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Hypercyclic and chaotic weighted shifts

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Abstract. Extending previous results of H. Salas we obtain a characterisation of hypercyclic weighted shifts on an arbitrary F-sequence space in which the canonical unit vectors (e_n) form a Schauder basis. If the basis is unconditional we give a characterisation of those hypercyclic weighted shifts that are even chaotic.

0. Introduction. A continuous linear operator T on a topological vector space X is called *hypercyclic* if there is an element x in X whose orbit $\{T^n x : n \in \mathbb{N}_0\}$ under T is dense in X. The vector x is then also called *hypercyclic*.

The best known, and historically earliest, examples of hypercyclic operators are due to G. D. Birkhoff [7], G. R. MacLane [24] and S. Rolewicz [29]. Each of these papers had a profound influence on the literature on hypercyclicity. Birkhoff's result on the hypercyclicity of the operator of translation on the space $H(\mathbb{C})$ of entire functions has led to an extensive study of hypercyclic composition operators (cf. [14, Section 4a]), while MacLane's result on the hypercyclicity of the differentiation operator $Df = f'$ on $H(\mathbb{C})$ initiated the study of hypercyclic (partial) differential operators (cf. [14, Section 4c]).

On the other hand, the operator D can also be regarded as a particular weighted backward shift operator since we have

$$De_n = ne_{n-1} \quad (n \in \mathbb{N}_0),$$

where $(e_n)_{n\in\mathbb{N}_0}$ denotes the canonical basis in $H(\mathbb{C})$ given by $e_n(z) = z^n$, and where $e_{-1} = 0$. Shift operators in a Banach space setting were first studied by Rolewicz who showed that for any $c > 1$ the multiple cB of the backward shift B on the sequence spaces l^p, $1 \le p < \infty$, or c_0 is hypercyclic. Since then the hypercyclicity of shift operators has been studied by several authors (cf. [19], [10], [2], [31], [11], [3], [26], [32], [15], [22] and [6], see also [14, Section 5]). Shift operators are of interest because many classical

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operators can be viewed as such operators and also because they have been "a favorite testing ground for operator-theorists" [32, p. 994].

In the same recent paper [32] H. Salas has extended Rolewicz' result by completely characterising the hypercyclic weighted shifts on l^2. He notes that his proof also works in any of the sequence spaces l^p, $1 \le p < \infty$. A simplified proof of the sufficiency of Salas' conditions was given by León and Montes [22]. In the present paper we show that Salas' result is capable of a far-reaching generalisation. Among other results we shall obtain the following characterisations that also contain MacLane's theorem as a special case. The notation and terminology will be explained below.

THEOREM 1. *Let X be an F-sequence space in which the canonical unit vectors e_n $(n \in \mathbb{N})$ form a basis. Let $T : X \to X$ be a unilateral weighted backward shift with weight sequence $(a_n)_{n\in\mathbb{N}}$. Then T is hypercyclic if and only if there is an increasing sequence (n_k) of positive integers such that*

$$\Big(\prod_{\nu=1}^{n_k} a_\nu\Big)^{-1} e_{n_k} \to 0 \quad \text{in } X$$

as $k \to \infty$.

We add that unilateral weighted *forward* shifts can never be hypercyclic (see Proposition 1 below).

COROLLARY. *Let $T : H(\mathbb{C}) \to H(\mathbb{C})$ be a weighted backward shift with weight sequence $(a_n)_{n\in\mathbb{N}_0}$. Then T is hypercyclic if and only if there is an increasing sequence (n_k) of positive integers such that*

$$\Big|\prod_{\nu=0}^{n_k} a_\nu\Big|^{1/n_k} \to \infty$$

as $k \to \infty$.

THEOREM 2. *Let X be a bilateral F-sequence space in which the canonical unit vectors e_n $(n \in \mathbb{Z})$ form a basis (in some ordering). Let $T : X \to X$ be a bilateral weighted backward shift with weight sequence $(a_n)_{n\in\mathbb{Z}}$. Then T is hypercyclic if and only if there is an increasing sequence (n_k) of positive integers such that, for every $j \in \mathbb{Z}$,*

$$\Big(\prod_{\nu=1}^{n_k} a_{j+\nu}\Big)^{-1} e_{j+n_k} \to 0 \quad \text{and} \quad \Big(\prod_{\nu=0}^{n_k-1} a_{j-\nu}\Big) e_{j-n_k} \to 0 \quad \text{in } X$$

as $k \to \infty$.

For *bilateral* shifts the distinction between forward and backward shifts is immaterial and reduces to a question of notation. Thus, Theorem 2 also immediately gives a characterisation of hypercyclic bilateral forward shifts.

The two theorems are obtained in Section 4 as consequences of our main result, derived in Sections 2 and 3, which characterises the hypercyclicity of the so-called pseudo-shift operators. In Section 5 we obtain a characterisation of those hypercyclic weighted shifts that are even chaotic. Here we have to assume that (e_n) is an unconditional basis.

Special cases of our results in Sections 4 and 5 were obtained independently and with different proofs by Martínez and Peris [25]. In another recent paper [20], deLaubenfels and Emamirad have studied when functions $f(B)$ of the (unweighted) backward shift operator B are chaotic on weighted l^p-spaces.

1. Preliminaries

1a. *F-spaces.* It had been our original intention to generalise Salas' results to Fréchet sequence spaces. An inspection of the proofs, however, showed that local convexity was nowhere required, which allowed us to work in F-spaces, that is, in completely metrisable topological vector spaces. In this way we can also include interesting spaces like the sequence spaces l^p for $0 < p < 1$ in our study. For the theory of F-spaces we refer to [17] and [30].

It will be convenient to assume that the topology of an F-space is induced by an F-norm, which is always possible (cf. [17, pp. 2–5]). Recall that a mapping $X \to \mathbb{R}$, $x \mapsto \|x\|$, on a vector space X is called an *F-norm* if the following conditions are satisfied for $x, y \in X$ and $c \in \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$:

$$\begin{aligned}
&\|x\| > 0 \quad \text{if } x \neq 0,\\
&\|cx\| \le \|x\| \quad \text{for } |c| \le 1,\\
&\lim_{c\to 0} \|cx\| = 0,\\
&\|x + y\| \le \|x\| + \|y\|.
\end{aligned}$$

In a Banach space the norm provides us with an F-norm, in a Fréchet space one can take the Fréchet combination $\|x\| = \sum_n (1/2^n) p_n(x)/(1 + p_n(x))$ of an increasing sequence (p_n) of seminorms that defines the topology of the space.

1b. *Sequence spaces.* In this paper we shall work in the setting of sequence spaces because they are the natural domains of shift operators. We need to emphasise, however, that our results are not limited to sequence spaces, as already exemplified by the differentiation operator on the function space $H(\mathbb{C})$. Throughout this paper we shall identify $H(\mathbb{C})$ and similar spaces of holomorphic functions in a canonical way with a sequence space by identifying each function f with the sequence $(c_n)_{n\in\mathbb{N}_0}$ of its Taylor coefficients at 0. In this interpretation the operator of differentiation becomes a weighted shift operator.

A *sequence space* is a (linear) subspace of the space $\omega = \omega(\mathbb{N}) = \mathbb{K}^{\mathbb{N}}$ of all scalar sequences. Apart from these unilateral sequence spaces we shall also consider *bilateral sequence spaces*, that is, subspaces of $\omega(\mathbb{Z}) = \mathbb{K}^{\mathbb{Z}}$. It turns out to be convenient—though, strictly speaking, not necessary—to allow arbitrary countably infinite sets I as index sets. Then a *sequence space over* I is a subspace of the space $\omega(I) = \mathbb{K}^I$ of all scalar families $(x_i)_{i\in I}$. The space $\omega(I)$ is endowed with its natural product topology. By e_i $(i \in I)$ we denote the canonical unit vectors $e_i = (\delta_{ik})_{k\in I}$.

