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Fractional Sobolev norms and structure of Carnot–Carathéodory balls for Hörmander vector fields

by

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Abstract. We study the notion of fractional L^p-differentiability of order $s \in (0,1)$ along vector fields satisfying the Hörmander condition on $\mathbb{R}^n$. We prove a modified version of the celebrated structure theorem for the Carnot–Carathéodory balls originally due to Nagel, Stein and Wainger. This result enables us to demonstrate that different $W^{s,p}$-norms are equivalent. We also prove a local embedding $W^{1,p} \subset W^{s,q}$, where q is a suitable exponent greater than p.

1. Introduction. It is well known that the classical theory of Sobolev spaces plays an important role in many problems concerning partial differential equations. It has also been realized in the last years that an essential tool in the study of second order differential operators arising from degenerate vector fields on $\mathbb{R}^n$ is the construction of generalized Sobolev spaces suitably related to the fields.

To motivate our discussion we recall some simple features of first order Sobolev spaces. Given a family $X_1, \ldots, X_m$ of (at least Lipschitz continuous) vector fields on $\mathbb{R}^n$, $X_j = \sum_{k=1}^n a_{j,k}(x)\partial/\partial x_k$, a natural generalization of the usual $W^{1,p}$ space can be defined by means of the norm

$$\|u\|_{W_X^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)},$$

where $\Omega \subset \mathbb{R}^n$ is an open set and $Xu = (X_1u, \ldots, X_mu)$ denotes the "degenerate gradient", $X_ju = \sum a_{j,k}\partial_k u$. If we assume that the fields are smooth and satisfy the Hörmander condition (see (5)), then a Sobolev-type embedding holds for the space $W_X^{1,p}$. Namely, representing a function u as a "convolution" by means of the fundamental solution Γ of $\sum X_j^2$, using the estimates of Γ and $X\Gamma$ (see Nagel, Stein and Wainger [47] and Sánchez-Calle [52]), together with the continuity of some "fractional integration op-

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erators" (see Capogna, Danielli and Garofalo [7]), one can show that, if p is greater than 1 and Ω is a bounded set, then

$$\|u\|_{L^q(\Omega)} \le c\|Xu\|_{L^p(\Omega)}, \qquad u \in C_0^\infty(\Omega),$$

for suitable $q = q(\Omega, X, p) > p$.

Several recent papers are devoted to the study of geometric and embedding properties of first order Sobolev spaces in various degenerate situations. We refer to Rothschild and Stein [49], Franchi and Lanconelli [22, 23] Jerison [35], Saloff-Coste [51], Varopoulos, Saloff-Coste and Coulhon [54], Capogna, Danielli and Garofalo [7, 8], Biroli and Mosco [4], Franchi, Lu and Wheeden [24, 25], Hajłasz and Koskela [29, 30], Maheux and Saloff-Coste [46], Garofalo and Nhieu [27, 28], Franchi, Serapioni and Serra Cassano [26], Berhanu and Pesenson [3] and to the references of those papers.

The aim of this paper is to give some properties of a family of spaces which are "intermediate" between L^p and $W^{1,p}_X$. It seems natural to define the fractional (semi)norm of order s, $0 < s < 1$, as a sum of fractional derivatives along the fields, setting

$$(1)\qquad [u]_{W^{s,p}(\Omega)} = \bigg(\sum_{j=1}^m \int_\Omega dx \int_{\{e^{tX_j}(x)\in\Omega\}} \frac{dt}{|t|^{1+ps}}\, |u(e^{tX_j}(x)) - u(x)|^p\bigg)^{1/p},$$

where Ω is a bounded set, $1 \le p < \infty$, and $t \mapsto e^{tX_j}(x)$ denotes the integral curve of the field X_j, starting from x at $t = 0$.

One of the results of this paper (Section 4) is that if Hörmander's condition is satisfied, then the norm (1) is locally equivalent to

$$(2)\qquad [u]^*_{W^{s,p}(\Omega)} := \bigg(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{d(x,y)^{ps}|B(x,d(x,y))|}\,dx\,dy\bigg)^{1/p}.$$

In (2), d denotes the Carnot–Carathéodory distance associated with the fields (see Section 2) and $|B|$ is the Lebesgue measure of the d-ball B (we let $B(x,r) := \{y : d(x,y) < r\}$). This equivalence result shows that the norm (1) is determined only by the distance d. A similar phenomenon occurs for first order Sobolev spaces (see Hajłasz and Koskela [30, Theorem 11.11]).

The main tool in the proof of the equivalence between (1) and (2) (Section 3) consists of a new structure theorem for d-balls. Our result is a modified version of a "classical" theorem by Nagel, Stein and Wainger [47, Theorem 7]. Roughly speaking, we prove that for any d-ball $B = B(x,r)$, there exists a C^1 diffeomorphism E defined on a neighborhood of the origin in $\mathbb{R}^n$ such that

$$E(c_1Q) \subset B \subset E(c_2Q),$$

where Q is a suitable box in $\mathbb{R}^n$, c_1 and c_2 are positive constants and $c_jQ := \{c_jx : x \in Q\}$, $j = 1,2$, is the homothetic box. The precise statement of our result is given in Theorem 3.1. We only remark that the difference between this theorem and Nagel–Stein–Wainger's original result is an alternative choice of the "exponential" maps E. The new feature of our maps is that they can be easily factorized as a composition of a finite number of elementary translations along integral curves of the vector fields $X_1, \ldots, X_m$. These maps are already known in the literature (they appeared in Nagel, Stein and Wainger [47], Lanconelli [41], Varopoulos, Saloff-Coste and Coulhon [54] and Danielli [16]). However their properties have not been completely exploited. The results of Section 3 give a contribution in this direction. We also remark that our Theorem 3.1 has been used in the new proof of Jerison's Poincaré inequality by E. Lanconelli and the author in [42].

We actually consider more general "anisotropic" norms, of the form

$$(3)\qquad [u]_{W^{s,p}(\Omega)} = \bigg(\sum_{j=1}^m \int_\Omega dx \int_{\{e^{tX_j}(x)\in\Omega\}} \frac{dt}{|t|^{1+ps/d_j}}\, |u(e^{tX_j}(x)) - u(x)|^p\bigg)^{1/p},$$

where the integer $d_j = d(X_j)$ is the *formal degree* (in the sense of [47]) of the field X_j and $0 < s < 1$. In this weighted situation we prove the equivalence between (3) and (2) provided the distance d is suitably defined taking account of the degrees of the fields. The interest of this generalization stems from the fact that (3) is related to the "parabolic" operator $X_0 + \sum_{j=1}^m X_j^2$ if we let $d(X_0) = 2$ and $d(X_1) = \ldots = d(X_m) = 1$.

In Section 5 we prove an embedding result of the form

$$(4)\qquad [u]_{W^{s,q}(\Omega)} \le c\|Xu\|_{L^p(\Omega)}, \qquad u \in C_0^\infty(\Omega),$$

where $p > 1$ and $q > p$ is suitable. We also give a "parabolic version" of (4). The proofs of these results rely on some properties of the fundamental solutions of Hörmander operators, essentially established by Sánchez-Calle [52] and Nagel, Stein and Wainger [47].

Before closing this introduction we quote some papers partially related to ours. Bakry, Coulhon, Ledoux and Saloff-Coste [1, Section 9] prove, as an application of their results, that, in a general situation, the space $W^{s,p}$ defined by (2) embeds in L^q for a suitable $q > p$. A similar embedding, for $p = 2$ and for Hörmander fields of type 2, is proved by Chemin and Xu [11]. Their spaces $W^{s,2}$, $s > 0$, are constructed by means of pseudodifferential techniques.

We finally remark that several results concerning fractional Sobolev spaces, in the particular situation of Carnot groups (all the fields have degree one and are left invariant on a nilpotent stratified Lie group) are given in Folland [19] and Saka [50]. The cited papers develop a quite rich theory. Here we obtain only partial results, but we work in a more general setting.

The paper is organized as follows. In Section 2 we recall some known results about Hörmander vector fields. In Section 3 we prove the structure theorem for balls. In Section 4 we study the equivalence between different $W^{s,p}$-norms. Section 5 is devoted to some embedding results.

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2. Notations and known results. In this section we recall some known properties of Hörmander's vector fields and we introduce the notations used in what follows.

The Hörmander condition. Consider a family of m vector fields $X_1, \ldots, X_m$ on $\mathbb{R}^n$, where $X_j = \sum_{k=1}^n a_{j,k}\partial/\partial x_k$ and the functions $a_{j,k} = a_{j,k}(x)$ are smooth on $\mathbb{R}^n$. Denote by $[X,Y]$ the commutator of the fields X and Y. Setting $\operatorname{ad}(X)(Y) = [X,Y]$, we can write the commutators of higher order by means of the following standard notation: if $I = (i_1, \ldots, i_p)$ is a multi-index ($p \in \mathbb{N}$ and $1 \le i_j \le m$), we set

$$X_{[I]} = \operatorname{ad}(X_{i_1})\operatorname{ad}(X_{i_2})\ldots\operatorname{ad}(X_{i_{p-1}})(X_{i_p}) = [X_{i_1}, [\ldots[X_{i_{p-1}}, X_{i_p}]\ldots]].$$

We say that the commutator $X_{[I]}$ has *length* p and we write $|I| = p$. The original fields X_j are commutators of length 1.

In what follows we assume that the fields satisfy the following *Hörmander condition* ([33]): for any $x \in \mathbb{R}^n$ there exists an integer r such that

$$\operatorname{span}\{X_{[I]}(x) : |I| \le r\} = \mathbb{R}^n. \tag{5}$$

The properties of vector fields satisfying (5) have been widely studied in the last years. Many papers cited in the introduction deal with Hörmander's vector fields. Some more references related to this topic are Bony [5], Hörmander and Melin [34], Jerison and Sánchez-Calle [36], Kusuoka and Strook [39, 40], Lu [43, 44, 45], Xu [56], Citti, Garofalo and Lanconelli [15], Franchi, Gallot and Wheeden [20], Citti and Di Fazio [14], Buckley, Koskela and Lu [6], Vodop'yanov and Markina [55], Capogna, Danielli and Garofalo [9, 10], Chernikov and Vodop'yanov [12], Krylov [38], Ben Arous and Gradinaru [2], Hajłasz and Strzelecki [31] and the references of those papers. (We have not mentioned here papers dealing with analysis on Carnot groups).

Carnot–Carathéodory distances. We introduce some noneuclidean distances associated with a family of vector fields (cf. [17], [21] and [47]; see also [37]). Let $X_1, \ldots, X_m$ be Hörmander vector fields. Attach to any field X_j a *degree* $d(X_j) \in \mathbb{N}$ (see [47]). Assign to the commutator $X_{[I]}$ the degree $d(X_{[I]}) = \sum d(X_{i_j})$. Denote by $Y_1, \ldots, Y_q$ an enumeration of all the commutators of length at most r, where r is an integer large enough to ensure that $Y_1, \ldots, Y_q$ span $\mathbb{R}^n$ at each point of a fixed bounded set $\Omega_0 \subset \mathbb{R}^n$. Denote also by $\Gamma_{x,y}$ the space of absolutely continuous paths $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Define

$$d(x,y) = \inf\Big\{r > 0 : \exists\gamma \in \Gamma_{x,y} \text{ such that } \dot\gamma(t) = \sum_{j=1}^m a_j(t)X_j(\gamma(t)) \\ \text{and } |a_j(t)| < r^{d(X_j)} \text{ a.e. in } [0,1]\Big\}.$$

Set also

$$\varrho(x,y) = \inf\Big\{r > 0 : \exists\gamma \in \Gamma_{x,y} \text{ such that } \dot\gamma(t) = \sum_{j=1}^q a_j(t)Y_j(\gamma(t)) \\ \text{and } |a_j(t)| < r^{d(Y_j)} \text{ a.e. in } [0,1]\Big\},$$

$$\varrho_2(x,y) = \inf\Big\{r > 0 : \exists\gamma \in \Gamma_{x,y} \text{ such that } \dot\gamma(t) = \sum_{j=1}^q a_jY_j(\gamma(t)) \\ \text{and } |a_j| < r^{d(Y_j)} \text{ a.e. in } [0,1]\Big\}.$$

It is not elementary to prove that $d < \infty$, that is, given two points x and y there exists *at least* a path which connects x and y and whose tangent vector lies in $\operatorname{span}\{X_j\}$. The existence of such a path (under Hörmander's condition) is a classical reachability result due to Chow [13]. Various "quantitative versions" of this result are contained in [41], [47] and [54]. Also our results of Section 3 give a proof of Chow's Theorem.

