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Representations of the spaces $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$

by

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Abstract. We give a representation of the spaces $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$ as spaces of vector-valued sequences and use it to investigate their topological properties and isomorphic classification. In particular, it is proved that $C^\infty(\mathbb{R}^N) \cap H^{k,2}(\mathbb{R}^N)$ is isomorphic to the sequence space $s^{\mathbb{N}} \cap \ell^2(\ell^2)$, thereby showing that the isomorphy class does not depend on the dimension N if $p = 2$.

1. Introduction. The present paper has its motivation in the articles [B, BT, MT, T2] and continues the study undertaken in [AMM1, AMM2]. In the latter papers it was proved that the Fréchet–Sobolev space $C^m(\Omega) \cap H^{k,p}(\Omega)$, with Ω an open subset of $\mathbb{R}^N$, $1 \le p \le \infty$ and $m, k \in \mathbb{N}_0$ (for Ω a proper subset of $\mathbb{R}^N$, k depends on N), $k \le m$, has a representation as a space of vector-valued sequences. Isomorphic classifications, topological properties and existence of bases were derived quite easily from such a representation, improving some results in [BT, MT]. The natural question arises, then, if a similar representation is valid also for the spaces $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$, by which a satisfactory study of their structure could be done in a rather direct way.

In this direction, we give here a representation for the Fréchet–Sobolev spaces $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$ (§3) in the same spirit as in [AMM2], thereby covering the case $m = \infty$. We then derive some consequences in §3, 4, including the existence of bases for $1 \le p < \infty$. In particular, in §3 we show that the space $C^\infty(\mathbb{R}^N) \cap H^{k,2}(\mathbb{R}^N)$ is isomorphic to the sequence space $s^{\mathbb{N}} \cap \ell^2(\ell^2)$. This result reveals the surprising fact that the isomorphy class does not depend on the dimension N if $p = 2$. In §4 we study the topological properties of these spaces and we characterize their Montel subspaces.

2. Notation. We assume that $1 \le p \le \infty$, $k \in \mathbb{N}_0$ and $N \in \mathbb{N}$. We also denote by I the closed interval [0,1] and by $Q_N = I^N$ the N-dimensional, closed, unit cube.

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If $q_{l,N} = \binom{N}{l}$ for $l = 0,\dots,N$, we let $\{I_{j,l}^N : j = 1,\dots,q_{l,N}\}$ be the collection of all subsets of $\{1,\dots,N\}$ containing exactly l elements such that $I_{j,l}^N = I_{j,l}^{N-1}$ for $j = 1,\dots,q_{l,N-1}$ and $l = 1,\dots,N-1$. Now for every $l = 0,\dots,N$ and $j = 1,\dots,q_{l,N}$ we denote by $C_{j,l}^\infty(Q_N)$ (resp. $C_{j,l}^k(Q_N)$) the closed subspace of the usual nuclear Fréchet space $C^\infty(Q_N)$ (resp. of the usual Banach space $C^k(Q_N)$) of all functions f such that, for every multi-index $\alpha \in \mathbb{N}_0^N$ (resp. with $|\alpha| = \alpha_1 + \dots + \alpha_k \le k$),

$$D^\alpha f(x_1,\dots,x_N) = 0 \quad \text{if } x_i \in \{0,1\} \text{ for some } i \in I_{j,l}^N.$$

Note that, if $l = 0$, then $q_{0,N} = j = 1$ and $I_{1,0}^N = \emptyset$, so that $C_{1,0}^\infty(Q_N) = C^\infty(Q_N)$ ($C_{1,0}^k(Q_N) = C^k(Q_N)$). Next, if $1 \le p < \infty$ we denote, for $l = 0,\dots,N$ and $j = 1,\dots,q_{l,N}$, by $H_{j,l}^{k,p}(Q_N)$ the completion of $C_{j,l}^k(Q_N)$ with respect to the Sobolev norm $f \mapsto \|f\|_{k,p,S} = (\sum_{|\alpha|\le k} \int_S |D^\alpha f(x)|^p\,dx)^{1/p}$, where $S = Q_N$, while for $p = \infty$ we set $H_{j,l}^{k,\infty}(Q_N) = H^{k,\infty}(Q_N)$.

We recall that $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$, $k \in \mathbb{N}_0$, $1 \le p \le \infty$, is a Fréchet space with its natural intersection topology given by the sequence of norms

$$p_r(\cdot) = \|\cdot\|_{r,\infty,J_r} + \|\cdot\|_{k,p,\mathbb{R}^N}, \tag{2.1}$$

where $\|\cdot\|_{k,p,\mathbb{R}^N}$ is the Sobolev norm defined as above with $S = \mathbb{R}^N$ if $p < \infty$ and by

$$\|f\|_{k,\infty,\mathbb{R}^N} = \sup_{|\alpha|\le k} \sup_{\mathbb{R}^N} |D^\alpha f(x)| \quad \text{if } p = \infty,$$

and

$$\|f\|_{r,\infty,J_r} = \sup_{|\alpha|\le r} \sup_{x\in J_r} |D^\alpha f(x)|$$

(for any subset S of $\mathbb{R}^N$, we use $\|f\|_{r,\infty,S} = \sup_{|\alpha|\le r} \sup_{x\in S} |D^\alpha f(x)|$). Here $(J_r)_r$ is a sequence of compact subsets of $\mathbb{R}^N$ such that $J_r = \overline{\mathring{J}_r} \subset \mathring{J}_{r+1}$ and $\bigcup_r J_r = \mathbb{R}^N$.

We also recall that, if $(X, \|\cdot\|)$ is a Banach space, the space $\ell^p(X)$ is defined as follows (with the usual change for $p = \infty$):

$$\ell^p(X) = \Big\{(x_n)_n \in X^{\mathbb{N}} : \|(x_n)_n\|_p = \Big(\sum_n \|x_n\|^p\Big)^{1/p} < \infty\Big\},$$

which is Banach with the obvious norm.

