measurable functions which has the Lacunary Convergence Property but is not m-boundedly sequentially closed in its L_0-space?

(c) If μ is an order continuous submeasure such that the space $L_0(\mu)$ has the Lacunary Convergence Property, is then μ equivalent to a finite measure?

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Extreme points of the complex binary trilinear ball

by

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Abstract. We characterize all the extreme points of the unit ball in the space of trilinear forms on the Hilbert space $\mathbb{C}^2$. This answers a question posed by R. Grząślewicz and K. John [7], who solved the corresponding problem for the real Hilbert space $\mathbb{R}^2$. As an application we determine the best constant in the inequality between the Hilbert–Schmidt norm and the norm of trilinear forms.

It is well known that the extreme points of the unit ball in the space $\mathcal{L}(H)$ of all bounded linear operators on a Hilbert space H are just isometries or coisometries (see [8]). For real Hilbert spaces H, $\mathcal{L}(H)$ can be identified with the space $\mathcal{B}(H,H)$ of all bounded bilinear forms on H. This leads in a natural way to the problem of characterizing extreme points of the unit ball of multilinear forms. In the case of trilinear forms on $H=\mathbb{R}^2$ this question was solved by R. Grząślewicz and K. John [7]. The complex case, where $H=\mathbb{C}^2$, was left there as an open problem (see [7], Remark 5). Accordingly, we prove here such a result. As an application we compute the exact value of the best constant d in the inequality $\|T\|_2\le d\|T\|$ between the Hilbert–Schmidt norm and the norm of a trilinear form T in the binary case, thus complementing our previous results in [3] for the n-ary case, where the asymptotic behaviour of these constants was investigated. For more background material about trilinear forms we refer to [4].

Let $\mathcal{B}(H,H,H)$ be the space of all trilinear forms $T: H\times H\times H\to\mathbb{C}$ equipped with the norm

$$\|T\|=\sup\{|T(x,y,z)|:\|x\|=\|y\|=\|z\|=1\}.$$

Our main results are the following.

THEOREM 1. *For a trilinear form $T: H\times H\times H\to\mathbb{C}$ on the Hilbert space $H=\mathbb{C}^2$ one has $\|T\|=1$ if and only if there are three orthonormal bases*

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