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Orbit equivalence and Kakutani equivalence with Sturmian subshifts

by

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Abstract. Using dimension group tools and Bratteli–Vershik representations of minimal Cantor systems we prove that a minimal Cantor system and a Sturmian subshift are topologically conjugate if and only if they are orbit equivalent and Kakutani equivalent.

1. Preliminaries. In the last decade concepts and techniques coming from the theory of C^*-algebras have been exhaustively used in topological dynamics in order to explain various phenomena appearing mainly in Cantor dynamical systems. In particular, those concepts together with the description of minimal Cantor systems by means of Bratteli–Vershik transformations [HPS], [V1], [V2] gave rise to a complete invariant of orbit and strong orbit equivalence for this class of maps [GPS], [HPS]. In the same vein the authors of [BH] obtained new results about flow equivalence and orbit equivalence for non-minimal Cantor systems. In particular they obtained new conjugacy invariants for subshifts of finite type. The study of substitution systems and Toeplitz systems in this scope was undertaken in [F], [DHS] and [GJ] respectively.

If we consider two (strong) orbit equivalent Cantor systems, their Bratteli–Vershik representation without considering the order is in some sense the same [GPS], [HPS]. Therefore, we can ask which additional property could imply topological conjugacy, in other words how to recover the order. In this direction it is proved in [BT] that with a continuity condition on the cocycles involved in the orbit equivalence we get flip conjugacy. In general, (strong) orbit equivalence is not enough. It is known [O], [Su] that in the same class of orbit equivalence we can have all possible entropies. In the case

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of odometers it is easy to show that orbit equivalence implies topological conjugacy.

Among the different conditions we can consider, Kakutani equivalence appears as a natural one which is intimately related to the order. In that case the systems can be represented by diagrams that are the same up to a finite number of edges and vertices.

In this paper we solve this question when one of the systems is a Sturmian subshift.

THEOREM 1.1. *Let $0 < \overline{\alpha} < 1$ be an irrational number and (X,T) a minimal Cantor system. The Sturmian subshift $(\Omega_{\overline{\alpha}}, \sigma)$ and the system (X,T) are Kakutani and orbit equivalent if and only if they are topologically conjugate.*

The proof is based upon a detailed study of a Bratteli–Vershik representation of Sturmian systems. In that case we only need orbit equivalence and Kakutani equivalence because there are no infinitesimals in their dimension groups. We remark that for two Sturmian systems only orbit equivalence is needed. Also, there is a simple condition for having topological conjugacy of Sturmian subshifts (Ω_α, σ) and (Ω_β, σ): $\alpha = \beta$ or $\alpha = 1-\beta$. Indeed, both systems have the same eigenvalues $\{n\alpha + m \mid n,m \in \mathbb{N}\} = \{n\beta + m \mid n,m \in \mathbb{N}\}$. Since $0 < \alpha, \beta < 1$ we get the result.

In the case of general Cantor minimal systems the answer to our question is not known, even for substitution dynamical systems. From [DHS] we get a detailed representation of these systems by means of Bratteli–Vershik diagrams which does not give us the group of automorphisms of their dimension groups.

The paper is organized in three sections and two appendices. In the present section we give the background for what we will need later. The construction of a particular Bratteli–Vershik representation for Sturmian subshifts is done in Section 2. In Section 3 we prove Theorem 1.1. In the appendices we prove a matrix proposition needed for the proof of the main theorem and we compute the dimension group of a Sturmian subshift.

In what follows, we give some definitions and notation that will be used in the paper.

1.0.1. *Topological dynamical systems and subshifts.* A *topological dynamical system*, or just dynamical system, is a compact Hausdorff space X together with a homeomorphism $T : X \to X$. We use the notation (X,T). If X is a Cantor set we say that the system is *Cantor*. That is, X has a countable basis of closed and open sets and it has no isolated points. A dynamical system is *minimal* if all orbits are dense in X, or equivalently, the only non-empty closed invariant set is X.

A particular class of Cantor systems is the class of *subshifts*. They are defined as follows. Take a finite set or alphabet A. The set $A^{\mathbb{Z}}$ consists of infinite sequences $(x_i)_{i\in\mathbb{Z}}$ with coordinates $x_i \in A$. With the product topology $A^{\mathbb{Z}}$ is a compact Hausdorff Cantor space. We define the *shift transformation* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by $(\sigma(x))_i = x_{i+1}$ for any $x \in A^{\mathbb{Z}}$, $i \in \mathbb{Z}$. The pair $(A^{\mathbb{Z}}, \sigma)$ is called a *full shift*. A *subshift* is a pair (X, σ) where X is any σ-invariant closed subset of $A^{\mathbb{Z}}$. A classical procedure to construct subshifts is by considering the closure of the orbit under the shift of a single sequence $x \in A^{\mathbb{Z}}$, $\Omega(x) = \overline{\{\sigma^i(x) \mid i \in \mathbb{Z}\}}$.

Let $(x_i)_{i\in\mathbb{N}}$ be an element of $A^{\mathbb{N}}$. Another classical procedure is to consider the set $\Omega(x)$ of infinite sequences $(y_i)_{i\in\mathbb{Z}}$ such that for all $i \le j$ there exists $k \ge 0$ such that $y_i y_{i+1} \ldots y_j = x_k x_{k+1} \ldots x_{k+j-i}$. In both cases we say that $(\Omega(x), \sigma)$ is the *subshift generated* by x.

In a minimal subshift any finite sequence of symbols appears with bounded gaps in any sequence of the system.

In this paper we consider two kinds of minimal subshifts: substitution subshifts and Sturmian subshifts. Let us first describe Sturmian subshifts.

Let $0 < \alpha < 1$ be an irrational number. We define the map $R_\alpha : [0,1[\to [0,1[$ by $R_\alpha(t) = t + \alpha \pmod 1$ and the map $I_\alpha : [0,1[\to \{0,1\}$ by $I_\alpha(t) = 0$ if $t \in [0, 1-\alpha[$ and $I_\alpha(t) = 1$ otherwise. Let Ω_α be the set $\overline{\{(I_\alpha(R^n_\alpha(t)))_{n\in\mathbb{Z}} \mid t \in [0,1[\}} \subseteq \{0,1\}^{\mathbb{Z}}$. The subshift (Ω_α, σ) is called a *Sturmian subshift* (generated by α) and its elements are called *Sturmian sequences*. There exists a factor map (see [HM]) $\gamma : (\Omega_\alpha, \sigma) \to ([0,1[, R_\alpha)$ such that

$$|\gamma^{-1}(\{\beta\})| = \begin{cases} 2 & \text{if } \beta \in \{n\alpha \mid n \in \mathbb{Z}\}, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\beta \in [0,1[$. It is well known that $\Omega_\alpha = \Omega_\beta$ if and only if $\alpha = \beta$ and also that (Ω_α, σ) is a non-periodic uniquely ergodic minimal subshift. Sometimes we will write $(\Omega_\alpha, \sigma, \mu)$ instead of (Ω_α, σ) where μ is the unique ergodic measure of (Ω_α, σ). We give later a useful characterization of Sturmian subshifts to obtain Bratteli–Vershik representations of these systems. For more details and properties of Sturmian sequences and subshifts the reader can refer to [BS] and [HM].

A *substitution* is a map $\tau : A \to A^+$, where A^+ is the set of finite sequences with values in A. We associate with τ a $A \times A$ square matrix $M_\tau = (m_{a,b})_{a,b\in A}$ such that $m_{a,b}$ is the number of times the letter b appears in $\tau(a)$. We say that τ is *primitive* if M_τ is primitive, i.e. if some power of M_τ has strictly positive entries only. A substitution τ can be naturally extended by concatenation to A^+, $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$. We say that a subshift of $A^{\mathbb{Z}}$ is *generated by the substitution* τ if it is generated by a fixed point for τ in $A^{\mathbb{N}}$ (see [Q] for more details).

