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Stochastic representation of reflecting diffusions corresponding to divergence form operators

by

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Abstract. We obtain a stochastic representation of a diffusion corresponding to a uniformly elliptic divergence form operator with co-normal reflection at the boundary of a bounded C^2-domain. We also show that the diffusion is a Dirichlet process for each starting point inside the domain.

0. Introduction and notation. Let D be the following non-empty bounded domain in $\mathbb{R}^d$:

$$D=\{x\in\mathbb{R}^d:\Phi(x)>0\}\quad\text{with}\quad \partial D=\{x\in\mathbb{R}^d:\Phi(x)=0\}, \tag{0.1}$$

where $\Phi\in C_b^2(\mathbb{R}^d)$ satisfies $|\nabla\Phi(x)|\ge 1$ for all $x\in\partial D$, and let $a:\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d$ belong to the class $\mathcal{A}(\lambda,\Lambda)$ of all measurable, symmetric matrix-valued functions which satisfy the ellipticity condition

$$\lambda|\xi|^2\le a^{ij}(x)\xi_i\xi_j\le\Lambda|\xi|^2,\qquad x,\xi\in\mathbb{R}^d \tag{0.2}$$

for some $0<\lambda\le\Lambda$ (we employ the summation convention over repeated indices). Consider the operator

$$A=D_j\big(\tfrac12 a^{ij}(\cdot)D_i\big)$$

and let p be a weak Neumann function for A on D (see Section 2). Using the estimates on p proved in Gushchin [13] we first construct a family $\{P^x:x\in D\}$ of probability measures on $C([0,T];\overline{D})$ such that the finite-dimensional distributions of P^x are determined by p and then we investigate the structure of the canonical process X under the measures P^x.

More precisely, let γ_a denote the co-normal vector field on ∂D, i.e. $\gamma_a^i(x)=(1/2)a^{ij}(x)\boldsymbol{n}_j(x)$ for $i=1,\dots,d$, where $\boldsymbol{n}(x)=\nabla\Phi(x)/|\nabla\Phi(x)|$ is the unit inward normal to ∂D. We prove that X is a Dirichlet process in the sense of Föllmer [5] under P^x for every $x\in D$ and its components admit

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the decomposition

$$X_t^i - x^i = \frac{1}{2}(M_t^i + N_{T-t}^i - N_T^i) - \frac{1}{2}\int_0^t \mathbf{1}_D a^{ij}\frac{D_j p}{p}(u,x,X_u)\,du + K_t^i \tag{0.3}$$

$$= M_t^i + A_t^i, \quad t \in [0,T],\ i=1,\ldots,d, \tag{0.4}$$

where M^i (resp. N^i) is a square-integrable martingale with respect to the filtration $\{\mathcal{F}_t\}$ generated by X (resp. $\{\overline{\mathcal{F}}_t\}$ generated by the time-reversed process $\{\overline{X}_t = X_{T-t} : t \in [0,T]\}$) with

$$\langle M^i\rangle_t = \int_0^t a^{ii}(X_u)\,du \quad \Big(\text{resp. } \langle N^i\rangle_t = \int_0^t a^{ii}(\overline{X}_u)\,du\Big), \quad t \in [0,T],$$

K^i is an $\{\mathcal{F}_t\}$-adapted process such that $K_0^i = 0$ and K^i increases only when $X \in \partial D$, and A^i is an $\{\mathcal{F}_t\}$-adapted process of 0-quadratic variation. Actually, X belongs to the class $\mathcal{D}^2$ considered in Coquet and Słomiński [4], which is strictly smaller than the class of Dirichlet processes. If, in addition, a is continuous, then there is an $\{\mathcal{F}_t\}$-adapted non-decreasing process K which increases only when $X \in \partial D$ such that

$$K_t^i = \int_0^t \gamma_a^i(X_u)\,dK_u, \quad t \in [0,T]. \tag{0.5}$$

By using a theory of Dirichlet forms, in Fukushima and Tomisaki [10, 11] a diffusion associated with A on much more general domains is constructed and a strict Fukushima decomposition of $X - X_0$ into a martingale additive functional of finite energy and an additive functional of zero energy is proved. Moreover, it is shown that if a^{ij}'s have bounded partial derivatives in the sense of distributions then X is a semimartingale under P^x for every $x \in \overline{D}$ and a Skorokhod representation of X is obtained. From Fukushima's decomposition it follows in particular that there is a sequence of partitions of $[0,T]$ into intervals of equal length such that X is a Dirichlet process along it under P^x for almost every $x \in \overline{D}$. Our assumptions on D are rather restrictive. We know, however, that for every $x \in D$ the process X is under P^x a Dirichlet process along any sequence of partitions of $[0,T]$ whose mesh-size tends to zero. Secondly, our method of construction of (X, P^x) based on estimates on p allows us to obtain a Lyons–Zheng–Skorokhod representation (0.3) without any regularity assumptions on a. Of course, it would be desirable to prove (0.3) for $x \in \partial D$ and (0.5) for $a \in \mathcal{A}(\lambda,\Lambda)$. Unfortunately, we do not know how to do this.

In case $D = \mathbb{R}^d$ we have $K^i = 0$ for $i = 1,\ldots,d$, and so (0.3) specializes to the decomposition proved in Lyons and Zheng [22], Rozkosz [25], Rozkosz and Słomiński [28]. The representation (0.4) corresponds to the one proved in [25, 28]. See also Fukushima [8], Fukushima, Oshima and Takeda [9] and Rozkosz [26], where connections between Fukushima's decomposition and a decomposition in the sense of Föllmer are examined in detail.

The decompositions (0.3), (0.4) allow us to develop some stochastic calculus against X. In the present paper we confine ourselves to showing that for any $x \in D$ and $\varphi \in C^2(\overline{D})$ the stochastic integral $\int D_i\varphi(X)\,dX^i$ and the mutual quadratic variation $\langle D_i\varphi(X), X^i\rangle$ exist as limits in P^x of Riemann sums and

$$\varphi(X_t) = \varphi(x) + \int_0^t D_i\varphi(X_u)\,dX_u^i + \tfrac{1}{2}\langle D_i\varphi(X), X^i\rangle_t, \quad t \in [0,T], \tag{0.6}$$

P^x-a.s. This extends Itô's formula proved in [25, 28] in case $D = \mathbb{R}^d$ but the basic ideas of proof had appeared previously in Föllmer, Protter and Shiryaev [6], Lyons and Zhang [20], Lyons and Zheng [21]. Note also that the fact that $X \in \mathcal{D}^2$ can be used to define integrals $\int Y\,dX$, $\int X\,dY$ for $\{\mathcal{F}_t\}$-adapted processes of the class $\mathcal{D}^p$ with $p \in [1,2)$ (see [4]).

The paper is organized as follows. In Section 1 we show that under the measure P^x associated with a smooth $a \in \mathcal{A}(\lambda,\Lambda)$ the time-reversed process $\overline{X}$ is again a diffusion with reflection in the co-normal direction, and we identify its coefficients. In Section 2 we recall some facts from the PDE theory that are used in Section 3 to construct a diffusion process associated with $a \in \mathcal{A}(\lambda,\Lambda)$. Section 4 contains the proof of the main result. In Section 5 we define stochastic integrals and we prove Itô's formula. Finally, in the Appendix we prove a general theorem on convergence of strong Markov processes satisfying the condition UTD introduced in [4]. This result was proved essentially in [28] but in a form not directly applicable to our situation.

We will use the following notation:

$$D_T = (0,T)\times D, \quad D_{\delta T} = (\delta,T)\times D, \quad S_T = (0,T)\times\partial D.$$

$D_i = \partial/\partial x^i$ is the partial derivative in the distribution sense. $\mathcal{A}^\infty(\lambda,\Lambda)$ is the subset of $\mathcal{A}(\lambda,\Lambda)$ consisting of all functions having bounded continuous derivatives of all orders in $\overline{D}$.

$C([0,T];\mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued continuous functions on $[0,T]$. Given a process Y with trajectories in $C([0,T];\mathbb{R}^d)$ and $f:\mathbb{R}^d \to \mathbb{R}$ we write $\overline{Y}_t = Y_{T-t}$, $\widetilde{Y}_t = Y_{T-t} - Y_T$, ${}^\delta Y_t = Y_{t\vee\delta} - Y_\delta$, $\langle Y\rangle_s^t = \langle Y\rangle_t - \langle Y\rangle_s$ and

$$(f\cdot Y)_t = \int_0^t f(X_u)\,dY_u, \quad (f * Y)_t = -\int_{T-t}^T f(\overline{X}_u)\,dY_u, \quad t\in[0,T],$$

whenever the integrals make sense. $\operatorname{Var} Y_T$ is the variation of Y on $[0,T]$.

Further,

$$\mathcal{F}_t = \sigma(X_u : u \in [0,t]), \quad \overline{\mathcal{F}}_t = \sigma(\overline{X}_u : u \in [0,t]), \quad t \in [0,T].$$

$\mathcal{M}$ (resp. $\widetilde{\mathcal{M}}$) is the space of square-integrable $(\{\mathcal{F}_t\}, P^x)$ (resp. $(\{\overline{\mathcal{F}}_t\}, P^x)$) continuous martingales on $[0,T]$ vanishing at zero equipped with the usual norm $(E^x M_T^2)^{1/2} = (E^x\langle M\rangle_T)^{1/2}$.

$\mathcal{L}[Y \mid P^x]$ is the law of Y under P^x. By E^x, E_n^x we denote expectations with respect to P^x and P_n^x, respectively.

$C(\overline{D})$ is the set of continuous functions in $\overline{D}$, and $C^k(\overline{D})$, $k=1,2$, is the set of all continuous functions in $\overline{D}$ having derivatives up to order k inclusive that are continuous in $\overline{D}$. Next, $C_b^2(\mathbb{R}^d)$ is the set of all continuous functions in $\mathbb{R}^d$ having bounded continuous derivatives up to order 2 and $C_0^\infty(\mathbb{R}^d)$ is the set of all smooth functions in $\mathbb{R}^d$ having compact support. $W_p^1(D)$ is the Banach space consisting of all elements u of $\mathbb{L}_p(D)$ having generalized derivatives $D_i u$ from $\mathbb{L}_p(D)$. We denote by $W_p^{0,1}(D_T)$ the Banach space consisting of all elements u of $\mathbb{L}_p(D_T)$ having generalized derivatives $D_i u$ from $\mathbb{L}_p(D_T)$, and $W_p^{1,1}(D_T)$ is the Banach space consisting of all elements u of $\mathbb{L}_p(D_T)$ having generalized derivatives $\partial u/\partial t$ and $D_i u$ from $\mathbb{L}_p(D_T)$.

1. Time reversal. Suppose $a \in \mathcal{A}^\infty(\lambda, \Lambda)$ and consider the operator

$$A = \tfrac{1}{2}a^{ij}(\cdot)D_iD_j + \theta^i(\cdot)D_i, \quad \text{where} \quad \theta^i(x) = \tfrac{1}{2}D_j a^{ij}(x).$$

Due to results by Stroock and Varadhan [30], for each $x \in \overline{D}$ there is a unique solution P^x, starting from x at time 0, to the submartingale problem on D for a, θ and γ_a, and we call (X, P^x) a *diffusion corresponding to A with reflection along γ_a*.

It is possible to construct P^x analytically by first constructing a Neumann function p for A on D (see [7, Exercise V.5]) and then a Markov semigroup $\{P^t : 0 \le t \le T\}$ on $C(\overline{D})$ by

$$P^t\varphi(x) = \int_D \varphi(y)p(t,x,y)\,dy, \quad \varphi \in C(\overline{D}),$$

which gives rise to the strong Markov family $\{P^x : x \in \overline{D}\}$ with p as the transition density. The Markov and the semigroup properties ensure that for each $x \in \overline{D}$ the measure P^x is a solution to the submartingale problem for a, θ, γ_a starting from x. Note also that for given $\varphi \in C(\overline{D})$, $u : [0,T] \times \overline{D} \to \mathbb{R}$ defined by $u(t,x) = P^t\varphi(x)$ is a unique classical solution to the Neumann problem (see [7])

$$(1.1) \quad \begin{cases} (\frac{\partial}{\partial t} - A)u = 0 & \text{on } (0,T] \times D, \\ \lim_{t \searrow 0} u(t,x) = \varphi(x) & \text{on } \overline{D}, \\ \langle \gamma_a, \nabla u\rangle = 0 & \text{on } S_T. \end{cases}$$

For fixed $x \in \overline{D}$ put $p(t,y) = p(t,x,y)$ for $(t,y) \in (0,T] \times \overline{D}$ and define

$$\mathcal{A}_t\varphi = D_j\big(\tfrac{1}{2}a^{ij}(\cdot)D_i\varphi\big) + (a^{ij}p^{-1}D_jp)(t,\cdot)D_i\varphi, \quad \overline{\mathcal{A}}_t = \mathcal{A}_{T-t},$$

with the convention that $p^{-1}D_jp(t,y)$ equals 0 if $p(t,y)=0$. In what follows we use ideas from Haussmann and Pardoux [14] and from [30] to show that for each $x \in \overline{D}$, $(\overline{X}, P^x)$ is a diffusion corresponding to $\overline{\mathcal{A}}$ with reflection along γ_a.

THEOREM 1.1. *Assume $a \in \mathcal{A}^\infty(\lambda, \Lambda)$ and let $\theta^i = D_j a^{ij}$, $i = 1,\dots,d$. Let P^x be a solution to the submartingale problem on D for a, θ, γ_a starting from x. Then there exists a continuous non-decreasing process $K : [0,T] \times \Omega \to \mathbb{R}$ with the property that*

$$(1.2) \quad K \text{ is } \{\mathcal{F}_t\}\text{-adapted}, \ K_0 = 0, \ E^x K_T < \infty, \ K_t = \int_0^t \mathbf{1}_{\partial D}(X_u)\,dK_u,$$

and for any $\varphi \in C^2(\overline{D})$,

$$M_t^\varphi = \varphi(X_t) - \varphi(X_0) - \int_0^t \mathbf{1}_D A\varphi(X_u)\,du - \int_0^t \langle\gamma_a, \nabla\varphi\rangle(X_u)\,dK_u,$$

$$N_t^\varphi = \varphi(\overline{X}_t) - \varphi(\overline{X}_0) - \int_0^t \mathbf{1}_D \overline{\mathcal{A}}_u\varphi(\overline{X}_u)\,du - \int_0^t \langle\gamma_a, \nabla\varphi\rangle(\overline{X}_u)\,d\widetilde{K}_u$$

are $(\{\mathcal{F}_t\}, P^x)$- and $(\{\overline{\mathcal{F}}_t\}, P^x)$-martingales on $[0,T]$ respectively. Moreover,

$$(1.3) \quad \langle M^\varphi\rangle_t = \int_0^t \mathbf{1}_D a^{ij} D_i\varphi D_j\varphi(X_u)\,du,$$

$$(1.4) \quad \langle N^\varphi\rangle_t = \int_0^t \mathbf{1}_D a^{ij} D_i\varphi D_j\varphi(\overline{X}_u)\,du$$

and $\langle M^\varphi, \widetilde{N}^\varphi\rangle_t = \langle M^\varphi\rangle_t$ for $t \in [0,T]$. Finally,

$$(1.5) \quad \varphi(X_t) - \varphi(X_0) = \frac{1}{2}(M_t^\varphi + \widetilde{N}_t^\varphi - V_t^\varphi) + \int_0^t \langle\gamma_a, \nabla\varphi\rangle(X_u)\,dK_u$$

for $t \in [0,T]$, where $V_t^\varphi = \int_0^t \mathbf{1}_D a^{ij}p^{-1}D_jpD_i\varphi(u, X_u)\,du$.