The functions d, ϱ and ϱ_2 just introduced are trivially symmetric. Moreover d and ϱ clearly satisfy the triangle inequality. The distance ϱ_2 satisfies locally $\varrho_2(x,y) \le c(\varrho_2(x,z) + \varrho_2(y,z))$, where c can be greater than 1. This inequality is a consequence of the local equivalence between ϱ and ϱ_2 [47, Theorem 7].

A remarkable property of the distance arising from a family of Hörmander vector fields is a local estimate of the form

$$|x-y| \le c_1 d(x,y) \le c_2|x-y|^{\varepsilon}. \tag{6}$$

Here $\varepsilon < 1$ depends on the geometric properties of the vector fields. The proof of (6) is easy for the distance ϱ (see [47, Proposition 1.1]). The fact that (6) holds for d (in particular the second inequality) is a consequence of Theorem 3.1 (or of the results of the papers [47], [41] and [54], if all the fields have degree one).

We now introduce some notations in order to recall the results of [47]. Given a multi-index $I = (i_1, \ldots, i_n)$, $i_j \le q$, we set

(7)
$$\begin{aligned}&\Phi_{I,x}(h)=\exp\Big(\sum_{j=1}^{n}h_jY_{i_j}\Big)(x),\quad h\in\mathbb{R}^n,\ \text{small},\\&\lambda_I(x)=\det[Y_{i_1}(x),\dots,Y_{i_n}(x)],\qquad \|h\|_I=\max_{j=1,\dots,n}|h_j|^{1/d(Y_{i_j})},\\&d(I)=d(Y_{i_1})+\dots+d(Y_{i_n}).\end{aligned}$$

Nagel, Stein and Wainger proved the following theorem.

THEOREM 2.1. *Let $X_1,\dots,X_m$ be Hörmander fields of degrees $d_1,\dots,d_m$ and let $K\subset\Omega_0$ be a compact set. Then there exist $r_0>0$ and $\eta_2<\eta_1<1$ such that, if $x\in K$ and $r<r_0$, it is possible to find a multi-index I so that:*

(i) $|\lambda_I(x)|r^{d(I)}\geq\frac12\max_J|\lambda_J(x)|r^{d(J)}$, *where the maximum is taken over the set* $\{J=(j_1,\dots,j_n):j_k\leq q,\ k=1,\dots,n\}$;

(ii) *if* $\|h\|_I<\eta_1r$, *then* $\frac14|\lambda_I(x)|\leq|\det\partial\Phi_{I,x}(h)/\partial h|\leq4|\lambda_I(x)|$;

(iii) *if B_ϱ denotes the ball with respect to the distance ϱ, then we have the inclusion* $B_\varrho(x,\eta_2r)\subset\Phi_{I,x}(\{\|h\|_I<\eta_1r\})\subset B_\varrho(x,\eta_1r)$;

(iv) *the function $\Phi_{I,x}$ is one-to-one on* $\{\|h\|_I<\eta_1r\}$.

An easy consequence of Theorem 2.1 is the polynomial behavior of the measure of the ball, i.e. $|B(x,r)|\sim\sum_I|\lambda_I(x)|r^{d(I)}$. This equivalence is uniform in x in each compact set K and $r\leq r_0(K)$. Moreover there exists $c>0$ such that the following *doubling property* holds:

(8)
$$|B(x,2r)|\leq c|B(x,r)|,\quad x\in K,\ r\leq r_0.$$

Fundamental solutions. It has been proved by Sánchez-Calle [52] (see also [47]) that, given a family $X_1,\dots,X_m$ of Hörmander vector fields on $\mathbb{R}^n$, $n\geq3$, and a bounded set Ω, there exists a kernel $\Gamma(x,y)$ smooth off the diagonal of $\Omega\times\Omega$ which is a fundamental solution of the differential operator $\Delta:=\sum_{j=1}^mX_j^*X_j$ ([1]), i.e. the equation $\Delta f=\phi$, $\phi\in C_0^\infty(\Omega)$, is solved by $f(x)=\int\Gamma(x,y)\phi(y)\,dy$. The kernel also satisfies the estimates

$$\frac1c\frac{d(x,y)^2}{|B(x,d(x,y))|}\leq\Gamma(x,y)\leq c\frac{d(x,y)^2}{|B(x,d(x,y))|},$$

(9)
$$|X_j\Gamma(x,y)|\leq c\frac{d(x,y)}{|B(x,d(x,y))|},\qquad |X_iX_j\Gamma(x,y)|\leq\frac{c}{|B(x,d(x,y))|},$$

$x,y\in\Omega$, $d(x,y)\leq r_0=r_0(\Omega)$. In (9) each derivative can act both on the first and on the second argument.

Multiplying the equation $\Delta f=\phi$ by a function $u\in C_0^\infty(\Omega)$ and integrating by parts we obtain the representation formula

(10)
$$u(x)=\int X_y\Gamma(y,x)\cdot Xu(y)\,dy.$$

([1]) $X_j^*=-X_j-\operatorname{div}X_j$ denotes the formal adjoint (in L^2) of X_j.

In the "parabolic" case (see [47] and [18]), given a family $X_0,X_1,\dots,X_m$ of Hörmander vector fields, the representation formula (10) becomes $u(x)=\int H(y,x)Lu(y)\,dy$, where $L=X_0+\sum_{j=1}^mX_j^*X_j$, while H is a kernel satisfying the growth estimates

$$|H(x,y)|\leq c\frac{d(x,y)^2}{|B(x,d(x,y))|},\qquad |X_jH(x,y)|\leq c\frac{d(x,y)^{2-d(X_j)}}{|B(x,d(x,y))|},$$

where $j=0,1,\dots,m$, $d(X_0)=2$, while $d(X_1)=\dots=d(X_m)=1$.

Riesz potentials. We give the generalization to our context of the classical continuity result concerning "fractional integration operators". For any point x denote by $D(x)$ the "pointwise homogeneous dimension" defined by

(11)
$$D(x)=\min\{d(Y_{i_1})+\dots+d(Y_{i_n}):\lambda_I(x)\neq0\}.$$

Recall that $|B(x,r)|\sim\sum_I|\lambda_I(x)|r^{d(I)}$. Thus $|B(x,r)|$ behaves as $r^{D(x)}$ as $r\to0$. We will need the following result (see [7] for a proof).

THEOREM 2.2. *Let K be a compact set. Then there exists $r_0>0$ so that, for every ball $B=B(x,r)$, $x\in K$, $r\leq r_0$, if we set $\bar D=\max_{x\in\bar B}D(x)$, and, for a fixed $a\in\,]0,\bar D[$,*

$$I_af(x):=\int_Bf(y)\frac{d(x,y)^a}{|B(x,d(x,y))|}\,dy,$$

then for any $p\in\,]1,\bar D/a[$ there exists $c>0$ such that

$$\|I_af\|_{L^q(B)}\leq c\|f\|_{L^p(B)},\qquad f\in C_0^\infty(B),\ q=\frac{\bar Dp}{\bar D-ap}.$$

The Campbell–Hausdorff formula. Given a smooth vector field X, we denote by $t\mapsto e^{tX}(x)$ the integral curve of X starting from x at $t=0$. The map $x\mapsto e^{tX}(x)$ is a diffeomorphism between suitable subsets of $\mathbb{R}^n$. We denote this function by $e^{tX}\equiv\exp(tX)$. We will get useful information on the composition of e^{tX} and e^{sY} by means of this version of the Campbell–Hausdorff formula.

PROPOSITION 2.3. *Let X and Y be smooth vector fields on the open set $\Omega\subset\mathbb{R}^n$. Then the following formal equality holds:*

(12)
$$\exp(sX)\exp(tY)=\exp\Big(sX+tY-\frac{st}{2}[X,Y]+\sum_{k+j>1}s^kt^jC_{k,j}\Big).$$

Here $C_{k,j}$ denotes a finite linear combination of commutators. Any commutator contains k times the field X and j times the field Y. The meaning of (12) is the following: for any fixed compact set $K\subset\Omega$, given two integers

k_0 and j_0, there exists $r_0 > 0$ such that, if $|s|, |t| < r_0$, then

$$e^{sX}e^{tY}(x) = \exp\Big(sX + tY - \frac{st}{2}[X,Y] + \sum_{k\le k_0,\, j\le j_0} s^k t^j C_{k,j}\Big)(x) + O(s^{k_0+1}) + O(t^{j_0+1}),$$

where $|O(t^m)| \le ct^m$, uniformly in $x \in K$.

Classical references on the Campbell–Hausdorff formula are Hochschild [32] and Serre [53]. The applications of this tool to our context are discussed in Hörmander [33], Rothschild and Stein [49], Nagel, Stein and Wainger [47] and Varopoulos, Saloff-Coste and Coulhon [54].

3. Structure of balls. In this section we construct some modified versions of the exponential maps used by Nagel, Stein and Wainger in the proof of their representation result [47, Theorem 7]. We consider a class of "almost exponential" maps, defined in (16), which can be factorized as a composition of a finite number of elementary translations along integral curves of the fields $X_1, \ldots, X_m$. We prove in Theorem 3.1 that our maps give a good representation of the Carnot–Carathéodory balls. Our result also gives (not surprisingly) a proof of the equivalence between the distances d, ϱ and ϱ_2, for any choice of the degrees of the fields.

Let now $S_1, \ldots, S_l$ be fields belonging to the family $X_1, \ldots, X_m$. Set $d_j = d(S_j)$, $j = 1, \ldots, l$. Keeping the notations of [47], we define for $a \in \mathbb{R}$,

$$\begin{aligned} C_1(a, S_1) &= \exp(a^{d_1}S_1),\\ C_2(a; S_1, S_2) &= \exp(-a^{d_2}S_2)\exp(-a^{d_1}S_1)\exp(a^{d_2}S_2)\exp(a^{d_1}S_1),\\ &\vdots\\ C_l(a; S_1, \ldots, S_l) &= C_{l-1}(a; S_2, \ldots, S_l)^{-1}\exp(-a^{d_1}S_1)\\ &\quad\cdot C_{l-1}(a; S_2, \ldots, S_l)\exp(a^{d_1}S_1). \end{aligned}$$

By the Campbell–Hausdorff formula and the Jacobi identity (a commutator of commutators is a sum of commutators), we get the following equality of formal series:

$$C_2(a, S_1, S_2) = \exp\Big(a^{d_1+d_2}[S_1, S_2] + \sum_{d(I)>d_1+d_2} c_I a^{d(I)} S_{[I]}\Big),$$

where c_I is a suitable number, $I = (i_1, \ldots, i_p)$ is a multi-index, $i_j \in \{1, 2\}$, $p \in \mathbb{N}$, $S_{[I]}$ is the commutator $[S_{i_1}, [\ldots, [S_{i_{p-1}}, S_{i_p}]\ldots]]$ and $d(I) = d(i_1) + \ldots + d(i_p)$ is the degree of $S_{[I]}$. Iterating and using again the Campbell–Hausdorff formula and the Jacobi identity we have

$$C_l(a; S_1, \ldots, S_l) = \exp\Big(a^{d_1+\ldots+d_l}S_{[(1,\ldots,l)]} + \sum_{d(I)>d_1+\ldots+d_l} c_I a^{d(I)} S_{[I]}\Big), \tag{13}$$

where $i_j \in \{1, \ldots, l\}$ and $p \in \mathbb{N}$. If $d_j = 1$ for each j, then (13) is contained in [47, Lemma 2.21].

Set now $d = d(S_1) + \ldots + d(S_l)$. We can define, for $\sigma \in \mathbb{R}$ small,

$$\exp^*(\sigma S_{[(1,\ldots,l)]}) = \begin{cases} C_l(\sigma^{1/d}; S_1, \ldots, S_l), & \sigma > 0,\\ C_l(|\sigma|^{1/d}; S_1, \ldots, S_l)^{-1}, & \sigma < 0. \end{cases} \tag{14}$$

Then, by means of (13), we discover that

$$\exp^*(\sigma S_{[(1,\ldots,l)]}) = \exp\Big(\sigma S_{[(1,\ldots,l)]} + \operatorname{sgn}(\sigma)\sum_{d(I)>d} c_I|\sigma|^{d(I)/d}S_{[I]}\Big). \tag{15}$$

Roughly speaking, the map $\exp^*(S)$ can be thought of as an "approximate exponential" of a commutator S. It is also factorizable in paths which are piecewise integral curves of the original fields. A computation of the derivative of the function $\sigma \mapsto \exp^*(\sigma[S_1, [S_2, [\ldots[S_{l-1}, S_l]\ldots]]])(x)$ is contained in Lemma 3.2 and is the fundamental step in the proof of Theorem 3.1.