In this paper we are concerned with three notions of equivalence between dynamical systems. Let (X, T) and (Y, S) be dynamical systems. We say that they are *topologically conjugate* if there is a homeomorphism $\phi : X \to Y$ such that $\phi \circ T = S \circ \phi$. We say that they are *orbit equivalent* (OE) if there is a homeomorphism $\phi : X \to Y$ and integer functions $n : X \to \mathbb{Z}$ and $m : X \to \mathbb{Z}$ such that for any $x \in X$, $\phi \circ T^{n(x)}(x) = S \circ \phi(x)$ and $\phi \circ T(x) = S^{m(x)} \circ \phi(x)$. Now assume the systems are minimal; then the maps n, m are uniquely determined. Under this hypothesis we say that the systems are *strong orbit equivalent* (SOE) if the maps n, m have at most one point of discontinuity. Finally, for Cantor systems, we say they are *Kakutani equivalent* (KE) if both have subsets that are closed and open (*clopen*) such that the corresponding induced systems are topologically conjugate.

1.0.2. *Bratteli–Vershik representations.* A *Bratteli diagram* is an infinite graph (V, E) which consists of a vertex set V and an edge set E, both of which are divided into levels $V = V_0 \cup V_1 \cup \ldots$, $E = E_1 \cup E_2 \cup \ldots$ and all levels are pairwise disjoint. The set V_0 is a singleton $\{v_0\}$, and for $k \geq 1$, E_k is the set of edges joining vertices in V_{k-1} to vertices in V_k. It is also required that every vertex in V_k is the "end-point" of some edge in E_k for $k \geq 1$, and an "initial point" of some edge in E_{k+1} for $k \geq 0$. We define the *level* k to be the subgraph consisting of the vertices in $V_k \cup V_{k+1}$ and the edges E_{k+1} between these vertices. Level 0 will be called the *hat* of the Bratteli diagram and it is uniquely determined by an integer vector

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_{|V_1|} \end{pmatrix} \in \mathbb{N}^{|V_1|},$$

where each component represents the number of edges joining v_0 and a vertex of V_1.

We describe the edge set E_k using a $V_k \times V_{k-1}$ incidence matrix whose (i, j)-entry is the number of edges in E_k joining vertex $j \in V_{k-1}$ to vertex $i \in V_k$.

An *ordered* Bratteli diagram $B = (V, E, \preceq)$ is a Bratteli diagram (V, E) together with a partial ordering $\preceq$ on E. Edges e and e' are comparable if and only if they have the same end-point.

Let $k < l$ in $\mathbb{N} \setminus \{0\}$ and let $E_{k,l}$ be the set of all paths in the graph joining vertices of V_{k-1} to vertices of V_l. The partial ordering of E induces another in $E_{k,l}$ given by $(e_k, \ldots, e_l) \prec (f_k, \ldots, f_l)$ if and only if there is $k \leq i \leq l$ such that $e_j = f_j$ for $i < j \leq l$ and $e_i \prec f_i$.

Given a strictly increasing sequence $(m_n)_{n \geq 0}$ of integers with $m_0 = 0$ we define the *contraction* of $B = (V, E, \preceq)$ (with respect to $(m_n)_{n \geq 0}$) as $((V_{m_n})_{n \geq 0}, (E_{m_n+1, m_{n+1}})_{n \geq 0}, \preceq)$, where $\preceq$ is the order induced in each set $E_{m_n+1, m_{n+1}}$ of edges.

We say that an ordered Bratteli diagram is *stationary* if for any $k \geq 1$ the incidence matrix and order are the same (after labeling the vertices appropriately).

Given an ordered Bratteli diagram $B = (V, E, \preceq)$ we define X_B as the set of infinite paths $(e_1, e_2, \ldots)$ starting at v_0 such that for all $i \geq 1$ the end-point of $e_i \in E_i$ is the initial point of $e_{i+1} \in E_{i+1}$. We topologize X_B by postulating a basis of open sets, namely the family of *cylinder sets*

$$U(e_1, \ldots, e_k) = \{(f_1, f_2, \ldots) \in X_B \mid f_i = e_i \text{ for } 1 \leq i \leq k\}.$$

Each $U(e_1, \ldots, e_k)$ is also closed, as is easily seen, and so X_B is a compact, totally disconnected metrizable space.

When there is a unique $x = (x_1, x_2, \ldots) \in X_B$ such that x_i is maximal for any $i \geq 1$ and a unique $y = (y_1, y_2, \ldots) \in X_B$ such that y_i is minimal for any $i \geq 1$, we say that $B = (V, E, \preceq)$ is a *properly ordered* Bratteli diagram. Call these particular points $x_{\max}$ and $x_{\min}$ respectively. In this case we can define a dynamics V_B over X_B called the *Vershik map*. The map V_B is defined as follows: let $(e_1, e_2, \ldots) \in X_B \setminus \{x_{\max}\}$ and let $k \geq 1$ be the smallest integer so that e_k is not a maximal edge. Let f_k be the successor of e_k and $(f_1, \ldots, f_{k-1})$ be the unique minimal path in $E_{1,k-1}$ connecting v_0 with the initial point of f_k. We set $V_B(x) = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, \ldots)$ and $V_B(x_{\max}) = x_{\min}$. The dynamical system (X_B, V_B) is called the *Bratteli–Vershik system* generated by $B = (V, E, \preceq)$. The dynamical system induced by any contraction of B is topologically conjugate to (X_B, V_B). In [HPS] it is proved that any minimal Cantor system (X, T) is topologically conjugate to a Bratteli–Vershik system (X_B, V_B). We say that (X_B, V_B) is a *Bratteli–Vershik representation* of (X, T).

1.0.3. *The notion of a dimension group.* Let (X, T) be a minimal Cantor system. Its *dimension group* is defined as $K^0(X, T) = C(X, \mathbb{Z})/\partial_T C(X, \mathbb{Z})$, where $C(X, \mathbb{Z})$ is the countable additive Abelian group of continuous functions on X with values in $\mathbb{Z}$ and $\partial_T : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ is the coboundary operator $\partial_T(f) = f \circ T - f$. The *positive cone* of $K^0(X, T)$ is the set $K^0(X, T)^+$ of equivalence classes of positive functions. We also distinguish an *order unit* [1] which is the equivalence class of the constant function equal to 1.

Let (V, E) be a Bratteli diagram and $(M_i)_{i \geq 0}$ be the corresponding incidence matrix of levels. Recall that $M_0 = \mathbf{u}$ corresponds to the hat of the Bratteli diagram. We define $K_0(V, E)$ as the inductive limit of the system of ordered groups

$$\mathbb{Z} \xrightarrow{\mathbf{u}} \mathbb{Z}^{|V_1|} \xrightarrow{M_1} \mathbb{Z}^{|V_2|} \xrightarrow{M_2} \ldots,$$

that is, $K_0(V,E) = \varinjlim(M_i, \mathbb{Z}^{|V_i|})$. This group carries a natural order given by a cone $K_0(V,E)^+$. We also distinguish an order unit $\mathbf{1}$ which is the element of $K_0(V,E)^+$ corresponding to $1 \in \mathbb{Z} = \mathbb{Z}^{|V_0|}$. For more details we refer the reader to [GPS].

In [HPS, Th. 5.4, Cor. 6.3] it is proved that if (X,T) is a Cantor minimal system and (X_B, V_B) its Bratteli–Vershik representation, then the ordered groups with distinguished order units $K^0 = (K^0(X,T), K^0(X,T)^+, [1])$ and $K_0 = (K_0(V,E), K_0(V,E)^+, \mathbf{1})$ are isomorphic. In [GPS, Th. 2.1] it is proved that K^0 is a complete SOE invariant. They also proved that the quotient group $K^0/\mathrm{Inf}(K^0)$ is a complete invariant of OE, where $\mathrm{Inf}(K^0)$ is the subgroup of $K^0(X,T)$ consisting of elements $a \in K^0(X,T)$ such that $-\varepsilon[1] \le a \le \varepsilon[1]$ for all $0 < \varepsilon \in \mathbb{Q}$.