If Y is a Fréchet space with a fundamental system of continuous seminorms $\|\cdot\|_1 \le \|\cdot\|_2 \le \dots$ on Y, and $f : Y \to X$ is a linear continuous map, we define the Fréchet space $Y^{\mathbb{N}} \cap \ell^p(X)$, $1 \le p \le \infty$, as the space

$$Y^{\mathbb{N}} \cap \ell^p(X) = \{(y_n)_n \in Y^{\mathbb{N}} : (f(y_n))_n \in \ell^p(X)\}$$

endowed with the intersection topology defined by the sequence of seminorms

$$q_r((y_n)_n) = \sup_{n\le r} \|y_n\|_r + \Big(\sum_{n=1}^\infty \|f(y_n)\|^p\Big)^{1/p} \quad \text{if } 1 \le p < \infty, \tag{2.2}$$

or

$$q_r((y_n)_n) = \sup_{n\le r} \|y_n\|_r + \sup_{\mathbb{N}} \|f(y_n)\| \quad \text{if } p = \infty. \tag{2.2$'$}$$

Clearly, the Fréchet space $Y^{\mathbb{N}} \cap \ell^p(X)$ is canonically topologically isomorphic to the projective limit of the projective sequence $E_r = \prod_{n<r} Y \oplus \ell^p((X)_{n\ge r})$ (with the corresponding product topology) and linking maps $E_{r+1} \to E_r$, $(y_n)_n \mapsto ((y_n)_{n<r}, f(y_r), (y_n)_{n>r})$. Moreover, the Fréchet spaces $Y^{\mathbb{N}} \cap \ell^p(X)$ are isomorphic to their squares, an isomorphism map being given by $(y_n)_n \mapsto ((y_{2n})_n, (y_{2n-1})_n)$.

Such spaces are a particular case of a general construction introduced in [M] and studied systematically in [BD] (see also [AM] and [DK]).

We find it convenient to introduce the spaces

$$C_N^m = \prod_{l=0}^N \prod_{j=1}^{q_{l,N}} [C_{j,l}^m(Q_N)]^{\mathbb{N}}, \qquad H_N^{k,p} = \prod_{l=0}^N \prod_{j=1}^{q_{l,N}} \ell^p[H_{j,l}^{k,p}(Q_N)]$$

(each endowed with the corresponding product topology) where $m \in \mathbb{N}_0$ or $m = \infty$, $k \in \mathbb{N}_0$ and $1 \le p \le \infty$.

We write $E \simeq F$ and $E < F$ when the space E is topologically isomorphic to F or to a topologically complemented subspace of F respectively.

Finally, we recall that a Fréchet space E is *distinguished* if its strong dual E'_β is barrelled. A Fréchet space E satisfies the *density condition* of Heinrich if every bounded subset of the strong dual E'_β is metrizable (see [BB] for more details).

3. Representation theorems and isomorphic classifications. The aim of this section is to represent the spaces $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$ as spaces of vector-valued sequences and then to study their isomorphic classification.

The first result, which is the proper extension of Theorem 1 of [AMM2] to the case $m = \infty$, is:

THEOREM 3.1. *Let $k \in \mathbb{N}_0$ and $1 \le p \le \infty$. Then*

$$C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N) \simeq (C^\infty(Q_N))^{\mathbb{N}} \cap \ell^p(H^{k,p}(Q_N)).$$

Its proof rests on the following result, which is interesting in itself:

THEOREM 3.2. *There exists an isomorphism T_N from $C^\infty(\mathbb{R}^N)$ onto C_N^∞ which extends to an isomorphism from $C^k(\mathbb{R}^N)$ onto C_N^k and to an isomorphism from $H^{k,p}(\mathbb{R}^N)$ onto $H_N^{k,p}$ for all $k \in \mathbb{N}_0$ and $1 \le p \le \infty$.*

Proof. It is by induction and is similar to the one of Theorem 3 of [AMM2], with the exception of the construction of a total and simultaneous extension operator on $\mathbb{R}^N$. Here we give such a construction.

Suppose that the assertion is true for $N-1$ and put, for any $k \in \mathbb{N}_0$ or $k=\infty$,

$$F_1^k = \prod_{n\in\mathbb{Z}} C^k(\mathbb{R}^{N-1}\times[2n,2n+1]),$$

$$C_0^k(\mathbb{R}^{N-1}\times[2n-1,2n]) = \{f \in C^k(\mathbb{R}^{N-1}\times[2n-1,2n]) : D^\alpha f(x) = 0$$
$$\text{for all } |\alpha|\le k \text{ (for all } \alpha\in\mathbb{N}_0^N \text{ if } k=\infty) \text{ if } x_N \in \{2n-1,2n\}\},$$

and

$$F_2^k = \prod_{n\in\mathbb{Z}} C_0^k(\mathbb{R}^{N-1}\times[2n-1,2n]).$$

The restriction map $R: C^k(\mathbb{R}^N) \to F_1^k$ is linear, continuous and $\ker R = F_2^k$ for every $k\in\mathbb{N}_0$ or $k=\infty$. Also, R has a continuous right-inverse L for every $k\in\mathbb{N}_0$ or $k=\infty$, which is defined as follows.

Let $(a_h)_{h\in\mathbb{N}_0}$ be a sequence of real numbers such that, for every $j\in\mathbb{N}_0$,

$$\sum_{h=0}^\infty 2^{jh}a_h = (-1)^j \quad\text{and}\quad \sum_{h=0}^\infty 2^{jh}|a_h| < \infty$$

(see [S, Lemma] or [A, Lemma 4.27]).