From now on we fix an open bounded set Ω_0 and we denote by $Y_1, \ldots, Y_q$ a fixed enumeration of the commutators of length at most r, where r is so large that $\operatorname{span}\{Y_1(x), \ldots, Y_q(x)\} = \mathbb{R}^n$ at every $x \in \Omega_0$. Consider also a multi-index $I = (i_1, \ldots, i_n)$, $i_j \le q$, and denote by $U_1 = Y_{i_1}, \ldots, U_n = Y_{i_n}$ the associated commutators. Set, for $h \in \mathbb{R}^n$ small enough,

$$E_I(x, h) \equiv E_{I,x}(h) = \exp^*(h_1U_1)\ldots\exp^*(h_nU_n)(x). \tag{16}$$

We are now ready to state the main result of this section.

THEOREM 3.1. *Let $K \subset \Omega_0$ be a compact set. Then there exists $\delta_0 > 0$ and positive numbers a and b, $b < a < 1$, so that, given any n-tuple I of commutators such that*

$$|\lambda_I(x)|\delta^{d(I)} \ge \tfrac{1}{2}\max_J|\lambda_J(x)|\delta^{d(J)} \tag{17}$$

for $x \in K$ and $\delta < \delta_0$, we have

(i) *if $\|h\|_I < a\delta$ and $J_hE_I(x,h)$ is the jacobian determinant of $E_I(x,\cdot)$, then $\frac{1}{4}|\lambda_I(x)| \le |J_hE_I(x,h)| \le 4|\lambda_I(x)|$;*

(ii) *if B_ϱ and B_d are the balls with respect to the metrics ϱ and d, then $B_\varrho(x, b\delta) \subset E_{I,x}(\{\|h\|_I < a\delta\}) \subset B_d(x, \delta)$;*

(iii) *the function $E_{I,x} \equiv E_I(x,\cdot)$ is one-to-one on $\{\|h\|_I < a\delta\}$.*

The proof of the theorem is organized as follows: (i) is an easy consequence of Lemmas 3.2–3.4. The proof of (ii) is contained in Lemma 3.5, while assertion (iii) is proved in Lemma 3.6.

Lemmas 3.2 and 3.3 below give the analogues of [47, Lemma 2.12].

Lemma 3.2. *Let $S_1, \ldots, S_l$ be vector fields of degrees $d_1, \ldots, d_l$. Set $U = [S_1, [S_2, [\ldots [S_{l-1}, S_l] \ldots]]]$. If $K \subset\subset \Omega_0$ is a compact set and M is a fixed integer, then there exists $\delta_0 > 0$ such that, if $0 < \lambda < \delta_0$ and $x \in K$, then*

$$\frac{\partial}{\partial \lambda} \exp^*(\lambda U)(x) = U(\exp^*(\lambda U)(x)) + \sum_{d<k\le M} \lambda^{k/d-1} Z_k(\exp^*(\lambda U)(x)) + R_M(\lambda, x),$$

where $d = d(U)$ is the degree of U, Z_k is a linear combination of commutators of degree k of the fields $S_1, \ldots, S_l$, while $|R_M(\lambda, x)| \le c|\lambda|^{(M+1)/d(U)-1}$, $x \in K$, $0 < \lambda < \delta_0$. An analogous result holds for $-\delta_0 < \lambda < 0$. In that case both Z_k and R_M have to be replaced by different expressions Z'_k and R'_M, with the same properties.

Proof. Set $u^d = \lambda$ and $\exp^*(\lambda U)(x) = \xi$. Then

$$\exp^*((u+\sigma)^d U)(x) - \exp^*(u^d U)(x) = \exp^*((u+\sigma)^d U) \exp^*(-u^d U)(\xi) - \xi.$$

From now on, R denotes any formal series of the form $\sum_{j,k\ge0} \sigma^j u^k V_{j,k}$, where $V_{j,k}$ is a vector field, and by Z_k any finite linear combination of commutators of degree k. The R and Z_k may not be the same at each occurrence. We can also assume that $0 < \sigma < u/2$. By means of (15) and the Campbell–Hausdorff formula, we get

$$\begin{aligned}
&\exp^*((u+\sigma)^d U) \exp^*(-u^d U) \\
&= \exp\Big((u+\sigma)^d U + \sum_{k>d} (u+\sigma)^k Z_k\Big) \exp\Big(-u^d U - \sum_{k>d} u^k Z_k\Big) \\
&= \exp\Big(u^d U + \sum_{k>d} u^k Z_k + \sigma d u^{d-1} U + \sigma \sum_{k>d} k u^{k-1} Z_k + \sigma^2 R\Big) \\
&\quad \cdot \exp\Big(-u^d U - \sum_{k>d} u^k Z_k\Big) \\
&= \exp\Big\{\sigma\Big(d u^{d-1} U + \sum_{k>d} k u^{k-1} Z_k\Big) \\
&\quad + \sigma\alpha_1 \operatorname{ad}\Big(u^d U + \sum_{k>d} u^k Z_k\Big)\Big(d u^{d-1} U + \sum_{k>d} k u^{k-1} Z_k\Big) + \ldots \\
&\quad + \sigma\alpha_{M-1}\Big(\operatorname{ad}\Big(u^d U + \sum_{k>d} u^k Z_k\Big)\Big)^{M-1}\Big(d u^{d-1} U + \sum_{k>d} k u^{k-1} Z_k\Big) \\
&\quad + \sigma^2 R + \sigma u^{Md} R\Big\},
\end{aligned}$$

where the α_j's are constants coming from the Campbell–Hausdorff formula.

Each term of the finite sum can be written as

$$\begin{aligned}
\sigma\alpha_j \Big(\operatorname{ad}\Big(u^d U + \sum_{k>d} u^k Z_k\Big)\Big)^j \Big(d u^{d-1} U + \sum_{k>d} k u^{k-1} Z_k\Big) \\
= \sigma\alpha_j \sum_{d<p\le M} c_{j,p} u^{p-1} Z_p + \sigma u^M R,
\end{aligned}$$

where Z_p is a linear combination of commutators of degree p, while $c_{j,p}$ is a constant. Then

$$\begin{aligned}
(18)\qquad &\exp^*((u+\sigma)^d U) \exp^*(-u^d U) \\
&= \exp\Big(\sigma d u^{d-1} U + \sigma \sum_{d<k\le M} u^{k-1} Z_k + \sigma^2 R + \sigma u^M R\Big).
\end{aligned}$$

In other words

$$\begin{aligned}
&\frac{1}{\sigma}\{\exp^*((u+\sigma)^d U)(x) - \exp^*(u^d U)(x)\} \\
&\quad = \frac{1}{\sigma}\Big\{\exp\Big(\sigma d u^{d-1} U + \sigma \sum_{d<k\le M} u^{k-1} Z_k\Big)(\xi) - \xi \\
&\qquad + O(\sigma^2) + \sigma O(u^M)\Big\} \\
&\quad \to d u^{d-1} U(\xi) + \sum_{d<k\le M} u^{k-1} Z_k(\xi) + O(u^M)
\end{aligned}$$

as $\sigma \to 0$. Finally keeping in mind that we set $u^d = \lambda$,

$$\begin{aligned}
\frac{\partial}{\partial \lambda} \exp^*(\lambda U)(x) &= \frac{1}{d u^{d-1}} \frac{\partial}{\partial u} \exp^*(u^d U)(x) \\
&= U(\exp^*(\lambda U)(x)) + \sum_{d<k\le M} \lambda^{k/d-1} Z_k(\exp^*(\lambda U)(x)) \\
&\quad + O(|\lambda|^{((M+1)/d-1)})
\end{aligned}$$

which ends the proof if $\lambda > 0$. In the case $\lambda < 0$ the proof is analogous. ■

The following lemma is an adapted version of [47, Lemma 2.12].

Lemma 3.3. *Let $E_I(x, h)$ be the map defined in (16). Let also K be a compact set. Then there exists $\delta_0 > 0$ such that, if $x \in K$ and $|h| < \delta_0$, then*

$$\begin{aligned}
(19)\qquad \frac{\partial}{\partial h_j} E_I(x,h) &= U_j(E_I(x,h)) \\
&\quad + \sum_{\substack{k_1+\ldots+k_j > d_j \\ k_1\le M,\ldots,k_j\le M}} |h_1|^{k_1/d_1} \ldots |h_{j-1}|^{k_{j-1}/d_{j-1}} |h_j|^{k_j/d_j - 1} Z_k(E_I(x,h)) \\
&\quad + O(\|h\|_I^{M+1-d_j}),
\end{aligned}$$

where $Z_k = Z_k^{(h)}$ denotes a finite linear combination (with constant coefficients) of commutators of degree $k_1 + \ldots + k_j$, which may possibly change if the coordinates of h change their sign.

Proof. The case $j = 1$ is a consequence of Lemma 3.2. If $j = 2$ we need a computation of the derivative $\frac{\partial}{\partial h_2}\exp^*(h_1U_1)\exp^*(h_2U_2)(x)$. We can assume $h_1 \geq 0$ and $h_2 \geq 0$. Set $h_i = u_i^{d_i}$, where d_1 and d_2 are the degrees of U_1 and U_2. Write $\exp^*(u_1^{d_1}U_1)\exp^*(u_2^{d_2}U_2)(x) = \xi$. Then

$$\frac{\partial}{\partial u_2}\exp^*(u_1^{d_1}U_1)\exp^*(u_2^{d_2}U_2)(x) = \lim_{\sigma\to 0}\frac{1}{\sigma}(\exp^*(u_1^{d_1}U_1)\exp^*((u_2+\sigma)^{d_2}U_2) \times \exp^*(-u_2^{d_2}U_2)\exp^*(-u_1^{d_1}U_1)(\xi) - \xi). \tag{20}$$

Let us now introduce the following notation:

$$\begin{aligned} H &= u_1^{d_1}U_1 + \sum_{k>d_1} u_1^k Z_k, \\ W &= d_2 u_2^{d_2-1}U_2 + \sum_{d_2<k_2\le M} u_2^{k_2-1}Z_{k_2}. \end{aligned} \tag{21}$$

In view of (15), we can assert that $\exp^*(u_1^{d_1}U_1) = e^H$. Then, by means of (18) (choosing $U = U_2$) and (21),

$$\begin{aligned} &\exp^*(u_1^{d_1}U_1)\exp^*((u_2+\sigma)^{d_2}U_2)\exp^*(-u_2^{d_2}U_2)\exp^*(-u_1^{d_1}U_1) \\ &= e^H\exp(\sigma W + \sigma^2 R + \sigma u_2^M R)e^{-H} \\ &= \exp\Big\{H + \sigma W + \sum_{k\ge 1}\alpha_k \operatorname{ad}(H)^k(\sigma W) + \sigma^2 R + \sigma u_2^M R\Big\}e^{-H} \\ &= \exp\Big\{H + \sigma W + \sigma\sum_{k\ge1}\alpha_k\operatorname{ad}(H)^k(W) - H \\ &\quad + \beta_2\operatorname{ad}(H)\Big(\sigma W + \sigma\sum_{k\ge1}\alpha_k\operatorname{ad}(H)^k(W)\Big) \\ &\quad + \beta_3\operatorname{ad}(H)^2\Big(\sigma W + \sigma\sum_{k\ge1}\alpha_k\operatorname{ad}(H)^k(W)\Big) \\ &\quad + \ldots + \sigma^2 R + \sigma u_2^M R\Big\} \\ &= \exp\Big\{\sigma W + \sigma\sum_{k\ge1}\gamma_k\operatorname{ad}(H)^k(W) + \sigma^2 R + \sigma u_2^M R\Big\}, \end{aligned} \tag{22}$$

where α_k, β_k and γ_k are suitable numbers. But

$$\begin{aligned} \operatorname{ad}(H)(W) &= \Big[u_1^{d_1}U_1 + \sum_{k_1>d_1} u_1^{k_1}Z_{k_1},\ d_2u_2^{d_2-1}U_2 + \sum_{d_2<k_2\le M} u_2^{k_2-1}Z_{k_2}\Big] \\ &= \sum_{\substack{d_2\le k_2\le M\\ d_1\le k_1\le M}} u_1^{k_1}u_2^{k_2-1}Z_{k_1+k_2} + u_1^{M+1}R, \end{aligned}$$

where $Z_{k_1+k_2}$ is a linear combination of commutators of degree k_1+k_2. An analogous argument shows that all the other terms of (22) (those containing $\operatorname{ad}(H)^k(W)$) can be written in the same form. The explicit form (21) of W enables us to rewrite (22) as follows:

$$\begin{aligned} &\exp^*(u_1^{d_1}U_1)\exp^*((u_2+\sigma)^{d_2}U_2)\exp^*(-u_2^{d_2}U_2)\exp^*(-u_1^{d_1}U_1) \\ &= \exp\Big\{\sigma W + \sigma\sum_{\substack{d_2\le k_2\le M\\ d_1\le k_1\le M}} u_1^{k_1}u_2^{k_2-1}Z_{k_1+k_2} + \sigma^2R + \sigma u_2^M R + \sigma u_1^{M+1}R\Big\} \\ &= \exp\Big\{\sigma d_2u_2^{d_2-1}U_2 + \sigma\sum_{\substack{k_1+k_2>d_2\\ k_1\le M,\, k_2\le M}} u_1^{k_1}u_2^{k_2-1}Z_{k_1+k_2} \\ &\quad + \sigma^2R + \sigma u_2^M R + \sigma u_1^{M+1}R\Big\}. \end{aligned}$$