In this paper we are particularly concerned with computations of dimension groups that are direct limits of sequences of integer matrices in $\mathrm{GL}(2,\mathbb{Z})$. That is, $K_0(V,E)$ is computed from the sequence

$$\mathbb{Z} \xrightarrow{\mathbf{u}} \mathbb{Z}^2 \xrightarrow{M_1} \mathbb{Z}^2 \xrightarrow{M_2} \ldots,$$

where $M_i \in \mathrm{GL}(2,\mathbb{Z})$ for $i \ge 1$. In this case and under some other conditions (see [ES]), the ordered group $(K_0(V,E), K_0(V,E)^+) = \varinjlim(M_i, \mathbb{Z}^2, \mathbb{Z}^2_+)$ is isomorphic to $(\mathbb{Z}^2, P_\alpha)$ where

$$P_\alpha = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \;\middle|\; x\cdot\alpha + y \ge 0 \right\}$$

for some $\alpha \in \mathbb{R}^+$.

We will say that a matrix $M \in \mathrm{GL}(2,\mathbb{Z})$ is an *automorphism* of $(\mathbb{Z}^2, P_\alpha)$ if $M \cdot P_\alpha = P_\alpha$.

Finally, let us agree on some notation. The 2×2 identity matrix will be denoted by $I_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. If M is a matrix with real entries, the notation $M \ge 0$ (respectively $M \le 0$, $M > 0$, $M < 0$) will mean that all entries of M are ≥ 0 (respectively ≤ 0, > 0, < 0).

2. Bratteli–Vershik representations of Sturmian subshifts. A morphism $f : \{0,1\} \to \{0,1\}^* = \{0,1\}^+ \cup \{\varepsilon\}$, where ε is the empty word, is called *Sturmian* if the image under f of each Sturmian sequence is a Sturmian sequence. In [MS] it is proved that a morphism is Sturmian if and only if it is an element of the free monoid $\mathcal{S}t$ generated by the morphisms E, ϕ and $\widetilde{\phi}$ from $\{0,1\}$ to $\{0,1\}^*$, where

$$E(0) = 1, \quad \phi(0) = 01, \quad \widetilde{\phi}(0) = 10,$$
$$E(1) = 0, \quad \phi(1) = 0, \quad \widetilde{\phi}(1) = 0.$$

In what follows the morphisms ϱ_n and γ_n, $n \in \mathbb{N}\setminus\{0\}$, from $\{0,1\}$ to $\{0,1\}^*$ defined by

$$\varrho_n(0) = 01^{n+1}, \quad \gamma_n(0) = 10^{n+1},$$
$$\varrho_n(1) = 01^{n}, \quad \gamma_n(1) = 10^{n},$$

will play a very important role. For $n \ge 1$, $\gamma_n = (\widetilde{\phi}E)^{n-1}\widetilde{\phi}\phi$ and $\varrho_n = E\gamma_n$, therefore both belong to $\mathcal{S}t$. The following theorem is due to Hedlund and Morse [HM].

THEOREM 2.1. *Let x be a Sturmian sequence.*

(i) *There is $n \ge 1$ such that $x = \ldots v_{-1}v_0v_1\ldots$ where $(v_i)_{i\in\mathbb{Z}}$ is a sequence taking values in $\{01^{n+1}, 01^n\}$ or in $\{10^{n+1}, 10^n\}$.*

(ii) *If $x = \varrho_n(z)$ or $x = \gamma_n(z)$, for some $n \ge 1$ and $z \in \{0,1\}^{\mathbb{Z}}$, then z is Sturmian.*

Proof. Assertion (i) follows from Theorem 7.1 of [HM], and (ii) is Theorem 8.1 of [HM]. ∎

Let (X,σ) be a Sturmian subshift and $a \in \{0,1\}$. We denote by $[a]$ the set $\{(x_i)_{i\in\mathbb{Z}} \in X \mid x_0 = a\}$, which turns out to be a clopen subset of X.

PROPOSITION 2.2. *Let (X,σ) be a Sturmian subshift. There exists a sequence $(\zeta_n)_{n\in\mathbb{N}}$ taking values in $\{\varrho_1,\gamma_1,\varrho_2,\gamma_2,\ldots\}$ such that*

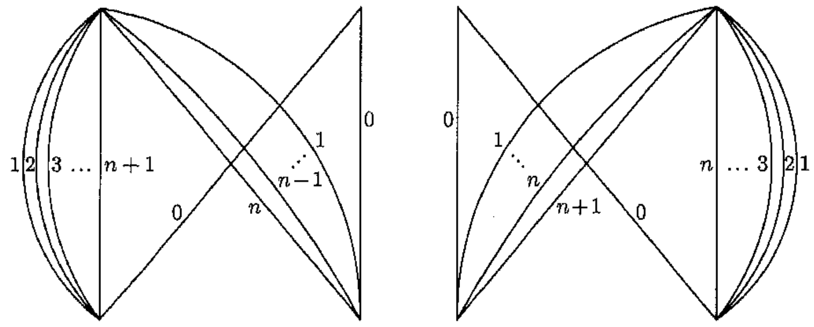
(i) *$y = \lim_{n\to\infty}\zeta_1\ldots\zeta_n(00\ldots)$ generates (X,σ).*

(ii) *The sequence $\{P_n\}_{n\in\mathbb{N}}$ of partitions of X, given by $P_0 = \{[0],[1]\}$ and $P_n = \{\sigma^k\zeta_1\ldots\zeta_n([a]) \mid 0 \le k < |\zeta_1\ldots\zeta_n(a)|,\ a \in \{0,1\}\}$ for $n \ge 1$, has the following properties:*

(a) *$\zeta_1\ldots\zeta_{n+1}([0]) \cup \zeta_1\ldots\zeta_{n+1}([1]) \subseteq \zeta_1\ldots\zeta_n([0]) \cup \zeta_1\ldots\zeta_n([1])$,*
(b) *$P_n \prec P_{n+1}$ as partitions,*
(c) *the set $\bigcap_{n\in\mathbb{N}}(\zeta_1\ldots\zeta_n([0]) \cup \zeta_1\ldots\zeta_n([1]))$ consists of one point only,*
(d) *this sequence of partitions generates the topology of X.*

Proof. Assertion (i) follows from Theorem 2.1, and (ii) comes from the fact (which can be proved by induction) that for all $n \in \mathbb{N}$ and all $x \in X$, x has a unique decomposition into a concatenation of elements of $\{\zeta_1\ldots\zeta_n(a) \mid a \in \{0,1\}\}$. ∎

Let (X,σ) be a Sturmian subshift and $\{P_n\}_{n\in\mathbb{N}}$ be the sequence of partitions given by Proposition 2.2. With such a sequence there is associated an ordered Bratteli–Vershik diagram $B = (V,E,\preceq)$ which can be described as follows: For all $n \in \mathbb{N}\setminus\{0\}$, V_n consists of two vertices and E_{n+1} is given by ζ_n and described in Figure 2.1, the hat is determined by P_0 and it is $\binom{1}{1}$, and this ordered Bratteli–Vershik diagram is isomorphic to (X,σ) (for more details see [HPS] or [DHS]).



(a) Level n for γ_n (b) Level n for ϱ_n

Fig. 2.1

3. Proof of Theorem 1.1. The assertion of Theorem 1.1 will be a consequence of Proposition 3.1 stated below and the construction of the Bratteli–Vershik representation for Sturmian subshifts given in Section 2.