Proof. Existence of K and the fact that M^φ is a martingale are well known (see [30, Theorem 2.4]). To prove that N^φ is a martingale we first show that

$$S_t^\varphi = \varphi(\overline{X}_t) - \varphi(\overline{X}_0) - \int_0^t \mathbf{1}_D \overline{\mathcal{A}}_u\varphi(\overline{X}_u)\,du, \quad t \in [0,T],$$

is an $(\{\overline{\mathcal{F}}_t\}, P^x)$ supermartingale for any $\varphi \in C^2(\overline{D})$ satisfying $\langle \gamma_a, \nabla\varphi\rangle \ge 0$ on ∂D. To this end, fix a non-negative $g \in C(\overline{D})$ and set

$$w(u,x) = E^x g(X_{t+s-u}) = \int_D g(y)p(t+s-u,x,y)\,dy, \qquad u \in [0,t+s).$$

Then w is the unique classical solution to the Neumann problem

$$\Big(\frac{\partial}{\partial u} + A\Big)w = 0 \quad \text{on } [0,t+s)\times D,$$
$$\lim_{u\nearrow t+s} w(u,x) = g(x), \qquad \langle \gamma_a, \nabla w\rangle|_{S_{t+s}} = 0.$$

Since X is a Markov process under P^x,

$$\begin{aligned} E^x\varphi(X_t)g(X_{t+s}) &= E^x\{E^x(\varphi(X_t)g(X_{t+s})\,|\,\mathcal{F}_t)\} \\ &= E^x\{\varphi(X_t)E^{X_t}g(X_s)\} = E^x\varphi(X_t)w(t,X_t) \end{aligned}$$

for $t,s \ge 0$. Therefore

$$\begin{aligned} &E^x\varphi(X_{t+s})g(X_{t+s}) - E^x\varphi(X_t)g(X_{t+s}) \\ &\quad = E^x\varphi(X_{t+s})w(t+s,X_{t+s}) - E^x\varphi(X_t)w(t,X_t) \\ &\quad \ge E^x\int_t^{t+s}\mathbf{1}_D\Big(\frac{\partial}{\partial u}+A\Big)(\varphi w)(u,X_u)\,du \equiv I, \end{aligned}$$

because $M^{\varphi w}$ is a martingale and $\langle\gamma_a,\nabla(\varphi w)\rangle = \langle\gamma_a, w\nabla\varphi\rangle \ge 0$ on S_T. Elementary computations show that

$$\begin{aligned} (1.6)\qquad I &= E^x\int_t^{t+s}\mathbf{1}_D\Big\{\varphi\Big(\frac{\partial}{\partial u}+A\Big)w \\ &\qquad + a^{ij}\Big(\frac12 wD_iD_j\varphi + D_i\varphi D_jw\Big) + \frac12 wD_ja^{ij}D_i\varphi\Big\}(u,X_u)\,du \\ &= \int_t^{t+s}du\int_D \frac12 a^{ij}(D_iD_j\varphi)wp(u,y)\,dy \\ &\qquad + \int_t^{t+s}du\int_D a^{ij}(D_i\varphi D_jw)p(u,y)\,dy \\ &\qquad + \int_t^{t+s}du\int_D \frac12(D_ja^{ij}D_i\varphi)wp(u,y)\,dy. \end{aligned}$$

Let σ denote the surface measure on ∂D. Integrating by parts gives

$$\begin{aligned} (1.7)\qquad &\int_t^{t+s}du\int_D a^{ij}(D_i\varphi D_jw)p(u,y)\,dy \\ &= -\int_t^{t+s}du\int_D D_j(a^{ij}pD_i\varphi)w(u,y)\,dy + 2\int_t^{t+s}du\int_{\partial D}\langle\gamma_a,\nabla\varphi\rangle wp(u,y)\,d\sigma(y) \\ &\ge -\int_t^{t+s}du\int_D a^{ij}(D_iD_j\varphi)wp(u,y)\,dy - \int_t^{t+s}du\int_D D_j(a^{ij}p)D_i\varphi w(u,y)\,dy. \end{aligned}$$

Combining (1.6) with (1.7) and taking into account that by [14, Lemma A.2], we have $a^{ij}D_jp(u,y) = 0$ a.e. on $\{(u,y) : p(u,y) = 0\}$ we obtain

$$\begin{aligned} I &\ge \int_t^{t+s}du\int_D\Big\{-\frac12 a^{ij}D_iD_j\varphi + \Big(\frac12 D_ja^{ij} - p^{-1}D_j(a^{ij}p)\Big)D_i\varphi\Big\}wp(u,y)\,dy \\ &= -\int_t^{t+s}du\int_D(\mathcal{A}_u\varphi)wp(u,y)\,dy = -\int_t^{t+s}E^x\mathbf{1}_D(\mathcal{A}_u\varphi)w(u,X_u)\,du \\ &= -E^x\Big\{\int_t^{t+s}\mathbf{1}_D\mathcal{A}_u\varphi(X_u)\,du\,g(X_{t+s})\Big\}. \end{aligned}$$

By the above,

$$E^x\Big\{\Big[\varphi(X_{t+s}) - \varphi(X_t) + \int_t^{t+s}\mathbf{1}_D\mathcal{A}_u\varphi(X_u)\,du\Big]g(X_{t+s})\Big\} \ge 0,$$

and hence

$$E^x\Big\{\Big[\varphi(\overline{X}_{T-t}) - \varphi(\overline{X}_{T-t-s}) - \int_{T-t-s}^{T-t}\mathbf{1}_D\overline{\mathcal{A}}_u\varphi(\overline{X}_u)\,du\Big]g(\overline{X}_{T-t-s})\Big\} \le 0.$$

In other words, $E^x[(S^\varphi_{t+s} - S^\varphi_t)g(\overline{X}_t)] \le 0$ for $t,s \ge 0$ such that $t+s \le T$. Since $(\overline{X}, P^x)$ is a Markov family and g is an arbitrary non-negative function from $C(\overline{D})$, it follows that $E^x((S^\varphi_{t+s} - S^\varphi_t)\,|\,\overline{\mathcal{F}}_t) \le 0$. Thus, $\{S^\varphi_t : t \in [0,T]\}$ is an $(\{\overline{\mathcal{F}}_t\}, P^x)$-supermartingale for any $\varphi \in C^2(\overline{D})$ satisfying $\langle\gamma_a,\nabla\varphi\rangle \ge 0$ on ∂D. We can now apply arguments from the proof of [30, Theorem 2.4] to show that there is an $\{\overline{\mathcal{F}}_t\}$-adapted non-decreasing process $L : [0,T]\times\Omega \to \mathbb{R}$ such that L is $\{\overline{\mathcal{F}}_t\}$-adapted, $L_0 = 0$, $E^xL_T < \infty$, $L_t = \int_0^t \mathbf{1}_{\partial D}(\overline{X}_u)\,dL_u$ and for any $\varphi \in C^2(\overline{D})$,

$$N^\varphi_t = S^\varphi_t + \int_0^t \langle\gamma_a,\nabla\varphi\rangle(\overline{X}_u)\,dL_u$$

is a martingale on $[0,T]$ with $\langle N^\varphi\rangle_t = \int_0^t \mathbf{1}_Da^{ij}D_i\varphi D_j\varphi(\overline{X}_u)\,du$ for $t \in [0,T]$.

We check at once that

$$(1.8)\quad \widetilde{N}_t^\varphi = \varphi(X_t) - \varphi(X_0) + \int_0^t \mathbf{1}_D \mathcal{A}_u\varphi(X_u)\,du + \int_0^t \langle \gamma_a, \nabla\varphi\rangle(X_u)\,d\widetilde{L}_u$$

$$= M_t^\varphi + 2\int_0^t \mathbf{1}_D A\varphi(X_u)\,du + V_t^\varphi + \int_0^t \langle \gamma_a, \nabla\varphi\rangle(X_u)\,d(K_u + \widetilde{L}_u).$$

Therefore

$$\langle M^\varphi\rangle_t = \langle \widetilde{N}^\varphi\rangle_t = \langle N^\varphi\rangle_{T-t}^T = \int_0^t \mathbf{1}_D a^{ij} D_i\varphi D_j\varphi(X_u)\,du$$

and $\langle M^\varphi, \widetilde{N}^\varphi\rangle_t = \langle M^\varphi\rangle_t$ for $t \in [0,T]$. Thus, what is left is to show that $\widetilde{L} = -K$. For this purpose, write

$$Y_t^\varphi \equiv -M_t^\varphi + \widetilde{N}_t^\varphi - V_t^\varphi - 2\int_0^t \mathbf{1}_D A\varphi(X_u)\,du = \int_0^t \langle \gamma_a, \nabla\varphi\rangle(X_u)\,d(K_u + \widetilde{L}_u)$$

and observe that (1.3), (1.4) lead to

$$(1.9)\quad M_t^\varphi = \int_0^t \mathbf{1}_D(X_u)\,dM_u^\varphi, \quad N_t^\varphi = \int_0^t \mathbf{1}_D(\overline{X}_u)\,dN_u^\varphi, \qquad t \in [0,T].$$

Hence $Y_t^\varphi = \int_0^t \mathbf{1}_D(X_u)\,dY_u^\varphi = 0$ for $t \in [0,T]$, since $K, \widetilde{L}$ increase only when $X \in \partial D$. Therefore, setting Y^i for Y^φ with $\varphi(x) = x_i$ we obtain

$$0 = \sum_{i=1}^d \int_0^t \frac{\gamma_a^i}{|\gamma_a|^2}(X_u)\,dY_u^i = \int_0^t d(K_u + \widetilde{L}_u) = K_t + \widetilde{L}_t, \qquad t \in [0,T],$$

which is the desired conclusion. ■

2. Weak solutions to the Neumann problem. In this section we recall, in a form appropriate for our purposes, some analytical facts concerning existence and basic properties of a weak Neumann function for A on D and we prove a limit theorem which will be needed in the next sections.

THEOREM 2.1. *Let $a \in \mathcal{A}(\lambda, \Lambda)$. Then:*

(i) *There exists a unique Markov semigroup $\{P^t : t \geq 0\}$ of positive operators on $\mathbb{L}_2(D)$ such that for every $T > 0$,*

(a) *$P^\cdot\varphi(\cdot) \in W_2^{0,1}(D_T)$ for $\varphi \in \mathbb{L}_2(D)$,*

(b) *for any $\eta \in W_2^{1,1}(D_T)$ vanishing at $t = T$ and any $\varphi \in \mathbb{L}_2(D)$,*

$$\int_D \varphi(x)\eta(0,x)\,dx = -\int_{D_T} P^t\varphi(x)\frac{\partial}{\partial t}\eta(t,x)\,dt\,dx + \int_{D_T} \frac{1}{2}a^{ij}(x)D_iP^t\varphi(x)D_j\eta(t,x)\,dt\,dx.$$

(ii) *There is a $p(t,x,y)$, $t > 0$, $x, y \in D$, with the following properties:*

(a) *there are $C_1, C_2 > 0$ depending only on λ, Λ, d such that for any fixed $t > 0$ and $x \in D$,*

$$(2.1)\quad \int_D |p(t,x,y)|^2\,dy \leq C_1(\min\{t, C_2(\mathrm{dist}(x,\partial D))^2\})^{-d/2},$$

(b) *for every $\delta \in (0,T]$ and every $K \subset D$ such that* $\mathrm{dist}(K,\partial D) > 0$ *there is $C_3 > 0$ depending only on λ, Λ, d, δ, T and* $\mathrm{dist}(K,\partial D)$ *such that*

$$(2.2)\quad \sup_{\delta\leq t\leq T,\, x,y\in K} p(t,x,y) \leq C_3,$$

(c) *for any $\varphi \in \mathbb{L}_2(D)$,*

$$(2.3)\quad P^t\varphi(x) = \int_D \varphi(y)p(t,x,y)\,dy, \qquad x \in D.$$

Proof. Let $G(t,x,s,y)$ be the Green function for the problem (1.1) constructed in [13, §4] and let $p(t,x,y) = G(t,x,0,y)$ for $t > 0$, $x, y \in D$. Then Theorem 2.1 is a reformulation of some results of [13, §4]. To see this it suffices to observe that $G(t,x,s,y) = G(t-s,x,y)$ for all $t > s$ and $x, y \in D$, because the coefficients of A do not depend on time. ■

In what follows we will call the function p of Theorem 2.1 a *weak Neumann function* for A on D.

Given $a_n \in \mathcal{A}(\lambda, \Lambda)$ let p_n denote a weak Neumann function for

$$(2.4)\quad A^n = D_j\big(\tfrac{1}{2}a_n^{ij}(\cdot)D_i\big)$$

on D and let

$$P_n^t\varphi(x) = \int_D \varphi(y)p_n(t,x,y)\,dy, \qquad \varphi \in \mathbb{L}_2(D),\ x \in D.$$

LEMMA 2.2. *Let $\{a, a_n\} \subset \mathcal{A}(\lambda, \Lambda)$ and let $a_n^{ij} \to a^{ij}$ a.e. for $i, j = 1, \ldots, d$. Then for any $T > 0$,*

(i) *$P_n^\cdot\varphi(\cdot) \to P^\cdot\varphi(\cdot)$ uniformly on compact sets in D_T for every $\varphi \in \mathbb{L}_2(D)$,*

(ii) *for any fixed $x, y \in D$, $p_n(\cdot, x, \cdot) \to p(\cdot, x, \cdot)$ and $p_n(\cdot, \cdot, y) \to p(\cdot, \cdot, y)$ uniformly on compact sets in $(0,T] \times D$,*

(iii) $\{p_n(\cdot, x, \cdot)\}$ *is bounded in* $W_2^{0,1}(D_{\delta T})$ *for any fixed* $x \in D$, $\delta \in (0,T)$ *and* $p_n(\cdot, x, \cdot) \to p(\cdot, x, \cdot)$ *in* $W_2^{0,1}(K_{\delta T})$ *for any* $K \subset D$ *such that* $\operatorname{dist}(K, \partial D) > 0$.

Proof. By Nash's continuity theorem (see, e.g., [2, 18]), $\{P_n^{\cdot}\varphi(\cdot)\}$ is equibounded and equicontinuous on any compact subset of D_T. At the same time, by [12, Proposition 1], $P_n^{\cdot}\varphi(\cdot) \to P^{\cdot}\varphi(\cdot)$ in $\mathbb{L}_2(D_T)$, which proves (i).