A *topological sequence space* X *over* I (also called a *K-space over* I (cf. [36])) is a sequence space over I that is endowed with a linear topology in such a way that the inclusion mapping $X \hookrightarrow \omega(I)$ is continuous or, equivalently, that every *coordinate functional* $f_i : X \to \mathbb{K}$, $(x_k)_{k\in I} \mapsto x_i$ $(i \in I)$, is continuous. A *Banach* (*Fréchet*, *F-*, ...) *sequence space over* I is a topological sequence space over I that is a Banach (Fréchet, F-, ...) space.

The family $(e_i)_{i\in I}$ of unit vectors is called an *M-basis* (or *Markushevich basis*) in a topological sequence space X over I if $\operatorname{span}\{e_i : i \in I\}$ is a dense subspace of X (cf. [34]). We shall call $(e_i)_{i\in I}$ an *OP-basis* (or *Ovsepian–Pełczyński basis*) if it is an M-basis and if the family of *coordinate projections* $x \mapsto x_i e_i$ $(i \in I)$ on X is equicontinuous. This terminology was suggested by a well known theorem of Ovsepian and Pełczyński [27, Theorem 1] (cf. [23, 1.f.4]), which can be stated equivalently as saying that every separable Banach space is isomorphic to a Banach sequence space in which $(e_n)_{n\in\mathbb{N}}$ is an OP-basis (note that in a Banach sequence space over I the family of coordinate projections is equicontinuous if and only if $\sup_{i\in I} \|e_i\| \, \|f_i\| < \infty$).

Now suppose that X is an F-sequence space. Then, clearly, if $(e_n)_{n\in\mathbb{N}}$ is a basis in X it is also an OP-basis, and the converse is true under the stronger assumption that the sequence of partial sum operators $x \mapsto \sum_{k=1}^n x_k e_k$ $(n \in \mathbb{N})$ is equicontinuous (cf. [33, Chapter I, Theorem 4.1] or [36, 10.3.19]). We give a standard example to show that not every OP-basis is a basis.

EXAMPLE. Let $C_{2\pi}$ be the Banach space of continuous 2π-periodic functions, endowed with the supremum norm. Writing $e_n(t) = e^{int}$ $(n \in \mathbb{Z})$ we can regard this space as a bilateral Banach sequence space when we identify a function $f \in C_{2\pi}$ with its sequence $(c_n)_{n\in\mathbb{Z}}$ of Fourier coefficients. It follows from Fejér's theorem that $(e_n)_{n\in\mathbb{Z}}$ is an M-basis in $C_{2\pi}$, and since $\|c_n e_n\| = |c_n| \le \|f\|$ for all n we see that it is also an OP-basis. On the other hand, it is well known not to be a basis.

Other interesting examples of spaces in which (e_n) is an OP-basis without being a basis are the Hardy space H^1 and the disc algebra A, both considered in the canonical way as sequence spaces (cf. [9] and [23, p. 37]). We add that in the Hardy spaces H^p with $0 < p < 1$ the sequence (e_n) is an M-basis but not an OP-basis ([17, p. 35] and [9, Theorem 6.4]), while for $1 < p < \infty$ the e_n even form a basis in H^p (cf. [35, (12.88)]).

1c. *Shift operators.* We turn to the definition of shift and pseudo-shift operators. Let X be a bilateral topological sequence space. Then a continuous linear operator T on X is called a *bilateral weighted backward* or *forward shift* if for some sequence $(a_n)_{n\in\mathbb{Z}}$ of non-zero scalars we have

$$T(x_n)_{n\in\mathbb{Z}} = (a_{n+1}x_{n+1})_{n\in\mathbb{Z}}$$

or

$$T(x_n)_{n\in\mathbb{Z}} = (a_{n-1}x_{n-1})_{n\in\mathbb{Z}},$$

respectively. The sequence $(a_n)_{n\in\mathbb{Z}}$ is called the *weight sequence*.

The *unilateral weighted backward* and *forward shifts* are defined analogously on topological sequence spaces (over $\mathbb{N}$), where we set $a_0 = x_0 = 0$.

In Sections 2 and 3 we shall in fact study more general kinds of operators that were suggested by Bernal's notion of a Taylor shift (cf. [4]).

DEFINITION. Let X and Y be topological sequence spaces over I and J, respectively. Then a continuous linear operator $T : X \to Y$ is called a *weighted pseudo-shift* if there is a sequence $(b_j)_{j\in J}$ of non-zero scalars and an injective mapping $\varphi : J \to I$ such that

$$T(x_i)_{i\in I} = (b_j x_{\varphi(j)})_{j\in J}$$

for $(x_i) \in X$. We then write $T = T_{b,\varphi}$, and $(b_j)_{j\in J}$ is called the *weight sequence.*

We note that if X and Y are F-sequence spaces then, by a standard argument using the closed graph theorem, the continuity of T is automatic once it is known that T maps each sequence from X into Y (cf. also [36, 4.2.8]).

REMARKS. (1) Every unilateral or bilateral weighted backward shift is a weighted pseudo-shift with $b_n = a_{n+1}$ and $\varphi(n) = n+1$, and every bilateral weighted forward shift is a weighted pseudo-shift with $b_n = a_{n-1}$ and $\varphi(n) = n-1$. In contrast, unilateral weighted forward shifts are never pseudo-shifts due to their definition in the first component. Also, Bernal's Taylor shifts are particular weighted pseudo-shifts (cf. [4, 3.1 and 3.2]).

(2) If $X = Y$ then the identity operator on X defines a pseudo-shift, albeit not a very interesting one.

(3) The term "pseudo-shift" was suggested by one of the referees, who also pointed out that if we regard X and Y as spaces of functions on the sets I and J, respectively, then in the language of function space theory weighted pseudo-shifts are the same as weighted composition operators.

In order to describe the action of a pseudo-shift on the sequences e_i ($i \in I$) we need to consider the inverse

$$\psi = \varphi^{-1} : \varphi(J) \to J$$

of the mapping φ. In addition we set

$$b_{\psi(i)} = 0 \quad \text{and} \quad e_{\psi(i)} = 0 \quad \text{if } i \in I \setminus \varphi(J),$$

that is, if $\psi(i)$ is "undefined". With these definitions we have, for all $i \in I$,

$$T_{b,\varphi} e_i = b_{\psi(i)} e_{\psi(i)}.$$

1d. *Universality.* We recall ([12]) that a sequence $(T_n)_{n\in\mathbb{N}_0}$ of continuous mappings $T_n : X \to Y$ between topological spaces X and Y is called *universal* if there is an element x in X such that the set $\{T_n x : n \in \mathbb{N}_0\}$ is dense in Y. The element x is then called *universal for* (T_n). Clearly, an operator $T : X \to X$ is hypercyclic if and only if the sequence $(T^n)_{n\in\mathbb{N}_0}$ is universal.

We refer to [14] for a comprehensive survey of universality and hypercyclicity.

2. Universal sequences of weighted pseudo-shifts. In this section we want to characterise the universality of sequences of weighted pseudo-shifts. The following definition will turn out to be important. We have borrowed the terminology from Bernal and Montes [5] who show that a related concept for automorphisms of domains in $\mathbb{C}$ is crucial for the universality of sequences of composition operators on spaces of holomorphic functions (cf. [5, Theorems 3.1 and 3.6]).