The following standard notation in number theory will be exhaustively used. The simple continued fraction expansion of $\alpha \in \mathbb{R}$ is denoted by $\alpha = [c_0 : c_1, c_2, \ldots]$. If α is quadratic irrational this expansion is *ultimately periodic*, that is, there exist $T \geq 1$ and $p \in \mathbb{N}$ such that $c_i = c_{i+kT}$ for $i \in \{p, \ldots, p+T-1\}$, $k \in \mathbb{N}$. In this last case we will use the notation $\alpha = [c_0 : c_1, \ldots, c_{p-1}, \overline{c_p, \ldots, c_{p+T-1}}]$. For $\alpha = [\overline{c_0 : c_1, \ldots, c_{T-1}}]$ we define the matrix

$$M_\alpha = N_{c_{T-1}} \cdot N_{c_{T-2}} \cdot \ldots \cdot N_{c_0},$$

where $N_n = \left[\begin{smallmatrix} n & 1 \\ 1 & 0 \end{smallmatrix}\right]$ for $n \in \mathbb{N}$.

PROPOSITION 3.1. *Let α be a positive quadratic irrational number with periodic simple continued fraction expansion $\alpha = [\overline{d_0 : d_1, \ldots, d_{T-1}}]$ such that T is a minimal length period and $d_i > 0$ for $i \in \{0, \ldots, T-1\}$. If M is an automorphism of $(\mathbb{Z}^2, P_\alpha)$ such that $M\binom{u}{v} > 0$ for some $\binom{u}{v} > 0$ in $\mathbb{Z}^2$, then there exists $k \in \mathbb{Z}$ such that $M = M_\alpha^k$.*

We will devote Appendix A to the proof of Proposition 3.1.

For the remainder of this section, we consider a fixed Sturmian subshift $(\Omega_{\overline{\alpha}}, \sigma)$. First, we study in detail the Bratteli–Vershik diagram of the Sturmian subshift found in the last section. By Proposition 2.2 there exists a sequence $(\zeta_n)_{n\in\mathbb{N}}$ in $\{\varrho_1, \gamma_1, \varrho_2, \gamma_2, \ldots\}$ such that $y = \lim_{n\to\infty} \zeta_1 \ldots \zeta_n(00\ldots)$ generates $(\Omega_{\overline{\alpha}}, \sigma)$.

For all $n \geq 1$ the matrices associated with γ_n and ϱ_n are respectively

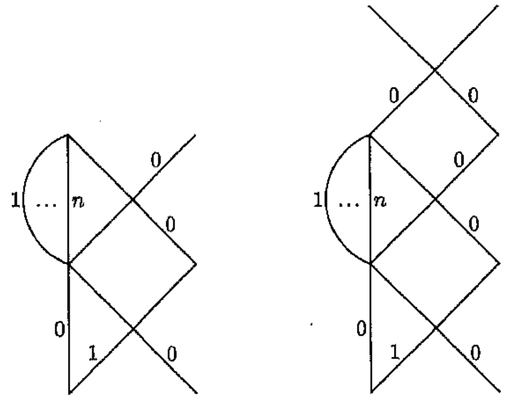
$$M^{(\gamma_n)} = \begin{bmatrix} n+1 & 1 \\ n & 1 \end{bmatrix} \quad \text{and} \quad M^{(\varrho_n)} = \begin{bmatrix} 1 & n+1 \\ 1 & n \end{bmatrix}. \tag{3.1}$$

These matrices can be factored as

$$\begin{aligned} M^{(\gamma_n)} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} n & 1 \\ 1 & 0 \end{bmatrix}, \\ M^{(\varrho_n)} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} n & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \end{aligned} \tag{3.2}$$

that is, $M^{(\gamma_n)} = N_1 \cdot N_n$, $M^{(\varrho_n)} = N_1 \cdot N_n \cdot N_0$.

We order the edges of the factor blocks in (3.2) as shown in Figure 3.1. These orderings are compatible with the ones the original matrices had, in the sense that when we contract we recover the orderings required for γ_n and ϱ_n.



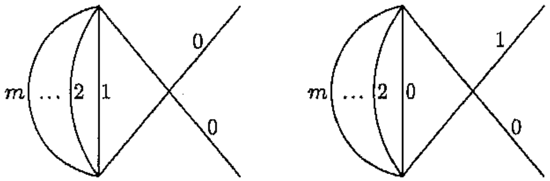
(a) Ordering inducing γ_n (b) Ordering inducing ϱ_n

Fig. 3.1

In view of the discussion above, we conclude that a Bratteli–Vershik representation associated with $(\Omega_{\overline{\alpha}}, \sigma)$ can be obtained as a concatenation of blocks associated with matrices N_n, $n \geq 0$. Since $N_n \cdot N_0 \cdot N_m = N_{n+m}$, we can contract the diagram to obtain a new one in which the matrices associated with the blocks are of the form N_d with $d > 0$. (If the first matrix is N_0, we contract it with the vector $\binom{1}{1}$ associated with the top edges in E_1, or the hat, and the new diagram will also have $\binom{1}{1}$ as its hat and no N_0 matrix any longer.)

Let us analyze the order structure in this ordered Bratteli–Vershik diagram. Notice that a level with incidence matrix N_m can appear with two possible different orderings. We will use the notation $O_m^{(0)}$ to indicate a level with incidence matrix N_m ordered as shown in Figure 3.2(a), and $O_m^{(1)}$ for a level with incidence matrix N_m ordered as shown in Figure 3.2(b). If $m > 1$, the second ordering appears exactly when it comes from a level contraction.

If $m = 1$, $O_1^{(1)}$ is the ordering for N_1 when it represents a bottom block in Figure 3.1, and $O_1^{(0)}$ the block on top of it.



(a) Ordering $O_m^{(0)}$ (b) Ordering $O_m^{(1)}$

Fig. 3.2

Summarizing, with any Sturmian subshift $(\Omega_{\overline{\alpha}}, \sigma)$ we associate a Bratteli–Vershik representation whose incidence matrix and ordering for any level $k \geq 2$ are $N_{d_k}, O_{d_k}^{(i_k)}$, with $d_k > 0, i_k \in \{0,1\}$. We will call this representation *standard*. We state the following corollary.

COROLLARY 3.2. *Let $(\Omega_{\overline{\alpha}}, \sigma)$ be a Sturmian subshift. There are a sequence $(d_k)_{k\geq 1}$ of positive integers and a sequence $(i_k)_{k\geq 1} \in \{0,1\}^{\mathbb{N}}$ such that the Bratteli–Vershik system defined by the ordered Bratteli diagram with level k given by $O_{d_k}^{(i_k)}$ for $k \geq 1$, and hat given by the vector $\binom{1}{1}$, is topologically conjugate to $(\Omega_{\overline{\alpha}}, \sigma)$.*

The following technical lemma will be useful later.

LEMMA 3.3. *Let $((N_{d_k})_{k\geq 0}, (O_{d_k}^{(i_k)})_{k\geq 0})$ and $((N_{d_k})_{k\geq 0}, (O_{d_k}^{(j_k)})_{k\geq 0})$ be two sequences of matrices and orderings coming from standard Bratteli–Vershik representations of Sturmian subshifts.*

- *If $(d_k)_{k\geq 0}$ is not the constant sequence $(111\ldots)$, then $i_k = j_k$ for all k large enough.*
- *If $(d_k)_{k\geq 0}$ is $(111\ldots)$, then either $i_k = j_k$ for all k large enough, or $i_k \neq j_k$ for all k large enough, and in both cases $i_{k+1} \neq i_k$ for k large enough.*

Proof. Suppose there is $k > 0$ such that $i_k \neq j_k$. Without loss of generality we can suppose that $i_k = 0$ and $j_k = 1$.