Now fix $x \in D$, $0 < \delta < T$ and define $u_n, u : D_{\delta T} \to \mathbb{R}$ as $u_n(t,y) = p_n(t,x,y)$, $u(t,y) = p(t,x,y)$. Then $u_n \in W_2^{0,1}(D_{\delta T})$ is a weak solution to the Neumann problem

$$\Big(\frac{\partial}{\partial t} - A^n\Big)u_n = 0 \quad \text{on } D_{\delta T},$$

$$\langle \gamma_{a_n}, \nabla u_n\rangle = 0 \quad \text{on } (\delta, T) \times \partial D, \qquad u_n(\delta, \cdot) = \psi_n,$$

where $\psi_n = p_n(\delta, x, \cdot)$, whereas $u \in W_2^{0,1}(D_{\delta T})$ is a weak solution to the problem

$$\Big(\frac{\partial}{\partial t} - A\Big)u = 0 \quad \text{on } D_{\delta T},$$

$$\langle \gamma_a, \nabla u\rangle = 0 \quad \text{on } (\delta, T) \times \partial D, \qquad u(\delta, \cdot) = \psi$$

with $\psi = p(\delta, x, \cdot)$. An elementary computation shows that $v_n = u_n - u$ satisfies

$$(2.5) \qquad \int_D (\psi_n - \psi)\eta(\delta, x)\,dx$$
$$= \int_{D_{\delta T}} \Big\{-v_n \frac{\partial}{\partial t}\eta(t,x) + \frac{1}{2}a_n^{ij} D_i u_n D_j \eta(t,x)\Big\}\,dt\,dx$$
$$- \int_{D_{\delta T}} \frac{1}{2} a^{ij} D_i u D_j \eta(t,x)\,dt\,dx$$
$$= \int_{D_{\delta T}} \Big\{-v_n \frac{\partial}{\partial t}\eta(t,x) + \frac{1}{2}a^{ij} D_i v_n D_j \eta(t,x)\Big\}\,dt\,dx$$
$$+ \int_{D_{\delta T}} \frac{1}{2}(a_n^{ij} - a^{ij}) D_i u_n D_j \eta(t,x)\,dt\,dx$$

for all $\eta \in W_2^{1,1}(D_{\delta T})$ with $\eta(T, \cdot) = 0$. By (2.1), $\{\psi_n - \psi\}$ is bounded in $\mathbb{L}_2(D)$, so the energy inequality for solutions to the Neumann problem (see remarks in §III.4 of [17]) implies that $\{v_n\}$ is bounded in $W_2^{0,1}(D_{\delta T})$. This proves the first statement of (iii). Moreover, by (i) and Nash's continuity theorem, $\psi_n \to \psi$ pointwise in D, so $\psi_n \to \psi$ weakly in $\mathbb{L}_2(D)$. Therefore, if $v_n \to v$ weakly in $W_2^{0,1}(D_{\delta T})$, then letting $n \to \infty$ in (2.5) gives

$$\int_{D_{\delta T}} \Big\{-v \frac{\partial}{\partial t}\eta(t,x) + \frac{1}{2}a^{ij} D_i v D_j \eta(t,x)\Big\}\,dt\,dx = 0,$$

which forces $v = 0$, by the energy inequality. Thus, $v_n \to 0$ weakly in $W_2^{0,1}(D_{\delta T})$. On the other hand, from Nash's continuity theorem and (2.2) we conclude that $\psi_n \to \psi$ in $\mathbb{L}_2(K)$ and $v_n \to 0$ in $\mathbb{L}_2(K_{\delta T})$ for any $\delta \in (0,T)$ and $K \subset D$ such that $\operatorname{dist}(K, \partial D) > 0$. Therefore $v_n \to 0$ in $W_2^{0,1}(K_{\delta T})$ by the inequality (2.18) in Chapter III of [18] and the fact that $v_n \in V_2^{0,1}(D_T)$ (see remarks at the end of §III.4 in [17]).

Finally, (ii) is a consequence of Nash's continuity theorem, (iii) and the fact that $p_n(t,x,y) = p_n(t,y,x)$ for $(t,x,y) \in (0,T] \times \mathbb{R}^{2d}$, $n \in \mathbb{N}$. ■

3. Construction of diffusion processes. Suppose we are given $a \in \mathcal{A}(\lambda, \Lambda)$, $\{a_n\} \subset \mathcal{A}^\infty(\lambda, \Lambda)$ such that $a_n^{ij} \to a^{ij}$ a.e. For $n \in \mathbb{N}$ let (X, P_n^x) denote a reflecting diffusion on D associated with A^n defined by (2.4) starting from $x \in D$. We are going to show that $\{P_n^x\}$ converges weakly in $C([0,T]; \mathbb{R}^d)$ to the measure P^x whose finite-dimensional distributions are determined by a weak Neumann function for A. To this end, we denote by K^n the process of Theorem 1.1 corresponding to a_n and given $\varphi \in C^2(\overline{D})$ we write

$$(3.1) \qquad K_t^{n,\varphi} = \int_0^t \langle \gamma_{a_n}, \nabla\varphi\rangle(X_u)\,dK_u^n,$$

$$(3.2) \qquad V_t^{n,\varphi} = \int_0^t \mathbf{1}_D a_n^{ij} p_n^{-1} D_j p_n D_i \varphi(u, X_u)\,du$$

and

$$(3.3) \qquad M_t^{n,\varphi} = \varphi(X_t) - \varphi(X_0) - \int_0^t \mathbf{1}_D A^n \varphi(X_u)\,du - \int_0^t \langle \gamma_{a_n}, \nabla\varphi\rangle(X_u)\,dK_u^n,$$

$$(3.4) \qquad N_t^{n,\varphi} = \varphi(\overline{X}_t) - \varphi(\overline{X}_0) - \int_0^t \mathbf{1}_D \overline{\mathcal{A}}_u^n \varphi(\overline{X}_u)\,du - \int_0^t \langle \gamma_{a_n}, \nabla\varphi\rangle(\overline{X}_u)\,d\widetilde{K}_u^n$$

for $t \in [0,T]$. Here

$$\mathcal{A}_t^n \varphi = A^n \varphi + (a_n^{ij} p_n^{-1} D_j p_n)(t, \cdot) D_j \varphi, \qquad \overline{\mathcal{A}}_t^n = \mathcal{A}_{T-t}^n,$$

and $p_n(u,y) = p_n(u,x,y)$, where $p_n(\cdot,\cdot,\cdot)$ is a Neumann function for A^n on D. Then we have

LEMMA 3.1. *Let $\varphi \in C^2(\overline{D})$. Then for each starting point $x \in D$ the sequences $\{M^{n,\varphi}\}$, $\{N^{n,\varphi}\}$, $\{\operatorname{Var} V^{n,\varphi}\}$ are tight in $C([0,T];\mathbb{R})$.*

Proof. Let $\{\tau_n\}$, $\{\overline{\tau}_n\}$ be sequences of $\{\mathcal{F}_t\}$, $\{\overline{\mathcal{F}}_t\}$-stopping times, respectively, and let $\{\delta_n\}$ be a sequence of positive numbers such that $\delta_n \searrow 0$. From (1.3), (1.4) it follows immediately that $\{\langle M^{n,\varphi}\rangle_T\}$, $\{\langle N^{n,\varphi}\rangle_T\}$ are uniformly bounded in $n \in \mathbb{N}$ and that

$$\lim_{n\to\infty} E_n^x \langle M^{n,\varphi}\rangle_{T\wedge\tau_n}^{T\wedge(\tau_n+\delta_n)} = \lim_{n\to\infty} E_n^x \langle N^{n,\varphi}\rangle_{T\wedge\overline{\tau}}^{T\wedge(\overline{\tau}_n+\delta_n)} = 0.$$

Hence, by a well known criterion proved in Aldous [1], we deduce that $\{\langle M^{n,\varphi}\rangle\}$, $\{\langle N^{n,\varphi}\rangle\}$ are tight in $C([0,T];\mathbb{R})$, and consequently, $\{M^{n,\varphi}\}$, $\{N^{n,\varphi}\}$ are tight in $C([0,T];\mathbb{R})$ as well.

To prove tightness of $\{\operatorname{Var} V^{n,\varphi}\}$ first fix $\delta \in (0,T)$ and given $\beta \geq 0$ and $u \in [\delta, T]$ set

$$F_\beta^n(u) = \{y \in D : p_n(u,y) > \beta\} \tag{3.5}$$

and

$$Z_t^{n,\beta} = \int_\delta^{\delta\vee t} \mathbf{1}_{F_\beta^n(u)} p_n^{-1} h_n(u, X_u)\,du, \qquad h_n = a_n^{ij} D_j p_n D_i\varphi \tag{3.6}$$

for $t \in [0,T]$. Since D is bounded and, by Lemma 2.2, $\{h_n\}$ is bounded in $\mathbb{L}_2(D_{\delta T})$, we have

$$\sup_{n\geq 1} E_n^x \operatorname{Var}(Z^{n,\beta})_T \leq \sup_{n\geq 1} \int_{D_{\delta T}} |h_n(u,y)|\,du\,dy < \infty$$

and

$$\lim_{n\to\infty} E_n^x\{(\operatorname{Var} Z^{n,\beta})_{\tau_n+\delta_n} - (\operatorname{Var} Z^{n,\beta})_{\tau_n}\} \leq \lim_{n\to\infty} \beta^{-1/2}\delta_n^{1/2}\|h_n\|_{\mathbb{L}_2(D_{\delta T})} = 0.$$

Therefore $\{\operatorname{Var} Z^{n,\beta}\}_{n\in\mathbb{N}}$ is tight in $C([0,T];\mathbb{R})$ by Aldous' criterion. Moreover, since $\{h_n\}$ is uniformly integrable on $D_{\delta T}$, for every $\varepsilon > 0$ there are $K \subset D$ and $\alpha > 0$ such that $\operatorname{dist}(K, \partial D) > 0$ and

$$\begin{aligned} I_n^\beta &\equiv E_n^x \sup_{t\in[0,T]} |(\operatorname{Var} Z^{n,\beta})_t - (\operatorname{Var} Z^{n,0})_t| = \int_{D_{\delta T}} \mathbf{1}_{G_\beta^n(u)} |h_n(u,y)|\,du\,dy \\ &\leq \varepsilon + \int_{K_{\delta T}} \mathbf{1}_{G_\beta^n(u)} (\alpha \wedge |h_n(u,y)|)\,du\,dy, \end{aligned}$$

where $G_\beta^n(u) = \{y \in D : p_n(u,y) \in (0,\beta]\}$. Set $G_\beta(u) = \{y \in D : p(u,y) \in (0,\beta]\}$ and

$$H_\beta(u) = \{y \in D : p(u,y) = \beta\}. \tag{3.7}$$

By Lemma 2.2, $\limsup_{n\to\infty} \mathbf{1}_{G_\beta^n(u)}(y) \leq \mathbf{1}_{H_0(u)\cup G_\beta(u)}(y)$ for every $(u,y) \in D_{\delta T}$ and $h_n \to a^{ij} D_j p D_i\varphi$ in $\mathbb{L}_2(K_{\delta T})$. Hence

$$\lim_{\beta\searrow 0} \limsup_{n\to\infty} I_n^\beta \leq \varepsilon + \lim_{\beta\searrow 0} \int_{D_{\delta T}} \mathbf{1}_{H_0(u)\cup G_\beta(u)}(y) |a^{ij} D_j p D_i\varphi|(u,y)\,du\,dy$$

for every $\varepsilon > 0$. The right-hand side above equals ε, because mesh $G_\beta(u) \to 0$ as $\beta \searrow 0$ for every $u \in [\delta, T]$ and $D_j p(u,\cdot) = 0$ a.e. on $H_0(u)$ by [23, Lemma A.2]. Thus, $\lim_{\beta\searrow 0}\limsup_{n\to\infty} I_n^\beta = 0$, and so

$$\{\operatorname{Var}{}^\delta V^{n,\varphi}\}_{n\in\mathbb{N}} \text{ is tight in } C([0,T];\mathbb{R}) \tag{3.8}$$

by [15, Lemma VI.3.32] and the fact that $Z^{n,0} = V^{n,\varphi}$.

Now set $\tau = \inf\{t \geq 0 : X_t \notin D\}$ and observe that the law of $(\tau, X_{\cdot\wedge\tau})$ under P_n^x is the same as under the diffusion measure Q_n^x of an unreflected process associated with A^n. Therefore, for any $\varepsilon > 0$,

$$\begin{aligned} I_n^\delta &= P_n^x(\sup_{t\in[0,T]} |(\operatorname{Var}{}^\delta V^{n,\varphi})_t - (\operatorname{Var} V^{n,\varphi})_t| > \varepsilon) \\ &\leq P_n^x((\operatorname{Var} V^{n,\varphi})_\delta > \varepsilon,\ \tau > \delta) + P_n^x(\tau \leq \delta) \\ &\leq Q_n^x\Big(\int_0^\delta \mathbf{1}_D |a_n^{ij} q_n^{-1} D_j q_n D_i\varphi(u, X_u)|\,du > \varepsilon\Big) + Q_n^x(\tau \leq \delta), \end{aligned}$$

where $q_n(u,y) = q_n(u,x,y)$ and $q_n(\cdot,\cdot,\cdot)$ is a transition density of (X, Q_n^x) or, in the language of PDE's, is a weak fundamental solution of $(\partial/\partial t - A^n)u = 0$ in $(0,T] \times \mathbb{R}^d$. By [2, Theorem 5], $\{q_n(\cdot,\cdot)\}$ is bounded in $W_1^{0,1}((0,T) \times \mathbb{R}^d)$. Therefore the first term on the right-hand side of the above inequality tends to 0 as $\delta \searrow 0$ uniformly in $n \in \mathbb{N}$. To deal with the second term, we note that by [27, Lemma 2] there is $0 < r < \operatorname{dist}(x, \partial D)$ such that τ^r defined by $\tau^r = \inf\{t \geq 0 : X_t \in D^r\}$, where $D^r = \{y \in D : \operatorname{dist}(y, \partial D) \leq r\}$ is continuous Q^x-a.s. Hence

$$\limsup_{n\to\infty} Q_n^x(\tau \leq \delta) \leq \limsup_{n\to\infty} Q_n^x(\tau^r \leq \delta) = Q^x(\tau^r \leq \delta),$$

since by [24, Theorem 7.1], $\{Q_n^x\}$ converges weakly to the law Q^x of an unreflected process associated with A. Of course, $Q^x(\tau^r > 0) = 1$, so $Q^x(\tau^r \leq \delta) \to 0$ as $\delta \searrow 0$. Thus, $\lim_{\delta\searrow 0}\limsup_{n\to\infty} I_n^\delta = 0$, which gives tightness of $\{\operatorname{Var} V^{n,\varphi}\}$ when combined with (3.8) and [15, Lemma VI.3.32]. ■

The above lemma, Lemma 2.2 and results of Lions and Sznitman [19] concerning tightness of solutions to the Skorokhod problem with oblique reflecting boundary conditions lead to the following.

THEOREM 3.2. *Let $a \in \mathcal{A}(\lambda, \Lambda)$, $\{a_n\} \subset \mathcal{A}^\infty(\lambda, \Lambda)$ and for $n \in \mathbb{N}$ let (X, P_n^x) be a diffusion corresponding to A^n with reflection along γ_{a_n} starting from $x \in D$. If $a_n^{ij} \to a^{ij}$ a.e. for $i,j = 1,\ldots,d$ then $\{P_n^x\}$ converges weakly* *in* $C([0,T];\mathbb{R}^d)$ *to the measure* P^x *such that for any* $0 < t_1 < \ldots < t_k \le T$, $k \in \mathbb{N}$ *and any continuous* $f : (\overline{D})^k \to \mathbb{R}$,

$$(3.9)\quad E^x f(X_{t_1}, \ldots, X_{t_k}) = \int_D p(t_1, x, y_1)\, dy_1 \int_D \ldots \int_D p(t_k, y_{k-1}, y_k) f(y_1, \ldots, y_k)\, dy_k,$$

where p *is a weak Neumann function for* A *on* D.

Proof. From Lemma 2.2 it follows that the finite-dimensional distributions of any weak limit point of $\{P_n^x\}$ are determined by (3.9), so what is left is to show that $\{P_n^x\}$ is weakly relatively compact. To see this, set

$$k_t^n = (k_t^{n,1}, \ldots, k_t^{n,d}), \quad k_t^{n,i} = \int_0^t \gamma_{a_n}^i(X_u)\, dK_u^n, \quad t \in [0,T].$$

One can check that $\operatorname{Var} k_t^n = \int_0^t |\gamma_{a_n}|(X_u)\, dK_u^n$. Hence

$$k_t^{n,i} = \int_0^t \frac{\gamma_{a_n}^i}{|\gamma_{a_n}|}(X_u)\, d\operatorname{Var} k_u^n, \quad t \in [0,T],$$

for $i = 1, \ldots, d$. Also, by (1.2),

$$\operatorname{Var} k_t^n = \int_0^t \mathbf{1}_{\partial D}(X_u)\, d\operatorname{Var} k_u^n, \quad t \in [0,T].$$

Finally, taking $\varphi(x) = x^i$ in (1.5) gives

$$X_t^i = x^i + \tfrac{1}{2}(M_t^{n,x_i} + \widetilde{N}_t^{n,x_i} - V_t^{n,x_i}) + k_t^{n,i}, \quad t \in [0,T], \quad P_n^x\text{-a.s.},$$

for $i = 1, \ldots, d$. Consequently, for each $n \in \mathbb{N}$ the pair (X, k^n) is under P_n^x a solution to the Skorokhod problem for $\{x^i + \frac{1}{2}(M^{n,x_i} + \widetilde{N}^{n,x_i} - V^{n,x_i})\}_{i=1}^d$ with reflection along $\gamma_{a_n}/|\gamma_{a_n}|$ on ∂D. That $\{P_n^x\}$ is relatively compact now follows from Lemma 3.1 and [19, Theorem 4.1]. ■

Let $\{P^x : x \in D\}$ be a family of measures constructed in Theorem 3.2 associated with some $a \in \mathcal{A}(\lambda, \Lambda)$. Combining (2.3) with (3.9) we see that for any $\varphi \in C(\overline{D})$, $(t,x) \mapsto E^x\varphi(X_t)$ is a unique (in $W_2^{0,1}(D_T)$) weak solution to the Neumann problem (1.1), which justifies the name of diffusion corresponding to A with reflection along γ_a on ∂D for (X, P^x). In what follows we call it for short a *reflecting diffusion corresponding to* a or we say that P^x is *associated with* a.