It is now easy to compute the derivative (20). Keeping in mind that we have written $\xi = \exp^*(u_1^{d_1}U_1)\exp^*(u_2^{d_2}U_2)(x)$ and $u_j^{d_j} = h_j$, we get

$$\begin{aligned} &\frac{\partial}{\partial h_2}\exp^*(h_1U_1)\exp^*(h_2U_2)(x) \\ &= \frac{\partial u_2}{\partial h_2}\frac{\partial}{\partial u_2}\exp^*(u_1^{d_1}U_1)(u_2^{d_2}U_2)(x) \\ &= \frac{h_2^{1/d_2-1}}{d_2}\Big\{d_2u_2^{d_2-1}U_2(\xi) + \sum_{\substack{k_1+k_2>d_2\\ k_1\le M,\, k_2\le M}} u_1^{k_1}u_2^{k_2-1}Z_{k_1+k_2}(\xi) \\ &\quad + O(u_2^M) + O(u_1^{M+1})\Big\} \\ &= U_2(\xi) + \sum_{\substack{k_1+k_2>d_2\\ k_1\le M,\, k_2\le M}} h_1^{k_1/d_1}h_2^{k_2/d_2-1}Z_{k_1+k_2}(\xi) \\ &\quad + O(h_2^{M/d_2+1/d_2-1}) + O(h_2^{1/d_2-1}h_1^{M/d_1+1/d_1}). \end{aligned}$$

That proves the lemma if $j = 2$ (as $h_k^{1/d_k} \le \|h\|$).

Finally, in order to compute the derivative $\partial_j E(x,\cdot)$ in the case $j > 2$ it is enough to remark that, by means of the Campbell–Hausdorff formula, we have

$$\exp^*(u_1^{d_1}U_1)\exp^*(u_2^{d_2}U_2)\dots\exp^*(u_{j-1}^{d_{j-1}}U_{j-1})$$
$$= \exp\Big(\sum_{k_1\ge d_1,\dots,k_{j-1}\ge d_{j-1}} u_1^{k_1}u_2^{k_2}\dots u_{j-1}^{k_{j-1}}Z_{k_1+\dots+k_{j-1}}\Big) =: \exp(\widetilde{H}),$$

where $Z_{k_1+\dots+k_{j-1}}$ is a linear combination of commutators of degree $k_1+\dots+k_{j-1}$. The proof follows by the same argument, on setting $\widetilde{H}$ instead of H. ∎

LEMMA 3.4. *Let $\chi>0$, let K be a compact set and $t>0$. Then there exists $\varepsilon=\varepsilon(t)>0$, depending also on χ and K, such that, if*

$$(23)\qquad |\lambda_I(x)|\delta^{d(I)} \ge t\max_J|\lambda_J(x)|\delta^{d(J)}$$

for some $x\in K$, $\delta>0$ and an n-tuple I, then, as soon as $\|h\|_I<\varepsilon(t)\delta$, we have

$$(24)\qquad \partial_{h_j}E_I(x,h) = U_j(E_I(x,h)) + \sum_{s=1}^n b_{js}(E_I(x,h))U_s(E_I(x,h)),$$

where $|b_{js}(E_I(x,h))|\le\chi\delta^{d(U_s)-d(U_j)}$.

Proof. Let I be an n-tuple of commutators $Y_{i_1}=U_1,\dots,Y_{i_n}=U_n$ of degrees $d_1,\dots,d_n$ and such that (23) holds for suitable $t>0$, $x\in K$ and $\delta>0$. By the triangle inequality, if $\varepsilon(t)$ is small and $\|h\|_I=\max|h_j|^{1/d(U_j)}<\varepsilon(t)\delta$, it is easy to see that the point $E_I(x,h)$ belongs to the ball $B_2(x,c\|h\|_I)$, where c is an absolute constant. Then, by [47, Lemma 2.10] we get

$$(25)\qquad \tfrac12|\lambda_I(x)|\le|\lambda_I(E_I(x,h))|\le2|\lambda_I(x)|.$$

Moreover,

$$|\lambda_I(x)|\delta^{d(I)}\ge t\max_J|\lambda_J(x)|\delta^{d(J)}\ge t(\max_J|\lambda_J(x)|)\delta^{\max_L d(L)}\ge tc\delta^{\max_L d(L)},$$

by Hörmander's condition. Thus $|\lambda_I(x)|\ge ct\delta^{N_0}$ provided N_0 is large enough (say, $N_0=\max_{J,K}|d(J)-d(K)|$). We remark that the vectors $U_j(E_I(x,h))$, $j=1,\dots,n$, are independent, by (25). Therefore the remainder in (19) can be written as follows:

$$O(\|h\|_I^{M+1-d_j}) = \sum_{s=1}^n \mu_{j,s}(E_I(x,h))U_s(E_I(x,h)),$$

where $\mu_{j,s}$ are suitable functions. Taking (25) into account and solving for the $\mu_{j,s}$'s, we have

$$\begin{aligned}|\mu_{j,s}(E_I(x,h))| &\le \frac{c}{|\lambda_I(E_I(x,h))|}O(\|h\|_I^{M+1-d_j})\\ &\le\frac{c}{|\lambda_I(x)|}O(\|h\|_I^{M+1-d_j})\le\frac{c\|h\|_I^{M+1-d_j}}{t\delta^{N_0}}\\ &\le c\frac{(\varepsilon(t)\delta)^{M+1-d_j}}{t}\delta^{-N_0}\le\frac{\chi}{2}\delta^{N_0}\end{aligned}$$

provided $M+1=2N_0+\max_J d(J)$ and $\varepsilon=\varepsilon(t)$ is small enough.

Again by the independence of the U_k's we can write each term of the finite sum in (19) as $Z_k=\sum_{s=1}^n a_k^sU_s$. But Z_k is a finite sum of commutators of degree $k_1+\dots+k_j$. Therefore by [47, Theorem 6], we have $|a_k^s(E_I(x,h))|\le(c/t)\delta^{d_s-(k_1+\dots+k_j)}$ provided $\|h\|_I<\varepsilon(t)\delta$. Then the finite sum in (19) is estimated as follows:

$$\begin{aligned}&\Big|\sum_{\substack{k_1\le M,\dots,k_j\le M\\ k_1+\dots+k_j>d_j}}|h_1|^{k_1/d_1}\dots|h_{j-1}|^{k_{j-1}/d_{j-1}}|h_j|^{k_j/d_j-1}a_k^s(E_I(x,h))\Big|\\ &\qquad\le c\sum_{\substack{k_1\le M,\dots,k_j\le M\\ k_1+\dots+k_j>d_j}}\|h\|_I^{k_1+\dots+k_{j-1}+k_j-d_j}\frac{c}{t}\delta^{d_s-(k_1+\dots+k_j)}\\ &\qquad\le c\varepsilon(t)\frac{c}{t}\delta^{d_s-d_j}\le\frac{\chi}{2}\delta^{d_s-d_j}\end{aligned}$$

as soon as $\varepsilon(t)$ is small depending on the compact set K, t and χ. That proves the lemma. ∎

It is now easy to prove part (i) of Theorem 3.1 in the following way. By the properties of the determinant, if δ_{jk} denotes the Kronecker delta, then

$$\det(\delta_{js}+b_{js}(E_I(x,h)))=\det(\delta_{js}+b_{js}(E_I(x,h))\delta^{d_j-d_s})\in[1/2,2],$$

provided we apply Lemma 3.4 for a suitably small χ (depending on the dimension n). Computing the determinant in (24) and using (25) we get (i).

We can now prove assertion (ii) of Theorem 3.1.

LEMMA 3.5. *Let K be a compact set and $t>0$. Then there exist positive numbers m, $\varepsilon=\varepsilon(t)$, $\eta=\eta(t)$ and δ_0 (also depending on K) such that, if $|\lambda_I(x)|\delta^{d(I)}\ge t\max_J|\lambda_J(x)\delta^{d(J)}|$, then*

$$B_\varrho(x,\eta(t)\varepsilon(t)\delta)\subset E_I(x,\{\|h\|_I<\varepsilon(t)\delta\})\subset B_d(x,m\varepsilon(t)\delta).$$

Proof. The second inclusion is a trivial consequence of the definition of the distance d. The other one has been proved in [47, Lemma 2.16]. For the convenience of the reader we give a proof. The argument is very similar (but not identical) to the one of the cited authors.

Let $y\in B_\varrho(x,\eta(t)\varepsilon(t)\delta)$ (η and ε will be made precise later). Then there exists an absolutely continuous path φ (we can assume that φ is one-to-one) such that $\varphi'(s)=\sum_{j=1}^q b_j(s)Y_j(\varphi)$ if $0<s<1$, $\varphi(0)=x$, $\varphi(1)=y$ and

$|b_j(s)| \leq (\eta(t)\varepsilon(t)\delta)^{d(Y_j)}$. In order to prove the lemma it is sufficient to construct a (unique) "lifted path" θ such that $\theta(0) = x$ and

$$\begin{cases} E_I(x,\theta(s)) = \varphi(s), & 0 \leq s \leq 1, \\ \|\theta(s)\|_I < \frac{1}{2}\varepsilon(t)\delta, & 0 \leq s \leq 1. \end{cases} \tag{26}$$

The existence of the lifted path θ will also be used in Lemma 3.6 below to prove that the map E is one-to-one. Some arguments here and in the lemma are taken from [48].

We begin by proving that, if $\theta : [0,\overline{s}] \to \mathbb{R}^n$ is absolutely continuous, $\theta(0) = x$ and $E_I(x,\theta) \equiv \varphi$ on $[0,\overline{s}]$, $\overline{s} \leq 1$, then for any $s \in [0,\overline{s}]$,

$$\|\theta(s)\|_I < \tfrac{1}{2}\varepsilon(t)\delta. \tag{27}$$

Assume by contradiction that for some $\widetilde{s} \leq \overline{s}$ (we choose the "smallest") we have $\|\theta(\widetilde{s})\|_I = \frac{1}{2}\varepsilon(t)\delta$. For any $s < \widetilde{s}$ there is a local inverse Ψ^s of $E_I(x,\cdot)$ such that $\Psi^s(\varphi(s)) = \theta(s)$ and $\Psi^s(E_I(x,h)) = h$ if h is near $\theta(s)$. For a suitable integer $\mu \leq n$ we have (we denote by Ψ^s_μ the μth component of the map Ψ^s)

$$\begin{aligned}
\left(\frac{1}{2}\varepsilon(t)\delta\right)^{d(U_\mu)} &= \theta_\mu(\widetilde{s}) = \int_0^{\widetilde{s}} \frac{d}{d\sigma}\Psi^s_\mu(\varphi(s+\sigma))\Big|_{\sigma=0} ds \\
&= \int_0^{\widetilde{s}} (\nabla\Psi^s_\mu)(\varphi(s))\cdot(\varphi'(s))\,ds \\
&= \int_0^{\widetilde{s}} (\nabla\Psi^s_\mu)(\varphi(s))\cdot\sum_{j=1}^q b_j(s)Y_j(\varphi(s))\,ds \\
&= \int_0^{\widetilde{s}} (\nabla\Psi^s_\mu)(\varphi(s))\cdot\sum_{j=1}^q b_j(s)\sum_{k=1}^n a^k_j(\varphi(s))U_k(\varphi(s)),
\end{aligned} \tag{28}$$

where the U_k's are, as usual, the fields associated with the n-tuple I. We now use the fact that ϱ and ϱ_2 are locally equivalent (see [47, Theorem 7]). If η is small, depending on the compact set K, then $\varphi(s)$ belongs to the ball $B_{\varrho_2}(x,\varepsilon(t)\delta)$. Then $|a^k_j(\varphi(s))| \leq (c/t)\delta^{d(U_k)-d(Y_j)}$, as proved in [47, Theorem 6]. Finally, in view of [47, Lemma 2.15], $|(U_k\Psi^s_\mu)(\varphi(s))| \leq c\delta^{d(U_\mu)-d(U_k)}$. Then the last line of (28) can be estimated by

$$\begin{aligned}
\sum_{j,k} c\int_0^{\widetilde{s}} (\eta\varepsilon(t)\delta)^{d(Y_j)}\,\frac{\delta^{d(U_k)-d(Y_j)}}{t}\,\delta^{d(U_\mu)-d(U_k)}\,ds &\leq c\eta\,\frac{\varepsilon(t)}{t}\,\delta^{d(U_\mu)} \\
&< \left(\frac{1}{2}\varepsilon(t)\delta\right)^{d(U_\mu)}
\end{aligned}$$

if $\eta = \eta(\varepsilon(t))$ is small enough. That is a contradiction. Thus (27) is proved.