First we assume that $d_k \neq 1$. Since $i_k = 0$ (this corresponds to a non-contracted level), we have $i_{k-1} = 1$ and the incidence matrix at level $k-1$ is N_1. On the other hand, since $j_k = 1$, level k for the second diagram is contracted, and thus the order associated with the previous level must be $j_{k-1} = 0$. By the same argument, $j_{k-2} = 1$ with incidence matrix N_1, which implies that $i_{k-2} = 0$ and the corresponding incidence matrix is also N_1. This way, we prove inductively that $d_0 = d_1 = \ldots = d_{k-1} = 1$, with $i_k = i_{k-2} = \ldots = 0 = j_{k-1} = j_{k-3} = \ldots$ and $i_{k-1} = i_{k-3} = \ldots = 1 = j_k = j_{k-2} = \ldots$ If $d_k = 1$, the same procedure implies that $d_0 = d_1 = \ldots = d_k = 1$ with similar conditions for the orderings in each level of both diagrams. ■

We will now try to get more information about this Bratteli–Vershik diagram by studying its associated dimension group, and in particular we will be interested in the automorphisms of the group. For that we need the following proposition whose proof is given in Appendix B.

PROPOSITION 3.4. *Let $(\Omega_{\overline{\alpha}}, \sigma)$ be a Sturmian subshift. Then the ordered group $(K^0(\Omega_{\overline{\alpha}}, \sigma), K^0(\Omega_{\overline{\alpha}}, \sigma)^+)$ is isomorphic to $(\mathbb{Z}^2, P_{(1-\overline{\alpha})/\overline{\alpha}})$.*

Let $\{N_{d_k} \mid k \in \mathbb{N}\}$ be the collection of matrices associated with the standard Bratteli–Vershik representation of $(\Omega_{\overline{\alpha}}, \sigma)$. From Proposition 3.4, $(K^0(\Omega_{\overline{\alpha}}, \sigma), K^0(\Omega_{\overline{\alpha}}, \sigma)^+)$ is isomorphic to $(\mathbb{Z}^2, P_{(1-\overline{\alpha})/\overline{\alpha}})$. On the other hand, from the Bratteli–Vershik diagram one can compute this ordered dimension group as $\varinjlim(\mathbb{Z}^2, N_{d_k})$. Set $\alpha = [d_0 : d_1, d_2, \ldots]$. By [ES, Th. 3.2] we have $(\mathbb{Z}^2, P_\alpha) = \varinjlim(\mathbb{Z}^2, N_{d_k})$. We conclude that $(\mathbb{Z}^2, P_{(1-\overline{\alpha})/\overline{\alpha}}) \cong (\mathbb{Z}^2, P_\alpha)$. Again by [ES, Th. 3.2], the simple continued fraction expansion of α is eventually equal to that of $(1-\overline{\alpha})/\overline{\alpha}$. Since the simple continued fraction expansions of $\overline{\alpha}$ and $(1-\overline{\alpha})/\overline{\alpha}$ are eventually equal we have $(K^0(\Omega_{\overline{\alpha}}, \sigma), K^0(\Omega_{\overline{\alpha}}, \sigma)^+) \cong (\mathbb{Z}^2, P_{\overline{\alpha}}) \cong (\mathbb{Z}^2, P_\alpha)$.

The following lemma implies that if $\overline{\alpha}$ is not a quadratic algebraic number, the identity is the only automorphism of the dimension group $(K^0(\Omega_{\overline{\alpha}}, \sigma), K^0(\Omega_{\overline{\alpha}}, \sigma)^+)$. For the sake of completeness we give the proof. Similar results appear in [S].

LEMMA 3.5. *Let $\beta > 0$ and $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, \mathbb{Z}) \setminus \{I_2\}$ be an automorphism of $(\mathbb{Z}^2, P_\beta)$. Then:*

(i) *β is a quadratic algebraic number.*
(ii) *M does not have any column ≤ 0.*
(iii) *If moreover β is irrational, then $b \neq 0, c \neq 0$ and the irreducible polynomial for β in $\mathbb{Q}[X]$ is*

$$X^2 + \frac{d-a}{b}X - \frac{c}{b}.$$

Proof. It is clear that if $M \cdot P_\beta = P_\beta$, then there exists an integer k such that the vector $M \cdot \binom{1}{-\beta} \in \mathbb{R}^2$ is equal to $k\binom{1}{-\beta}$. This implies that $(-a+b\beta) \cdot (-\beta) = (-c+d\beta) \cdot 1$, which is in turn equivalent to $b\beta^2 + (d-a)\beta - c = 0$. This proves (i) and (iii).

To prove (ii), assume that the first column of M is negative. Then M would fail to be an automorphism of P_β, since it would send the vector $\binom{1}{0} \in P_\beta$ into $\binom{a}{c} \notin P_\beta$ (an identical argument works if the second column is negative). ■

Let us study the case of $\overline{\alpha}$ quadratic irrational. We recall that $\alpha = [d_0 : d_1, d_2, \ldots]$ is the number defined above from the standard Bratteli–Vershik representation of $(\Omega_{\overline{\alpha}}, \sigma)$. Thus, the continued fraction expansions of α and $\overline{\alpha}$ are ultimately periodic, and since they are eventually equal, the period in both expansions is the same. Moreover, by Lemma 3.3, the sequence $(O_{d_k}^{(i_k)})_{k\geq 0}$ of orderings is eventually periodic, with the same period. Without loss of generality we suppose that α and the orderings are periodic: $\alpha = [\overline{d_0 : d_1, \ldots, d_{T-1}}]$ (otherwise we multiply up all matrices in the Bratteli–Vershik diagram that appear before they become periodic, and we get a new "hat" and a periodic diagram), and we let T be the length of the least period of the simple continued fraction expansion of α. Recall that $M_\alpha = N_{d_{T-1}} \cdot \ldots \cdot N_{d_1} \cdot N_{d_0}$.

We get

COROLLARY 3.6. *Let $\overline{\alpha} = [c_0 : c_1, \ldots, c_p, \overline{d_0, d_1, \ldots, d_{T-1}}]$ be a quadratic irrational number and $\alpha = [\overline{d_0 : d_1, \ldots, d_{T-1}}]$, where $d_0, d_1, \ldots, d_{T-1} > 0$ and T is the length of a minimal period of the simple continued fraction expansion of $\overline{\alpha}$. Then:*

(i) *If $\alpha \neq [\overline{1}]$, then $(\Omega_{\overline{\alpha}}, \sigma)$ is topologically conjugate to a stationary Bratteli–Vershik system with stationary incidence matrix $M_\alpha = N_{d_{T-1}} \cdot \ldots \cdot N_{d_1} \cdot N_{d_0}$.*

(ii) *If $\alpha = [\overline{1}]$, then $(\Omega_{\overline{\alpha}}, \sigma)$ is topologically conjugate to a stationary Bratteli–Vershik system with stationary incidence matrix $M_\alpha = N_1 \cdot N_1$ and order induced by $O_1^{(0)}$ followed by $O_1^{(1)}$.* ■

Before giving the proof of Theorem 1.1 let us remark that Theorem 1 of [DHS] and the last corollary imply for a quadratic number $\overline{\alpha}$ that $(\Omega_{\overline{\alpha}}, \sigma)$ is a substitutive system. The converse is also true. In fact, if $(\Omega_{\overline{\alpha}}, \sigma)$ is a substitutive system it is clear that there are non-trivial automorphisms [P]; then using Lemma 3.5 we conclude that $\overline{\alpha}$ is quadratic. We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $0 < \overline{\alpha} < 1$ be an irrational number such that $(\Omega_{\overline{\alpha}}, \sigma)$ and (X, T) are Kakutani and orbit equivalent. From the Kakutani equivalence, given a Bratteli–Vershik representation of $(\Omega_{\overline{\alpha}}, \sigma)$, by deleting and adding a finite number of arrows, we get a representation of (X, T). Let $\alpha = [d_0 : d_1, d_2, \ldots]$ be the real number coming from the standard Bratteli–Vershik representation of the Sturmian system. By contracting both diagrams we assume that they are the same, up to the corresponding hats

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

respectively. Then

$$(K^0(\Omega_{\overline{\alpha}}, \sigma), K^0(\Omega_{\overline{\alpha}}, \sigma)^+) \cong (K^0(X, T), K^0(X, T)^+) \cong (\mathbb{Z}^2, P_\alpha).$$

It is easy to see that the unique infinitesimal of the ordered group $(\mathbb{Z}^2, P_\alpha)$ is $\binom{0}{0}$. Consequently, $(\Omega_{\overline{\alpha}}, \sigma)$ and (X, T) are strong orbit equivalent. It follows that there is an automorphism M of $(\mathbb{Z}^2, P_\alpha)$ such that $M \cdot \mathbf{u} = \mathbf{v}$. If α is not a quadratic irrational, then by Lemma 3.5, $M = I_2$ and $\mathbf{u} = \mathbf{v}$, which implies that both representations are the same, and the systems are topologically conjugate.