Let φ be a C^∞-function on $[0,\infty[$ satisfying $\varphi(t)=1$ if $0\le t\le 1/2$, $\varphi(t)=0$ if $t\ge 1$.

Now, following Seeley [S, Theorem] (cf. [A, Theorem 4.28]), if $f = (f_n)_n \in F_1^k$ we define

$$(3.1)\quad L_n^+ f(x,t) = \begin{cases} \sum_{h=0}^\infty a_h\varphi(2^h(t-2n+1))f_{n-1}(x,2n-1-2^h(t-2n+1)) \\ \qquad\qquad \text{if } (x,t)\in\mathbb{R}^{N-1}\times[2n-1,2n], \\ f_{n-1}(x,t) \qquad \text{if } (x,t)\in\mathbb{R}^{N-1}\times[2n-2,2n-1], \end{cases}$$

$$(3.2)\quad L_n^- f(x,t) = \begin{cases} \sum_{h=0}^\infty a_h\varphi(2^h(2n-t))f_n(x,2n+2^h(2n-t)) \\ \qquad\qquad \text{if } (x,t)\in\mathbb{R}^{N-1}\times[2n-1,2n], \\ f_n(x,t) \qquad \text{if } (x,t)\in\mathbb{R}^{N-1}\times[2n,2n+1], \end{cases}$$

for every $n\in\mathbb{Z}$.

By [S, Theorem] we see that, for every $k\in\mathbb{N}_0$ or $k=\infty$, the functions $L_n^+ f$ and $L_n^- f$ are C^k-extensions onto $\mathbb{R}^{N-1}\times[2n-2,2n]$ and $\mathbb{R}^{N-1}\times[2n-1,2n+1]$ of f_{n-1} and f_n respectively, the extensions being linear. Moreover, by a direct computation, we obtain

$$(3.3)\quad \begin{aligned} \|L_n^+ f\|_{k,p,\mathbb{R}^{N-1}\times[2n-1,2n]} &\le d_k\|f\|_{k,p,\mathbb{R}^{N-1}\times[2(n-1),2n-1]}, \\ \|L_n^- f\|_{k,p,\mathbb{R}^{N-1}\times[2n-1,2n]} &\le d_k\|f\|_{k,p,\mathbb{R}^{N-1}\times[2n,2n+1]}, \end{aligned}$$

where d_k depends only on k, N, p and φ.

Next, let g be a non-negative C^∞-function on $\mathbb{R}$ satisfying $\operatorname{supp} g\subset[0,1]$ and $\int_{\mathbb{R}} g(t)\,dt = \int_0^1 g(t)\,dt = 1$ and consider $g_n(t) = \int_{-\infty}^{t-(2n-1)} g(s)\,ds$, $t\in\mathbb{R}$, $n\in\mathbb{Z}$.

If $f=(f_n)_n\in F_1^k$ we define

$$(3.4)\quad Lf(x,t) = \begin{cases} f_n(x,t) & \text{if } (x,t)\in\mathbb{R}^{N-1}\times[2n,2n+1], \\ L_n^+f(x,t)+g_n(t)[L_n^-f(x,t)-L_n^+f(x,t)] & \\ & \text{if } (x,t)\in\mathbb{R}^{N-1}\times[2n-1,2n]. \end{cases}$$

It is easy to verify that, for every $k\in\mathbb{N}_0$ or $k=\infty$, Lf is a C^k-extension on $\mathbb{R}^N$ of f, the extension being linear. Moreover, by (3.3), it follows that

$$\|Lf\|_{k,p,\mathbb{R}^N} \le c_k\|f\|_{k,p,\bigcup_{n\in\mathbb{Z}}\mathbb{R}^{N-1}\times[2n,2n+1]},$$

where c_k depends only on k, N, p and φ, g.

At this point, it suffices to proceed exactly as in [AMM2, Theorem 3] to complete the proof.

Proof of Theorem 3.1. This follows from Theorem 3.2, by repeating the proof of Theorem 1 of [AMM2].

REMARK 3.3. Theorem 3.1 yields that the spaces $C^\infty(\mathbb{R}^N)\cap H^{k,p}(\mathbb{R}^N)$ are isomorphic to their squares, being isomorphic to vector-valued sequence spaces of the type introduced in Section 2.

Moreover, by Theorem 3.1 we have

THEOREM 3.4. *Let $k\in\mathbb{N}_0$ and $1\le p<\infty$. The space $C^\infty(\mathbb{R}^N)\cap H^{k,p}(\mathbb{R}^N)$ has a basis.*

Proof. By Theorem 3.1 it suffices to prove that the space $[C^\infty(Q_N)]^{\mathbb{N}}\cap \ell^p[H^{k,p}(Q_N)]$ has a basis. By Lemma 4 of [AMM1] this is done by finding a common basis for $C^\infty(Q_N)$ and $H^{k,p}(Q_N)$.

Let $N=1$. For $1<p<\infty$ the trigonometric system $(e_n)_n = (1,\cos 2\pi x, \sin 2\pi x, \cos 4\pi x, \sin 4\pi x,\ldots)$ (in the indicated order) is a common basis for $C^\infty(I)$ and $H^{k,p}(I)$. For $p=1$ the system of Chebyshev polynomials $(e_n)_n = (1, x, \cos(2\arccos x),\ldots,\cos[(n-1)\arccos x],\ldots)$ is a basis for $C^\infty(I)$ and for $H^{k,1}(I)$.

Let $N>1$. For each $q=(q_1,\ldots,q_N)\in\mathbb{N}^N$ and $x=(x_1,\ldots,x_N)\in Q_N$ put $\widetilde{e}_q(x) = \prod_{j=1}^N e_{q_j}(x_j)$. Then the elements $(\widetilde{e}_q)_q$ form a basis of $C^\infty(Q_N)$ and of $H^{k,p}(Q_N)$, as is easy to verify.