DEFINITION. A sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ of mappings $\varphi_n : J \to I$ is called a *run-away sequence* if for each pair of finite subsets $I_0 \subset I$ and $J_0 \subset J$ there exists an $n_0 \in \mathbb{N}_0$ such that, for every $n \ge n_0$, $\varphi_n(J_0) \cap I_0 = \emptyset$.

With this we can now state the basic result of this paper; note the definition of ψ_n as inverses of φ_n (cf. Section 1c).

THEOREM 3. *Let X and Y be F-sequence spaces over I and J, respectively, in which $(e_i)_{i\in I}$ and $(e_j)_{j\in J}$ are OP-bases. Let $T_n = T_{b_n,\varphi_n} : X \to Y$ ($n \in \mathbb{N}_0$) be weighted pseudo-shifts with weights $b_n = (b_{n,j})_{j\in J}$. If (φ_n) is a run-away sequence, then the following assertions are equivalent*:

(i) *the sequence (T_n) has a dense set of universal elements*;

(ii) *there exists an increasing sequence (n_k) of positive integers such that*

$$\text{(U1)} \qquad b_{n_k,j}^{-1} e_{\varphi_{n_k}(j)} \to 0 \quad \text{in } X, \text{ for every } j \in J,$$

$$\text{(U2)} \qquad b_{n_k,\psi_{n_k}(i)} e_{\psi_{n_k}(i)} \to 0 \quad \text{in } Y, \text{ for every } i \in I,$$

as $k \to \infty$.

If one of these conditions holds then the set of universal elements is a dense G_δ-subset of X, hence residual in X.

Proof. For the proof of sufficiency we rely on a generalisation of the Hypercyclicity Criterion of Kitai, Gethner and Shapiro: A sequence (T_n) of continuous linear operators $T_n : X \to Y$ between an F-space X and a separable metrisable topological vector space Y has a dense G_δ-set of universal elements if we can find dense subsets X_0 of X and Y_0 of Y, mappings $S_n : Y_0 \to X$ ($n \in \mathbb{N}_0$) and a sequence (n_k) such that (T_{n_k}) and (S_{n_k}) converge pointwise to 0 on X_0 and Y_0, respectively, and $T_n S_n$ is the identity on Y_0 for all $n \in \mathbb{N}_0$ (see [13], [11, Corollary 1.4] and [14, Theorem 2]). In our case we set $X_0 = \operatorname{span}\{e_i : i \in I\}$ and $Y_0 = \operatorname{span}\{e_j : j \in J\}$, which are dense sets since $(e_i)_{i\in I}$ and $(e_j)_{j\in J}$ are M-bases, and we define linear mappings $S_n : Y_0 \to X$ by $S_n e_j = b_{n,j}^{-1} e_{\varphi_n(j)}$ ($n \in \mathbb{N}_0$, $j \in J$). Since $T_n e_i = b_{n,\psi_n(i)} e_{\psi_n(i)}$ and $T_n S_n e_j = e_j$ for $n \in \mathbb{N}_0$, $i \in I$ and $j \in J$, we see that, under condition (ii), (T_n) has a dense G_δ-set of universal elements, and hence condition (i) holds.

We turn to the converse implication and assume that (i) holds. It suffices to show that for every pair of finite subsets I_0 of I and J_0 of J, for every $\varepsilon > 0$ and every $N \in \mathbb{N}$ there exists an $n > N$ such that

$$\text{(U1}'\text{)} \qquad \|b_{n,j}^{-1} e_{\varphi_n(j)}\| < \varepsilon \quad \text{in } X, \text{ for } j \in J_0,$$

$$\text{(U2}'\text{)} \qquad \|b_{n,\psi_n(i)} e_{\psi_n(i)}\| < \varepsilon \quad \text{in } Y, \text{ for } i \in I_0,$$

where $\|\cdot\|$ denotes the F-norm in X and Y, respectively. To see this we fix enumerations (i_k) and (j_k) of I and J, respectively, and set $I_k = \{i_1, \ldots, i_k\}$ and $J_k = \{j_1, \ldots, j_k\}$. We then define inductively an increasing sequence (n_k) of positive integers by letting n_k be a number n satisfying (U1′) and (U2′) for $I_0 = I_k$, $J_0 = J_k$, $\varepsilon = 1/k$ and $N = n_{k-1}$, where we set $n_0 = 0$. It is clear that the sequence (n_k) satisfies (U1) and (U2), so that condition (ii) holds.

We therefore have to prove (U1′) and (U2′) under the assumption of (i). Let $\varepsilon > 0$, finite subsets $I_0 \subset I$, $J_0 \subset J$, and $N \in \mathbb{N}$ be given. By the equicontinuity of the coordinate projections in X and Y there is some $\delta > 0$ so that we have, for $x \in X$ and $y \in Y$,

$$\text{(2.1)} \qquad \|x_i e_i\| < \varepsilon/2 \quad \text{for } i \in I, \text{ if } \|x\| < \delta,$$

and

$$\text{(2.2)} \qquad \|y_j e_j\| < \varepsilon/2 \quad \text{for } j \in J, \text{ if } \|y\| < \delta.$$

Since by (i) the universal elements of (T_n) are dense in X there exist $x \in X$ and $n \in \mathbb{N}_0$, $n > N$, with

$$\text{(2.3)} \qquad \Big\|x - \sum_{i\in I_0} e_i\Big\| < \delta \quad \text{and} \quad \Big\|T_n x - \sum_{j\in J_0} e_j\Big\| < \delta.$$

By continuous inclusion of X and Y into $\omega(I)$ and $\omega(J)$ we can in addition achieve

$$\sup_{i\in I_0}|x_i-1|\le 1/2 \quad\text{and}\quad \sup_{j\in J_0}|y_j-1|\le 1/2, \tag{2.4}$$

where $y:=T_n x$. Since (φ_n) is a run-away sequence we can moreover have

$$\varphi_n(J_0)\cap I_0=\emptyset. \tag{2.5}$$

By (2.1), the first inequality in (2.3) implies that

$$\|x_i e_i\|<\varepsilon/2 \quad \text{if } i\notin I_0,$$

hence by (2.5),

$$\|x_{\varphi_n(j)}e_{\varphi_n(j)}\|<\varepsilon/2 \quad \text{for } j\in J_0. \tag{2.6}$$

By the second inequality in (2.4) we have, for $j\in J_0$,

$$|b_{n,j}x_{\varphi_n(j)}-1|\le 1/2,$$

hence $x_{\varphi_n(j)}\neq 0$ and

$$\left|\frac{1}{b_{n,j}x_{\varphi_n(j)}}-1\right|\le 1. \tag{2.7}$$

Now, (2.6) and (2.7) imply that, for $j\in J$,

$$\begin{aligned}\|b_{n,j}^{-1}e_{\varphi_n(j)}\| &= \left\|\frac{1}{b_{n,j}x_{\varphi_n(j)}}x_{\varphi_n(j)}e_{\varphi_n(j)}\right\| \\ &\le \|x_{\varphi_n(j)}e_{\varphi_n(j)}\|+\left\|\left(\frac{1}{b_{n,j}x_{\varphi_n(j)}}-1\right)x_{\varphi_n(j)}e_{\varphi_n(j)}\right\| \\ &\le 2\|x_{\varphi_n(j)}e_{\varphi_n(j)}\|<\varepsilon,\end{aligned}$$

where we have used properties of F-norms. Hence condition (U1′) holds.