When α is a quadratic irrational we can assume it is periodic with expansion $\alpha = [\overline{d_0 : d_1, \ldots, d_{T-1}}]$. Then, by Proposition 3.1, there is $k \in \mathbb{Z}$ such that $M = M_\alpha^k$. Thus, $M_\alpha^k \cdot \mathbf{u} = \mathbf{v}$. Without loss of generality we can assume $k \geq 0$.

We consider two cases. First assume $\alpha \neq [\overline{1}]$. Then by Corollary 3.6(i), M_α is the stationary matrix in the Bratteli–Vershik representation of the system. Then by contracting the first k levels of the diagram with unit $\mathbf{u}$ we get the diagram of the system with unit $\mathbf{v}$. This proves they are topologically conjugate.

We now suppose that $\alpha = [\overline{1}]$. There exists an integer k such that $M = N_1^k$. On the other hand, by Corollary 3.6(ii), the stationary matrix of the Bratteli–Vershik representation of the system is $M_\alpha = N_1 \cdot N_1$. We contract the ordered Bratteli diagram starting with $\mathbf{v}$, to get a new ordered Bratteli diagram with hat $\mathbf{w} = M^k \mathbf{v} = \mathbf{u} = \binom{a}{b}$ and with stationary matrix $M_\alpha = N_1 N_1$. In this way we have two stationary ordered Bratteli diagrams, $B_{\mathbf{u}}$ and $B_{\mathbf{w}}$, which can only differ in the orderings of M_α. If the orderings are the same then the proof is finished, hence we can suppose that the orderings are given by Figure 3.3. Let B_1 and B_2 be respectively the

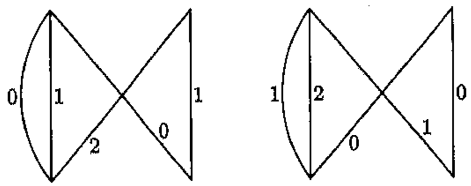


Fig. 3.3. The two possible orders for N_1^2

stationary ordered Bratteli diagrams with the same incidence matrices and orderings as $B_{\mathbf{u}}$ and $B_{\mathbf{w}}$, for levels $k \geq 2$, but with hat $\binom{1}{1}$. In [DHS] it is proved that (X_{B_1}, V_{B_1}) and (X_{B_2}, V_{B_2}) are respectively isomorphic to the subshifts (X_1, σ) and (X_2, σ) generated by the substitutions $\tau_1 : \{0,1\} \to \{0,1\}^*$ and $\tau_2 : \{0,1\} \to \{0,1\}^*$ defined by

$$\tau_1(0) = 001, \quad \tau_2(0) = 100,$$
$$\tau_1(1) = 01, \quad \tau_2(1) = 10.$$

Let $\phi : \{0,1\} \to \{(0,i) \mid 0 \le i \le a-1\}^+ \cup \{(1,i) \mid 0 \le i \le b-1\}^+$ be the map defined by

$$\phi(0) = (0,0)(0,1)\ldots(0,a-1), \quad \phi(1) = (1,0)(1,1)\ldots(1,b-1).$$

Let $x \in X_1$ and $y \in X_2$. We see that the subshift (Y_1,σ) (resp. (Y_2,σ)) generated by $\phi(x)$ (resp. $\phi(y)$) is isomorphic to $(X_{B_{\mathbf{u}}}, V_{B_{\mathbf{u}}})$ (resp. $(X_{B_{\mathbf{w}}}, V_{B_{\mathbf{w}}})$). But we can prove that $X_1 = X_2$ (the proof is left to the reader), hence using the minimality of (X_1,σ) and (X_2,σ) and the fact that $\phi(X_i) \subseteq Y_i$, $i \in \{1,2\}$, it follows that $Y_1 = Y_2$ and that $(X_{B_{\mathbf{u}}}, V_{B_{\mathbf{u}}})$ is isomorphic to $(X_{B_{\mathbf{w}}}, V_{B_{\mathbf{w}}})$. This completes the proof. ■

Appendix A: Proof of Proposition 3.1. The results of this section are closely related to the ones found in [S]. We write down whole proofs here for the sake of completeness.

LEMMA 3.7. *If $M \in \mathrm{GL}(2,\mathbb{Z})\setminus\{I_2\}$ is an automorphism of $(\mathbb{Z}^2, P_\alpha)$ such that $M\binom{u}{v} > 0$ for some $\binom{u}{v} > 0$ in $\mathbb{Z}^2$, then either $M \ge 0$ or $M^{-1} \ge 0$.*

Proof. Since $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is an element of $\mathrm{GL}(2,\mathbb{Z})$, it is invertible, $\det M = \pm 1$ and $M^{-1} = \det M \left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$. Let us make a couple of remarks:

By Lemma 3.5, neither M nor M^{-1} (since M^{-1} is also an automorphism of $(\mathbb{Z}^2, P_\alpha)$) can have a column ≤ 0.

The matrix M cannot have a row ≤ 0: if for instance its first row were non-positive, M would send $\binom{u}{v} > 0$ into a vector with first coordinate would be $au + bv \le 0$, which is not possible by the hypothesis on the automorphism M.

It follows from the above remarks about the impossibility for M to have non-positive columns and rows that M has at most two non-positive entries. If it has two, they must be either a and d, or b and c (and the other two entries must be > 0). But in view of the computation of M^{-1}, in that case all entries of this inverse have the same sign, which must be positive in view of a previous remark. If M has only one strictly negative entry there are only four cases possible:

$$\begin{bmatrix} 0 & 1 \\ 1 & -|d| \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ -|c| & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -|b| \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -|a| & 1 \\ 1 & 0 \end{bmatrix}.$$

The inverse of each of these matrices is non-negative. This completes the proof. ■

Let M be an automorphism of $(\mathbb{Z}^2, P_\alpha)$ satisfying the hypothesis of Lemma 3.7. Then either M or its inverse must be positive. Without loss of generality we will assume that $M \ge 0$. Let $K_0(M) = (\mathbb{Z}^2, K^+(M))$ be the ordered dimension group of M, that is to say,

$$K^+(M) = \{\mathbf{v} \in \mathbb{Z}^2 \mid M^k\mathbf{v} \ge 0 \text{ for some } k \in \mathbb{N}\}.$$

LEMMA 3.8. *Let α be an irrational number. If $M \ge 0$ is an automorphism of $(\mathbb{Z}^2, P_\alpha)$ with $M \ne I_2$, then M is a primitive matrix and $K_0(M) = (\mathbb{Z}^2, P_\alpha)$.*

Proof. Set $K^+ = K^+(M)$. The proof consists in showing that $K^+ = P_\alpha$.

Notice first that it follows easily from Lemma 3.5 that M can have at most one 0 entry, and it would be on the diagonal, which implies that in any case $M^2 > 0$ and M is primitive.

Thus, by the Perron–Frobenius theorem, M has an eigenvalue $\lambda_1 > 1$ with a strictly positive eigenvector $\mathbf{v}_1 > 0$. On the other hand, since $M \cdot P_\alpha = P_\alpha$, $\mathbf{v}_2 = \binom{1}{-\alpha}$ must be an eigenvector of M, and since $|\det M| = 1$, the corresponding eigenvalue $\lambda_2 = (\det M)/\lambda_1$ satisfies $|\lambda_2| < 1$.