THEOREM 3.5. *Let $\mathcal{F}$ be the family of all subsets of $\mathbb{R}$ which are unions of subsets of the following types:* (a) *an unbounded interval;* (b) *the union of* *a sequence of disjoint intervals J_n with* $\sup_n \operatorname{length}(J_n) < \infty$; (c) *the union of a sequence of disjoint bounded intervals J_n with* $\sup_n \operatorname{length}(J_n) = \infty$. *Then, for every $k \in \mathbb{N}_0$, $1 \le p \le \infty$ and $\Omega \in \mathcal{F}$,*

$$C^\infty(\Omega) \cap H^{k,p}(\Omega) \simeq [C^\infty(I)]^{\mathbb{N}} \cap \ell^p[L^p(I)].$$

This result is the proper analogue of Theorem 5 of [AMM1] for $m = \infty$ and its proof is the same as for Theorems 4 and 5 of [AMM1] with C^∞-extension operators as in (3.4).

By combining Theorems 3.1 and 3.5 we obtain the following isomorphic classification result:

COROLLARY 3.6. *Let $k \in \mathbb{N}$, $1 \le p \le \infty$ and $\Omega \in \mathcal{F}$. Then*

$$C^\infty(\Omega) \cap H^{k,p}(\Omega) \simeq C^\infty(\mathbb{R}) \cap L^p(\mathbb{R}).$$

Thus, for $\Omega \in \mathcal{F}$, the isomorphy class depends only on p.

Now, we introduce the space

$$P_N = \{f \in C^\infty(\mathbb{R}^N) : f \text{ is } 2\pi\text{-periodic with respect to each variable}\},$$

which is a Fréchet space when endowed with the topology given by the sequence of norms

$$|f|_r = \sup_{\substack{|\alpha| \le r\\ \alpha \in \mathbb{N}_0^N}} \sup_{x \in S_N} |D^\alpha f(x)| \quad (f \in P_N),$$

where $S_N = \{x \in \mathbb{R}^N : -\pi \le x_i \le \pi,\ i = 1, \ldots, N\}$. Then, for all $k \in \mathbb{N}_0$ and $1 \le p \le \infty$, the restriction map $R : P_N \to H^{k,p}(S_N)$, $f \mapsto f|_{S_N}$, is linear, continuous and one-to-one. So, as in Section 2, we can define the Fréchet space

$$\mathcal{P}_{N,k,p} = [P_N]^{\mathbb{N}} \cap \ell^p[H^{k,p}(S_N)],$$

whose topology is given by the sequence of norms

$$q_r(f) = \sup_{n \le r} |f_n|_r + \Big(\sum_{n=1}^\infty \|f_n\|^p_{k,p,S_N}\Big)^{1/p} \quad (f = (f_n)_n \in \mathcal{P}_{N,k,p}).$$

Note that $\mathcal{P}^2_{N,k,p} \simeq \mathcal{P}_{N,k,p}$.

We can now represent the spaces $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$ in the following way:

THEOREM 3.7. *Let $k \in \mathbb{N}_0$ and $1 \le p \le \infty$. Then*

$$C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N) \simeq \mathcal{P}_{N,k,p}.$$

Proof. By Theorem 3.1, $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$ is isomorphic to the space $[C^\infty(Q_N)]^{\mathbb{N}} \cap \ell^p[H^{k,p}(Q_N)]$ and hence, by a simple change of variables, to $[C^\infty(S)]^{\mathbb{N}} \cap \ell^p[H^{k,p}(S)]$, with $S = \{x \in \mathbb{R}^N : -2\pi \le x_i \le 2\pi,\ i = 1, \ldots, N\}$.

Now, $\mathcal{P}_{N,k,p}$ is clearly isomorphic to the closed subspace $E = \{f = (f_n)_n :$ there exist $g = (g_n)_n \in \mathcal{P}_{N,k,p}$ so that $f_n = g_n|_S$ for all $n \in \mathbb{N}\}$ of the space $[C^\infty(S)]^{\mathbb{N}} \cap \ell^p[H^{k,p}(S)]$. We show that E is complemented in the latter.

Following [V1, Ch. 3, 1.11], we consider, for all $q \in \mathbb{Z}^N$, the map $g_q : \mathbb{R}^N \to \mathbb{R}^N$ given by $g_q(x) = (x_1 + (2q_1+2)\pi, \ldots, x_N + (2q_N+2)\pi)$ (whence each g_q maps S onto $S_q = \{x \in \mathbb{R}^N : 2q_i\pi \le x_i \le (2q_i+4)\pi,\ i = 1, \ldots, N\}$) and we fix a map $\psi \in C_0^\infty(S)$ such that $\psi > 0$ in $\mathring{S}$. We define $\varphi = \psi / \sum_{q \in \mathbb{Z}^N} \psi \circ g_q^{-1} \in C_0^\infty(S)$; for $f = (f_n)_n \in [C^\infty(S)]^{\mathbb{N}} \cap \ell^p[H^{k,p}(S)]$, we set

$$\widetilde{T} f = \Big(\sum_{q \in \mathbb{Z}^N} (f_n\varphi) \circ g_q^{-1}\Big)_n.$$