As for (U2′), we deduce from (2.5) and the definition of the ψ_n that

$$\psi_n(I_0\cap\varphi_n(J))\cap J_0=\emptyset. \tag{2.8}$$

By (2.2), the second inequality in (2.3) implies that

$$\|b_{n,j}x_{\varphi_n(j)}e_j\|<\varepsilon/2 \quad \text{if } j\notin J_0,$$

hence by (2.8),

$$\|b_{n,\psi_n(i)}x_i e_{\psi_n(i)}\|<\varepsilon/2 \quad \text{for } i\in I_0; \tag{2.9}$$

note that $e_{\psi_n(i)}=0$ if $i\notin\varphi_n(J)$. By the first inequality in (2.4) we have

$$|x_i|\ge 1/2 \quad \text{for } i\in I_0, \tag{2.10}$$

in particular $x_i\neq 0$. Now, (2.9) and (2.10) imply that

$$\|b_{n,\psi_n(i)}e_{\psi_n(i)}\|=\left\|\frac{1}{2x_i}2b_{n,\psi_n(i)}x_i e_{\psi_n(i)}\right\|\le\|2b_{n,\psi_n(i)}x_i e_{\psi_n(i)}\|<\varepsilon$$

for all $i\in I_0$, where we have again used properties of F-norms. Hence also (U2′) holds. ∎

REMARKS. (1) The condition that (φ_n) be a run-away sequence is clearly not necessary for universality. It is not difficult to construct a sequence (T_n) of weighted pseudo-shifts with a dense set of universal elements even for the mappings $\varphi_n=\mathrm{id}_I$, the identity on I. However, in the next section we shall see that in the case of greatest interest to us, namely when the T_n arise as iterates of a weighted pseudo-shift, the run-away condition is indeed also necessary (cf. Theorem 5 and its proof).

(2) We note for possible future applications that our assumptions on the canonical unit vectors can be relaxed. As the proof shows, for the implication (ii)⇒(i) it suffices that $(e_i)_{i\in I}$ and $(e_j)_{j\in J}$ are M-bases in X and Y, respectively, while for the implication (i)⇒(ii) we need only assume the equicontinuity of the coordinate projections in X and Y. A similar remark applies to Theorems 5, 6 and 7 below.

In many situations, each $i\in I$ lies outside $\varphi_n(J)$ for all sufficiently large n, which implies in particular that the sequence (φ_n) is run-away. This is so, for instance, if the T_n are iterates of a unilateral weighted backward shift. In this case the statement of Theorem 3 can be simplified and strengthened. By the additional assumption we see that

$$T_n e_i=b_{n,\psi_n(i)}e_{\psi_n(i)}\to 0 \quad \text{in } Y \tag{2.11}$$

for every $i\in I$ because $e_{\psi_n(i)}$ is eventually 0 by the definition of ψ_n (cf. Section 1c). This shows that condition (U2) is automatically satisfied, which allows one to modify the proof in that equation (2.2) is no longer needed (it was only needed to obtain (U2′), hence (U2)). As a consequence, the assumptions of completeness and the equicontinuity of the coordinate projections for the space Y can be dropped. In addition, (2.11) and the fact that $(e_i)_{i\in I}$ is an M-basis in X imply that (T_n) converges pointwise on a dense subset of X and hence, by Satz 1.4.2 of [12], that a single universal element suffices to make the set of universal elements residual in X. We therefore have the following result.

THEOREM 4. *Let X be an F-sequence space over I in which $(e_i)_{i\in I}$ is an OP-basis, and let Y be a metrisable sequence space over J in which $(e_j)_{j\in J}$ is an M-basis. Let $T_n=T_{b_n,\varphi_n}:X\to Y$ $(n\in\mathbb{N}_0)$ be weighted pseudo-shifts with weights $b_n=(b_{n,j})_{j\in J}$ so that each $i\in I$ lies outside $\varphi_n(J)$ for all sufficiently large n. Then the following assertions are equivalent*:

(i) *the sequence (T_n) has a universal element*;

(ii) *there exists an increasing sequence (n_k) of positive integers such that*

$$b_{n_k,j}^{-1}e_{\varphi_{n_k}(j)}\to 0 \quad \textit{in } X, \textit{ for every } j\in J, \tag{U1}$$

as $k\to\infty$.

If one of these conditions holds then the set of universal elements is a dense G_δ-subset of X, hence residual in X.

REMARK. We shall see in the Example after Theorem 7 that it does not suffice if condition (U1) is only satisfied for one element $j \in J$.

We consider the special situation where $X = Y = H(\mathbb{D}_R)$, the space of holomorphic functions on $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$, $0 < R \leq \infty$, endowed, as usual, with the topology induced by the seminorms $p_r(f) = \sup\{|f(z)| : |z| \leq r\}$, $0 < r < R$. In the canonical way, $H(\mathbb{D}_R)$ becomes a Fréchet sequence space over $\mathbb{N}_0$ in which $(e_n)_{n\in\mathbb{N}_0}$ is a basis. We then obtain the following as an application of Theorem 4.

COROLLARY. *Let $T_n = T_{b_n,\varphi_n} : H(\mathbb{D}_R) \to H(\mathbb{D}_R)$ $(0 < R \leq \infty)$ be weighted pseudo-shifts with weights $b_n = (b_{n,j})_{j\in\mathbb{N}_0}$ so that $\min_{j\in\mathbb{N}_0} \varphi_n(j) \to \infty$ as $n \to \infty$. Then the following assertions are equivalent:*

(i) *the sequence (T_n) has a universal element;*

(ii) *there exists an increasing sequence (n_k) of positive integers such that*

$$\liminf_{k\to\infty} |b_{n_k,j}|^{1/(1+\varphi_{n_k}(j))} \geq R \quad \text{for all } j \in \mathbb{N}_0.$$

If one of these conditions holds then the set of universal elements is a dense G_δ-subset of $H(\mathbb{D}_R)$, hence residual in $H(\mathbb{D}_R)$.

This corollary covers the Taylor shifts considered by Bernal [4] and thus contains his Theorem 4.2 and improves on it. In addition, it implies Corollary 1 and part of Theorem 4 of [3]. We also remark that Propositions 3.1 and 3.4 of Bès and Peris [6] follow from Theorems 4 and 3, respectively.

3. Hypercyclic weighted pseudo-shifts. Let $T = T_{b,\varphi} : X \to X$ be a weighted pseudo-shift. In this section we characterise the universality of the sequence (T_n) of iterates $T_n = T^n$ of T, that is, the hypercyclicity of T. We first note that each T^n $(n \in \mathbb{N}_0)$ is a weighted pseudo-shift. More precisely, we have

$$T^n(x_i)_{i\in I} = (b_{n,i}x_{\varphi_n(i)})_{i\in I}$$

with

$$\varphi_n(i) := \varphi^n(i) = (\varphi \circ \ldots \circ \varphi)(i) \quad (n\text{-fold}),$$

$$b_{n,i} := b_i b_{\varphi(i)} \ldots b_{\varphi^{n-1}(i)} = \prod_{\nu=0}^{n-1} b_{\varphi^\nu(i)}.$$

For the meaning of ψ we refer to Section 1c, where we now add that $b_{\psi^n(i)} = 0$ and $e_{\psi^n(i)} = 0$ whenever $\psi^n(i)$ is undefined.

THEOREM 5. *Let X be an F-sequence space over I in which $(e_i)_{i\in I}$ is an OP-basis. Let $T = T_{b,\varphi} : X \to X$ be a weighted pseudo-shift. Then the following assertions are equivalent:*

(i) *T is hypercyclic;*

(ii) (α) *the mapping $\varphi : I \to I$ has no periodic points;*

(β) *there exists an increasing sequence (n_k) of positive integers such that, for every $i \in I$,*

$$\text{(H1)} \qquad \Big(\prod_{\nu=0}^{n_k-1} b_{\varphi^\nu(i)}\Big)^{-1} e_{\varphi^{n_k}(i)} \to 0,$$

$$\text{(H2)} \qquad \Big(\prod_{\nu=1}^{n_k} b_{\psi^\nu(i)}\Big) e_{\psi^{n_k}(i)} \to 0$$

in X, as $k \to \infty$.