Therefore any $\mathbf{v} \in \mathbb{Z}^2$ can be written in a unique way as $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$ with $x_1, x_2 \in \mathbb{R}$. Notice that for such a $\mathbf{v}$, $\mathbf{v} \in P_\alpha \Leftrightarrow x_1 \ge 0$. For any $k \in \mathbb{N}$ we get $M^k\mathbf{v} = \lambda_1^k x_1\mathbf{v}_1 + \lambda_2^k x_2\mathbf{v}_2$, and since $\mathbf{v}_1 > 0$, if $x_2 \ne 0$, $M^k\mathbf{v}$ will eventually become positive if and only if $x_1 > 0$. If $x_2 = 0$, the condition becomes $x_1 \ge 0$, and noticing that the only point $\mathbf{v}$ with integer coordinates on the line of equation $x_1 = 0$ is the origin, we conclude that $\mathbf{v} \in K^+ \Leftrightarrow x_1 \ge 0$. Thus $K^+ = P_\alpha$. ■

Consider now a positive matrix $M \in \mathrm{GL}(2,\mathbb{Z})$. From [ES, Lemma 4.1], $M = N_{c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_0}$ with $c_i \ge 0, i = 0, \ldots, l$. The following lemma tells us exactly what the dimension group for M is (not just a characterization up to isomorphism).

LEMMA 3.9. *Let $M = N_{c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_0} \ge 0$, with $c_i \ge 0, i = 0, \ldots, l$, be a non-negative invertible primitive 2×2 matrix, and let $\beta = [\overline{c_0 : c_1, \ldots, c_l}]$. Then M is an automorphism of $(\mathbb{Z}^2, P_\beta)$ and $K_0(M) = (\mathbb{Z}^2, P_\beta)$.*

Proof. Since $N_d \cdot N_0 \cdot N_{d'} = N_{d+d'}$ for all d, d', we can suppose that in the decomposition $M = N_{c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_0}$ all numbers c_i are strictly positive, except perhaps c_0 and/or c_l. Thus we have four cases, and in each of them we can compute the simple continued fraction expansion for the associated irrational number β:

(i) $c_0 \ge 1$, $c_l \ge 1$, $M = N_{c_l} \cdot \ldots \cdot N_{c_1} \cdot N_{c_0}$ and
$$\beta = [\overline{c_0 : c_1, \ldots, c_l}], \quad l \ge 0.$$

(ii) $c_0 = 0$, $c_l \ge 1$, $M = N_{c_l} \cdot \ldots \cdot N_{c_1} \cdot N_0$ and
$$\beta = [0 : c_1, \overline{c_2, \ldots, c_{l-1}, c_1 + c_l}], \quad l \ge 2.$$

(iii) $c_0 \ge 1,\ c_l = 0,\ M = N_0 \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_1} \cdot N_{c_0}$ and
$$\beta = [c_0 : \overline{c_1, \ldots, c_{l-2}, c_0 + c_{l-1}}], \qquad l \ge 2.$$
(iv) $c_0 = c_l = 0,\ M = N_0 \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_1} \cdot N_0$ and
$$\beta = [0 : \overline{c_1, \ldots, c_{l-1}}], \qquad l \ge 2.$$

Notice that in cases (ii), (iii) and (iv), M fails to be primitive if $l = 1$.

Let us prove the lemma in Case (i); the other three cases will follow from it later.

For $r_0, r_1, \ldots, r_m > 0$, the notation $r = [r_0 : r_1, \ldots, r_m]$ will stand for the positive number
$$r = r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cfrac{}{\ddots \, r_{m-1} + \cfrac{1}{r_m}}}}.$$
As in the theory of continued fractions presented in [HW], recursive matrix equations can be written for computing r. Namely, if we write
$$F_0 = N_{r_0}, \qquad F_k = \begin{bmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{bmatrix},$$
and set the recursion $F_k = N_{r_k} \cdot F_{k-1}$ for $1 \le k \le m$, we easily get $F_m = N_{r_m} \cdot N_{r_{m-1}} \cdot \ldots \cdot N_{r_0}$, and $r = p_m/q_m$. Now, since $\beta = [\overline{c_0 : c_1, \ldots, c_l}]$, we have
$$\beta = c_0 + \cfrac{1}{c_1 + \cfrac{}{\ddots \, c_l + \cfrac{1}{\beta}}}.$$
In other words, $\beta = [c_0 : c_1, \ldots, c_l, \beta]$.

Therefore β will be of the form $\beta = x/y$, with the matrix
$$F_{l+1} = \begin{bmatrix} x & y \\ p_l & q_l \end{bmatrix}$$
satisfying
$$\begin{bmatrix} x & y \\ p_l & q_l \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} p_l & q_l \\ p_{l-1} & q_{l-1} \end{bmatrix} \tag{3.3}$$
and $F_l = \left[\begin{smallmatrix} p_l & q_l \\ p_{l-1} & q_{l-1} \end{smallmatrix}\right]$ coming from the finite (rational) continued fraction $r = [c_0 : c_1, \ldots, c_l] = p_l/q_l$, which, incidentally, implies that
$$F_l = N_{c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_0} = M.$$

From the matrix equation (3.3),
$$\beta = \frac{x}{y} = \frac{p_l\beta + p_{l-1}}{q_l\beta + q_{l-1}},$$
that is,
$$\beta^2 + \frac{q_{l-1} - p_l}{q_l}\beta - \frac{p_{l-1}}{q_l} = 0. \tag{3.4}$$
Notice that (3.4) defines the irreducible polynomial for β in $\mathbb{Q}[X]$, and since the constant term is strictly negative, β is the only positive root of this equation.

Using the fact that $M \ge 0$ is primitive and following a similar argument to the one in the proof of Lemma 3.8, we show that M has an associated ordered dimension group $K_0(M) = (\mathbb{Z}^2, P_\mu)$, $\mu \in \mathbb{R}^+ \setminus \mathbb{Q}$. Moreover P_μ can be computed as $P_\mu = \{\mathbf{v} \in \mathbb{Z}^2 \mid M^k\mathbf{v} \ge 0 \text{ for some } k \in \mathbb{N}\}$, and therefore $M \cdot P_\mu = P_\mu$ (that is, M is an automorphism of $(\mathbb{Z}^2, P_\mu)$). Since
$$M = F_l = \begin{bmatrix} p_l & q_l \\ p_{l-1} & q_{l-1} \end{bmatrix},$$
it follows from Lemma 3.5 that μ satisfies
$$\mu^2 + \frac{q_{l-1} - p_l}{q_l}\mu - \frac{p_{l-1}}{q_l} = 0.$$
Since $\mu > 0$ is a root for (3.4), we get $\mu = \beta$, and Case (i) is proved.

Let us deduce Case (ii) from what we just proved. Since $\beta = [0 : c_1, \overline{c_2, \ldots, c_{l-1}, c_1 + c_l}]$, we can write
$$\beta = 0 + \frac{1}{c_1 + \frac{1}{\mu}} = \frac{\mu}{c_1\mu + 1} \quad \text{with} \quad \mu = [\overline{c_2 : \ldots, c_{l-1}, c_1 + c_l}].$$
Let
$$M_\mu = N_{c_1+c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_2} = N_{c_1} \cdot N_0 \cdot N_{c_l} \cdot N_{c_{l-1}} \cdot \ldots \cdot N_{c_2}$$
be the matrix having μ as its associated irrational number. Then
$$M = N_{c_l} \cdot \ldots \cdot N_{c_1} \cdot N_0 = (N_{c_1} \cdot N_0)^{-1} \cdot M_\mu \cdot (N_{c_1} \cdot N_0).$$
We know from Case (i) that $M_\mu \cdot \left(\begin{smallmatrix} 1 \\ -\mu \end{smallmatrix}\right)$ is parallel to $\left(\begin{smallmatrix} 1 \\ -\mu \end{smallmatrix}\right)$. Therefore
$$M \cdot (N_{c_1} \cdot N_0)^{-1} \cdot \begin{pmatrix} 1 \\ -\mu \end{pmatrix} = (N_{c_1} \cdot N_0)^{-1} \cdot M_\mu \cdot \begin{pmatrix} 1 \\ -\mu \end{pmatrix}$$
is parallel to
$$(N_{c_1} \cdot N_0)^{-1} \begin{pmatrix} 1 \\ -\mu \end{pmatrix} = (c_1\mu + 1) \begin{pmatrix} 1 \\ -\frac{\mu}{c_1\mu + 1} \end{pmatrix},$$
which means that $M \cdot \left(\begin{smallmatrix} 1 \\ -\beta \end{smallmatrix}\right)$ is parallel to $\left(\begin{smallmatrix} 1 \\ -\beta \end{smallmatrix}\right)$. Since M is positive, Case (ii) is established. The remaining two cases are proved in a similar fashion. ∎