Then, by [V1, Ch. 3, 1.11.1–1.11.2], $\widetilde{T} f \in [P_N]^{\mathbb{N}}$ and, for all n and $r \in \mathbb{N}$,

$$|(\widetilde{T} f)_n|_r \le c_r |f_n|_r, \tag{3.5}$$

where c_r depends only on φ, r and N. Moreover, by observing that $g_q^{-1}(S) \cap S \ne \emptyset$ if, and only if, $q \in \{-2,-1,0\}^N$ and $\operatorname{supp}(f_n\varphi) \subset S$ for all $n \in \mathbb{N}$, it follows that

$$\begin{aligned}
\|(\widetilde{T} f)_n\|_{k,p,S} &= \Big(\sum_{|\alpha| \le k} \int_S \Big| \sum_{q \in \{-2,-1,0\}^N} D^\alpha (f_n\varphi)(g_q^{-1}(x))\Big|^p dx\Big)^{1/p}\\
&\le c \sum_{q \in \{-2,-1,0\}^N} \Big(\sum_{|\alpha| \le k} \int_S |D^\alpha (f_n\varphi)(g_q^{-1}(x))|^p dx\Big)^{1/p}\\
&\le c \sum_{q \in \{-2,-1,0\}^N} \Big(\sum_{|\alpha| \le k} \int_S |D^\alpha (f_n\varphi)(y)|^p dy\Big)^{1/p}\\
&\le 3^N c' \|f_n\|_{k,p,S},
\end{aligned}$$

where c' depends only on φ, k and N. Hence

$$\|\widetilde{T} f\|_p = \Big(\sum_{n=1}^\infty \|(\widetilde{T} f)_n\|^p_{k,p,S}\Big)^{1/p} \le 3^N c' \Big(\sum_{n=1}^\infty \|f_n\|^p_{k,p,S}\Big)^{1/p} = 3^N c' \|f\|_p.$$

Put $Tf = ((\widetilde{T} f)_{n|S})$; the above inequality together with (3.5) implies that T is a continuous linear map from $[C^\infty(S)]^{\mathbb{N}} \cap \ell^p[H^{k,p}(S)]$ into E. By [V1, Ch. 3, 1.11.3], $Tf = f$ for all $f \in E$. Therefore, T is a projection onto E.

Next, we show that $\mathcal{P}_{N,k,p}$ contains a complemented copy of the space $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$.

Let

$$E = \{(f_n)_n \in \mathcal{P}_{N,k,p} : \forall n \in \mathbb{N},\ f_n|_{\partial S_N} \equiv 0\}$$

and

$$F = [C^\infty(J_N)]^{\mathbb{N}} \cap \ell^p[H^{k,p}(J_N)],$$

with $J_N = \{x \in \mathbb{R}^N : 1 \le x_i \le 2,\ i = 1, \ldots, N\}$. Then E is a closed subspace of $\mathcal{P}_{N,k,p}$ and hence a Fréchet space with the induced topology, and F is a Fréchet space with its intersection topology.

Now, by using suitable reflections as in the proof of Theorem 3.2 (see (3.1) and (3.2)), we can construct a continuous linear extension operator $V : F \to E \subset \mathcal{P}_{N,k,p}$. It follows that, if $R : \mathcal{P}_{N,k,p} \to F$ is the restriction map, i.e. $R(f_n)_n = (f_n|_{J_N})_n$, the composition map $VR : \mathcal{P}_{N,k,p} \to E \subset \mathcal{P}_{N,k,p}$ is a projection from $\mathcal{P}_{N,k,p}$ into itself such that $V(R(\mathcal{P}_{N,k,p})) = V(F)$. Therefore, the subspace $V(F)$ of $\mathcal{P}_{N,k,p}$ is complemented in $\mathcal{P}_{N,k,p}$ and isomorphic to F. But $F \simeq [C^\infty(Q_N)]^{\mathbb{N}} \cap \ell^p[H^{k,p}(Q_N)]$ and hence, by Theorem 3.1, we get $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N) < \mathcal{P}_{N,k,p}$.

Finally, from $\mathcal{P}_{N,k,p} < C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N) < \mathcal{P}_{N,k,p}$ and from the fact that the spaces considered are isomorphic to their squares (see Remark 3.3 and Section 2), we deduce that $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N) \simeq \mathcal{P}_{N,k,p}$. The proof of Theorem 3.7 is complete.

We can now study isomorphisms of the spaces $C^\infty(\mathbb{R}^N) \cap H^{k,2}(\mathbb{R}^N)$ to the Fréchet space

$$s^{\mathbb{N}} \cap \ell^2(\ell^2) = \{\xi = (\xi_n)_n \in s^{\mathbb{N}} : (\|\xi_n\|_2)_n \in \ell^2\}$$

(s denotes, as usual, the nuclear Fréchet space of rapidly decreasing sequences), whose topology is defined by the following sequence of norms:

$$|\xi|_r = \sup_{n \le r} \|(\xi_{nj} j^r)_j\|_1 + \|(\|\xi_n\|_2)_n\|_2 \quad (\xi = (\xi_n)_n,\ \xi_n = (\xi_{nj})_j).$$

REMARK 3.8. If we consider the spaces

$$\ell_k^2(\mathbb{Z}^N) = \Big\{\xi = (\xi_q)_q : \|\xi\|_{k,2}^2 = \sup_{\substack{|\alpha| \le k \\ \alpha \in \mathbb{N}_0}} \sum_{q \in \mathbb{Z}^N} |q^{2\alpha}| \cdot |\xi_q|^2 < \infty\Big\}$$

and

$$s(\mathbb{Z}^N) = \Big\{\xi = (\xi_q)_q : \|(\xi_q q^r)_q\|_1 = \sum_{q \in \mathbb{Z}^N} |q^r| \cdot |\xi_q| < \infty \text{ for all } r \in \mathbb{N}_0\Big\}$$

(here and below, for all $q \in \mathbb{Z}^N$ and $\beta \in \mathbb{N}_0^N$, $r \in \mathbb{N}_0$, we use $q^\beta := \prod_{i=1}^N q_i^{\beta_i}$ and $q^r := \prod_{i=1}^N q_i^r$), with their natural topologies, then the inclusion map $s(\mathbb{Z}^N) \hookrightarrow \ell_k^2(\mathbb{Z}^N)$ is continuous and hence we can define the Fréchet space

$$[s(\mathbb{Z}^N)]^{\mathbb{N}} \cap \ell^2[\ell_k^2(\mathbb{Z}^N)] = \{\xi = (\xi_n)_n \in [s(\mathbb{Z}^N)]^{\mathbb{N}} : (\|\xi_n\|_{k,2})_n \in \ell^2\},$$

whose topology is given by the sequence of norms

$$|\xi|_{r,N} = \sup_{n \le r} \|(\xi_q q^r)_q\|_1 + \|(\|\xi_n\|_{k,2})_n\|_2$$

(compare with (2.2)).