Proof. First assume that φ has a periodic point, that is, $\varphi^N(i) = i$ for some $i \in I$ and $N \in \mathbb{N}$. Since the entry of $T^n x$ at position i is

$$\Big(\prod_{\nu=0}^{n-1} \beta_\nu\Big)\xi_n \quad \text{with } \beta_\nu = b_{\varphi^\nu(i)} \text{ and } \xi_n = x_{\varphi^n(i)},$$

and since both $(\beta_\nu)_\nu$ and $(\xi_n)_n$ are periodic sequences we see that for no x in X can these entries form a dense set in $\mathbb{K}$ as n varies. Since X contains e_i and is continuously included in $\mathbb{K}^I$ this shows that $\{T^n x : n \in \mathbb{N}_0\}$ cannot be dense in X for any element x, hence that T is not hypercyclic.

On the other hand, if φ has no periodic points then for every finite subset I_0 of I and any $i \in I$ there is an $n_0 \in \mathbb{N}$ with $\varphi^n(i) \notin I_0$ for $n \geq n_0$. This shows that (φ^n) is a run-away sequence. Thus we can apply Theorem 3 to obtain the present theorem; note that every hypercyclic operator has a dense set of hypercyclic vectors because, obviously, any vector in the orbit of a hypercyclic vector is again hypercyclic. ∎

REMARKS. (1) The proofs of Theorems 3 and 5 show that in the situation of Theorem 5 every hypercyclic weighted pseudo-shift satisfies the Hypercyclicity Criterion in the weak form given in [21, pp. 526–527] or in [6] (cf. also [14]). It is an open problem (cf. [22, Section 6], [6]) if every hypercyclic operator satisfies this criterion.

(2) Condition (H2) is always satisfied if $\bigcap_{n=1}^\infty \varphi^n(I) = \emptyset$, for example if T is a unilateral weighted backward shift; see our discussion before Theorem 4.

4. Hypercyclic weighted shifts. We turn to the study of hypercyclic weighted (backward or forward) shifts. In particular, we shall give here the proofs of the results that we stated in the Introduction.

We begin with the case of bilateral weighted backward shifts. As we have already mentioned in Section 1c, these are weighted pseudo-shifts $T_{b,\varphi}$ on bilateral sequence spaces X with $\varphi(n) := n+1$ and $b_n := a_{n+1}$ for $n \in \mathbb{Z}$.

Following Salas [32] we shall consider more generally the direct sum $T = \bigoplus_{\mu=1}^m T_\mu$ of m such operators on the space X^m. We recall that while the hypercyclicity of each T_μ is necessary for the hypercyclicity of the direct sum, it is not a sufficient condition (cf. [31, Remark 2(a)]). The hypercyclicity of $\bigoplus_{\mu=1}^m T_\mu$ is of interest because it says that there is an m-tuple $(x_1,\ldots,x_m)$ of vectors in X such that the vectors $(T_1^n x_1,\ldots,T_m^n x_m)$, $n \in \mathbb{N}_0$, get arbitrarily close to any preassigned m-tuple $(y_1,\ldots,y_m)$, that is, these vectors approximate every given "configuration" in X^m, with a uniform exponent n.

THEOREM 6. *Let X be a bilateral F-sequence space in which $(e_n)_{n\in\mathbb{Z}}$ is an OP-basis. Let $T_\mu : X \to X$, $\mu = 1,\ldots,m$, be bilateral weighted backward shifts with weight sequences $(a_{\mu,n})_{n\in\mathbb{Z}}$. Then the direct sum operator $T = \bigoplus_{\mu=1}^m T_\mu$ is hypercyclic on X^m if and only if there is an increasing sequence (n_k) of positive integers such that, for every $j \in \mathbb{Z}$ and $1 \le \mu \le m$,*

$$\Big(\prod_{\nu=1}^{n_k} a_{\mu,j+\nu}\Big)^{-1} e_{j+n_k} \to 0 \quad \text{and} \quad \Big(\prod_{\nu=0}^{n_k-1} a_{\mu,j-\nu}\Big) e_{j-n_k} \to 0 \quad \text{in } X$$

as $k \to \infty$.

Proof. In order to be able to apply Theorem 5 we show that $T = \bigoplus_{\mu=1}^m T_\mu$ is a particular weighted pseudo-shift. First, identifying the m-tuple $((x_{1,n})_{n\in\mathbb{Z}},\ldots,(x_{m,n})_{n\in\mathbb{Z}})$ of sequences with the family $(x_{\mu,n})_{1\le\mu\le m,\,n\in\mathbb{Z}}$ we see that we can regard X^m as a sequence space over $\{1,\ldots,m\}\times\mathbb{Z}$. In this interpretation, $T = \bigoplus_{\mu=1}^m T_\mu$ is the operator given by

$$T(x_{\mu,n})_{1\le\mu\le m,\,n\in\mathbb{Z}} = (T_\mu(x_{\mu,n})_{n\in\mathbb{Z}})_{1\le\mu\le m} = (a_{\mu,n+1}x_{\mu,n+1})_{1\le\mu\le m,\,n\in\mathbb{Z}}.$$

Hence T is a weighted pseudo-shift $T_{b,\varphi}$ with

$$b_{\mu,n} = a_{\mu,n+1} \quad \text{and} \quad \varphi(\mu,n) = (\mu,n+1)$$

for $(\mu,n) \in \{1,\ldots,m\}\times\mathbb{Z}$. We then have

$$\varphi^\nu(\mu,n) = (\mu,n+\nu) \quad \text{and} \quad \psi^\nu(\mu,n) = (\mu,n-\nu).$$

Now the theorem follows from Theorem 5 when we note that with X also X^m is an F-sequence space in which the canonical unit vectors form an OP-basis. ■

REMARKS. (1) Taking $m = 1$ we obtain Theorem 2 as a special case. Of course, the assumption there that (e_n) is a basis can be replaced by the weaker assumption that it is an OP-basis.

(2) The characterising condition for the bilateral weighted forward shifts follows similarly, or it can be deduced from that for the backward shift after realising that a forward shift for (e_n) is a backward shift for (e_{-n}). The condition turns out to be the following: There exists an increasing sequence (n_k) of positive integers such that, for every $j \in \mathbb{Z}$ and $1 \le \mu \le m$,

$$\Big(\prod_{\nu=1}^{n_k} a_{\mu,j-\nu}\Big)^{-1} e_{j-n_k} \to 0 \quad \text{and} \quad \Big(\prod_{\nu=0}^{n_k-1} a_{\mu,j+\nu}\Big) e_{j+n_k} \to 0 \quad \text{in } X$$

as $k \to \infty$. This contains and generalises Theorems 2.1 and 2.5 of Salas [32]. In particular, we see that Salas' results remain true verbatim for all spaces l^p, $0 < p < \infty$, and for the Hardy spaces H^p, $1 \le p < \infty$; see our discussion at the end of Section 1b.

We next consider unilateral weighted backward shifts. They are weighted pseudo-shifts $T_{b,\varphi}$ on sequence spaces (over $\mathbb{N}$) with $\varphi(n) := n+1$ and $b_n := a_{n+1}$ for $n \in \mathbb{N}$. We shall see that for unilateral shifts the characterising conditions simplify considerably, which is a consequence of the following.