We are now ready to prove Proposition 3.1. If $M \neq I_2$ is a positive automorphism of $(\mathbb{Z}^2, P_\alpha)$, then by Lemma 3.8, M is primitive and $M \cdot P_\alpha = P_\alpha$. It follows from Lemma 3.9 that $M \cdot P_\beta = P_\beta$, with β the irrational number associated with M. But then it is clear that $\beta = \alpha$. Recall that $\alpha = [\overline{d_0 : d_1, \ldots, d_{T-1}}]$ and $M_\alpha = N_{d_{T-1}} \cdot \ldots \cdot N_{d_1} \cdot N_{d_0}$ with $d_i > 0$ for all $i = 0, \ldots, T-1$, and T is the minimal length of a period in the simple continued fraction expansion of α. Thus the simple continued fraction expansion for β is periodic and must be of the form of Case (i) in the proof of Lemma 3.9, that is to say, $\beta = [\overline{c_0 : c_1, \ldots, c_l}]$ and $M = N_{c_l} \cdot \ldots \cdot N_{c_1} \cdot N_{c_0}$ with all $c_i > 0$. Finally, from the minimality of the period of α, $l+1 = k \cdot T$ for some $k \in \mathbb{N}$, and $M = M_\alpha^k$.

Appendix B: Proof of Proposition 3.4. We will make use of the following lemma whose proof can be found in [H].

LEMMA 3.10. *Let (X,T) be a minimal Cantor system and $f \in C(X, \mathbb{Z})$.*

(i) *There exists $g \in C(X,\mathbb{Z})$ such that $f + g \circ T - g \geq 0$ if and only if for every $x \in X$ the sequence $(f(T^n x) + \ldots + f(Tx) + f(x))_{n\in\mathbb{N}}$ is bounded from below.*

(ii) *f is a coboundary if and only if the sequence $(\sum_{i=0}^{n} f(T^i(x)))_{n\in\mathbb{N}}$ is bounded for all $x \in X$.*

Let us also state a technical lemma:

LEMMA 3.11. *Let $(\Omega_\alpha, \sigma, \mu)$ be a Sturmian subshift.*

(i) [K] *For all clopen subsets U of Ω_α and all elements $x \in \Omega_\alpha$, the sequence $(\sum_{i=0}^{n-1}(\mathbf{1}_U(\sigma^i(x)) - \mu(U)))_{n\in\mathbb{N}}$ is bounded.*

(ii) [HM] *$\{\mu(U) \mid U$ is a clopen set in $\Omega_\alpha\} \subseteq \{m\alpha + n \mid m, n \in \mathbb{Z}\}$.*

(For example we have $\mu([0]) = 1 - \alpha$ and $\mu([1]) = \alpha$.)

We start the proof of Proposition 3.4 with some notations. For all $n \in \mathbb{N}$ let

$$L_n = \{x_i \ldots x_{i+n-1} \mid i \in \mathbb{Z},\ (x_m)_{m\in\mathbb{Z}} \in \Omega_\alpha\}$$

(this set is usually called the *language* of Ω_α) and $Q_n = \{U(u) \mid u \in L_{2n+1}\}$ where for all $u = u_0 \ldots u_{2n} \in L_{2n+1}$,

$$\begin{aligned} U(u) &= [u_0 \ldots u_{n-1}.u_n u_{n+1} \ldots u_{2n}] \\ &= \{(y_m)_{m\in\mathbb{Z}} \in \Omega_\alpha \mid y_{i-n} = u_i,\ 0 \leq i \leq 2n\}. \end{aligned}$$

It is classical that this last set is clopen, that Q_n is a partition and that $\bigcup_{n\in\mathbb{N}} Q_n$ is a basis for the topology of Ω_α.

Let $f \in C(\Omega_\alpha, \mathbb{Z})$. There exists an integer n such that f is constant on each set of Q_n. Hence there exists $\{f_u \mid u \in L_{2n+1}\} \subseteq \mathbb{Z}$ such that

$$f = \sum_{u\in L_{2n+1}} f_u \mathbf{1}_{U(u)}.$$

From Lemma 3.11 there exist $p, q \in \mathbb{Z}$ such that $\sum_{u\in L_{2n+1}} f_u \mu(\{U(u)\}) = p\alpha + q$. Hence there exist two integers g_0 and g_1 (uniquely determined) such that

$$p\alpha + q = g_0(1-\alpha) + g_1\alpha = g_0\mu([0]) + g_1\mu([1]).$$

We remark that g_0 and g_1 do not depend on Q_n in the sense that if f is constant on each clopen set of Q_m, for some $m \in \mathbb{N}$, then $\sum_{u\in L_{2m+1}} f_u \mu(\{U(u)\}) = p\alpha + q$.

We define $g \in C(\Omega_\alpha, \mathbb{Z})$ by

$$g(x) = g_0 \mathbf{1}_{[0]}(x) + g_1 \mathbf{1}_{[1]}(x) \quad \text{for all } x \in \Omega_\alpha.$$

We now show that $(\sum_{i=0}^{N-1}(f-g)(\sigma^i(x)))_{N\in\mathbb{N}}$ is bounded for all $x \in \Omega_\alpha$. Let $x \in \Omega_\alpha$. Then

$$\begin{aligned} \sum_{i=0}^{N-1}(f-g)(\sigma^i(x)) = {} & \sum_{i=0}^{N-1}\Big(\sum_{u\in L_{2n+1}} f_u \mathbf{1}_{U(u)}(\sigma^i(x)) - g_0\mu([0]) - g_1\mu([1])\Big) \\ & + \sum_{i=0}^{N-1}(g_0\mu([0]) + g_1\mu([1]) - g(\sigma^i(x))). \end{aligned}$$

Using Lemma 3.11 we clearly see that the second sum is bounded independently of N, and using the definition of g_0 and g_1 together with the same lemma it is not difficult to see that so is the first sum. It follows from Lemma 3.10 that $f - g$ is a coboundary.

We set $\psi(f) = (g_0, g_1)$. It is not difficult to see that this defines a group homomorphism $\psi : C(\Omega_\alpha, \mathbb{Z}) \to \mathbb{Z}^2$.

If $\psi(f) = 0$ then using 3.11 one can prove that f is bounded and hence $\operatorname{Ker}\psi = \partial_\sigma C(\Omega_\alpha, \mathbb{Z})$, consequently $K^0(\Omega_\alpha, \sigma)$ is isomorphic to $\mathbb{Z}^2$. Moreover if f is positive then we obtain $g_0(1-\alpha) + g_1\alpha \geq 0$, that is to say, $(g_0, g_1) \in P_{(1-\alpha)/\alpha}$. And conversely, if (a, b) belongs to $P_{(1-\alpha)/\alpha}$ then Lemmas 3.10 and 3.11 show that the function $h = a\mathbf{1}_{[0]} + b\mathbf{1}_{[1]}$ is cohomologous to a positive function. Finally $(K^0(\Omega_\alpha, \sigma), K^0(\Omega_\alpha, \sigma)^+)$ is isomorphic to $(\mathbb{Z}^2, P_{(1-\alpha)/\alpha})$ as an ordered group.

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