Clearly, for all $k \in \mathbb{N}$,

$$[s(\mathbb{Z}^N)]^{\mathbb{N}} \cap \ell^2[\ell_k^2(\mathbb{Z}^N)] \simeq s^{\mathbb{N}} \cap \ell^2(\ell^2).$$

Now, we can state and prove

THEOREM 3.9. *Let $k \in \mathbb{N}_0$. Then*

$$C^\infty(\mathbb{R}^N) \cap H^{k,2}(\mathbb{R}^N) \simeq s^{\mathbb{N}} \cap \ell^2(\ell^2).$$

Proof. By Theorem 3.7 and Remark 3.8 it suffices to prove that $\mathcal{P}_{N,k,2} \simeq [s(\mathbb{Z}^N)]^{\mathbb{N}} \cap \ell^2[\ell_k^2(\mathbb{Z}^N)]$.

For a given $f \in P_N$ and $q \in \mathbb{Z}^N$, let $\widehat{f}_q$ be the qth Fourier coefficient of f. Since

$$\|f\|_{k,2,S_N} = \Big(\sum_{|\alpha| \le k} \|D^\alpha f\|_{2,S_n}^2\Big)^{1/2} = (2\pi)^{N/2}\Big(\sum_{|\alpha| \le k} \sum_{q \in \mathbb{Z}^N} |q^\alpha|^2 |\widehat{f}_q|^2\Big)^{1/2}$$

and for all $\alpha \in \mathbb{N}_0$,

$$\|(q^\alpha \widehat{f}_q)_q\|_\infty \le \sup_{x \in S_N} |D^\alpha f(x)|,$$

the map

$$F : \mathcal{P}_{N,k,2} \to [s(\mathbb{Z}^N)]^{\mathbb{N}} \cap \ell^2[\ell_k^2(\mathbb{Z}^N)], \quad (f_n)_n \mapsto (((\widehat{f}_n)_q)_q)_n$$

is well-defined, linear, continuous and clearly one-to-one. We show that F is onto.

Let $\xi \in [s(\mathbb{Z}^N)]^{\mathbb{N}} \cap \ell^2[\ell_k^2(\mathbb{Z}^N)]$. Then, for all $n \in \mathbb{N}$, the series

$$f_n(x) = \sum_{q \in \mathbb{Z}^N} \xi_{nq} e^{-i(q,x)}$$

converges in P_N so that $f_n \in P_N$ and, for all $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^N$,

$$\|D^\alpha f_n\|_{2,S_N} = (2\pi)^{N/2}\Big(\sum_{q \in \mathbb{Z}^N} |q^\alpha|^2 |\xi_{nq}|^2\Big)^{1/2}$$

and hence

$$\sum_{n=1}^\infty \|f_n\|_{k,2,S_N}^2 = (2\pi)^N \sum_{n=1}^\infty \sum_{|\alpha| \le k} \sum_{q \in \mathbb{Z}^N} |q^\alpha|^2 |\xi_{nq}|^2 < \infty.$$

This implies that $f = (f_n)_n \in \mathcal{P}_{N,k,2}$. Since $Ff = \xi$, we have $\mathcal{P}_{N,k,2} \simeq [s(\mathbb{Z}^N)]^{\mathbb{N}} \cap \ell^2[\ell_k^2(\mathbb{Z}^N)]$ and the result follows by Remark 3.8.

Finally, Theorem 3.9 yields

THEOREM 3.10. *For each $k \in \mathbb{N}_0$ and $N \in \mathbb{N}$,*

$$C^\infty(\mathbb{R}^N) \cap H^{k,2}(\mathbb{R}^N) \simeq C^\infty(\mathbb{R}) \cap L^2(\mathbb{R}).$$

This result is quite surprising because it shows that the isomorphy class does not depend on the dimension N if $p = 2$.

4. Topological properties and Montel subspaces. In this final section we study some topological properties and Montel subspaces of the spaces $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$. To do this we shall use the representation obtained in Theorem 3.1.

PROPOSITION 4.1. *Let $k \in \mathbb{N}_0$ and $1 < p < \infty$. Then the spaces $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$ are totally reflexive, hence distinguished.*

Proof. By Theorem 3.1 it suffices to show that the space $[C^\infty(Q_N)]^{\mathbb{N}} \cap \ell^p[H^{k,p}(Q_N)]$ is totally reflexive for $1 < p < \infty$.

The latter space is canonically topologically isomorphic to the projective limit of the sequence $E_r = [C^\infty(Q_N)]^{r-1} \oplus \ell^p[(H^{k,p}(Q_N))_{n\ge r}]$ with linking maps $E_{r+1} \hookrightarrow E_r$, $(f_n)_n \mapsto (f_n)_n$ (see Section 2). Then, E_r being reflexive for all $r \in \mathbb{N}$ if $1 < p < \infty$, the result follows from [V2, Theorem 4].

In a similar simple way and using results from [DK] one can see that:

- *The spaces $C^\infty(\mathbb{R}^N) \cap H^{k,1}(\mathbb{R}^N)$ are not distinguished ($k \in \mathbb{N}_0$).*
- *The spaces $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$ do not satisfy the density condition, hence they are not quasinormable ($k \in \mathbb{N}_0$, $1 \le p \le \infty$).*

The latter facts were proved in a different way in [BT, Theorems 2 and 4].