In the next section we will need the following convergence result.

LEMMA 3.3. *Let* $\{a, a_n\} \in \mathcal{A}(\lambda, \Lambda)$ *and let* (X, P^x), (X, P_n^x) *be reflecting diffusions corresponding to* a *and* a_n, *respectively, starting from* $x \in D$. *Let* $\{Y^n\}$ *be a sequence of* m*-dimensional continuous processes on* $[0,T]$ *and let* f, f_n, g, g_n *be real measurable functions on* $\overline{D}_T$. *Assume that*

$$\mathcal{L}[(Y^n, X) \mid P_n^x] \to \mathcal{L}[(Y, X) \mid P^x]$$

in $C([0,T];\mathbb{R}^{m+d})$ *and either that* $f_n \to f$, $g_n \to g$ *in* $\mathbb{L}_2(D_T)$ *or that* $f_n \to f$, $g_n \to g$ *in* $\mathbb{L}_1(D_T)$ *and there is* $K \subset D$ *such that* $\operatorname{dist}(K, \partial D) > 0$ *and* $\operatorname{supp} f_n, \operatorname{supp} g_n \subset K_T$ *for* $n \in \mathbb{N}$. *Then for every* $\delta \in (0,T)$,

$$\mathcal{L}\Big[\Big(Y^n, X, \int_\delta^{\cdot\vee\delta} f_n(u, X_u)\, du, \int_0^{\cdot\wedge(T-\delta)} g_n(u, \overline{X}_u)\, du\Big) \Big| P_n^x\Big] \to \mathcal{L}\Big[\Big(Y, X, \int_\delta^{\cdot\vee\delta} f(u, X_u)\, du, \int_0^{\cdot\wedge(T-\delta)} g(u, \overline{X}_u)\, du\Big) \Big| P^x\Big]$$

in $C([0,T];\mathbb{R}^{m+d+2})$.

Proof. First observe that by (2.1),

$$(3.10)\quad E^x \int_\delta^T |h(u, X_u)|\, du \vee \sup_{n\ge1} E_n^x \int_\delta^T |h(u, X_u)|\, du \le C_1^{1/2}(\min\{\delta, C_2(\operatorname{dist}(x, \partial D))^2\})^{-d/4} \|h\|_{\mathbb{L}_2(D_{\delta T})}$$

for $h \in \mathbb{L}_2(D_{\delta T})$, whereas by (2.2),

$$(3.11)\quad E^x \int_\delta^T |h(u, X_u)|\, du \vee \sup_{n\ge1} E_n^x \int_\delta^T |h(u, X_u)|\, du \le C_3 \|h\|_{\mathbb{L}_1(K_{\delta T})}$$

for any $h \in \mathbb{L}_1(D_{\delta T})$ such that $\operatorname{supp} h \subset K$. Suppose now that $f_n \to f$, $g_n \to g$ in $\mathbb{L}_2(D_T)$. Choose $\{F_k\}, \{G_k\} \subset C(\overline{D}_T)$ so that $F_k \to f$, $G_k \to g$ in $\mathbb{L}_2(D_T)$. Then for each $k \in \mathbb{N}$,

$$\mathcal{L}\Big[\Big(Y^n, X, \int_\delta^{\cdot\vee\delta} F_k(u, X_u)\, du, \int_0^{\cdot\wedge(T-\delta)} G_k(u, \overline{X}_u)\, du\Big) \Big| P_n^x\Big] \to \mathcal{L}\Big[\Big(Y, X, \int_\delta^{\cdot\vee\delta} F_k(u, X_u)\, du, \int_0^{\cdot\wedge(T-\delta)} G_k(u, \overline{X}_u)\, du\Big) \Big| P^x\Big]$$

in $C([0,T];\mathbb{R}^{m+d+2})$. Moreover, by (3.10),

$$\lim_{k\to\infty} E^x\Big\{\int_\delta^T |F_k - f|(u, X_u)\, du + \int_0^{T-\delta} |G_k - g|(u, \overline{X}_u)\, du\Big\} = \lim_{k\to\infty} \iint_{D_{\delta T}} \{|F_k - f|(u,y) + |G_k - g|(T-u, y)\} p(u,y)\, dy = 0$$

and

$$\lim_{k\to\infty}\limsup_{n\to\infty} E_n^x\Big\{\int_\delta^T |F_k - f|(u,X_u)\,du + \int_0^{T-\delta} |G_k - g|(u,\overline{X}_u)\,du\Big\} = 0.$$

By the above and [3, Theorem 4.2],

$$\mathcal{L}\Big[\Big(Y^n, X, \int_\delta^{\cdot\vee\delta} f(u,X_u)\,du, \int_0^{\cdot\wedge(T-\delta)} g(u,\overline{X}_u)\,du\Big)\,\Big|\,P_n^x\Big]$$
$$\to \mathcal{L}\Big[\Big(Y, X, \int_\delta^{\cdot\vee\delta} f(u,X_u)\,du, \int_0^{\cdot\wedge(T-\delta)} g(u,\overline{X}_u)\,du\Big)\,\Big|\,P^x\Big]$$

in $C([0,T];\mathbb{R}^{m+d+2})$. Finally, again by (3.10),

$$\lim_{n\to\infty} E_n^x\Big\{\int_\delta^T |f_n - f|(u,X_u)\,du + \int_0^{T-\delta} |g_n - g|(u,\overline{X}_u)\,du\Big\} = 0,$$

and the lemma follows.

In the second case, one can find $K' \subset D$ and $\{F_k\},\{G_k\} \subset C(\overline{D}_T)$ such that $\mathrm{dist}(K',\partial D) > 0$, $\operatorname{supp} F_k \subset K'_T$, $\operatorname{supp} G_k \subset K'_T$ for $k\in\mathbb{N}$ and $F_k \to f$, $G_k \to g$ in $\mathbb{L}_1(K_{\delta T})$. Therefore we can proceed as before, the only difference being in the use of (3.11) instead of (3.10). ∎

4. Stochastic representation. Let $\mathcal{A}^\infty(\lambda,\Lambda;\mathbb{R}^d)$ denote the class of all mappings from $\mathbb{R}^d$ into $\mathbb{R}^d\otimes\mathbb{R}^d$ which have bounded continuous derivatives of all orders and satisfy (0.2) for $x\in\mathbb{R}^d$. Suppose we are given $\{a_n\}\subset\mathcal{A}^\infty(\lambda,\Lambda;\mathbb{R}^d)$ and let $q_n(\cdot,\cdot,\cdot)$, $n\in\mathbb{N}$, be a transition density of a diffusion in $\mathbb{R}^d$ associated with A^n defined by (2.4). In the next lemma we gather some properties of the resolvents

$$R_\alpha^n\varphi(x) = \int_0^\infty e^{-\alpha t}\,dt \int_{\mathbb{R}^d} \varphi(y) q_n(t,x,y)\,dy, \quad \alpha > 0,$$

corresponding to the operators A^n. These properties will be extremely useful in the proof of our main result.

LEMMA 4.1. *Assume $a\in\mathcal{A}(\lambda,\Lambda)$, $\{a_n\}\subset\mathcal{A}^\infty(\lambda,\Lambda;\mathbb{R}^d)$ and $\varphi\in C^2(\overline{D})$. Then*

(i) *for fixed $n\in\mathbb{N}$, $\alpha>0$, $R_\alpha^n\varphi\in C_b^2(\mathbb{R}^d)$ is a solution to the equation*

$$(\alpha - A^n)R_\alpha^n\varphi = \varphi \quad \text{on } \mathbb{R}^d,$$

(ii) *if $a_n^{ij}\to\widetilde{a}^{ij}$ a.e. on $\mathbb{R}^d$ for $i,j=1,\ldots,d$, where*

$$\widetilde{a}(x) = \begin{cases} a(x) & \text{if } x\in\overline{D},\\ \lambda I & \text{otherwise } (I \text{ is the identity matrix}),\end{cases}$$

then for every $\alpha \ge \lambda/4 + 4\Lambda^2/\lambda + \Lambda$, $\{R_\alpha^n\varphi\}$ converges to some $R_\alpha\varphi$ in $W_2^1(\mathbb{R}^d)$ as $n\to\infty$ and $\{\alpha R_\alpha\varphi\}$ converges to φ in $W_2^1(\mathbb{R}^d)$ as $\alpha\to\infty$.

Proof. See [28, Lemma 2.1]. ∎

In what follows, $\{\Pi_m\} = \{0 = t_0 < t_1 < \ldots < t_{k(m)} = T\}$ denotes an arbitrary but fixed sequence of partitions of $[0,T]$ such that $\|\Pi_m\| = \max_{1\le k\le k(m)}|t_k - t_{k-1}| \to 0$ as $m\to\infty$.

THEOREM 4.2. *Let $a\in\mathcal{A}(\lambda,\Lambda)$ and let (X,P^x) be a diffusion corresponding to A with reflection along γ_a on ∂D starting from $x\in D$ at time 0. Then for every Lipschitz-continuous $\varphi:\overline{D}\to\mathbb{R}$ there is a unique quadruple $(M^\varphi,N^\varphi,V^\varphi,K^\varphi)$ such that*

(i) *$M^\varphi\in\mathcal{M}$, $N^\varphi\in\widetilde{\mathcal{M}}$, V^φ, K^φ are continuous $\{\mathcal{F}_t\}$-adapted process of finite variation on $[0,T]$ satisfying*

$$V_0^\varphi = K_0^\varphi = 0, \quad K_t^\varphi = \int_0^t \mathbf{1}_{\partial D}(X_u)\,dK_u^\varphi, \quad V_t^\varphi = \int_0^t \mathbf{1}_D(X_u)\,dV_u^\varphi.$$

(ii) *M^φ, $\widetilde{N}^\varphi$ admit mutual quadratic variation along $\{\Pi_m\}$ and*

$$\langle M^\varphi,\widetilde{N}^\varphi\rangle_t = \langle M^\varphi\rangle_t, \quad t\in[0,T].$$

(iii) *$\varphi(X_t)-\varphi(X_0) = \frac12(M_t^\varphi + \widetilde{N}_t^\varphi - V_t^\varphi) + K_t^\varphi$, $t\in[0,T]$, P^x-a.s.*

In particular, $\varphi(X)$ is an $(\{\mathcal{F}_t\},P^x)$-Dirichlet process on $[0,T]$ with the decomposition

$$\varphi(X_t)-\varphi(X_0) = M_t^\varphi + A_t^\varphi, \quad A_t^\varphi = \tfrac12(-M_t^\varphi + \widetilde{N}_t^\varphi - V_t^\varphi) + K_t^\varphi. \tag{4.1}$$

Moreover,

$$V_t^\varphi = \lim_{\delta\searrow0} {}^\delta V_t^\varphi = \lim_{\delta\searrow0}\int_\delta^{t\vee\delta} \mathbf{1}_D a^{ij}p^{-1}D_jpD_i\varphi(u,X_u)\,du \quad \text{in } P^x, \tag{4.2}$$

$$\langle M^\varphi\rangle_t = \int_0^t \mathbf{1}_D a^{ij}D_i\varphi D_j\varphi(X_u)\,du, \tag{4.3}$$

$$\langle N^\varphi\rangle_t = \int_0^t \mathbf{1}_D a^{ij}D_i\varphi D_j\varphi(\overline{X}_u)\,du \tag{4.4}$$

for $t\in[0,T]$, and for any bounded measurable $f:\overline{D}\to\mathbb{R}$ the processes $f\cdot M^\varphi$, $f * N^\varphi$ admit mutual quadratic variation, and

$$\langle f\cdot M^\varphi, f * N^\varphi\rangle_t = \langle f\cdot M^\varphi\rangle_t, \quad t\in[0,T]. \tag{4.5}$$

Finally, if (M^i,N^i,V^i,K^i) is a quadruple corresponding to the function

$x \mapsto x_i$ and $\varphi \in C^1(\overline{D})$ then

(4.6) $$D_i\varphi \cdot M_t^i = M_t^\varphi, \quad D_i\varphi(\overline{X}_u) * N_t^i = \widetilde{N}_t^\varphi$$

and

(4.7) $$D_i\varphi \cdot V_t^i = V_t^\varphi, \quad D_i\varphi \cdot K_t^i = K_t^\varphi$$

for $t \in [0,T]$.

Proof. *Uniqueness.* Suppose that $({}_iM^\varphi, {}_iN^\varphi, {}_iV^\varphi, {}_iK^\varphi)$, $i = 1,2$, satisfy (i)–(iii) with respect to the same sequence of partitions of $[0,T]$. Then ${}_1M^\varphi + {}_1A^\varphi = {}_2M^\varphi + {}_2A^\varphi$, where ${}_iA^\varphi = (1/2)(-{}_iM^\varphi + {}_i\widetilde{N}^\varphi - {}_iV^\varphi) + {}_iK^\varphi$, $i = 1,2$, and hence ${}_1M^\varphi = {}_2M^\varphi$, ${}_1A^\varphi = {}_2A^\varphi$ due to uniqueness of the decomposition of Dirichlet processes. Consequently, ${}_1\widetilde{N}^\varphi - {}_2\widetilde{N}^\varphi = {}_1V^\varphi - {}_2V^\varphi + 2({}_2K^\varphi - {}_1K^\varphi)$. Thus ${}_1N^\varphi - {}_2N^\varphi = {}_1\widetilde{V}^\varphi - {}_2\widetilde{V}^\varphi + 2({}_2\widetilde{K}^\varphi - {}_1\widetilde{K}^\varphi)$ is an $(\{\overline{\mathcal{F}}_t\}, P^x)$-martingale of 0-quadratic variation, which forces ${}_1N^\varphi = {}_2N^\varphi$, ${}_1V^\varphi - {}_2V^\varphi = 2({}_1K^\varphi - {}_2K^\varphi)$. The last equality yields

$$\begin{aligned} {}_1V_t^\varphi - {}_2V_t^\varphi &= 2\int_0^t \mathbf{1}_D(X_u)\,d({}_1K_u^\varphi - {}_2K_u^\varphi) \\ &= 2\int_0^t \mathbf{1}_D\mathbf{1}_{\partial D}(X_u)\,d({}_1K_u^\varphi - {}_2K_u^\varphi) = 0. \end{aligned}$$

Accordingly, ${}_1V^\varphi = {}_2V^\varphi$ and ${}_1K^\varphi = {}_2K^\varphi$.