LEMMA. *Let M be a metric space and v_n $(n \in \mathbb{N})$ and v elements in M. If there is an increasing sequence (n_k) of positive integers such that*

$$v_{n_k-j} \to v \quad \text{as } k \to \infty, \text{ for every } j \in \mathbb{N},$$

then there exists an increasing sequence (n_k') of positive integers such that

$$v_{n_k'+j} \to v \quad \text{as } k \to \infty, \text{ for every } j \in \mathbb{N}.$$

Proof. It follows from the assumption that for every $k \in \mathbb{N}$ there is some $N \in \mathbb{N}$ with $d(v_{N-j}, v) < 1/k$ for $j = 0,\ldots,k$, where d is the metric in X. If we define $n_k' = N - k$ then, for every $j \in \mathbb{N}$, we have $d(v_{n_k'+j}, v) < 1/k$ if $k \ge j$. This implies the result when we pass to an increasing subsequence of (n_k'), if necessary. ■

THEOREM 7. *Let X be an F-sequence space in which $(e_n)_{n\in\mathbb{N}}$ is an OP-basis. Let $T_\mu : X \to X$, $\mu = 1,\ldots,m$, be unilateral weighted backward shifts with weight sequences $(a_{\mu,n})_{n\in\mathbb{N}}$. Then the direct sum operator $T = \bigoplus_{\mu=1}^m T_\mu$ is hypercyclic on X^m if and only if there is an increasing sequence (n_k) of positive integers such that, for $1 \le \mu \le m$,*

$$(4.1) \qquad \Big(\prod_{\nu=1}^{n_k} a_{\mu,\nu}\Big)^{-1} e_{n_k} \to 0 \quad \text{in } X$$

as $k \to \infty$.

Proof. As in the proof of Theorem 6 we regard X^m as an F-sequence space over $\{1,\ldots,m\}\times\mathbb{N}$ and identify T as a weighted pseudo-shift $T_{b,\varphi}$ on X^m with $b_{\mu,n} = a_{\mu,n+1}$ and $\varphi(\mu,n) = (\mu,n+1)$ for $(\mu,n) \in \{1,\ldots,m\}\times\mathbb{N}$. By an application of Theorem 5 we deduce that T is hypercyclic if and only if there is an increasing sequence (n'_k) of positive integers such that

$$\Big(\prod_{\nu=1}^{n'_k} a_{\mu,j+\nu}\Big)^{-1} e_{j+n'_k} \to 0 \quad \text{in } X, \text{ for } 1 \le \mu \le m \text{ and } j \in \mathbb{N},$$

as $k \to \infty$, which is the same as

$$\Big(\prod_{\nu=1}^{j+n'_k} a_{\mu,\nu}\Big)^{-1} e_{j+n'_k} \to 0 \quad \text{in } X, \text{ for } 1 \le \mu \le m \text{ and } j \in \mathbb{N}; \tag{4.2}$$

note that by Remark (2) to Theorem 5 condition (H2) always holds in our present situation. This condition, which is clearly stronger than the one given in the theorem, is in fact equivalent to it. To see this, assume that there is some (n_k) such that (4.1) holds. We then consider the elements

$$v_n = \Big(\Big(\prod_{\nu=1}^{n} a_{1,\nu}\Big)^{-1} e_n, \ldots, \Big(\prod_{\nu=1}^{n} a_{m,\nu}\Big)^{-1} e_n\Big), \quad n \in \mathbb{N},$$

in X^m. By (4.1) we have $v_{n_k} \to 0$ in X^m. It follows from the definition of the operator T that $T^m v_n = v_{n-1}$, and hence, since T is continuous, that we also have $v_{n_k-j} \to 0$ in X^m for every $j \in \mathbb{N}$. We can now apply the lemma to the sequence (v_n) and obtain a sequence (n'_k) such that $v_{n'_k+j} \to 0$ in X^m for every $j \in \mathbb{N}$, which implies (4.2). This had to be shown. ■

REMARKS. (1) Theorem 1 follows from Theorem 7 as the special case $m = 1$; cf. also Remark (1) to Theorem 6.

(2) The theorem contains and generalises Theorem 2.8 of Salas [32]. Again it also shows that Salas' result remains true verbatim for all spaces l^p, $0 < p < \infty$, and the Hardy spaces H^p, $1 \le p < \infty$.

Theorem 7 is remarkable in that considerably fewer conditions than in the previous theorems ensure universality. In fact, in the case of one operator T we have just one condition instead of countably many. We give an example to show that even for operators that are slightly more general than the ones considered in Theorem 7 the reduction is not possible.

EXAMPLE. Let $T : \omega \to \omega$ be the unilateral weighted backward shift with weight sequence $(a_n) = \big(1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots\big)$, where $\omega = \omega(\mathbb{N})$ is the space of all sequences, and define $T_n : l^1 \to \omega$ by $T_n = T^n|_{l^1}$ $(n \in \mathbb{N}_0)$. Then the T_n are weighted pseudo-shifts that satisfy the assumptions of Theorem 4 and also condition (U1) in Theorem 4 for $j = 1$ (corresponding to condition (4.1)). But (T_n) does not have a universal element because one cannot find a sequence (n_k) such that (U1) is satisfied for $j = 1$ and $j = 2$ simultaneously.

Salas [32, Remark after Corollary 2.6] has shown that no unilateral weighted *forward* shift on l^2 can be hypercyclic. This result, in fact, holds in much greater generality. We emphasise that unilateral forward shifts are never pseudo-shifts.

PROPOSITION 1. *No unilateral weighted forward shift on any topological sequence space $X \neq \{0\}$ can be hypercyclic.*

Proof. Since $X \neq \{0\}$ some coordinate functional $f_j : X \to \mathbb{K}$ $(j \in \mathbb{N})$ is surjective. Now, if $x \in X$ were a hypercyclic vector then the finite set $\{f_j(T^n x) : n \in \mathbb{N}_0\}$ would be dense in $\mathbb{K}$, which is impossible. ■

REMARK. The corollary to Theorem 1 follows immediately from Theorem 1 if one recalls that the topology of $H(\mathbb{C})$ is induced by the seminorms $p_r(f) = \sup\{|f(z)| : |z| \le r\}$, $0 < r < \infty$. This corollary contains some known results as special cases. For $a_n = n$ $(n \ge 1)$ we recover MacLane's theorem. More generally, a weighted backward shift on $H(\mathbb{C})$ is hypercyclic whenever $a_n \to \infty$. This was first obtained by Mathew [26]; his second condition, $\sup_n |a_n|^{1/n} < \infty$, is simply the characterising condition for T to map $H(\mathbb{C})$ into itself, as is easily seen. On the other hand, no constant sequence defines a hypercyclic backward shift, as was already observed by Godefroy and Shapiro [11, 5.4].

5. Chaotic weighted shifts. Godefroy and Shapiro [11, Section 6] have introduced the study of chaos to the theory of hypercyclicity. According to a definition proposed by Devaney [8] a continuous mapping $f : M \to M$ on a metric space M is called *chaotic* if

(i) it is topologically transitive, that is, if for any pair U, V of non-empty open sets in M there is some $n \in \mathbb{N}$ with $f^n(U) \cap V \neq \emptyset$,

(ii) it has a dense set of periodic points, that is, points x for which there is some $n \in \mathbb{N}$ with $f^n(x) = x$, and

(iii) it has a certain property called sensitive dependence on initial conditions.