In order to prove the existence of a path θ satisfying (26), we can set

$$\Sigma = \{s_0 \in [0,1] : \text{there exists } \theta \text{ a.c. such that } \theta(0) = x \text{ and } E_I(x,\theta) \equiv \varphi \text{ on } [0,s_0]\}.$$

We remark that, if $s_0 \in \Sigma$, then the function θ is unique (if we had θ_1 and θ_2 the set $\{s \in [0,s_0] : \theta_1(s) = \theta_2(s)\}$ would be open, closed and nonempty). We now prove that $\Sigma = [0,1]$. Σ is open: if $s_0 \in \Sigma$, then since $E_I(x,\cdot)$ is a diffeomorphism near $\theta(s_0)$, we can extend the map θ on an interval $[0,s_0+\sigma]$ for a small $\sigma > 0$. To prove that Σ is closed, we take a sequence $s_j \in \Sigma$, $s_j \to s_0$. By uniqueness we have a map $\theta : [0,s_0[\to \{\|h\|_I < \varepsilon(t)\delta/2\}$. But $\theta([0,s_0[)$ is contained in a set where E_I is "strictly" nonsingular. Then $\theta(s_j)$ is a Cauchy sequence. More precisely, for any $s < s_0$, we denote again by Ψ^s a local inverse of $E_{I,x}$ defined near $\varphi(s)$ (which sends $\varphi(s)$ to $\theta(s)$). We have

$$\begin{aligned}
|\theta(s_j) - \theta(s_k)| &= \Big|\int_{s_k}^{s_j} \theta'(s)\,ds\Big| = \Big|\int_{s_k}^{s_j} \frac{d}{d\sigma}\Psi^s(\varphi(s+\sigma))\Big|_{\sigma=0} ds\Big| \\
&= \Big|\int_{s_k}^{s_j} (dE_{I,x}(\theta(s)))^{-1}(\varphi'(s))\,ds\Big| \\
&\leq c\sup|\varphi'|\,\frac{c}{|\lambda_I(x)|}|s_j - s_k|.
\end{aligned}$$

Here we used the fact that the path θ lies in the set $\|h\|_I < \varepsilon(t)\delta/2$; on that set (25) holds, and gives easily the estimate $\|(dE_s(\theta(s)))^{-1}\| \leq c/|\lambda_I(x)|$. Then $\theta(s_j) \to h_0$ and $\|h_0\|_I < \varepsilon(t)\delta/2$. Thus $\Sigma = [0,1]$ and finally $E_I(x,\theta(1)) = y$, $\|\theta(1)\|_I < \varepsilon(t)\delta/2$. ∎

We now prove the injectivity of the map E (part (iii) of Theorem 3.1).

LEMMA 3.6. *For any compact set K, there exist $\overline{\delta}$ and $\alpha > 0$ depending on K such that, if $x \in K$, $\delta < \overline{\delta}$ and I are such that*

$$|\lambda_I(x)|\delta^{d(I)} \geq \tfrac{1}{2}\max_J|\lambda_J(x)|\delta^{d(J)}, \tag{29}$$

then the function $E_I(x,\cdot)$ is one-to-one on the set $\{\|h\|_I < \alpha\delta\}$.

Proof. The argument here is the same as that of [47, pp. 132–133]. We give some details for the reader who is not familiar with problems concerning global inversion of functions.

For any $\overline{x} \in K$, among the n-tuples such that $|\lambda_I(\overline{x})| > 0$ we first select all the n-tuples of minimal degree. Then we choose from this family an n-tuple $\overline{I}$ such that $|\lambda_{\overline{I}}(\overline{x})|$ is maximal. Under this choice, for suitable $\overline{\delta} > 0$ we have $|\lambda_{\overline{I}}(\overline{x})|\delta^{d(\overline{I})} \geq \max_J|\lambda_J(\overline{x})|\delta^{d(J)}$ for any $\delta < \overline{\delta}$.

Then by a compactness argument, we cover K by a finite family of open sets $U_k \supset \{x_k\}$, $k = 1,\ldots,p$, such that there exist $\delta_{0,k} > 0$ and an n-tuple

$I_{0,k}$ (of minimal degree) satisfying

$$(30)\qquad \begin{cases} |\lambda_{I_{0,k}}(x_k)| = \max_{d(J)=d(I_{0,k})} |\lambda_J(x_k)|, \\ |\lambda_{I_{0,k}}(x_k)|\delta^{d(I_{0,k})} \geq \max_J |\lambda_J(x_k)|\delta^{d(J)}, \quad 0<\delta<\delta_{0,k}, \\ |\lambda_{I_{0,k}}(x)|\delta_{0,k}^{d(I_{0,k})} > \frac{1}{2}\max_J |\lambda_{I_{0,k}}(x)|\delta_{0,k}^{d(J)}, \quad x \in U_k. \end{cases}$$

We can also require, by the Inverse Function Theorem, that $E_{I_{0,k}}(x,\cdot)$ is one-to-one on the set $\{\|h\|_I < \delta_{0,k}\}$ for any $x \in U_k$.

Set $\overline{\delta} = \min_k \delta_{0,k}$ and let $x \in K$ and I be respectively a point and an n-tuple such that (29) is satisfied for a suitable $\delta < \overline{\delta}$. We now fix a set U_k which contains x and denote it by U_0. Let also I_0 be the corresponding n-tuple such that (30) holds. We finally write δ_0 instead of $\delta_{0,k}$.

Given any fixed n-tuple J, the set $\Delta_{J,x} := \{\delta > 0 : |\lambda_J(x)|\delta^{d(J)} \geq \frac{1}{2}\max_K |\lambda_K(x)|\delta^{d(K)}\}$ is a closed interval. It is easy to see that the interval $[\delta,\delta_0]$ is covered by the union of the $\Delta_{J,x}$'s. Coming back to the n-tuple I and to the number δ of the lemma, we can write $\delta \in \Delta_{I,x} := [r_{I,x}, R_{I,x}]$ and $\delta_0 \in \Delta_{I_0,x}$, by (30). We can now connect δ and δ_0 by a sequence of intervals as follows: if $\Delta_{I,x}\cap\Delta_{I_0,x} = \emptyset$, we choose an index I_1 such that $\Delta_{I_1,x}\cap\Delta_{I_0,x} \neq \emptyset$ and $r_{I_1,x} < r_{I_0,x}$ and we set $\delta_1 = r_{I_0,x}$. Then, if $\Delta_{I,x}\cap\Delta_{I_1,x} = \emptyset$, we choose I_2 such that $\Delta_{I_2}\cap\Delta_{I_1}$ is nonempty and we set $\delta_2 = r_{I_1,x}$. Iterating we find a sequence $0 \leq \delta_{N+1} \leq \delta \leq \delta_N < \delta_{N-1} < \ldots < \delta_0$ and a sequence of n-tuples $I = I_N, I_{N-1}, \ldots, I_1, I_0$ such that

$$|\lambda_{I_j}(x)|r^{d(I_j)} \geq \tfrac{1}{2}\max_J |\lambda_J(x)|r^{d(J)}, \qquad r \in [\delta_{j+1},\delta_j],$$

In order to prove that $E_I(x,\cdot) = E_{I_N}(x,\cdot)$ is one-to-one we begin by proving that $E_{I_1}(x,\cdot)$ is one-to-one on $\{\|h\|_I < \alpha_1\delta_1\}$, where α_1 is suitable and depends on the compact set K. It is already known that the map E_{I_0} is one-to-one on $\{\|h\|_{I_0} < \delta_0\}$. Now, the sequence has been constructed in such a way that, if $k = 0,1$, then

$$(31)\qquad |\lambda_{I_k}(x)|\delta_1^{d(I_k)} \geq \tfrac{1}{2}\max_J |\lambda_J(x)|\delta_1^{d(J)}.$$

A double application of Lemma 3.5 gives

$$(32)\qquad \begin{aligned} E_{I_1}(x,\{\|h\|_{I_1} < \alpha_1\delta_1\}) &\subset E_{I_0}(x,\{\|h\|_{I_0} < \beta\delta_1\}) \\ &\subset E_{I_1}(x,\{\|h\|_{I_1} < \gamma\delta_1\}), \end{aligned}$$

where $\alpha_1 < \beta < \gamma$ are constants depending on the compact set.

We now deduce from (32) that $E_{I_1}(x,\cdot)$ is one-to-one on $\|h\|_{I_1} < \alpha_1\delta_1$. The following argument is standard (see e.g. [48]). Suppose by contradiction that there exist h and h' such that $E_{I_1}(h) = E_{I_1}(h') = y$. Then the segment $r(s) = (1-s)h + sh'$ would have as image a closed path contained in $E_{I_0}(\|h\|_{I_0} < \beta\delta_1)$, by (32). Keeping in mind that E_{I_0} is one-to-one, we can deform the closed path $E_{I_1}(r(s)) := \gamma(s)$ to a point, letting $q(\lambda,s) = E_{I_0}(\lambda E_{I_0}^{-1}(y) + (1-\lambda)E_{I_0}^{-1}(\gamma(s)))$, $(\lambda,s) \in [0,1]\times[0,1]$. Again by (32) we have $q(\lambda,s) \in \{\|h\|_{I_1} < \gamma\delta_1\}$. We can now use the same argument used in the proof of Lemma 3.5—essentially the nonsingularity of the map E_{I_1}—to construct, for any fixed $s \in [0,1]$, a (unique) lifted path $\lambda \mapsto p(\lambda,s)$, globally defined on $[0,1]$, such that $p(0,s) = r(s)$ and $E_{I_1}(p(\lambda,s)) = q(\lambda,s)$. A standard argument (see for example [48]) shows that the function p is actually continuous on $[0,1]\times[0,1]$. Moreover, $q(\lambda,0) = y$, $\lambda \in [0,1]$. Thus $E_{I_1}(p(\lambda,0)) = y$, $\lambda \in [0,1]$. But E_{I_1} is a local diffeomorphism. Therefore $p(\lambda,0) = \text{constant} = p(0,0) = h$. Analogously, $p(\lambda,1) = \text{constant} = p(0,1) = h'$. Finally, $E_{I_1}(p(1,s)) = q(1,s) = y$ if $s \in [0,1]$. Thus $p(1,s)$ is constant in s and takes the values h and h', respectively, for $s = 0$ and 1. Consequently, $h = h'$.

Now we know that E_{I_1} is one-to-one on $\|h\|_{I_1} < \alpha_1\delta_1$. Moreover (31) holds with δ_2 instead of δ_1, and I_2 instead of I_1. Thus the first inclusion in (32) becomes $E_{I_1}(\|h\|_{I_1} < \alpha_1\delta_2) \subset E_{I_2}(\|h\|_{I_2} < \beta\delta_2)$. Choose t such that $\varepsilon(t) < \alpha_1$. By Lemma 3.5 we have

$$\begin{aligned} E_{I_1}(\|h\|_{I_1} < \alpha_1\delta_2) &\supset E_{I_1}(\|h\|_{I_1} < \varepsilon(t)\delta_2) \supset B_\varrho(x,\eta(t)\varepsilon(t)\delta_2) \\ &\supset B_d(x,\eta(t)\varepsilon(t)\delta_2) \supset E_{I_2}(\|h\|_{I_2} < \eta(t)\varepsilon(t)\delta_2/m) \\ &= E_{I_2}(\|h\|_{I_2} < \alpha_2\delta_2). \end{aligned}$$

α_2 is defined by the last equality and depends only on the compact set K. Thus, by the same argument as before it is easy to prove that E_{I_2} is one-to-one on $\{\|h\|_{I_2} < \alpha_2\delta_2\}$.