Existence. First assume $\varphi \in C^2(\overline{D})$. Define $\widetilde{a}$ as in Lemma 4.1 and choose $\{a_n\} \subset \mathcal{A}^\infty(\lambda, \Lambda; \mathbb{R}^d)$ so that $a_n^{ij} \to \widetilde{a}^{ij}$ a.e. in $\mathbb{R}^d$. In turn, for $n \in \mathbb{N}$ define P_n^x, A^n, $\mathcal{A}_t^n$ and then $K^{n,\varphi}$, $V^{n,\varphi}$, $M^{n,\varphi}$, $N^{n,\varphi}$ by (3.1)–(3.4). Then by (1.5),

(4.8) $$\varphi(X_t) - \varphi(X_0) = M_t^{n,\varphi} + A_t^{n,\varphi}, \quad t \in [0,T], \quad P_n^x\text{-a.s.},$$

where

$$A^{n,\varphi} = \tfrac{1}{2}(-M^{n,\varphi} + \widetilde{N}^{n,\varphi} - V^{n,\varphi}) + K^{n,\varphi}.$$

We are going to show that

(4.9) $$\{M^{n,\varphi} + A^{n,\varphi}\}_{n\in\mathbb{N}} \text{ satisfies UTD}$$

for $\varphi \in C^2(\overline{D})$ (see Appendix). For this purpose we first prove that

(4.10) $$\{f \cdot M^{n,\varphi} - f * N^{n,\varphi}\}_{n\in\mathbb{N}} \text{ satisfies UTD}$$

for $\varphi \in C^2(\overline{D})$, $f \in C^2(\overline{D})$. We will follow rather closely the proof of [28, Theorem 2.2], but the lack of lower Aronson's estimates for p, p_n as well as upper estimates near ∂D causes some new technical difficulties.

As in the proof of (2.16) in [28], the submartingale inequality and (1.3), (1.4) imply that $\{\sup_{0\le t\le T} |f \cdot M_t^{n,\varphi} - f * N_t^{n,\varphi}|\}_{n\in\mathbb{N}}$ is tight in $\mathbb{R}$. Therefore

we only need to prove that

(4.11) $$\forall_{\varepsilon>0} \lim_{m\to\infty} \sup_{n\ge1} P_n^x(Q_T^m(f \cdot M_t^{n,\varphi} - f * N^{n,\varphi}) > \varepsilon) = 0.$$

Observe that (1.9) gives $f \cdot M^{n,\varphi} = \mathbf{1}_D f \cdot M^{n,\varphi}$, $f * N^{n,\varphi} = \mathbf{1}_D f * N^{n,\varphi}$. Therefore, if we take $\{f_k\} \subset C^2(\overline{D})$ such that $\operatorname{dist}(\operatorname{supp} f_k, \partial D) > 0$ for $k \in \mathbb{N}$ and $f_k \to \mathbf{1}_D f$ boundedly and pointwise, then for fixed $n, m \in \mathbb{N}$,

$$Q_T^m(f_k \cdot M_t^{n,\varphi} - f_k * N^{n,\varphi}) \to Q_T^m(f \cdot M_t^{n,\varphi} - f * N^{n,\varphi})$$

in P_n^x as $k \to \infty$. Thus, in order to get (4.11) we may assume without loss of generality that there is $K \subset D$ such that $\operatorname{dist}(K, \partial D) > 0$ and $\operatorname{supp} f \subset K$. Since $f(\overline{X})$, $N^{n,\varphi}$ are $(\{\overline{\mathcal{F}}_t\}, P_n^x)$-semimartingales and stochastic integrals with respect to semimartingales can be defined as limits of Riemann sums,

$$f * N^{n,\varphi} = \langle f(X), \widetilde{N}^{n,\varphi}\rangle + f \cdot \widetilde{N}^{n,\varphi},$$

as is easy to check. On the other hand, from (1.8) we see that $\widetilde{N}^{n,\varphi}$ is the sum of $M^{n,\varphi}$ and a process of finite variation on $[0,T]$. Therefore $\langle f \cdot M^{n,\varphi} - f * N^{n,\varphi}\rangle_T = 0$ and we have

$$\begin{aligned} Q_T^m(f \cdot M^{n,\varphi} - f * N^{n,\varphi}) \\ &= |Q_T^m(f \cdot M^{n,\varphi} - f * N^{n,\varphi}) - \langle f \cdot M^{n,\varphi} - f * N^{n,\varphi}\rangle_T| \\ &\le |Q_T^m(f \cdot M^{n,\varphi}) - \langle f \cdot M^{n,\varphi}\rangle_T| \\ &\quad + |Q_T^m(f * N^{n,\varphi}) - \langle f * N^{n,\varphi}\rangle_T| \\ &\quad + |Q_T^m(f \cdot M^{n,\varphi}, f * N^{n,\varphi}) - \langle f \cdot M^{n,\varphi}, f * N^{n,\varphi}\rangle_T| \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

For any fixed $0 < \delta < T$,

$$\begin{aligned} I_3 &\le |Q_\delta^m(f \cdot M^{n,\varphi}, f * N^{n,\varphi})| + |\langle f \cdot M^{n,\varphi}, f * N^{n,\varphi}\rangle_\delta| \\ &\quad + |Q_{\delta T}^m(f \cdot M^{n,\varphi}, f * N^{n,\varphi}) - \langle f \cdot M^{n,\varphi}, f * N^{n,\varphi}\rangle_\delta^T| \\ &\equiv I_{31} + I_{32} + I_{33}. \end{aligned}$$

As in the proof of [28, Theorem 2.2] we show that

(4.12) $$\lim_{m\to\infty} \sup_{n\ge1} P_n^x(I_1 + I_2 > \varepsilon) = 0,$$

(4.13) $$\lim_{\delta\searrow0} \lim_{m\to\infty} \sup_{n\ge1} E_n^x(I_{31} + I_{32}) = 0,$$

so it remains to evaluate I_{33}. To this end, for $k \in \mathbb{N}$ put $\varphi_k^n = kR_k^n\varphi$, where $\{R_\alpha^n\}_{\alpha>0}$ is the resolvent of Lemma 4.1 associated with a_n, and define M^{n,φ_k^n}, N^{n,φ_k^n} as in (3.3), (3.4) with φ replaced by φ_k^n. Clearly, for any $0<\delta<T$,

$$\begin{aligned}I_{33} &\le |Q^m_{\delta T}(f\cdot M^{n,\varphi}, f*N^{n,\varphi}) - Q^m_{\delta T}(f\cdot M^{n,\varphi^n_k}, f*N^{n,\varphi^n_k})| \\ &\quad + |Q^m_{\delta T}(f\cdot M^{n,\varphi^n_k}, f*N^{n,\varphi^n_k}) - \langle f\cdot M^{n,\varphi^n_k}, f*N^{n,\varphi^n_k}\rangle^T_\delta| \\ &\quad + |\langle f\cdot M^{n,\varphi^n_k}, f*N^{n,\varphi^n_k}\rangle^T_\delta - \langle f\cdot M^{n,\varphi}, f*N^{n,\varphi}\rangle^T_\delta| \\ &\equiv I_{331}+I_{332}+I_{333}.\end{aligned}$$

To deal with I_{331}, we note that by Lemma 4.1, $\{\varphi-\varphi^n_k\}_{n\in\mathbb{N}}$ is convergent in $W^1_2(D)$, whereas by (2.2), $p_n(\cdot\,,\cdot)$ is bounded on $K_{\delta T}$ uniformly in $n\in\mathbb{N}$. Therefore repeating the arguments used to prove (2.21) in [28] we deduce that

$$\lim_{k\to\infty}\lim_{m\to\infty}\sup_{n\ge1}E^x_n I_{331}=0 \tag{4.14}$$

for any fixed $0<\delta<T$. Define now V^{n,φ^n_k}, K^{n,φ^n_k} by (3.2), (3.1) with φ^n_k instead of φ. Since $(k-A^n)R^n_k\varphi=\varphi$, it follows from Theorem 1.1 that

$$M^{n,\varphi^n_k}_t = \varphi^n_k(X_t)-\varphi^n_k(X_0) - \int_0^t \mathbf{1}_D k(\varphi^n_k-\varphi)(X_u)\,du - K^{n,\varphi^n_k}_t,$$

$$\begin{aligned}N^{n,\varphi^n_k}_t &= \varphi^n_k(\overline{X}_t)-\varphi^n_k(\overline{X}_0) - \int_0^t \mathbf{1}_D k(\varphi^n_k-\varphi)(\overline{X}_u)\,du \\ &\quad - \int_0^t \mathbf{1}_D a^{ij}_n \frac{D_j\overline{p}_n}{\overline{p}_n} D_i\varphi^n_k(u,\overline{X}_u)\,du - \int_0^t \langle\gamma_{a_n},\nabla\varphi^n_k\rangle(\overline{X}_u)\,d\widetilde{K}^n_u\end{aligned}$$

for $t\in[0,T]$, where $\overline{p}_n(\cdot\,,\cdot)=p_n(T-\cdot\,,\cdot)$. An easy computation shows that

$$\begin{aligned}\widetilde{N}^{n,\varphi^n_k}_t &= \varphi^n_k(X_t)-\varphi^n_k(X_0) + \int_0^t \mathbf{1}_D k(\varphi^n_k-\varphi)(X_u)\,du + V^{n,\varphi^n_k}_t - K^{n,\varphi^n_k}_t \\ &= M^{n,\varphi^n_k}_t + Z^{n,k}_t\end{aligned}$$

for $t\in[0,T]$, where

$$Z^{n,k}_t = 2\int_0^t \mathbf{1}_D k(\varphi^n_k-\varphi)(X_u)\,du + V^{n,\varphi^n_k}_t,\qquad t\in[0,T].$$

In particular, if we take $h\in C^2(\overline{D})$ such that $h=1$ on K and $\operatorname{supp} h\subset D$, then

$$\langle f(X),\widetilde{N}^{n,\varphi^n_k}\rangle = \langle M^{n,f}, M^{n,\varphi^n_k}\rangle = \langle M^{n,f}, h\cdot M^{n,\varphi^n_k}\rangle,$$

the last equality being a consequence of (1.3) and the fact that $\operatorname{supp} f\subset K$. Since

$$f*N^{n,\varphi^n_k} = \langle f(X),\widetilde{N}^{n,\varphi^n_k}\rangle + f\cdot\widetilde{N}^{n,\varphi^n_k},$$

from what has already been proved it follows that

$$f*N^{n,\varphi^n_k} = f\cdot M^{n,\varphi^n_k} + R^{n,k},\quad \text{where}\quad R^{n,k}=\langle M^{n,f}, h\cdot M^{n,\varphi^n_k}\rangle + f\cdot Z^{n,k}.$$

By the assumptions and Lemma 4.1, $\{a^{ij}_n D_i f D_j f\}_{n\in\mathbb{N}}$, $\{a^{ij}_n D_i\varphi^n_k D_j\varphi^n_k\}_{n\in\mathbb{N}}$ are convergent in $\mathbb{L}_1(K)$, so combining Theorem 3.2 with Lemma 3.3 and (1.3) we see that $\{\langle{}^\delta M^{n,f}\rangle\}_{n\in\mathbb{N}}$, $\{\langle h\cdot{}^\delta M^{n,\varphi^n_k}\rangle\}_{n\in\mathbb{N}}$ are tight in $C([0,T];\mathbb{R})$, and hence that $\{\operatorname{Var}\langle{}^\delta M^{n,f}, h\cdot{}^\delta M^{n,\varphi^n_k}\rangle\}_{n\in\mathbb{N}}$ is tight for fixed $k\in\mathbb{N}$ and $\delta\in(0,T)$.

Define now $F^n_\beta(u)$ by (3.5) and $H_\beta(u)$ by (3.7). By Lemmas 2.2 and 4.1, for each sufficently large $k\in\mathbb{N}$, $fa^{ij}_n D_jp_n D_i\varphi^n_k\to fa^{ij}D_jpD_iR_k\varphi$ in $\mathbb{L}_1(D_{\delta T})$ as $n\to\infty$. Hence, for fixed $\beta>0$,

$$\begin{aligned}\lim_{n\to\infty} E^x_n \int_\delta^T \mathbf{1}_{H_\beta(u)}|fa^{ij}_n p_n^{-1}D_jp_nD_i\varphi^n_k|(u,X_u)\,du \\ = \lim_{n\to\infty}\int_\delta^T du\int_{H_\beta(u)} |fa^{ij}_n D_jp_nD_i\varphi^n_k|(u,y)\,dy=0,\end{aligned}$$

because a slight change in the proof of [23, Lemma A.2] actually shows that $D_jp=0$ a.e. on $H_\beta(u)$ for $u\in[\delta,T]$, $j=1,\ldots,d$. Applying once again Lemmas 2.2 and 4.1 we see that

$$\{f[2k(\varphi^n_k-\varphi)+\mathbf{1}_{F^n_\beta(\cdot)\setminus H_\beta(\cdot)}a^{ij}_np_n^{-1}D_jp_nD_i\varphi^n_k]\}_{n\in\mathbb{N}}$$

is convergent in $\mathbb{L}_1(K_{\delta T})$. By the above, Lemma 3.3 and [15, Lemma VI.3.32],

$$\Big\{\operatorname{Var}\Big(\int_\delta^{\delta\vee\cdot} f[\mathbf{1}_D 2k(\varphi^n_k-\varphi)+\mathbf{1}_{F^n_\beta(\cdot)}a^{ij}_np_n^{-1}D_jp_nD_i\varphi^n_k](u,X_u)\,du\Big)\Big\}_{n\in\mathbb{N}}$$

is tight in $C([0,T];\mathbb{R})$. Also, by the arguments used to prove (3.8),

$$\lim_{\beta\searrow0}\limsup_{n\to\infty}E^x_n\int_\delta^T(1-\mathbf{1}_{F^n_\beta(\cdot)})|fa^{ij}_np_n^{-1}D_jp_nD_i\varphi^n_k|(u,X_u)\,du=0.$$

By the above estimates, $\{\operatorname{Var} f\cdot{}^\delta Z^{n,k}\}_{n\in\mathbb{N}}$ is tight in $C([0,T];\mathbb{R})$ and so is $\{\operatorname{Var}{}^\delta R^{n,k}\}_{n\in\mathbb{N}}$. Therefore, arguing as in the proof of (2.32) in [28] we conclude that

$$\lim_{k\to\infty}\lim_{m\to\infty}\sup_{n\ge1}P^x_n(I_{332}+I_{333}>\varepsilon)=0$$

for fixed $0<\delta<T$, hence that

$$\lim_{m\to\infty}\sup_{n\ge1}P^x_n(I_3>\varepsilon)=0,$$

by (4.13), (4.14). This and (4.12) give (4.11), and the proof of (4.10) is complete.

In particular, taking $f \equiv 1$ we see that $\{M^{n,\varphi} - \widetilde{N}^{n,\varphi}\}$ satisfies UTD. Now note that by Lemma 3.1 and Theorem 3.2, $\{K^{n,\Phi}\}$ is tight in $C([0,T];\mathbb{R})$ and so is $\{\operatorname{Var} K^{n,\Phi}\}$, because $\langle \gamma_{a_n}, \nabla\Phi\rangle \ge \lambda/2$, which implies that $K^{n,\Phi}$ is an increasing process for each $n \in \mathbb{N}$. Therefore

$$\{\operatorname{Var} K^{n,\varphi}\}_{n\in\mathbb{N}} \text{ is tight in } C([0,T];\mathbb{R}), \tag{4.15}$$

since

$$K_t^{n,\varphi} = \int_0^t \frac{\langle \gamma_{a_n}, \nabla\varphi\rangle}{\langle \gamma_{a_n}, \nabla\Phi\rangle}(X_u)\, dK_u^{n,\Phi}, \quad t \in [0,T]. \tag{4.16}$$

From the above and Lemma 3.1 it follows that $\{\operatorname{Var}(-(1/2)V^{n,\varphi} + K^{n,\varphi})\}$ is tight in $C([0,T];\mathbb{R})$, which proves (4.9) when combined with (4.10) and [28, Lemma 1.4].