Banks *et al.* [1] have shown that sensitive dependence on initial conditions is a consequence of the other two conditions (cf. also [11, Proposition 6.1]), and it is well known that topological transitivity is the same as the existence of a dense orbit if the underlying space is a separable complete metric space without isolated points (cf. [14, Sections 1a and 1b]). Thus, an operator on a separable F-space is chaotic if and only if it is hypercyclic and it has a dense set of periodic points.

Godefroy and Shapiro [11, Theorem 6.3] have identified the chaotic operators among a certain class of weighted backward shifts on l^2. We shall here characterise completely the chaotic weighted shifts on any F-sequence space in which the e_n form an unconditional basis.

As a by-product we obtain the surprising fact that for the operators studied here the existence of a single non-trivial, that is, non-zero periodic point implies that the operator is hypercyclic and, indeed, chaotic. In general, not every hypercyclic operator that has a non-trivial periodic point is chaotic. Examples of such operators were first given by Herrero and Wang [16, Corollary 2.5] and Salas [31, Remark 3]. Another example will be given at the end of this section. Our example will also show that the assumption in the following theorem that the basis be unconditional cannot be dropped.

Recall that in a topological sequence space X over I the family $(e_i)_{i\in I}$ is an *unconditional basis* if for every $(x_i)_{i\in I} \in X$ and every bijection $\pi : \mathbb{N} \to I$ the series $\sum_{n=1}^{\infty} x_{\pi(n)} e_{\pi(n)}$ converges in X. The following collects some known facts (cf. [30, 3.8.2], [18, 3.3.8 and 3.3.9]).

PROPOSITION 2. *Let X be an F-sequence space over I. Consider the following assertions*:

(i) *$(e_i)_{i\in I}$ is an unconditional basis*;

(ii) *$(e_i)_{i\in I}$ is a basis in some ordering, and if $(x_i) \in X$ then also $(\varepsilon_i x_i) \in X$ whenever each ε_i is either* 0 *or* 1;

(iii) *$(e_i)_{i\in I}$ is a basis in some ordering and if $(x_i) \in X$ then also $(c_i x_i) \in X$ whenever (c_i) is a bounded sequence of scalars.*

Then we have the following implications: (i)$\Leftrightarrow$(ii)$\Leftarrow$(iii). *If X is in addition locally convex or locally bounded then all three assertions are equivalent.*

We remark that a sequence space that satisfies the second condition in (ii) is called *monotone*, one that satisfies the second condition in (iii) is called *solid*.

THEOREM 8. *Let X be an F-sequence space in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. Let $T : X \to X$ be a unilateral weighted backward shift with weight sequence $(a_n)_{n\in\mathbb{N}}$. Then the following assertions are equivalent*:

(i) *T is chaotic*;

(ii) *T is hypercyclic and has a non-trivial periodic point*;

(iii) *T has a non-trivial periodic point*;

(iv) *the series $\sum_{n=1}^{\infty}(\prod_{\nu=1}^{n} a_\nu)^{-1} e_n$ converges in X.*

Proof. Since the implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are trivial we need only show that (iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(i) hold.

(iii)$\Rightarrow$(iv). Let $x = (x_n) \in X$ be a non-trivial periodic point for T, that is, there are $N \in \mathbb{N}$ and $j \in \mathbb{N}$ with $T^N x = x$ and $x_j \neq 0$. Comparing the entries at positions $j + kN$, $k \in \mathbb{N}_0$, of x and $T^N x$ we find that

$$x_{j+kN} = \Big(\prod_{\nu=1}^{N} a_{j+kN+\nu}\Big) x_{j+(k+1)N},$$

so that we have, for $k \in \mathbb{N}_0$,

$$x_{j+kN} = \Big(\prod_{\nu=j+1}^{j+kN} a_\nu\Big)^{-1} x_j = c\Big(\prod_{\nu=1}^{j+kN} a_\nu\Big)^{-1}$$

with $c = (\prod_{\nu=1}^{j} a_\nu) x_j \neq 0$. Since (e_n) is an unconditional basis and $x \in X$ it follows from Proposition 2 that

$$\sum_{k=0}^{\infty} \frac{1}{\prod_{\nu=1}^{j+kN} a_\nu} e_{j+kN} = \frac{1}{c}\sum_{k=0}^{\infty} x_{j+kN} e_{j+kN}$$

converges in X. Without loss of generality we may assume that $j \geq N$. Applying the operators $T, T^2, \ldots, T^{N-1}$ to this series and noting that $Te_n = a_n e_{n-1}$ for $n \geq 2$ we deduce that

$$\sum_{k=0}^{\infty} \frac{1}{\prod_{\nu=1}^{j+kN-l} a_\nu} e_{j+kN-l}$$

converges in X for $l = 0, \ldots, N-1$. Adding these N series we see that condition (iv) holds.

(iv)$\Rightarrow$(i). It follows from Theorem 1 that under condition (iv) the operator T is hypercyclic. Hence it remains to show that T has a dense set of periodic points. Since (e_n) is an unconditional basis, condition (iv) with Proposition 2 implies that for each $j \in \mathbb{N}$ and $N \in \mathbb{N}$ the series

$$g^{(j,N)} := \sum_{k=0}^{\infty} \frac{1}{\prod_{\nu=j+1}^{j+kN} a_\nu} e_{j+kN} = \Big(\prod_{\nu=1}^{j} a_\nu\Big) \sum_{k=0}^{\infty} \frac{1}{\prod_{\nu=1}^{j+kN} a_\nu} e_{j+kN}$$

converges and defines an element in X. Moreover, if $N \geq j$ then

$$T^N g^{(j,N)} = g^{(j,N)}, \tag{5.1}$$

so that each $g^{(j,N)}$ $(j \leq N)$ is a periodic point for T.

We shall now show that T has a dense set of periodic points. Since (e_n) is a basis it suffices to show that for every element $x \in \operatorname{span}\{e_n : n \in \mathbb{N}\}$ there is a periodic point y arbitrarily close to it. To see this, let $x = \sum_{j=1}^{m} x_j e_j$ and $\varepsilon > 0$. We can assume without loss of generality that

$$\Big|x_j \prod_{\nu=1}^{j} a_\nu\Big| \leq 1 \quad \text{for } j = 1, \ldots, m. \tag{5.2}$$

Since (e_n) is an unconditional basis condition (iv) implies that there is an $N \geq m$ such that

$$\Big\|\sum_{n=N+1}^{\infty} \varepsilon_n \frac{1}{\prod_{\nu=1}^{n} a_\nu} e_n\Big\| < \frac{\varepsilon}{m} \tag{5.3}$$

for every sequence (ε_n) taking values 0 or 1 (cf. [30, p. 153]). By (5.1) the element

$$y = \sum_{j=1}^{m} x_j g^{(j,N)}$$

of X is a periodic point for T, and we have

$$\begin{aligned} \|y - x\| &= \Big\|\sum_{j=1}^{m} x_j (g^{(j,N)} - e_j)\Big\| \\ &= \Big\|\sum_{j=1}^{m}\Big(x_j \prod_{\nu=1}^{j} a_\nu\Big)\Big(\sum_{k=1}^{\infty} \frac{1}{\prod_{\nu=1}^{j+kN} a_\nu} e_{j+kN}\Big)\Big\| \\ &\le \sum_{j=1}^{m}\Big\|\Big(x_j \prod_{\nu=1}^{j} a_\nu\Big)\Big(\sum_{k=1}^{\infty} \frac{1}{\prod_{\nu=1}^{j+kN} a_\nu} e_{j+kN}\Big)\Big\| \\ &\le \sum_{j=1}^{m}\Big\|\sum_{k=1}^{\infty} \frac{1}{\prod_{\nu=1}^{j+kN} a_\nu} e_{j+kN}\Big\| \quad \text{by (5.2)} \\ &< \varepsilon \quad \text{by (5.3).} \end{aligned}$$

This had to be shown. ■

REMARKS. (1) Under the assumptions of the theorem condition (iv) can be stated succinctly as

(iv′) *the sequence* $(1/\prod_{\nu=1}^{n} a_\nu)_{n\in\mathbb{N}}$ *belongs to* X.