Iterating (at most) N times, where N is the number of n-tuples available, and setting $\alpha_N = \alpha$ we complete the proof. ∎

4. A characterization of the spaces. In this section (Propositions 4.1 and 4.2) we prove the local equivalence between the fractional norms (3) and (2).

PROPOSITION 4.1. *Let $X_1,\ldots,X_m$ be a family of Hörmander vector fields on $\mathbb{R}^n$ and denote by $d_1,\ldots,d_m$ their degrees. Let also $\widetilde{\Omega} \subset \mathbb{R}^n$ be an open bounded set and fix an open set $\Omega \subset\subset \widetilde{\Omega}$. Then there exist $r_0 > 0$ and $c > 0$ such that, for any $u \in C^1(\widetilde{\Omega})$,*

$$\int_{\substack{\Omega\times\Omega \\ d(x,y)<r_0}} \frac{|u(x)-u(y)|^p}{d(x,y)^{ps}|B(x,d(x,y))|}\,dx\,dy \leq c\sum_{j=1}^m \int_{\widetilde{\Omega}} dx \int_{e^{tX_j}(x)\in\widetilde{\Omega}} \frac{dt}{|t|^{1+ps/d_j}}|u(x)-u(e^{tX_j}(x))|^p.$$

Proof. The proof relies on the results of Section 3. Roughly speaking, by means of the maps constructed there we can connect two points x and y by suitable integral curves of the fields.

We begin by fixing $r_0 = b\delta_0$, where δ_0 and b are the constants appearing in Theorem 3.1, applied to the compact set $K = \overline{\Omega}$. Let $x \in \Omega$. For any n-tuple I we define

$$(33)\qquad M_{I,x} \equiv \Big\{ y \in \mathbb{R}^n : d(x,y) < r_0 \text{ and } |\lambda_I(x)| \Big(\frac{d(x,y)}{b}\Big)^{d(I)} > \frac{1}{2}\max_J |\lambda_J(x)| \Big(\frac{d(x,y)}{b}\Big)^{d(J)} \Big\}.$$

It is easy to see that the set $M_{I,x}$ is an annulus (possibly empty), i.e. for some $0 \le r_{I,x} \le R_{I,x}$ we have $M_{I,x} = \{y \in \mathbb{R}^n : r_{I,x} < d(x,y) < R_{I,x} \wedge r_0\}$, where $\alpha \wedge \beta = \min\{\alpha, \beta\}$. For any $x \in \Omega$ we also have $\bigcup_I M_{I,x} = B(x, r_0)$. Thus

$$(34)\qquad \int\limits_{\substack{\Omega\times\Omega\\ d(x,y)<r_0}} \frac{|u(x)-u(y)|^p}{d(x,y)^{ps}|B(x,d(x,y))|}\,dx\,dy = \int_\Omega dx \int\limits_{B(x,r_0)\cap\Omega} dy\, \frac{|u(x)-u(y)|^p}{d(x,y)^{ps}|B(x,d(x,y))|} \le \sum_I \int_\Omega dx \int\limits_{M_{I,x}\cap\Omega} dy\, \frac{|u(x)-u(y)|^p}{d(x,y)^{ps}|B(x,d(x,y))|}.$$

In order to estimate each term of the sum we remark that a significant property of $R_{I,x}$ is the following:

$$|\lambda_I(x)| \Big(\frac{R_{I,x}\wedge r_0}{b}\Big)^{d(I)} \ge \frac{1}{2}\max_J |\lambda_J(x)| \Big(\frac{R_{I,x}\wedge r_0}{b}\Big)^{d(J)}.$$

Furthermore, $(R_{I,x}\wedge r_0)/b \le \delta_0$. Then, invoking Theorem 3.1, we have

$$(35)\qquad \begin{gathered} |\lambda_I(x)| \sim |J_h E_I(x,h)| \quad \text{if } \|h\|_I < \frac{a}{b}(R_{I,x}\wedge r_0), \\ B(x, R_{I,x}\wedge r_0) \subset E_{I,x}\Big(\Big\{\|h\|_I < \frac{a}{b}(R_{I,x}\wedge r_0)\Big\}\Big). \end{gathered}$$

Moreover the map $E_{I,x}$ is one-to-one on $\{\|h\|_I < (a/b)(R_{I,x}\wedge r_0)\}$. We also need a lower estimate for $d(x,y)$. If $y \in M_{I,x}$, then condition (17) is satisfied if we choose $\delta = d(x,y)/b$. Then $B(x, d(x,y)) \subset E_{I,x}(\|h\|_I < (a/b)d(x,y))$. So, we can write $y = E_I(x,h)$ and we have

$$(36)\qquad \|h\|_I \le \frac{a}{b} d(x, E_{I,x}(h)), \qquad E_{I,x}(h) \in M_{I,x}.$$

Thus, performing the change of variable $y = E_{I,x}(h)$ and using (35) and (36), we can write the last line of (34) in the form

$$(37)\qquad \sum_I \int_\Omega dx \int\limits_{E^{-1}_{I,x}(M_{I,x}\cap\Omega)} dh\, \frac{|u(x)-u(E_I(x,h))|^p\,|J_hE_I(x,h)|}{d(x,E_I(x,h))^{ps}|B(x,d(x,E_I(x,h)))|} \le c\sum_I \int_\Omega dx \int\limits_{E^{-1}_{I,x}(M_{I,x}\cap\Omega)} dh\, \frac{|u(x)-u(E_I(x,h))|^p}{\|h\|_I^{d(I)+ps}}.$$

Here we have also used the equivalence $\|h\|_I^{d(I)}|\lambda_I(x)| \sim |B(x,\|h\|_I)|$, which holds on the set on which we are integrating.

Recall that if $U_1,\ldots,U_n$ are the commutators corresponding to the n-tuple I, then

$$E_I(x,h) = \prod_{j=1}^{N(I)} \exp(|h_{k_j}|^{d(X_{r_j})/d(U_{k_j})}\sigma_j X_{r_j})(x),$$

where $\sigma_j = \pm 1$. This last expression is actually incorrect: the definition of the map E_I depends on the sign of the h_j's (cf. (14)). This difficulty can be easily overcome by splitting the last integral of (37) in 2^n integrals such that in each integral the sign of the h_j's does not change.

Thus the right hand side of (37) can be estimated by means of a finite sum of terms of the form

$$(38)\qquad \int\limits_{\|h\|_I<ar_0/b} \frac{dh}{\|h\|_I^{d(I)+ps}} \times \int_\Omega dx\, |u(\exp(|h_{k_j}|^{d(X_{r_j})/d(U_{k_j})}\sigma_j X_{r_j})(z)) - u(z)|^p.$$

Since $d(x, \exp(|h_{k_j}|^{d(X_{r_j})/d(U_{k_j})}\sigma_j X_{r_j})(x)) \le |h_{k_j}|^{1/d(U_{k_j})} \le \|h\|_I < (a/b)r_0$, by (36), if r_0 is small enough, then the point

$$(39)\qquad z = \prod_{p=j+1}^{N} \exp(|h_{k_p}|^{d(X_{r_p})/d(U_{k_p})}\sigma_p X_{r_p})(x)$$

belongs to a fixed open set Ω^* such that $\Omega \subset\subset \Omega^* \subset\subset \widetilde{\Omega}$. For any fixed h, $x \mapsto z$ is a change of variable whose jacobian is bounded by geometric constants (depending on the fields and on the sets Ω and $\widetilde{\Omega}$).

Our next step is to split the variables in the integral (38) and to apply Fubini's Theorem. For fixed $j = 1,\ldots,n$, let $h = (h_j, \widehat{h}_j)$, $d_j = d(U_j)$ and $d(I) = \sum_k d_k$. Then $\|h\|_I = \max|h_k|^{1/d_k} \sim \sum_{k=1}^n |h_k|^{1/d_k} \sim |h_j|^{1/d_j} + \|\widehat{h}_j\|_I$

and for any measurable function $\psi \geq 0$ of a real variable, we have

$$
\begin{aligned}
(40)\qquad & \int\limits_{\|h\|_I<\delta_0} \frac{\psi(h_j)}{\|h\|_I^{d(I)+ps}}\, dh_j\, d\widehat{h}_j \\
&\sim \int\limits_{\|h\|_I<\delta_0} \frac{\psi(h_j)}{(|h_j|^{1/d_j}+\|\widehat{h}_j\|_I)^{d(I)+ps}}\, dh_j\, d\widehat{h}_j \\
&= \int\limits_{|h_j|<\delta_0^{d_j}} \frac{\psi(h_j)}{|h_j|^{(d(I)+ps)/d_j}}\, dh_j \int\limits_{\|\widehat{h}_j\|<\delta_0} \frac{d\widehat{h}_j}{(1+\|\widehat{h}_j\|_I/|h_j|^{1/d_j})^{d(I)+ps}}.
\end{aligned}
$$

Set now $h_k = |h_j|^{d_k/d_j} u_k$, $k \neq j$. Then $d\widehat{h}_j = |h_j|^{d(I)/d_j-1} d\widehat{u}_j$ and moreover (40) takes the form

$$
\begin{aligned}
(41)\qquad & \int\limits_{|h_j|<\delta_0^{d_j}} dh_j\, \frac{\psi(h_j)}{|h_j|^{(d(I)+ps)/d_j}} |h_j|^{d(I)/d_j-1} \int\limits_{\mathbb{R}^{n-1}} \frac{d\widehat{u}_j}{(1+\|\widehat{u}_j\|_I)^{d(I)+ps}} \\
&= c \int\limits_{|h_j|<\delta_0^{d_j}} \frac{\psi(h_j)}{|h_j|^{1+ps/d_j}}\, dh_j.
\end{aligned}
$$

We have also used the fact that $\int_{\mathbb{R}^{n-1}} (1+\|\widehat{u}_j\|_I)^{-(d(I)+ps)}\, d\widehat{u}_j < \infty$.

In order to estimate (38) we perform the change of variable $x \mapsto z$ (see (39)) and integrate in $d\widehat{h}_{k_j}$. Then (38) is less than

$$
\begin{aligned}
&\int\limits_{\|h\|_I<cr_0} \frac{dh}{\|h\|_I^{d(I)+ps}} \int\limits_{\Omega^*} dx\, |u(\exp(|h_{k_j}|^{d(X_{r_j})/d_{k_j}} \sigma_j X_{r_j})(x)) - u(x)|^p \\
&\leq c \int\limits_{\Omega^*} dx \int\limits_{|h_{k_j}|<r_0^{1/d_{k_j}}} \frac{dh_{k_j}}{|h_{k_j}|^{1+ps/d_{k_j}}} |u(\exp(|h_{k_j}|^{d(X_{r_j})/d_{k_j}} \sigma_j X_{r_j}))(x) - u(x)|^p \\
&\leq c \int\limits_{(x,\exp(tX_{r_j}(x)))\in\widetilde{\Omega}\times\widetilde{\Omega}} \frac{dx\, dt}{|t|^{1+ps/d(X_{r_j})}} |u(\exp(tX_{r_j})(x)) - u(x)|^p.
\end{aligned}
$$

In the last inequality we have again changed variable, letting $h_{k_j}^{d(X_{r_j})/d_{k_j}} = t$. We have also assumed that r_0 is small enough such that $e^{tX}(x) \in \widetilde{\Omega}$, for any field X of the family and for each $|t| < r_0^{d(X)}$. ∎

The counterpart of Proposition 4.1 is:

PROPOSITION 4.2. *Let $X_1, \ldots, X_m$ be a family of Hörmander vector fields on $\mathbb{R}^n$. Fix $x_0 \in \mathbb{R}^n$. Then there exists a neighborhood U of x_0 so that for any $O \subset\subset \widetilde{O} \subset\subset U$ there exist positive constants α, δ_0 and c such that*

$$
\begin{aligned}
&\sum_{j=1}^{m} \int_O dx \int\limits_{\substack{e^{\alpha t X_j}(x)\in O\\ |t|<\delta_0}} \frac{dt}{|t|^{1+ps/d(X_j)}} |u(e^{\alpha t X_j}(x)) - u(x)|^p \\
&\qquad\leq c \int\limits_{\widetilde{O}\times\widetilde{O}} \frac{|u(x)-u(y)|^p}{d(x,y)^{ps}|B(x,d(x,y))|}\, dx\, dy.
\end{aligned}
$$

The proof of Proposition 4.2 relies on the so-called lifting method, introduced by Rothschild and Stein. In [49, Theorem 4] the following is proved. Assume that $X_1, \ldots, X_m$ are smooth vector fields whose commutators of length at most r span $\mathbb{R}^n$ at a point $x_0 \in \mathbb{R}^n$. Then there exists an open set U containing x_0 so that it is possible to introduce new coordinates $\tau \in V \subset \mathbb{R}^d$ and new fields $\widetilde{X}_j = X_j + \sum_{l=1}^d a_{j,l}(x,\tau)\partial/\partial\tau_l$, on $U \times V \subset \mathbb{R}^{n+d} \equiv \mathbb{R}^q$, which are free up to step r. That means that the only linear relations (at any point of $U \times V$) between the commutators of length at most r of the fields $\widetilde{X}_j$ are given by antisymmetry and the Jacobi identity.