Consequently, by Lemma 2.2 and Theorems 3.2 and 6.1, there exist continuous processes M^φ, A^φ on $[0,T]$ such that

$$\mathcal{L}[(X, M^{n,\varphi}, A^{n,\varphi}) \mid P_n^x] \to \mathcal{L}[(X, M^\varphi, A^\varphi) \mid P^x]$$

in $C([0,T];\mathbb{R}^{d+2})$ and $\varphi(X)$ is an $(\{\mathcal{F}_t\}, P^x)$-Dirichlet process admitting the decomposition

$$\varphi(X_t) - \varphi(X_0) = M_t^\varphi + A_t^\varphi, \quad t \in [0,T], \qquad P^x\text{-a.s.}$$

Our next goal is to show that

$$\mathcal{L}[(X, M^{n,\varphi}, A^{n,\varphi}, V^{n,\varphi}) \mid P_n^x] \to \mathcal{L}[(X, M^\varphi, A^\varphi, V^\varphi) \mid P^x] \tag{4.17}$$

in $C([0,T];\mathbb{R}^{d+3})$. For this purpose, for given $\delta \in (0,T)$, $\beta \ge 0$ and $u \in [\delta, T]$ set

$$Z_t^\beta = \int_\delta^{\delta\vee t} \mathbf{1}_{F_\beta(u)} a^{ij} p^{-1} D_j p D_i \varphi(u, X_u)\, du, \qquad F_\beta(u) = \{y \in D : p(u,y) > \beta\}$$

and define $F_\beta^n(u)$, $Z^{n,\beta}$, H_β by (3.5)–(3.7). Choose also a sequence of nonnegative continuous functions $h_k : \overline{D} \to \mathbb{R}$ such that $\operatorname{dist}(\operatorname{supp} h_k, \partial D) > 0$ for $k \in \mathbb{N}$ and $h_k \nearrow \mathbf{1}_D$ as $k \nearrow \infty$. Due to Lemma 2.2, for fixed $\beta > 0$, $k \in \mathbb{N}$,

$$\mathbf{1}_{F_\beta^n(\cdot)\setminus H_\beta(\cdot)} h_k a_n^{ij} p_n^{-1} D_j p_n D_i \varphi \to \mathbf{1}_{F_\beta(\cdot)} h_k a^{ij} p^{-1} D_j p D_i \varphi$$

in $\mathbb{L}_2(D_{\delta T})$ as $n \to \infty$, and

$$\lim_{n\to\infty} E_n^x \int_\delta^T \mathbf{1}_{H_\beta(u)} |h_k a_n^{ij} p_n^{-1} D_j p_n D_i \varphi(u, X_u)|\, du \tag{4.18}$$

$$= \int_\delta^T du \int_{H_\beta(u)} |h_k a^{ij} D_j p D_i \varphi(u,y)|\, dy = 0,$$

because $h_k a_n^{ij} D_j p_n D_i \varphi \to h_k a^{ij} D_j p D_i \varphi$ in $\mathbb{L}_2(D_{\delta T})$ as $n \to \infty$ and $D_j p = 0$ a.e. on $H_\beta(u)$. Therefore, by Lemma 3.3,

$$\mathcal{L}[(X, M^{n,\varphi}, A^{n,\varphi}, h_k \cdot Z^{n,\beta}) \mid P_n^x] \to \mathcal{L}[(X, M^\varphi, A^\varphi, h_k \cdot Z^\beta) \mid P^x]. \tag{4.19}$$

Furthermore, analysis similar to that in the proof of Lemma 3.1 shows that

$$\lim_{\beta\searrow 0} \limsup_{n\to\infty} E_n^x \sup_{t\in[\delta,T]} |h_k \cdot Z_t^{n,\beta} - h_k \cdot Z_t^{n,0}| = 0, \tag{4.20}$$

whereas applying the Lebesgue dominated convergence theorem yields

$$\lim_{\beta\searrow 0} E^x \sup_{t\in[\delta,T]} |h_k \cdot Z_t^\beta - h_k \cdot Z_t^0| = 0. \tag{4.21}$$

Since $h_k \cdot Z^{n,0} = h_k \cdot {}^\delta V^{n,\varphi}$, $h_k \cdot Z^0 = h_k \cdot {}^\delta V^\varphi$, putting (4.19)–(4.21) together and applying [3, Theorem 4.2] we get

$$\mathcal{L}[(X, M^{n,\varphi}, A^{n,\varphi}, h_k \cdot {}^\delta V^{n,\varphi}) \mid P_n^x] \to \mathcal{L}[(X, M^\varphi, A^\varphi, h_k \cdot {}^\delta V^\varphi) \mid P^x]$$

for $k \in \mathbb{N}$. Since $\{a_n^{ij} D_j p_n D_i \varphi\}$ is bounded in $\mathbb{L}_2(D_{\delta T})$, we also have

$$\lim_{k\to\infty} \limsup_{n\to\infty} E_n^x \sup_{t\in[0,T]} |{}^\delta V_t^{n,\varphi} - h_k \cdot {}^\delta V_t^{n,\varphi}|$$

$$\le \lim_{k\to\infty} \limsup_{n\to\infty} \iint_{D_{\delta T}} (\mathbf{1}_D - h_k) |a_n^{ij} D_j p_n D_i \varphi(u,y)|\, du\, dy = 0.$$

Likewise,

$$\lim_{k\to\infty} E^x \sup_{t\in[0,T]} |{}^\delta V_t^\varphi - h_k \cdot {}^\delta V_t^\varphi| = 0,$$

so applying once again [3, Theorem 4.2] we conclude that

$$\mathcal{L}[(X, M^{n,\varphi}, A^{n,\varphi}, {}^\delta V^{n,\varphi}) \mid P_n^x] \to \mathcal{L}[(X, M^\varphi, A^\varphi, {}^\delta V^\varphi) \mid P^x] \tag{4.22}$$

in $C([0,T];\mathbb{R}^{d+3})$.

Our next claim is that

$$\{{}^\delta V^\varphi\} \text{ converges in } P^x \text{ as } \delta \searrow 0. \tag{4.23}$$

To see this, define $\tau = \inf\{t \ge 0 : X_t \notin D\}$. Then for any $0 < \delta < \varrho \le T$ and $\varepsilon > 0$,

$$P^x\big(\sup_{t\in[0,T]} |{}^\delta V_t^\varphi - {}^\varrho V_t^\varphi| > \varepsilon\big) \tag{4.24}$$

$$\le P^x\Big(\int_\delta^\varrho \mathbf{1}_D |a^{ij} p^{-1} D_j p D_i \varphi(u, X_u)|\, du > \varepsilon,\ \tau > \varrho\Big) + P^x(\tau \le \varrho).$$

Since $K_\varrho = 0$ on $\{\tau \ge \varrho\}$, the law of $X_{\cdot\wedge\tau}$ under P^x is the same as under the measure Q^x of an unreflected process associated with a. Furthermore, by [2, Theorem 5], a transition density $q(t,x,y)$ of (X, Q^x), which coincides with a weak fundamental solution of $(\partial/\partial t - A)u = 0$ in $[0,T) \times \mathbb{R}^d$, belongs, as a function of (t,y) for fixed x, to $W_1^{0,1}((0,T)\times\mathbb{R}^d)$. Therefore $a^{ij}D_jq(\cdot,x,\cdot)D_i\varphi$ is integrable on D_T and the first summand on the right-hand side of (4.24) is as small as desired when ϱ is sufficiently small. Also, since $x\in D$, $P^x(\tau>0)=1$, and hence $P^x(\tau\le\varrho)\to0$ as $\varrho\to0$. Accordingly, $\{{}^\delta V^\varphi\}$ is a Cauchy sequence with respect to the convergence in P^x, so converges in P^x as $\delta\searrow0$. In the same manner we can see that $\{(\operatorname{Var}{}^\delta V^\varphi)_T\}$ is bounded in P^x uniformly in $\delta\in(0,T)$. Consequently, the limit V^φ of $\{{}^\delta V^\varphi\}$ is of finite variation on $[0,T]$.

Observe now that

(4.25) $$\lim_{\delta\searrow0}\limsup_{n\to\infty}P_n^x(|V_\delta^{n,\varphi}|>\varepsilon)=0$$

for $\varepsilon>0$. Indeed, for $\delta\in(0,T)$, $n\in\mathbb{N}$ we have

$$P_n^x(|V_\delta^{n,\varphi}|>\varepsilon)\le P_n^x(|V_\delta^{n,\varphi}|>\varepsilon,\ \tau>\delta)+P_n^x(\tau\le\delta).$$

The first term on the right-hand side of the above inequality tends to 0 as $\delta\searrow0$ uniformly in $n\in\mathbb{N}$, because by [2, Theorem 5], the functions $q_n(\cdot,x,\cdot)$ defined as $q(\cdot,x,\cdot)$ but with a_n in place of a are bounded in $W_1^{0,1}((0,T)\times\mathbb{R}^d)$ uniformly in $n\in\mathbb{N}$. As for the second term, note that $\mathcal{L}[\tau\mid P_n^x]\to\mathcal{L}[\tau\mid P^x]$, so

$$\lim_{\delta\searrow0}\limsup_{n\to\infty}P_n^x(\tau\le\delta)\le\lim_{\delta\searrow0}P^x(\tau\le\delta)=0,$$

which concludes the proof of (4.25). Combining (4.22) with (4.23), (4.25) and using [3, Theorem 4.2] we get (4.17). By (4.8), (4.17) and the continuous mapping theorem there is a continuous process U^φ such that

(4.26) $$\mathcal{L}\big[\big(X,M^{n,\varphi},A^{n,\varphi},V^{n,\varphi},\tfrac12N^{n,\varphi}+\widetilde K^{n,\varphi}\big)\mid P_n^x\big]\to\mathcal{L}[(X,M^\varphi,A^\varphi,V^\varphi,U^\varphi)\mid P^x]$$

in $C([0,T];\mathbb{R}^{d+4})$.

As in the proof of [28, Theorem 2.2], the Markov property and the fact that $\varphi(X)$ is an $(\{\mathcal{F}_t\},P^x)$-Dirichlet process show that $\widetilde A^\varphi$ is an $\{\overline{\mathcal{F}}_t\}$-adapted process. On the other hand,

$$\varphi(\overline X_t)-\varphi(\overline X_0)=\widetilde M_t^\varphi+\widetilde A_t^\varphi=\tfrac12(M_t^\varphi-\widetilde V_t^\varphi)+U_t^\varphi,\qquad t\in[0,T],$$

and $\varphi(\overline X_\cdot)-\varphi(\overline X_0)$, $\widetilde V^\varphi$ are $\{\overline{\mathcal{F}}_t\}$-adapted. Therefore U^φ is $\{\overline{\mathcal{F}}_t\}$-adapted as well. Set

$$N_t^\varphi=\int_0^t\mathbf{1}_D(\overline X_u)\,dU_u^\varphi,\qquad t\in[0,T].$$

Then N^φ is $\{\overline{\mathcal{F}}_t\}$-adapted and

(4.27) $$\int_0^{\cdot}h_k(\overline X_u)\,dU_u^\varphi\to N^\varphi\quad\text{in } P^x$$

as $k\to\infty$.

Due to (4.15), (4.26) and [16, Lemma 3.1], $\{(1/2)N^{n,\varphi}+\widetilde K^{n,\varphi}\}$ satisfies the condition UT and hence, by [16, Corollary 2.7], for each $k\in\mathbb{N}$,

(4.28) $$\mathcal{L}\Big[\Big(X,M^{n,\varphi},A^{n,\varphi},V^{n,\varphi},\int_0^{\cdot}h_k(\overline X_u)\,d\Big(\frac12N_u^{n,\varphi}+\widetilde K_u^{n,\varphi}\Big)\Big)\Bigm| P_n^x\Big]\to\mathcal{L}\Big[\Big(X,M^\varphi,A^\varphi,V^\varphi,\int_0^{\cdot}h_k(\overline X_u)\,dU_u^\varphi\Big)\Bigm| P^x\Big]$$

in $C([0,T];\mathbb{R}^{d+4})$. Also, by (1.9) and Doob's $\mathbb{L}_2$-inequality,

$$\begin{aligned}I^{n,k}&\equiv E_n^x\sup_{0\le t\le T}\Big|\frac12N_t^{n,\varphi}-\int_0^th_k(\overline X_u)d\Big(\frac12N_u^{n,\varphi}+\widetilde K_u^{n,\varphi}\Big)\Big|^2\\&=\frac12E_n^x\sup_{0\le t\le T}\Big|\int_0^t(\mathbf{1}_D-h_k)(\overline X_u)\,dN_u^{n,\varphi}\Big|^2\\&\le2E_n^x\int_0^T(\mathbf{1}_D-h_k)^2(\overline X_u)\,d\langle N^{n,\varphi}\rangle_u.\end{aligned}$$

Hence

(4.29) $$\lim_{k\to\infty}\limsup_{n\to\infty}I^{n,k}=0.$$

From (4.15) and (4.26)–(4.29) we deduce that there is a continuous process K^φ of finite variation on $[0,T]$ such that $K_0^\varphi=0$ and

(4.30) $$\mathcal{L}[(X,M^{n,\varphi},N^{n,\varphi},V^{n,\varphi},K^{n,\varphi})\mid P_n^x]\to\mathcal{L}[(X,M^\varphi,N^\varphi,V^\varphi,K^\varphi)\mid P^x]$$

in $C([0,T];\mathbb{R}^{d+4})$. In particular, by the continuous mapping theorem,

(4.31) $$\varphi(X_t)-\varphi(X_0)=M_t^\varphi+A_t^\varphi=M_t^\varphi+\tfrac12(-M_t^\varphi+\widetilde N_t^\varphi-V_t^\varphi)+K_t^\varphi$$

for $t\in[0,T]$. From (4.30) and [15, Proposition IX.1.17] we conclude that N^φ is an $(\{\overline{\mathcal{F}}_t\},P^x)$-local martingale, whereas from (4.30) and [15, Corollary VI.6.6] it follows that

$$\mathcal{L}[(M^{n,\varphi},\langle M^{n,\varphi}\rangle)\mid P_n^x]\to\mathcal{L}[(M^\varphi,\langle M^\varphi\rangle)\mid P^x]$$

and

$$\mathcal{L}[(N^{n,\varphi},\langle N^{n,\varphi}\rangle)\mid P_n^x]\to\mathcal{L}[(N^\varphi,\langle N^\varphi\rangle)\mid P^x]$$

in $C([0,T];\mathbb{R}^2)$. On the other hand, by Lemma 3.3,

$$\mathcal{L}[(M^{n,\varphi},\langle M^{n,\varphi}\rangle)\mid P_n^x]\to\mathcal{L}\Big[\Big(M^\varphi,\int_0^{\cdot}\mathbf{1}_Da_n^{ij}D_i\varphi D_j\varphi(X_u)\,du\Big)\Bigm| P^x\Big]$$

and

$$\mathcal{L}[(N^{n,\varphi},\langle N^{n,\varphi}\rangle)\mid P_n^x]\to\mathcal{L}\Big[\Big(N^\varphi,\int_0^{\cdot}\mathbf{1}_Da^{ij}D_i\varphi D_j\varphi(\overline X_u)\,du\Big)\Bigm| P^x\Big]$$

in $C([0,T];\mathbb{R}^2)$. Accordingly, $\langle M^\varphi\rangle$, $\langle N^\varphi\rangle$ are given by (4.3), (4.4). In particular, we have $E^x\langle M^\varphi\rangle_T = E^x\langle N^\varphi\rangle_T < \infty$, which implies that M^φ, N^φ are square-integrable martingales on $[0,T]$. Furthermore, from (4.30) it follows that for $t\in(0,T]$ and $k\in\mathbb{N}$,

$$(4.32)\qquad \mathcal{L}\Big[0=\int_0^t h_k(X_u)\,dK_u^{n,\varphi}\;\Big|\;P_n^x\Big]\to\mathcal{L}\Big[\int_0^t h_k(X_u)\,dK_u^{\varphi}\;\Big|\;P^x\Big].$$

Letting $k\to\infty$ we obtain $\int_0^t \mathbf{1}_D(X_u)\,dK_u^\varphi=0$, $t\in(0,T]$. Thus, K^φ increases only when $X\in\partial D$.