(2) In the case when X is a *complex* and *locally convex* space one may follow the elegant proof of Godefroy and Shapiro [11, pp. 266–267] to obtain the implication (iv)⇒(i). The idea for the present proof was taken from Peris [28].

COROLLARY 1. *Let* $T : H(\mathbb{C}) \to H(\mathbb{C})$ *be a weighted backward shift with weight sequence* $(a_n)_{n\in\mathbb{N}_0}$. *Then* T *is chaotic if and only if*

$$\Big|\prod_{\nu=0}^{n} a_\nu\Big|^{1/n} \to \infty$$

as $n \to \infty$.

Thus, each of the weighted shifts on $H(\mathbb{C})$ considered by Mathew [26], namely those with weights tending to ∞, is chaotic. In particular, the differentiation operator D is chaotic, as was first shown by Godefroy and Shapiro [11, Theorem 6.2] (see also [28]). In view of Theorem 8 one might be tempted to pronounce that the existence of the exponential function as a fixed point makes the differentiation operator D chaotic.

COROLLARY 2. *Every unilateral weighted backward shift on the space* ω *of all sequences is chaotic.*

We consider another special case of Theorem 8.

EXAMPLE. Let s be the Fréchet space of all rapidly decreasing sequences,

$$s = \Big\{x = (x_n)_{n\in\mathbb{N}_0} : p_k(x) := \Big(\sum_{n=0}^{\infty} |x_n|^2 (n+1)^k\Big)^{1/2} < \infty \text{ for all } k \in \mathbb{N}\Big\},$$

whose topology is defined by the seminorms p_k $(k \in \mathbb{N})$. Then the weighted backward shift T defined by

$$Te_n = \sqrt{n}\, e_{n-1} \quad (n \ge 1)$$

is chaotic because

$$\sum_{n=0}^{\infty} \frac{1}{n!}(n+1)^k < \infty \quad \text{for all } k \in \mathbb{N}.$$

This result has recently been obtained by Gulisashvili and MacCluer [15] who have shown that T arises as the annihilation operator of the quantum harmonic oscillator in physics. Thus, as the authors note, *linear* chaos can occur in a quasi-physical system.

One might ask if the characterisation of chaotic shifts given in Theorem 8 remains true without assuming that the basis is unconditional. We show by a counterexample that this is not the case.

EXAMPLE. Let X be the Banach sequence space defined by

$$X = \Big\{x = (x_n) : \|x\| = \sum_{n=1}^{\infty}\Big|\frac{x_n}{n} - \frac{x_{n+1}}{n+1}\Big| < \infty \text{ and } \frac{x_n}{n} \to 0 \text{ as } n \to \infty\Big\};$$

that is, $x \in X$ if and only if (x_n/n) belongs to the familiar sequence space bv_0 (cf. [36, Section 7.3]). Then (e_n) is a basis in X, but it is not unconditional because X is not solid (cf. Proposition 2).

We want to show that the (unweighted) backward shift B on X given by $B(x_n) = (x_{n+1})$ is hypercyclic, has non-trivial periodic points and satisfies condition (iv) of Theorem 8 but is not chaotic.

First, a simple calculation shows that B indeed maps X into X and hence is a continuous operator. Now, since B has weights $a_n = 1$ for all n and since $\sum_{n=1}^{\infty} e_n$ converges in X, condition (iv) of Theorem 8 holds. By Theorem 1, B is therefore hypercyclic. And since all constant sequences belong to X, B has non-trivial periodic points.

But the constant sequences are the only periodic points, which shows that B cannot be chaotic. Indeed, let $x = (x_n)$ be a non-constant sequence that is periodic for B. Then there are $j, N \in \mathbb{N}$ with $x_{j+kN} = a$ and $x_{j+kN+1} = b$

for $k \in \mathbb{N}_0$ with constants $a \neq b$. We then have

$$\|x\| \geq \sum_{k=0}^{\infty} \left| \frac{x_{j+kN}}{j+kN} - \frac{x_{j+kN+1}}{j+kN+1} \right| = \sum_{k=0}^{\infty} \left| \frac{(j+kN)(a-b)+a}{(j+kN)(j+kN+1)} \right| = \infty,$$

which is a contradiction.

We next characterise chaotic bilateral shifts. The proof of the result is similar to that of Theorem 8 and is therefore omitted.

THEOREM 9. *Let X be a bilateral F-sequence space in which $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis. Let $T : X \to X$ be a bilateral weighted backward shift with weight sequence $(a_n)_{n\in\mathbb{Z}}$. Then the following assertions are equivalent*:

(i) *T is chaotic*;
(ii) *T is hypercyclic and has a non-trivial periodic point*;
(iii) *T has a non-trivial periodic point*;
(iv) *the series*

$$\sum_{n=0}^{\infty} \Big(\prod_{\nu=0}^{n-1} a_{-\nu}\Big) e_{-n} + \sum_{n=1}^{\infty} \frac{1}{\prod_{\nu=1}^{n} a_\nu} e_n$$

converges in X.

REMARK. The corresponding condition for bilateral weighted forward shifts reads: The series

$$\sum_{n=1}^{\infty} \frac{1}{\prod_{\nu=1}^{n} a_{-\nu}} e_{-n} + \sum_{n=0}^{\infty} \Big(\prod_{\nu=0}^{n-1} a_\nu\Big) e_n$$

converges in X.

In a recent paper [20] deLaubenfels and Emamirad obtain further classes of operators for which the existence of one non-trivial periodic point implies chaos.

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The L^p solvability of the Dirichlet problems for parabolic equations

by

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Abstract. For two general second order parabolic equations in divergence form in Lip(1, 1/2) cylinders, we give a criterion for the preservation of L^p solvability of the Dirichlet problems.

1. Introduction. The purpose of this article is to study the solvability of the L^p Dirichlet problem for second order parabolic divergence form operators with time dependent coefficients in a Lip(1, 1/2) cylinder Ω. The operators L we consider are of the form

$$Lu = \operatorname{div}(A(x,t)\nabla u) - \partial_t u \quad \text{in } \Omega_T \in \mathbb{R}^{n+1}$$

where Ω_T is a finite cylinder having lateral boundary S_T and parabolic boundary $\partial_{\mathrm{p}}\Omega_T$, and the matrix $A(x,t)$ is assumed to be symmetric, bounded, measurable and to satisfy the ellipticity condition; that is, there exists a positive constant λ such that for all $(x,t) \in \mathbb{R}^{n+1}$ and $\xi \in \mathbb{R}^n$,

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} A_{ij}\xi_i\xi_j \le \lambda|\xi|^2.$$

It is well known that if $f \in C(\partial_{\mathrm{p}}\Omega_T)$ is given, then the classical Dirichlet problem

$$Lu = 0 \quad \text{in } \Omega_T, \qquad u|_{\partial_{\mathrm{p}}\Omega_T} = f \in C(\partial_{\mathrm{p}}\Omega_T),$$

is solvable. The solvability of the L^p Dirichlet problem for L is related to the $D(p, S_T)$ property. If there exists a $p \in (1,\infty)$ such that the solution function u satisfies

$$\|N(u)\|_{L^p(S_T,d\sigma)} \le C\|f\|_{L^p(S_T,d\sigma)}, \qquad f \in C(S_T),$$

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