Following [49, p. 272], we can select q commutators

$$
(42)\qquad \widetilde{Y}_1 = \widetilde{X}_1, \ldots, \widetilde{Y}_m = \widetilde{X}_m, \widetilde{Y}_{m+1}, \ldots, \widetilde{Y}_q
$$

linearly independent at any point of $U \times V$. The remaining commutators can be expressed as linear combinations (with constant coefficients, given by antisymmetry and the Jacobi identity) of $\widetilde{Y}_1, \ldots, \widetilde{Y}_q$.

The proof of Proposition 4.2 relies on the following lemmas.

LEMMA 4.3. *Let $E \subset\subset U$ and $H \subset\subset V$. Then there exist $\delta_0 > 0$, $b < 1$ and $c > 0$ such that, if $x, y \in E$ and $d(x,y) < \delta_0$, we have*

$$
\int\limits_{\substack{H\times H\\ \widetilde{d}(x,y)<\delta_0}} \frac{d\tau\, d\sigma}{\widetilde{d}((x,\tau),(y,\sigma))^{Q+ps}} \leq c \frac{1}{d(x,y)^{ps}|B(x,d(x,y))|}.
$$

Here $\widetilde{d}$ denotes the distance defined by the $\widetilde{Y}_j$'s.

Proof. It is proved in [47, Lemma 3.2] (see also [35, Lemma 4.1]) that, for $Q = d(\widetilde{Y}_1) + \ldots + d(\widetilde{Y}_q)$, if $x \in E$ and $\delta < \delta_0$, then

$$
(43)\qquad |\{\sigma : (y,\sigma) \in \widetilde{B}((x,\tau),\delta)\}| \leq c \frac{|\widetilde{B}((x,\tau),\delta)|}{|B(x,\delta)|} \sim \frac{\delta^Q}{|B(x,\delta)|},
$$

with constants depending on the choice of the sets E, H, U, V.

Now $\widetilde{d}((x,\tau),(y,\sigma)) \geq d(x,y) \equiv d$. Take $x, y \in E$ with $d(x,y) \leq \delta_0$. From (43) we get

$$\int_{\substack{H\times H\\ \tilde d((x,s),(y,\sigma))\le\delta_0}} \frac{d\tau\,d\sigma}{\tilde d((x,\tau),(y,\sigma))^{Q+ps}}$$

$$\le c\int_H d\tau\sum_{k=0}^{\infty}\frac{1}{(2^kd)^{Q+ps}}\int_{2^kd\le\tilde d((x,\tau),(y,\sigma))\le\min\{2^{k+1}d,\delta_0\}} d\sigma$$

$$\le c\int_H d\tau\sum_{\substack{k\ge0\\ 2^{k+1}d<\delta_0}}\frac{1}{(2^kd)^{Q+ps}}\cdot\frac{(2^{k+1}d)^Q}{|B(x,2^{k+1}d)|}$$

$$\le \frac{c}{d^{ps}}\int_H d\tau\sum_{k=0}^{\infty}\frac{1}{2^{kps}}\cdot\frac{1}{|B(x,2^{k+1}d)|}$$

$$\le \frac{c}{d^{ps}}\sum_k\frac{1}{2^{kps}}\cdot\frac{1}{|B(x,d)|}=\frac{c}{d(x,y)^{ps}|B(x,d(x,y))|}.\ \blacksquare$$

LEMMA 4.4. *Let $\tilde Y_1,\dots,\tilde Y_q$ be the commutators introduced in* (42). *Fix open sets $G\subset\subset\tilde G\subset\subset U\times V\subset\mathbb R^q$. If $d_j=d(\tilde Y_j)$ is the degree of the commutator $\tilde Y_j$ and if $\tilde d$ denotes the distance in $U\times V$, then there exist positive numbers α, δ_0 and c such that, for any function u on $\tilde G$, we have*

$$\int_G d\xi\int_{\substack{\exp(\alpha t\tilde Y_j)(\xi)\in G\\ |t|<\delta_0^{d_j}}}\frac{dt}{|t|^{1+ps/d_j}}|u(e^{t\alpha\tilde Y_j}(\xi))-u(\xi)|^p$$

$$\le c\int_{\tilde G\times\tilde G}\frac{d\xi\,d\eta}{\tilde d(\xi,\eta)^{Q+ps}}|u(\xi)-u(\eta)|^p.$$

Proof. We first fix an open set G^* such that $G\subset\subset G^*\subset\subset\tilde G$. For any fixed $j\in\{1,\dots,q\}$ we write $h=(h_j,\hat h_j)\in\mathbb R\times\mathbb R^{q-1}$ and

$$\|h\|:=\max_{k=1,\dots n}|h_k|^{1/d_k}\sim|h_j|^{1/d_j}+\sum_{k\ne j}|h_k|^{1/d_k}\sim|h_j|^{1/d_j}+\|\hat h_j\|.$$

Set now $Q=\sum_{k=1}^q d_k$ and $\tilde\Phi(\xi,h)=\exp(h_1\tilde Y_1+\dots+h_q\tilde Y_q)(\xi)$. Then there exists δ_0 such that $\tilde\Phi(\xi,h)\in G^*$ and $\exp(\alpha h_jY_j)(\xi)\in G^*$ provided $\xi\in G$, $\|h\|<\delta_0=\delta_0(G,G^*)$ and $\alpha\le1$.

Now, keeping in mind (40) and (41) we get

$$\int_{\|h\|<\delta_0}\frac{dh}{\|h\|^{Q+ps}}\psi(h_j)\ge c\int_{|h_j|<\delta_0^{d_j}}\frac{dh_j}{|h_j|^{1+ps/d_j}}\psi(h_j).$$

Here $\psi\ge0$ is any measurable function. Then

$$\int_G d\xi\int_{\substack{\exp(\alpha h_j\tilde Y_j)(\xi)\in G\\ h_j<\delta_0^{d_j}}}\frac{dh_j}{|h_j|^{1+ps/d_j}}|u(e^{h_j\alpha\tilde Y_j}(\xi))-u(\xi)|^p \tag{44}$$

$$\le c\int_G d\xi\int_{\|h\|<\delta_0}\frac{dh}{\|h\|^{Q+ps}}|u(e^{h_j\alpha\tilde Y_j}(\xi))-u(\xi)|^p$$

$$\le c\Big\{\int_G d\xi\int_{\|h\|<\delta_0}\frac{dh}{\|h\|^{Q+ps}}|u(\xi)-u(\tilde\Phi(\xi,h))|^p$$

$$+\int_G d\xi\int_{\|h\|<\delta_0}\frac{dh}{\|h\|^{Q+ps}}|u(e^{h_j\alpha Y_j}(\xi))-u(\tilde\Phi(\xi,h))|^p\Big\}.$$

The first term can be handled by performing the change of variable $h\mapsto\eta=\tilde\Phi(\xi,h)$, whose jacobian is strictly nonsingular as soon as $\|h\|<\delta_0$ and δ_0 is small enough. Moreover by the properties of the distance, we have $\tilde d(\xi,\eta)\le\text{const}\cdot\|h\|$. Thus the integral is under control by means of

$$c\int_{G^*\times G^*}d\xi\,d\eta\,\frac{|u(\xi)-u(\eta)|^p}{\tilde d(\xi,\eta)^{Q+ps}}.$$

Concerning the second integral we write $\exp(h_j\alpha\tilde Y_j)(\xi)=\eta$ and we consider the function $\tilde\Phi_\alpha(\eta,h)=\exp(h_1\tilde Y_1+\dots+h_q\tilde Y_q)\exp(-\alpha h_j\tilde Y_j)(\eta)$. The function just introduced has continuous first derivatives in all the variables, h, η and α. By means of the Inverse Function Theorem, there exist $\varepsilon>0$ and $\delta_0>0$ such that, as soon as $0<\alpha<\varepsilon$, $\eta\in G^*$, the function $h\mapsto\tilde\Phi_\alpha(\eta,h)$ restricted to $\|h\|<\delta_0$ has an inverse. On the other hand, $\tilde d(\eta,\tilde\Phi_\alpha(\eta,h))\le\text{const}\cdot\|h\|$ provided α is small enough. Then the last line of (44) becomes

$$\int_{G\times\{\|h\|<\delta_0\}}\frac{d\xi\,dh}{\|h\|^{Q+ps}}|u(e^{h_j\alpha Y_j}(\xi))-u(\tilde\Phi(\xi,h))|^p$$

$$\le\int_{G^*\times\{\|h\|<\delta_0\}}\frac{d\eta\,dh}{\|h\|^{Q+ps}}|u(\eta)-u(\tilde\Phi_\alpha(\eta,h))|^p$$

$$\le c\int_{\tilde G\times\tilde G}d\zeta\,d\eta\,\frac{|u(\zeta)-u(\eta)|}{d(\zeta,\eta)^{Q+ps}}.\ \blacksquare$$

Proof of Proposition 4.2. Let $x_0\in\mathbb R^n$. Choose an open set U containing x_0 and small enough such that the results of Rothschild and Stein hold. We can denote by $(x,\tau)\in U\times V$ the new coordinates. Let now $O\subset\subset \widetilde{O}\subset\subset U$. We fix an open set $H_1\subset\subset H\subset\subset \widetilde{H}\subset\subset V$. If δ_0 is small enough (depending on the choice of the sets), as soon as $(x,\tau)\in O\times H_1$, $|t|<\delta_0$ and $e^{tX_j}(x)\in O$, then $e^{t\widetilde{X}_j}(x,\tau)\in O\times H$. Thus

$$(45)\qquad \int_O dx \int_{\substack{\exp(\alpha t X_j)(x)\in O\\ |t|<\delta_0}} \frac{dt}{|t|^{1+ps/d_j}}|u(e^{\alpha t X_j}(x)-u(x))|^p$$
$$\le c\int_{O\times H_1} dx\,d\tau \int_{\substack{\exp(\alpha t\widetilde{X}_j)(x,\tau)\in O\times H\\ |t|<\delta_0}} \frac{dt}{|t|^{1+ps/d_j}}|u(e^{\alpha t X_j}(x)-u(x))|^p.$$

This last expression can be handled by using first Lemma 4.4 (with $G=O\times H$ and $\widetilde{G}=\widetilde{O}\times\widetilde{H}$) and secondly Lemma 4.3 (with $E=\widetilde{O}$). The last term of (45) can be finally estimated by

$$\int_{\widetilde{O}\times\widetilde{H}} dx\,d\tau \int_{\widetilde{O}\times\widetilde{H}} dy\,d\sigma\,\frac{|u(x)-u(y)|^p}{\widetilde{d}((x,s),(y,\sigma))^{Q+ps}}$$
$$=\int_{\widetilde{O}\times\widetilde{O}} dx\,dy\,|u(x)-u(y)|^p\int_{\widetilde{H}\times\widetilde{H}}\frac{d\tau\,d\sigma}{\widetilde{d}((x,\tau),(y,\sigma))^{Q+ps}}$$
$$\le\int_{\widetilde{O}\times\widetilde{O}}\frac{|u(x)-u(y)|^p}{d(x,y)^{ps}|B(x,d(x,y))|}\,dx\,dy. \ \blacksquare$$

5. Some embedding results. In this section we give some embedding results (Theorems 5.1 and 5.2) concerning the spaces introduced before. We use the representation of smooth functions by means of fundamental solutions. The embedding results hold either in the case $d(X_j)=1$ (we then use the fundamental solution of $\sum X_j^2$), or in the case where only one among the fields (say X_0) has degree two, while $X_1,\ldots,X_m$ have degree one (in that case we use the fundamental solution of $X_0+\sum X_j^2$).