Finally, from what has already been proved we see that

$$\varphi(\overline{X}_t)-\varphi(\overline{X}_0)=N_t^\varphi+B_t^\varphi=N_t^\varphi+\tfrac12(-N_t^\varphi+\widetilde{M}_t^\varphi-\widetilde{V}_t^\varphi)+\widetilde{K}_t^\varphi,\qquad t\in[0,T],$$

is an $(\{\overline{\mathcal{F}}_t\},P^x)$-Dirichlet process (along $\{\Pi_m\}$) with martingale part N^φ. Therefore, by the arguments used to prove (1.11) in [28], for any $0\le t<t+\delta\le T$,

$$Y_t^m\equiv\sum_{T-t+\delta<t_k\le T,\,t_k\in\Pi_m}E^x(\varphi(\overline{X}_{t_k})-\varphi(\overline{X}_{t_{k-1}})\,|\,\overline{\mathcal{F}}_{t_{k-1}})\to B_T^\varphi-B_{T-t+\delta}^\varphi$$

in P^x as $m\to\infty$. Since $\overline{X}$ is a Markov process under P^x,

$$Y_t^m=\sum_{T-t+\delta<t_k\le T,\,t_k\in\Pi_m}E^x(\varphi(\overline{X}_{t_k})-\varphi(\overline{X}_{t_{k-1}})\,|\,\overline{X}_{t_{k-1}}),$$

and hence Y_t^m is $\mathcal{F}_t$-measurable for all sufficiently large m. Thus $B_T^\varphi-B_{T-t+\delta}^\varphi$ is $\mathcal{F}_t$-measurable for any $\delta\in(0,T-t)$. As a consequence, $\widetilde{B}^\varphi$ is $\{\mathcal{F}_t\}$-adapted, and so is $\widetilde{N}$, because $\varphi(X_t)-\varphi(X_0)=\widetilde{N}_t^\varphi+\widetilde{B}_t^\varphi$ for $t\in[0,T]$. Since M^φ and V^φ are $\{\mathcal{F}_t\}$-adapted, it now follows from (4.31) that K^φ is $\{\mathcal{F}_t\}$-adapted.

Finally, if $V^{n,i}$, $M^{n,i}$, $N^{n,i}$, $K^{n,i}$ are defined by (3.1)–(3.4) with $\varphi(x)=x_i$, then by (4.30),

$$\mathcal{L}[\varphi(X),(D_i\varphi(X),M^{n,i},N^{n,i},V^{n,i},K^{n,i})\,|\,P_n^x]\\ \to\mathcal{L}[\varphi(X),(D_i\varphi(X),M^i,N^i,V^i,K^i)\,|\,P^x]$$

in $C([0,T];\mathbb{R}^6)$ for $i=1,\dots,d$. Therefore

$$(4.33)\qquad \mathcal{L}[(\varphi(X),M^{n,\varphi},\widetilde{N}^{n,\varphi},V^{n,\varphi},K^{n,\varphi})\,|\,P_n^x]\\ \to\mathcal{L}[\varphi(X),D_i\varphi\cdot M^i,D_i\varphi*N^i,D_i\varphi V^i,\varphi K^i)\,|\,P^x]$$

in $C([0,T];\mathbb{R}^5)$, because $V^{n,\varphi}=D_i\varphi\cdot V^{n,i}$, $K^{n,\varphi}=D_i\varphi\cdot K^{n,i}$ and $M^{n,\varphi}=D_i\varphi\cdot M^{n,i}$, $\widetilde{N}^{n,\varphi}=D_i\varphi*N^{n,i}$, the last two equalities being a consequence of Itô's formula and uniqueness of the decomposition of semimartingales into a martingale and a finite variation parts.

Since (M^i,N^i,V^i,K^i) satisfies (i)–(ii) and, by (4.8), (4.33) and the continuous mapping theorem,

$$\varphi(X_t)-\varphi(X_0)=\tfrac12(D_i\varphi\cdot M^i+D_i\varphi*N^i-D_i\varphi V^i)+D_i\varphi K^i,\qquad t\in[0,T],$$

it follows that $(D_i\varphi\cdot M^i,\int_0^\cdot D_i\varphi(\overline{X}_u)\,dN_u^i,D_i\varphi\cdot V^i,D_i\varphi\cdot K^i)$ satisfies (i)–(iii). In view of uniqueness of the decomposition, this gives (4.6), (4.7) and the proof in case φ, $f\in C^2(\overline{D})$ is complete.

Now assume that φ is Lipschitz-continuous and f is bounded measurable. Let us extend φ to a Lipschitz-continuous function $\widetilde{\varphi}$ on $\mathbb{R}^d$ and f to a measurable bounded $\widetilde{f}$ on $\mathbb{R}^d$. For $x\in\mathbb{R}^d$ set

$$(4.34)\qquad \varrho_k(x)=k^d\varrho(kx),\qquad \varphi_k(x)=(\varrho_k*\widetilde{\varphi})(x),\qquad f_k(x)=(\varrho_k*\widetilde{f})(x)$$

($*$ denotes convolution), where $\varrho\in C_0^\infty(\mathbb{R}^d)$ is a non-negative function such that $\int_{\mathbb{R}^d}\varrho(x)\,dx=1$. Then for each $k\in\mathbb{N}$,

$$(4.35)\qquad \varphi_k(X_t)-\varphi_k(X_0)=M_t^{\varphi_k}+A_t^{\varphi_k}\\ =\tfrac12(M_t^{\varphi_k}+\widetilde{N}_t^{\varphi_k}-V_t^{\varphi_k})+K_t^{\varphi_k},\qquad t\in[0,T],$$

is an $(\{\mathcal{F}_t\},P^x)$-Dirichlet process and $\langle M^{\varphi_k}\rangle$, $\langle N^{\varphi_k}\rangle$, V^{φ_k} are given by (4.2)–(4.4) with φ in place of φ_k. Since the functions $D_i\varphi_k$ are bounded in $\overline{D}$ uniformly in $k\in\mathbb{N}$, $i=1,\dots,d$, and $D_i\varphi_k\to D_i\varphi$ a.e. in D for $i=1,\dots,d$, applying the dominated convergence theorem we deduce that

$$E^x\langle M^{\varphi_k-\varphi_l}\rangle_T=E^x\langle N^{\varphi_k-\varphi_l}\rangle_T\\ =E^x\int_0^T\mathbf{1}_Da^{ij}D_i(\varphi_k-\varphi_l)D_j(\varphi_k-\varphi_l)(X_u)\,du\to0$$

as $k,l\to\infty$.

On the other hand, by uniqueness of the decomposition of the form (i)–(iii), $M^{\varphi_k}-M^{\varphi_l}=M^{\varphi_k-\varphi_l}$, $N^{\varphi_k}-N^{\varphi_l}=N^{\varphi_k-\varphi_l}$ for $k,l\in\mathbb{N}$. Accordingly, $\{M^{\varphi_k}\}$, $\{N^{\varphi_k}\}$ are Cauchy sequences in $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, respectively, and in consequence, there are M^φ, N^φ such that $M^{\varphi_k}\to M^\varphi$ in $\mathcal{M}$ and $N^{\varphi_k}\to N^\varphi$ in $\widetilde{\mathcal{M}}$. Since

$$E^x\int_0^T\mathbf{1}_Da^{ij}D_i(\varphi_k-\varphi)D_j(\varphi_k-\varphi)(X_u)\,du\to0,$$

$\langle M^\varphi\rangle$, $\langle N^\varphi\rangle$ are given by (4.3), (4.4). Moreover, since

$$E^x\sup_{0\le t\le T}|V_t^{\varphi_k}-V_t^\varphi|\le\iint_{D_T}|D_jpD_i(\varphi_k-\varphi)(u,y)|\,dy\to0$$

and $\varphi_k(X)\to\varphi(X)$ in P^x, it follows from the above and the continuous mapping theorem that there is an $\{\mathcal{F}_t\}$-adapted process K^φ such that

$$(4.36)\qquad (\varphi_k(X),M^{\varphi_k},N^{\varphi_k},V^{\varphi_k},K^{\varphi_k})\to(\varphi(X),M^\varphi,N^\varphi,V^\varphi,K^\varphi).$$

in P^x. This gives (iii).

To prove that $\varphi(X)$ is a Dirichlet process with the decomposition (4.1) and K^φ has the desired properties, we first note that $\{\operatorname{Var} V^{\varphi_k}\}$, $\{\operatorname{Var} K^{\varphi_k}\}$ are tight in $C([0,T];\mathbb{R})$, because $V^{\varphi_k} = D_i\varphi\cdot V^i$, $K^{\varphi_k} = D_i\varphi\cdot K^i$ for $k\in\mathbb{N}$ and $D_i\varphi_k$ are bounded uniformly in $k\in\mathbb{N}$, $i=1,\ldots,d$. In particular, K^φ has finite variation on $[0,T]$ and, as in (4.32), for every $t\in(0,T]$ and $m\in\mathbb{N}$,

$$\mathcal{L}\Big[0=\int_0^t h_m(X_u)\,dK_u^{\varphi_k}\,\Big|\,P^x\Big]\to\mathcal{L}\Big[\int_0^t h_m(X_u)\,dK_u^{\varphi}\,\Big|\,P^x\Big]$$

as $k\to\infty$, which forces $K_t^\varphi=\int_0^t \mathbf{1}_{\partial D}(X_u)\,dK_u^\varphi$ for $t\in[0,T]$.

Furthermore, as in the proof of [28, Theorem 2.2] we check that $\{M^{\varphi_k}-\widetilde{N}^{\varphi_k}\}$ satisfies UTD, so taking into account (4.36) and applying [28, Lemma 1.4] and [4, Theorem 2] we conclude that A^φ is a 0-quadratic variation process on $[0,T]$. This completes the proof of (i)–(iii) and (4.1)–(4.4). Moreover, since $f_k\to f$ a.e. in D, (4.5) follows by the same method as at the end of the proof of [28, Theorem 2.2]. Finally, if $\varphi\in C^1(\overline{D})$, then $\varphi_k\to\varphi$ and $D_i\varphi_k\to D_i\varphi$ for $i=1,\ldots,d$ uniformly in $\overline{D}$. Therefore

$$(4.37)\qquad (\varphi_k(X), D_i\varphi_k\cdot M^i, D_i\varphi_k * N^i, D_i\varphi_k\cdot V^i, D_i\varphi_k\cdot K^i) \to (\varphi(X), D_i\varphi\cdot M^i, D_i\varphi * N^i, D_i\varphi\cdot V^i, D_i\varphi\cdot K^i)$$

in P^x. Since we already know that

$$(M^{\varphi_k},\widetilde{N}^{\varphi_k},V^{\varphi_k},K^{\varphi_k}) = (D_i\varphi_k\cdot M^i, D_i\varphi_k * N^i, D_i\varphi_k\cdot V^i, D_i\varphi_k\cdot K^i),$$

it follows from (4.37) and (4.35) that $(D_i\varphi\cdot M^i, D_i\varphi * N^i, D_i\varphi\cdot V^i, D_i\varphi\cdot K^i)$ satisfies (i)–(iii). By uniqueness, this gives (4.6), (4.7) and the proof of Theorem 4.2 is complete. ∎

REMARK 4.3. If in Theorem 4.2 we assume additionally that a is continuous then there is a continuous non-decreasing $\{\mathcal{F}_t\}$-adapted process K on $[0,T]$ such that $K_0=0$, $K_t=\int_0^t \mathbf{1}_{\partial D}(X_u)\,dK_u$ and

$$(4.38)\qquad K_t^\varphi=\int_0^t\langle\gamma_a,\nabla\varphi\rangle(X_u)\,dK_u,\qquad t\in[0,T].$$

Indeed, define $\widetilde{a}$ as in Lemma 4.1 and choose $a_n\subset\mathcal{A}^\infty(\lambda,\Lambda;\mathbb{R}^d)$ so that $a_n^{ij}\to\widetilde{a}^{ij}$ uniformly in compact subsets of $\mathbb{R}^d$. Then by (4.30) and the continuous mapping theorem,

$$(4.39)\qquad \mathcal{L}[(K^{n,\varphi},\langle\gamma_{a_n},\nabla\varphi\rangle(X),\langle\gamma_{a_n},\nabla\Phi\rangle(X),K^{n,\Phi})\mid P_n^x] \to \mathcal{L}[(K^{\varphi},\langle\gamma_{a},\nabla\varphi\rangle(X),\langle\gamma_{a},\nabla\Phi\rangle(X),K^{\Phi})\mid P^x]$$

in $C([0,T];\mathbb{R}^4)$. For $t\in[0,T]$ set

$$K_t^n=\int_0^t\frac{1}{\langle\gamma_{a_n},\nabla\Phi\rangle}(X_u)\,dK_u^{n,\Phi},\qquad K_t=\int_0^t\frac{1}{\langle\gamma_{a},\nabla\Phi\rangle}(X_u)\,dK_u^{\Phi}.$$

Clearly, $K_0=0$, K is $\{\mathcal{F}_t\}$-adapted, non-decreasing and increases only when $X\in\partial D$. Furthermore, by (4.39) and [16, Theorem 2.6],

$$\mathcal{L}\Big[\Big(K^{n,\varphi},\int_0^{\cdot}\langle\gamma_{a_n},\nabla\varphi\rangle(X)\,dK_u^n\Big)\,\Big|\,P_n^x\Big]\to\mathcal{L}\Big[\Big(K^{\varphi},\int_0^{\cdot}\langle\gamma_{a},\nabla\varphi\rangle(X)\,dK_u\Big)\,\Big|\,P^x\Big]$$

in $C([0,T];\mathbb{R}^2)$, which gives (4.38) by (4.16) and the continuous mapping theorem.

COROLLARY 4.4. *Under the assumptions of Theorem 4.2*, $\varphi(X)\in\mathcal{D}^2$.

Proof. We only need to show (6.1) with $p=2$ and A^φ in place of A. We have

$$E^x\sum_{j=1}^l(|M_{s_j}^\varphi-M_{s_{j-1}}^\varphi|^2+|\widetilde{N}_{s_j}^\varphi-\widetilde{N}_{s_{j-1}}^\varphi|^2)\le E^x\langle M^\varphi\rangle_T+E^x\langle N^\varphi\rangle_T$$

and

$$\sum_{j=1}^l|V_{s_j}^\varphi-V_{s_{j-1}}^\varphi|^2\le(\operatorname{Var}V_T^\varphi)^2,$$

$$\sum_{j=1}^l|K_{s_j}^\varphi-K_{s_{j-1}}^\varphi|^2\le(\operatorname{Var}K_T^\varphi)^2$$

for any $s_0<s_1<\ldots<s_l$, $s_j\in\Pi_m$, $1\le l\le k(m)$, $m\in\mathbb{N}$, so the desired result follows from Theorem 4.2. ∎

5. Stochastic calculus. In this section X^i, M^i, N^i, V^i, K^i denote the processes of Theorem 4.2 corresponding to the function $x\mapsto x^i$.