Finally, we state and prove the following general results which, by Theorem 3.1, immediately apply to the spaces $C^\infty(\mathbb{R}^N) \cap H^{k,p}(\mathbb{R}^N)$. The first one should be compared with [AA, Theorem 2].

PROPOSITION 4.2. *Let $1 \le p \le \infty$. Let $(Y, (\|\cdot\|_r)_r)$ be a nuclear Fréchet space continuously included in a Banach space $(X, \|\cdot\|)$. If E is a Montel subspace of $Y^{\mathbb{N}} \cap \ell^p(X)$, then E is nuclear.*

Proof. The proof is carried out for $1 \le p < \infty$, the case $p = \infty$ being similar.

First, we show that there exist $r_0 \in \mathbb{N}$ and $d > 0$ such that for all $y = (y_n)_n \in E$,

$$\|y\|_p = \Big(\sum_{n=1}^{\infty} \|y_n\|^p\Big)^{1/p} \le d \sup_{n\le r_0} \|y_n\|_{r_0}. \tag{4.1}$$

Suppose, by contradiction, that (4.1) is false. Then we can find a sequence $(y_r)_r$ of E, $y_r = (y_{nr})_n$, such that, for each $r \in \mathbb{N}$,

$$\|y_r\|_p = \Big(\sum_{n=1}^{\infty} \|y_{nr}\|^p\Big)^{1/p} = 1 \quad \text{and} \quad \sup_{n\le r} \|y_{nr}\|_r < 2^{-r}. \tag{4.2}$$

It follows that, for all $n \in \mathbb{N}$, $(y_{nr})_r$ is a sequence in Y converging to 0 and hence bounded in Y. This jointly with (4.2) implies that $(y_r)_r$ is a bounded sequence in E with limit point 0 in $Y^{\mathbb{N}}$. Since E is Montel, $(y_r)_r$ contains a subsequence converging to 0 in $Y^{\mathbb{N}} \cap \ell^p(X)$; this is a contradiction to (4.2). Now, by (4.1) the topology induced on E by $Y^{\mathbb{N}} \cap \ell^p(X)$ coincides with the one induced by $Y^{\mathbb{N}}$ and hence E is nuclear.

PROPOSITION 4.3. *Let $1 \le p < \infty$. Let $(Y, (\|\cdot\|_r)_r)$ be a nuclear Fréchet space continuously included in a Banach space $(X, \|\cdot\|)$. Let E be a closed subspace of $Y^{\mathbb{N}} \cap \ell^p(X)$. If E is not Montel, then E contains a complemented copy of ℓ^p.*

Proof. We first observe that, by the above proof,

A subspace F of $Y^{\mathbb{N}} \cap \ell^p(X)$ is Montel if, and only if, condition (4.1) is satisfied.

Since E is not Montel, it then contains a sequence $(y_r)_r$, $y_r = (y_{nr})_n$, satisfying condition (4.2) and hence, for all $n \in \mathbb{N}$, $(y_{nr})_r$ is convergent to 0 in Y. Using this together with the fact that, for all $n \in \mathbb{N}$, $\lim_{h\to\infty} \sum_{n\ge h} \|y_{nr}\|^p = 0$, we can find an increasing subsequence $(r_j)_j$ of positive integers such that, for each $j \in \mathbb{N}$,

$$\sum_{n<r_j} \|y_{nr_j}\|^p < 2^{-p(j+3)}, \qquad \sum_{n\ge r_{j+1}} \|y_{nr_j}\|^p < 2^{-p(j+3)}, \tag{4.3}$$

and hence, by (4.2),

$$\frac{3}{4} < \Big(\sum_{n=r_j}^{r_{j+1}-1} \|y_{nr_j}\|^p\Big)^{1/p} < 1. \tag{4.4}$$

Put, for all $j \in \mathbb{N}$,

$$\widetilde{y}_j = ((0)_{n<r_j}, (y_{nr_j})_{r_j\le n<r_{j+1}}, (0)_{n\ge r_{j+1}}).$$

Then, by (4.4), for every sequence $(a_j)_j$ of scalars which are eventually 0,

$$\begin{aligned}\frac{3}{4}\Big(\sum_{j=1}^{\infty} |a_j|^p\Big)^{1/p} &\le \Big\|\sum_{j=1}^{\infty} a_j \widetilde{y}_j\Big\|_p \\ &= \Big(\sum_{j=1}^{\infty} |a_j|^p \|\widetilde{y}_j\|_p^p\Big)^{1/p} \le \Big(\sum_{j=1}^{\infty} |a_j|^p\Big)^{1/p}.\end{aligned} \tag{4.5}$$

So, $(\widetilde{y}_j)_j$ is equivalent to the unit vector basis of ℓ^p in $\ell^p(X)$ and its basis constant with respect to the norm $\|\cdot\|_p$ is 1. Moreover, it is easy to verify that the topology induced on the linear span of $(\widetilde{y}_j)_j$ by $Y^{\mathbb{N}} \cap \ell^p(X)$ coincides with that induced by $\ell^p(X)$ and hence $[\widetilde{y}_j : j \in \mathbb{N}]$ is a closed subspace of $Y^{\mathbb{N}} \cap \ell^p(X)$ isomorphic to ℓ^p.