THEOREM 5.1. *Let $X_1,\ldots,X_m$ be a family of Hörmander vector fields on $\mathbb{R}^n$. Denote by d the distance associated with $\sum X_j^2$. Fix a compact set $K\subset\mathbb{R}^n$. There exists $r_0=r_0(K)$ such that for every ball $B=B(x_0,r)$, $x_0\in K$, $r\le r_0$, the following holds:*

$$[u]_{W^{s,q}(B)}\le c\sum_{j=1}^m\|X_ju\|_{L^p(B)},\qquad u\in C_0^\infty(B).$$

Here $0<s<1$, $p>1$, $q=Dp/(D-(1-s)p)$ *and* $D=\max\{D(x):x\in\bar{B}\}$ (see (11)) *denotes the homogeneous dimension of B.*

THEOREM 5.2. *Let $X_0,X_1,\ldots,X_m$ be a family of Hörmander vector fields and let $d(X_0)=2$, $d(X_j)=1$, $j\ge1$. Let also $K\subset\mathbb{R}^n$ be a compact set. Denote by L the operator $\sum X_j^*X_j+X_0$. Let B be a ball centered at $x\in K$ and of radius $r\le r_0=r_0(K)$. Fix s and p, $0<s<1$, $1<p<D/(2-s)$, where D is the homogeneous dimension of B. Then*

$$[u]_{W^{s,q}(B)}\le c\|Lu\|_{L^p(B)},\qquad u\in C_0^\infty(B),\qquad q=Dp/(D-(2-s)p).$$

The proofs of the two results are actually very similar. We give only a proof of the first.

We need a lemma.

LEMMA 5.3. *Let $X_1,\ldots,X_m$ be Hörmander fields on $\mathbb{R}^n$. Let Ω be an open set and let $\Gamma:\Omega\times\Omega\to\mathbb{R}$ be a fundamental solution of the operator $\sum X_j^*X_j$* (see Section 2)*. Fix a compact set $K\subset\Omega$. Let $\xi,x,y\in K$ and assume that $d(\xi,x)\le\frac14 d(x,y)\le\delta_0$, where δ_0 is a suitable constant depending on K. Then*

$$(46)\qquad |\Gamma(\xi,y)-\Gamma(x,y)|\le c\frac{d(x,\xi)d(x,y)}{|B(x,d(x,y))|},$$
$$(47)\qquad |X_j\Gamma(\xi,y)-X_j\Gamma(x,y)|\le c\frac{d(x,\xi)}{|B(x,d(x,y))|}.$$

Here the derivatives can act both on the first and on the second argument.

Proof. The proof is an easy consequence of the deep results by Sánchez-Calle and Nagel, Stein and Wainger, on the estimate of the fundamental solution of a Hörmander sum-of-squares. Let ξ, x and y by such that $d(x,\xi)\le\frac14 d(x,y)$. By definition of distance there exists an absolutely continuous path $\gamma:[0,1]\to\mathbb{R}^n$ such that $\dot\gamma(t)=\sum_j a_j(t)X_j(\gamma(t))$ and $|a_j(t)|<2d(x,\xi)$ for a.e. $t\in[0,1]$. Then

$$|\Gamma(\xi,y)-\Gamma(x,y)|=\Big|\int_0^1\sum_{j=1}^m a_j(\tau)X_j\Gamma(\gamma(\tau),y)\,d\tau\Big|$$
$$\le cd(x,\xi)\Big|\int_0^1\frac{d(\gamma(\tau),y)}{|B(\gamma(\tau),d(\gamma(\tau),y))|}\,d\tau\Big|,$$

in view of (9). By the triangle inequality we have $d(\gamma(\tau),y)\sim d(x,y)$ with uniform constants. Moreover, if $d(x,y)<\delta_0$ we can also use the doubling property to get $|B(\gamma(\tau),d(\gamma(\tau),y))|\sim|B(x,d(x,y))|$. That gives trivially the proof of (46). The proof of (47) is quite similar.

Proof of Theorem 5.1. Let $B=B(x_0,r)$ be a ball. Then equation (10) gives

$$\int_{B\times B} \frac{dx\,dy}{d(x,y)^{qs}|B(x,d(x,y))|}|u(x)-u(y)|^q \tag{48}$$
$$= \int_{B\times B} \frac{dx\,dy}{d(x,y)^{qs}|B(x,d(x,y))|}\Big|\int_B \{X\Gamma(\xi,y)-X\Gamma(\xi,x)\}Xu(\xi)\,d\xi\Big|^q,$$

where the derivatives act on the variable ξ. In view of the equivalence $|B(x,d(x,y))| \sim |B(y,d(x,y))|$, we can also assume that we are integrating on the set

$$\{|X\Gamma(\xi,x)| \ge |X\Gamma(\xi,y)|\}. \tag{49}$$

Taking (34) and (37) into account we can write, for any $\psi(x,y)\ge 0$,

$$\int_{B\times B} \frac{\psi(x,y)}{d(x,y)^{qs}|B(x,d(x,y))|}\,dx\,dy \le \sum_I \int_B dx \int_{E_{I,x}^{-1}(M_{I,x}\cap B)} \frac{\psi(x,E_{I,x}(h))}{\|h\|^{d(I)+qs}}\,dh.$$

Then (48) is estimated by

$$\sum_I \int_B dx \int_{E_{I,x}^{-1}(M_{I,x}\cap B)} \frac{dh}{\|h\|_I^{d(I)+qs}}\Big|\int_B \{X\Gamma(\xi,E_{I,x}(h))-X\Gamma(\xi,x)\}Xu(\xi)\,d\xi\Big|^q.$$

Letting now $\{X\Gamma(\xi,E_{I,x}(h))-X\Gamma(\xi,x)\} =: \Delta(x,\xi,h)$, we get, by (47), $|\Delta(x,\xi,h)| \le d(x,E_{I,x}(h))/|B(x,d(x,\xi))|$ provided $d(x,E_{I,x}(h)) \le \frac{1}{4}d(x,\xi)$, while the estimate

$$|\Delta(x,\xi,h)| \le 2|X\Gamma(\xi,x)| \le \frac{d(x,\xi)}{|B(x,d(x,\xi))|} \tag{50}$$

always holds, in view of (49).

By the Minkowski inequality, we have

$$\sum_I \int_B dx \int_{E_{I,x}^{-1}(M_{I,x}\cap B)} \frac{dh}{\|h\|_I^{d(I)+qs}}\Big|\int_B \Delta(x,\xi,h)Xu(\xi)\,d\xi\Big|^q \tag{51}$$
$$\le \sum_I \int_B dx\bigg\{\int_B d\xi\bigg[\int_{E_{I,x}^{-1}(M_{I,x}\cap B)} dh\,\frac{|\Delta(x,\xi,h)|^q}{\|h\|_I^{d(I)+qs}}\bigg]^{1/q}|Xu(\xi)|\bigg\}^q.$$

In order to estimate (51) we remark that $E_{I,x}^{-1}(M_{I,x}\cap B) \subset \{\|h\|_I \le cr\}$, where r is the radius of the ball B and c is a suitable constant. Moreover, $\|h\|_I \sim d(x,E_{I,x}(h))$ on $E_{I,x}^{-1}(M_{I,x}\cap B)$. Thus, if λ is a positive number, small enough, but only depending on the compact set, we can assert that $\|h\|_I \le \lambda d(x,\xi) \Rightarrow d(x,E_{I,x}(h)) \le \frac{1}{4}d(x,\xi)$. Then the square bracket in (51) can be estimated as follows:

$$[\ldots] = \int_{\|h\|_I<\lambda d(x,\xi)} dh\,\frac{|\Delta(x,\xi,h)|^q}{\|h\|_I^{d(I)+qs}} + \int_{\lambda d(x,\xi)<\|h\|_I<cr_0} dh\,\frac{|\Delta(x,\xi,h)|^q}{\|h\|_I^{d(I)+qs}}$$
$$\le \int_{\|h\|_I<\lambda d(x,\xi)} dh\,\frac{c}{\|h\|_I^{d(I)+qs}}\cdot\frac{\|h\|_I^q}{|B(x,d(x,\xi))|^q}$$
$$+ \int_{\lambda d(x,\xi)<\|h\|_I<r_0} dh\,\frac{c}{\|h\|_I^{d(I)+qs}}\cdot\frac{d(x,\xi)^q}{|B(x,d(x,\xi))|^q}$$
$$= \int_0^{\lambda d(x,\xi)} \frac{\varrho^{d(I)-1}}{\varrho^{d(I)+qs}}\cdot\frac{\varrho^q}{|B(x,d(x,\xi))|}\,d\varrho$$
$$+ \int_{\lambda d(x,\xi)}^{\infty} \frac{\varrho^{d(I)-1}}{\varrho^{d(I)+qs}}\cdot\frac{d(x,\xi)^q}{|B(x,d(x,\xi))|}\,d\varrho$$
$$= c\frac{d(x,\xi)^{q(1-s)}}{|B(x,d(x,\xi))|^q}$$

(we have also used "polar coordinates"). Thus we have proved that

$$\int_{B\times B} \frac{|u(x)-u(y)|^q}{d(x,y)^{qs}|B(x,d(x,y))|}\,dx\,dy$$
$$\le \sum_I \int_B dx\bigg\{\int_B d\xi\bigg[\int_{E_{I,x}^{-1}(M_{I,x}\cap B)} dh\,\frac{|\Delta(x,\xi,h)|^q}{\|h\|^{d(I)+qs}}\bigg]^{1/q}|Xu(\xi)|\bigg\}^q$$
$$\le \int_B dx\bigg\{\int_B d\xi\,\frac{d(x,\xi)^{1-s}}{|B(x,d(x,\xi))|}|Xu(\xi)|\bigg\}^q \le c(\|Xu\|_{L^p(B)})^q,$$

in view of Theorem 2.2. ∎

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Composition operators: $\mathcal{N}_\alpha$ to the Bloch space to $\mathcal{Q}_\beta$

by

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Abstract. Let $\mathcal{N}_\alpha$, $\mathcal{B}$ and $\mathcal{Q}_\beta$ be the weighted Nevanlinna space, the Bloch space and the $\mathcal{Q}$ space, respectively. Note that $\mathcal{B}$ and $\mathcal{Q}_\beta$ are Möbius invariant, but $\mathcal{N}_\alpha$ is not. We characterize, in function-theoretic terms, when the composition operator $C_\phi f = f\circ\phi$ induced by an analytic self-map ϕ of the unit disk defines an operator $C_\phi : \mathcal{N}_\alpha \to \mathcal{B}$, $\mathcal{B}\to\mathcal{Q}_\beta$, $\mathcal{N}_\alpha\to\mathcal{Q}_\beta$ which is bounded resp. compact.

1. Introduction. Let Δ be the unit disk $\{z\in\mathbb{C} : |z|<1\}$ in the complex plane, and let $\mathcal{H}(\Delta)$ be the space of all analytic functions on Δ. Any analytic map $\phi:\Delta\to\Delta$ gives rise to an operator $C_\phi:\mathcal{H}(\Delta)\to\mathcal{H}(\Delta)$ defined by $C_\phi f = f\circ\phi$, the *composition operator* induced by ϕ.

One of the central problems on composition operators is to know when C_ϕ maps between two subclasses of $\mathcal{H}(\Delta)$ and in fact to relate function-theoretic properties of ϕ to operator-theoretic properties of C_ϕ. This problem is addressed here for the weighted Nevanlinna, the Bloch and the $\mathcal{Q}$ spaces with respect to boundedness and compactness of the operator. The related research has recently been done by various authors (see for example [JX], [MM], [RU], [SZ], [T] and [X2]). The present paper continues their work, but also solves two problems which remained open in [SZ].

For each $\alpha\in(-1,\infty)$, let $\mathcal{N}_\alpha$ be the space of all functions $f\in\mathcal{H}(\Delta)$ satisfying

$$T_\alpha(f) = \frac{1+\alpha}{\pi}\int_\Delta [\log^+|f(z)|](1-|z|^2)^\alpha\,dm(z) < \infty.$$

Here and afterwards, dm means the usual element of the area measure on Δ, and $\log^+ x$ is $\log x$ if $x>1$ and 0 if $0\le x\le 1$.

From $\log^+ x\le\log(1+x)\le 1+\log^+ x$ for $x\ge 0$ we see that a function $f\in\mathcal{H}(\Delta)$ belongs to $\mathcal{N}_\alpha$ if and only if

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