THEOREM 5.1. *Let (X,P^x) be a diffusion corresponding to $a\in\mathcal{A}(\lambda,\Lambda)$ with reflection along γ_a starting from $x\in D$ at time* 0. *Then for $i=1,\ldots,d$,*

$$\lim_{m\to\infty}\sum_{t_k\in\Pi_m,\,t_k<t}\psi(X_{t_k})(X_{t_{k+1}}^i-X_{t_k}^i)\equiv\int_0^t\psi(X_u)\,dX_u^i$$

$$\lim_{m\to\infty}\sum_{t_k\in\Pi_m,\,t_k<t}\psi(X_{t_{k+1}})(X_{t_{k+1}}^i-X_{t_k}^i)\equiv\int_0^t\psi(X_u)\,d^*X_u^i$$

exist as limits in P^x for any Lipschitz-continuous $\psi:\overline{D}\to\mathbb{R}$ and $t\in[0,T]$. In particular,

$$\langle\psi(X),X^i\rangle_t=\int_0^t\psi(X_u)\,d^*X_u^i-\int_0^t\psi(X_u)\,dX_u^i,\qquad t\in[0,T].$$

Actually,

$$\int_0^{\cdot}\psi(X_u)\,dX_u^i=\frac{1}{2}(\psi\cdot M^i+\psi*N^i-\langle\psi(\overline{X}),N^i\rangle_{T-\cdot}^T-\psi\cdot V^i)+\psi\cdot K^i,$$

$$\int_0^{\cdot}\psi(X_u)\,d^*X_u^i=\frac{1}{2}(\psi\cdot M^i+\psi*N^i+\langle\psi(X),M^i\rangle-\psi\cdot V^i)+\psi\cdot K^i$$

and both integrals define $(\{\mathcal{F}_t\},P^x)$*-Dirichlet processes on* $[0,T]$ *with martingale part* $\psi\cdot M^i$.

Proof. The proof is similar to that of [25, Theorem 3.1], so we omit it. ∎

THEOREM 5.2. *Let* (X,P^x) *be a diffusion corresponding to* $a\in\mathcal{A}(\lambda,\Lambda)$ *with reflection along* γ_a. *Then* (0.6) *holds for any* $\varphi\in C^2(\overline{D})$ *and* $x\in D$.

Proof. This follows immediately from Theorems 4.2 and 5.1, since $\langle D_i\varphi(\overline{X}),N^i\rangle_{T-t}^T=\langle D_i\varphi(X),X^i\rangle_t$ for $t\in[0,T]$. ∎

6. Appendix. Let $\{\Pi_m\}=\{0=t_0<t_1<\ldots<t_{k(m)}=T\}$ be a sequence of partitions of $[0,T]$ such that $\|\Pi_m\|\to0$ as $m\to\infty$. Let $\{X_t:t\in[0,T]\}$ be a continuous process on some filtered probability space $(\Omega,\mathcal{F},\mathbb{F},P)$. We call X an $(\mathbb{F},P)$*-Dirichlet process* (along $\{\Pi_m\}$) on $[0,T]$ if it admits a decomposition

$$X_t=X_0+M_t+A_t,\quad t\in[0,T],$$

where M is an $(\mathbb{F},P)$-local martingale with $M_0=0$ and A is an $\mathbb{F}$-adapted process of 0-quadratic variation along $\{\Pi_m\}$, i.e. $A_0=0$ and

$$Q_T^m(A)\equiv\sum_{t_k\in\Pi_m}|A_{t_k}-A_{t_{k-1}}|^2\quad\text{in } P \text{ as } m\to\infty.$$

If additionally

$$\lim_{R\to\infty}\sup_{m\ge1}\sup_{1\le l\le k(m)}\sup_{\substack{s_0<s_1<\ldots<s_l,\\ s_j\in\Pi_m}}P\Big(\sum_{j=1}^{l}|A_{s_j}-A_{s_{j-1}}|^p\ge R\Big)=0\tag{6.1}$$

for some $p\in[1,2]$, then following [4] we say that X belongs to the class $\mathcal{D}^p$.

For $n\in\mathbb{N}$ let X^n be an $(\mathbb{F}^n,P^n)$-Dirichlet process on $[0,T]$ along $\{\Pi_m\}$ with the decomposition $X_t^n=X_0^n+M_t^n+A_t^n$, $t\in[0,T]$, and suppose that $\{X^n\}$ is weakly convergent in $C([0,T];\mathbb{R}^d)$. Then following [4] (see also [28]) we will say that $\{X^n\}$ satisfies the *condition UTD* if

$$\{\sup_{0\le t\le T}|A_t^n|\}_{n\in\mathbb{N}}\ \text{ is tight in } \mathbb{R}$$

and

$$\forall_{\varepsilon>0}\ \lim_{m\to\infty}\sup_{n\ge1}P^n(Q_T^m(A^n)>\varepsilon)=0.\tag{6.2}$$

From now on, $\mathbb{F}^X=\{\mathcal{F}_t^X\}$, where $\mathcal{F}_t^X=\sigma(X_s:s\le t)$ for $t\in[0,T]$.

THEOREM 6.1. *Suppose* $\varphi\in C(\overline{D})$. *Let* X *be an* $\mathbb{F}^X$*-Markov process with transition density* p *and for* $n=1,2,\ldots$ *let* X^n *be an* $\mathbb{F}^{X^n}$*-strong Markov process with transition density* p_n *such that* $\varphi(X^n)$ *is an* $(\mathbb{F}^{X^n},P^n)$*-Dirichlet process along* $\{\Pi_m\}$ *with the decomposition* $\varphi(X_t^n)=\varphi(X_0^n)+M_t^{n,\varphi}+A_t^{n,\varphi}$, $t\in[0,T]$. *If* $\{\varphi(X^n)\}$ *satisfies UTD*,

$$\mathcal{L}[X^n\,|\,P^n]\to\mathcal{L}[X\,|\,P]\quad \text{in } C([0,T];\mathbb{R}^d)\tag{6.3}$$

and for each $y\in D$,

$$p_n(\cdot\,,\cdot\,,y)\to p(\cdot\,,\cdot\,,y)\tag{6.4}$$

uniformly on compact sets in $(0,T]\times D$, *then*

$$\mathcal{L}[(X^n,M^{n,\varphi},A^{n,\varphi})\,|\,P^n]\to\mathcal{L}[(X,M^\varphi,A^\varphi)\,|\,P]\quad\text{in } C([0,T];\mathbb{R}^{d+2})\tag{6.5}$$

and $\varphi(X)$ *is an* $(\mathbb{F}^X,P)$*-Dirichlet process along* $\{\Pi_m\}$ *admitting the decomposition* $\varphi(X_t)=\varphi(X_0)+M_t^\varphi+A_t^\varphi$, $t\in[0,T]$.

Proof. Let $(\Omega,\widehat{\mathcal{F}},\widehat{P})$ be a completion of the space $(\Omega,\mathcal{F},P)$ and for $n\in\mathbb{N}$ let $(\Omega^n,\widehat{\mathcal{F}}^n,\widehat{P}^n)$ be a completion of $(\Omega^n,\mathcal{F}^n,P^n)$. Set $\widehat{\mathbb{F}}^+=\{\widehat{\mathcal{F}}_t^+\}$, $\widehat{\mathbb{F}}^{n+}=\{\widehat{\mathcal{F}}_t^{n+}\}$, where

$$\widehat{\mathcal{F}}_t^+=\bigcap_{t<s}\widehat{\mathcal{F}}_s^X,\quad \widehat{\mathcal{F}}_t^X=\mathcal{F}_t^X\vee\mathcal{N},\quad \widehat{\mathcal{F}}_t^{n+}=\bigcap_{t<s}\widehat{\mathcal{F}}_s^{X^n},\quad \widehat{\mathcal{F}}_t^{X^n}=\mathcal{F}_t^{X^n}\vee\mathcal{N}^n$$

for $t\in[0,T]$ and $\mathcal{N}$ (resp. $\mathcal{N}^n$) denotes the collection of P- (resp. P^n-) null sets of $\mathcal{F}$ (resp. $\mathcal{F}^n$). In view of Theorem 1.1 and Lemma 1.2 in [28] we only need to show that

$$\mathcal{L}[(X^n,\varphi(X^n),\widehat{\mathbb{F}}^{n+})\,|\,\widehat{P}^n]\to\mathcal{L}[(X,\varphi(X),\widehat{\mathbb{F}}^+)\,|\,\widehat{P}]\tag{6.6}$$

in the sense of extended convergence (see [28, 29]). To this end, given $m\in\mathbb{N}$, $\mathcal{T}=(t_1,\ldots,t_m)\in[0,T]^m$ such that $t_1<\ldots<t_m$ and $\Theta=(\theta_1,\ldots,\theta_m)\in\mathbb{R}^{md}$ denote by $X^{\mathcal{T},\Theta,m}$ the regular version of the martingale

$$\Big\{\widehat{E}\Big(\exp\Big\{i\sum_{k=1}^m\theta_kX_{t_k}\Big\}\,\Big|\,\widehat{\mathcal{F}}_t^{X+}\Big):t\in[0,T]\Big\},$$

where $\widehat{E}$ stands for the expectation sign with respect to $\widehat{P}$. Due to Proposition 1 and Corollary 5 in [29], (6.6) will be proved once we prove that for every $m\in\mathbb{N}$, $\mathcal{T}\in[0,T]^m$ and $\Theta,\Theta^i\in\mathbb{R}^{md}$, $i=1,\ldots,m$,

$$\mathcal{L}[(X_{t_1}^{n,\mathcal{T},\Theta^1,m},\ldots,X_{t_m}^{n,\mathcal{T},\Theta^m,m})\,|\,\widehat{P}^n]\to\mathcal{L}[(X_{t_1}^{\mathcal{T},\Theta^1,m},\ldots,X_{t_m}^{\mathcal{T},\Theta^m,m})\,|\,\widehat{P}]\tag{6.7}$$

in $\mathbb{C}^m$ and

$$\{(X^n,X^{n,\mathcal{T},\Theta,m})\}\ \text{is tight in } \mathcal{D}([0,T];\mathbb{R}^d\times\mathbb{C}),\tag{6.8}$$

where $\mathbb{C}$ is the set of complex numbers.

For this purpose choose a sequence $\{h_\varepsilon\}$ of non-negative continuous functions $h_\varepsilon : \overline{D} \to \mathbb{R}$ such that $\operatorname{dist}(\operatorname{supp} h_\varepsilon, \partial D) > 0$ for $\varepsilon > 0$ and $h_\varepsilon \nearrow \mathbf{1}_D$ as $\varepsilon \searrow 0$. By (6.3),

$$\mathcal{L}[(X^n, h_\varepsilon(X^n)) \mid P^n] \to \mathcal{L}[(X, h_\varepsilon(X)) \mid P] \quad \text{in } C([0,T];\mathbb{R}^{2d}) \tag{6.9}$$

for every $\varepsilon > 0$. Since $P(X_t \in \partial D) = 0$ for $t > 0$, it follows from (6.9) that for every $\delta > 0$,

$$\lim_{\varepsilon \searrow 0} \limsup_{n\to\infty} P^n(|X_t^n - h_\varepsilon(X_t^n)X_t^n| \geq \delta) \leq \lim_{\varepsilon \searrow 0} P(|X_t - h_\varepsilon(X_t)X_t| \geq \delta) = 0 \tag{6.10}$$

for all $t > 0$.

For arbitrary but fixed $\varepsilon > 0$ denote by $X^{T,\Theta,m,\varepsilon}$ the regular version of the martingale

$$\Big\{\widehat{E}\Big(\exp\Big\{i\sum_{k=1}^m \theta_k h_\varepsilon(X_{t_k})X_{t_k}\Big\}\,\Big|\,\widehat{\mathcal{F}}_t^+\Big) : t \in [0,T]\Big\}$$

in case $t_1 > 0$ or the martingale

$$\Big\{\widehat{E}\Big(\exp\Big\{it_1X_0 + i\sum_{k=2}^m \theta_k h_\varepsilon(X_{t_k})X_{t_k}\Big\}\,\Big|\,\widehat{\mathcal{F}}_t^+\Big) : t \in [0,T]\Big\}$$

if $t_1 = 0$. Then by (6.10) and Doob's maximal inequality,

$$\begin{aligned}\lim_{\varepsilon\searrow 0} \widehat{E}\sup_{t\leq T}|X_t^{T,\Theta,m,\varepsilon} - X_t^{T,\Theta,m}|^2 &\leq 4\widehat{E}|X_T^{T,\Theta,m,\varepsilon} - X_T^{T,\Theta,m}|^2 \\ &\leq 4\widehat{E}\Big|\exp\Big\{\sum_{k:t_k>0}\theta_k(1-h_\varepsilon(X_{t_k}))X_{t_k}\Big\} - 1\Big|^2 \\ &\leq 4\sum_{k:t_k>0}\widehat{E}|\exp\{\theta_k(1-h_\varepsilon(X_{t_k}))X_{t_k}\} - 1|^2 = 0\end{aligned} \tag{6.11}$$

Similarly,

$$\lim_{\varepsilon\searrow 0}\limsup_{n\to\infty}\widehat{E}^n\sup_{t\leq T}|X_t^{n,T,\Theta,m,\varepsilon} - X_t^{n,T,\Theta,m}|^2 = 0. \tag{6.12}$$

On the other hand, by the arguments from the proof of [28, Theorem 1.3],

$$\mathcal{L}[(X_{t_1}^{n,T,\Theta^1,m,\varepsilon},\ldots,X_{t_m}^{n,T,\Theta^m,m,\varepsilon}) \mid \widehat{P}^n] \to \mathcal{L}[(X_{t_1}^{T,\Theta^1,m,\varepsilon},\ldots,X_{t_m}^{T,\Theta^m,m,\varepsilon}) \mid \widehat{P}]$$

in $\mathbb{C}^m$ and $\{(X^n, X^{n,T,\Theta,m,\varepsilon})\}$ is tight in $\mathcal{D}([0,T];\mathbb{R}^d\times\mathbb{C})$. Therefore (6.7), (6.8) follow from (6.11), (6.12) and [3, Theorem 4.2]. ■

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On absolutely representing systems in spaces of infinitely differentiable functions

by

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Abstract. The main part of the paper is devoted to the problem of the existence of absolutely representing systems of exponentials with imaginary exponents in the spaces $C^\infty(G)$ and $C^\infty(K)$ of infinitely differentiable functions where G is an arbitrary domain in $\mathbb{R}^p$, $p \ge 1$, while K is a compact set in $\mathbb{R}^p$ with non-void interior $\dot{K}$ such that $\overline{\dot{K}} = K$. Moreover, absolutely representing systems of exponents in the space $H(G)$ of functions analytic in an arbitrary domain $G \subseteq \mathbb{C}^p$ are also investigated.

1. Introduction. Let H be a linear topological space over the field $\mathbb{C}$. A sequence $X := (x_k)_{k=1}^\infty \subset H$ is called a *representing system* (RS) in H if each element x of H can be represented in the form of a series

$$x = \sum_{k=1}^\infty c_k x_k, \quad c_k \in \mathbb{C},\ k = 1, 2, \ldots, \tag{1.1}$$

converging in H. Let now H be a complete locally convex space (CLCS). A sequence X is said to be an *absolute representing system* (ARS) in H if each $x \in H$ can be represented in the form of a series (1.1) absolutely converging in H. It is evident that every ARS in H is a fortiori an RS. The problem of existence of such systems was investigated in [9].

Suppose that $H = \underrightarrow{\operatorname{ind}} H_n$ where for any $n \ge 1$, H_n is a CLCS, $H_n \hookrightarrow H_{n+1}$ and $x_k \in H_1$, $k \ge 1$. If X is an RS (or an ARS) in each H_n then X is an RS (respectively, an ARS) in H. This trivial fact is mentioned in [13, §3, point 1]; a far more difficult question is also posed there: whether X is an RS (or an ARS) in $H = \underleftarrow{\operatorname{proj}} H_n$ if X is an RS (respectively, an ARS) in each H_n.

A number of results in this direction for certain function spaces (mainly for the Fréchet space $H = H(G)$ of functions analytic in the domain G with the standard compact-open topology) and for some sequences x_k (mainly of

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