By (4.3), we also have

$$\sum_{j=1}^{\infty}\|\widetilde{y}_j - y_{r_j}\|_p \le \sum_{j=1}^{\infty} 2^{-(j+2)} = \frac{1}{4}. \tag{4.6}$$

This and (4.2) imply that the map $T:[\widetilde{y}_j : j\in\mathbb{N}]\to[y_{r_j} : j\in\mathbb{N}]$ given by $\sum_j a_j\widetilde{y}_j \mapsto \sum_j a_j y_{r_j}$, is an isomorphism onto, so that $[y_{r_j} : j\in\mathbb{N}]$ is a subspace of E isomorphic to ℓ^p. Indeed, from (4.5) and (4.6) it follows that, for every sequence of scalars $(a_j)_j$ which are eventually 0,

$$\frac{1}{2}\Big(\sum_{j=1}^{\infty}|a_j|^p\Big)^{1/p} \le \Big\|\sum_{j=1}^{\infty} a_j y_{r_j}\Big\|_p \le \frac{5}{4}\Big(\sum_{j=1}^{\infty}|a_j|^p\Big)^{1/p}. \tag{4.7}$$

Moreover, by (4.5) we get, for a given $r\in\mathbb{N}$ and for every sequence of scalars $(a_j)_j$ which are eventually 0,

$$\begin{aligned}\sup_{n\le r}\Big\|\sum_{j=1}^{\infty} a_j y_{nr_j}\Big\|_r &\le \sum_{j=1}^{\infty}|a_j|\sup_{n\le r}\|y_{nr_j}\|_r\\ &\le \Big(\sum_{j=1}^{\infty}|a_j|^p\Big)^{1/p}\Big[\sum_{j=1}^{\infty}(\sup_{n\le r}\|y_{nr_j}\|_r)^q\Big]^{1/q}\\ &\le \frac{4}{3}c_r\Big\|\sum_{j=1}^{\infty} a_j\widetilde{y}_j\Big\|_p,\end{aligned} \tag{4.8}$$

where $c_r^q = \sum_{j=1}^{\infty}(\sup_{n\le r}\|y_{nr_j}\|_r)^q < \infty$ because of (4.2) $(1/p+1/q=1)$.

Now, (4.7) and (4.8) imply that T is an isomorphism onto. This completes the first part of our proof.

For all $j\in\mathbb{N}$ and $r_j\le n<r_{j+1}$, let $y'_{nj}\in X'$ so that

$$\sum_{n=r_j}^{r_{j+1}-1}(\|y'_{nj}\|')^q = 1 \quad\text{and}\quad \sum_{n=r_j}^{r_{j+1}-1} y'_{nj}(y_{nr_j}) = \|\widetilde{y}_j\|_p.$$

Put, for all $j\in\mathbb{N}$, $y'_j = ((0)_{n<r_j}, (y'_{nj})_{r_j\le n<r_{j+1}}, (0)_{n\ge r_{j+1}}) \in \ell^q(X')$ and, for all $y\in Y^{\mathbb{N}}\cap\ell^p(X)$, $y=(y_n)_n$, let

$$(y, y'_j) = \sum_{n=r_j}^{r_{j+1}-1} y'_{nj}(y_{nr_j}).$$

Then, by (4.4), the map

$$Py = \sum_{j=1}^{\infty}\frac{(y,y'_j)}{(\widetilde{y}_j, y'_j)}\widetilde{y}_j$$

satisfies

$$\|Py\|_p \le \frac{4}{3}\|y\|_p \quad\text{and}\quad P\widetilde{y}_j = \widetilde{y}_j \text{ for all } j\in\mathbb{N}.$$

Therefore, $P: Y^N\cap\ell^p(X)\to[\widetilde{y}_j : j\in\mathbb{N}]$ is a continuous projection onto.

Finally, (4.6) implies that the composition map

$$S = T(P|_{[y_j:j\in\mathbb{N}]}) : [y_j : j\in\mathbb{N}] \xrightarrow{P} [\widetilde{y}_j : j\in\mathbb{N}] \xrightarrow{T} [y_j : j\in\mathbb{N}]$$

is invertible on $[y_j : j\in\mathbb{N}]$ and $S^{-1}TP$ is a projection from $Y^{\mathbb{N}}\cap\ell^p(X)$ onto $[y_j : j\in\mathbb{N}]$, and hence the result follows.

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Smooth operators for the regular representation on homogeneous spaces

by

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Abstract. A necessary and sufficient condition for a bounded operator on $L^2(M)$, M a Riemannian compact homogeneous space, to be smooth under conjugation by the regular representation is given. It is shown that, if all formal "Fourier multipliers with variable coefficients" are bounded, then they are also smooth. In particular, they are smooth if M is a rank-one symmetric space.

1. Introduction. Consider the two unitary representations of $\mathbb{R}$ on $L^2(\mathbb{R})$: $(T_z u)(x) = u(x-z)$ and $(E_\zeta)u(x) = e^{ix\zeta}u(x)$. A bounded operator A on $L^2(\mathbb{R})$ is such that both mappings $z \mapsto T_z A T_z^{-1}$ and $\zeta \mapsto E_\zeta A E_\zeta^{-1}$ are smooth in the norm topology if and only if it is a pseudodifferential operator with symbol having bounded derivatives of all orders in $\mathbb{R}^2$. This remarkable result was proven by Cordes ([3], Theorem 1.2; see also [4], Theorem VIII.2.1) and was closely related to a previous abstract characterization (involving boundedness of commutators) of pseudodifferential operators due to Beals [2]. Other descriptions of pseudodifferential operators as bounded operators which give rise to smooth mappings when conjugated by Lie-group unitary representations have been called *Beals–Cordes-type characterizations* [15]. A class of operators characterized by such a smoothness condition naturally becomes ([4], Theorem VIII.6.6) a Ψ^*-algebra, in the sense of Gramsch [8]. As observed further by Payne [13], it also becomes a smooth tame Fréchet algebra.

A Beals–Cordes-type characterization for operators on the circle $\mathbb{S}^1$ was given in [12], Theorem 2: a bounded operator A on $L^2(\mathbb{S}^1)$ defines a smooth function when conjugated by the regular representation of $\mathbb{S}^1$ if and only if it is given by

$$(1) \qquad Au(x) = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} e^{ijx}a_j(x)\int_{-\pi}^{\pi} e^{-ijy}u(y)\,dy,$$

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