ACTA ARITHMETICA
XCII.1 (2000)

On a system of two diophantine inequalities with prime numbers

by

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1. Introduction and results. In 1952 Piatetski-Shapiro [7] considered the following analogue of the Goldbach–Waring problem: Assume that $c > 1$ is not an integer and let ε be a small positive number. Let $H(c)$ denote the smallest natural number r such that the inequality

$$|p_1^c + \ldots + p_r^c - N| < \varepsilon \tag{1.1}$$

is solvable in prime numbers $p_1, \ldots, p_r$ for sufficiently large N. Then it is proved in [7] that

$$\limsup_{c\to\infty} \frac{H(c)}{c\log c} \le 4.$$

Piatetski-Shapiro also proved that $H(c) \le 5$ for $1 < c < 3/2$. In [8] Tolev first improved this result for c close to one. More precisely, he proved that if $1 < c < 15/14$, then the inequality

$$|p_1^c + p_2^c + p_3^c - N| < \varepsilon(N) \tag{1.2}$$

has prime solutions p_1, p_2, p_3 for large N, where

$$\varepsilon(N) = N^{-(1/c)(15/14-c)} \log^9 N.$$

This result was improved by several authors (see [1, 4, 5]).

In [9] Tolev first studied the system of two inequalities with primes

$$\begin{aligned} &|p_1^c + \ldots + p_5^c - N_1| < \varepsilon_1(N_1), \\ &|p_1^d + \ldots + p_5^d - N_2| < \varepsilon_2(N_2), \end{aligned} \tag{1.3}$$

where $1 < d < c < 2$ are different numbers and $\varepsilon_1(N_1)$ and $\varepsilon_2(N_2)$ tend to zero as N_1 and N_2 tend to infinity. Tolev proved that if c, d, α, β are real

1991 *Mathematics Subject Classification*: 11P55, 11D75.

This work is supported by National Natural Science Foundation of China (Grant No. 19801021) and Natural Science Foundation of Shandong Province (Grant No. Q98A02110).

numbers satisfying

$$(1.4) \qquad 1 < d < c < 35/34,$$
$$(1.5) \qquad 1 < \alpha < \beta < 5^{1-d/c},$$

then there exist numbers $N_1^{(0)}, N_2^{(0)}$, depending on c, d, α, β, such that for all real numbers N_1, N_2 satisfying $N_1 > N_1^{(0)}$, $N_2 > N_2^{(0)}$ and

$$(1.6) \qquad \alpha \le N_2/N_1^{d/c} \le \beta,$$

the system (1.3) has prime solutions $p_1, \ldots, p_5$ for

$$\varepsilon_1(N_1) = N_1^{-(1/c)(35/34-c)} \log^{12} N_1, \quad \varepsilon_2(N_2) = N_2^{-(1/d)(35/34-d)} \log^{12} N_2.$$

In this paper we shall prove

Theorem. *Suppose c and d are real numbers such that*

$$(1.7) \qquad 1 < d < c < 25/24,$$

and α and β are real numbers satisfying (1.5). *Then for all real numbers N_1, N_2 satisfying* (1.6), *the system* (1.3) *has prime solutions $p_1, \ldots, p_5$ for*

$$\varepsilon_1(N_1) = N_1^{-(1/c)(25/24-c)} \log^{335} N_1,$$
$$\varepsilon_2(N_2) = N_2^{-(1/d)(25/24-d)} \log^{335} N_2.$$

A short proof, which follows the argument of Tolev [9], will be given in Section 2. The main difficulty is to prove the Proposition of Section 2, which improves Lemma 13 of Tolev [9] and is the key to our result. In Section 3, some preliminary lemmas are given. A detailed proof of the Proposition is given in Section 4. The new idea of the proof combines elementary methods and van der Corput's classical estimates.

Notations. Throughout, c and d are real numbers satisfying (1.7), α and β are real numbers satisfying (1.5), and λ denotes a sufficiently small positive number determined precisely by Lemma 1 of Tolev [9], depending on c, d, α, β. N_1 and N_2 are large numbers satisfying (1.6). $X = N_1^{1/c}$, $\varepsilon_1(N_1) = N_1^{-(1/c)(25/24-c)} \log^{335} N_1$, $\varepsilon_2(N_2) = N_2^{-(1/d)(25/24-d)} \log^{335} N_2$, $K_1 = \varepsilon_1^{-1} \log X$, $K_2 = \varepsilon_2^{-1} \log X$, η is a sufficiently small positive number in terms of c and d, $\tau_1 = X^{3/4-c-\eta}$, $\tau_2 = X^{3/4-d-\eta}$, $e(t) = e^{2\pi i t}$, $\varphi(t) = e^{-\pi t}$, $\varphi_\delta(t) = \delta\varphi(\delta t)$, and $\chi(t)$ is the characteristic function of the interval $[-1, 1]$. We set

$$B = \sum_{\lambda X < p_1, \ldots, p_5 < X} \log p_1 \ldots \log p_5 \chi\bigg(\frac{p_1^c + \ldots + p_5^c - N_1}{\varepsilon_1 \log X}\bigg) \times \chi\bigg(\frac{p_1^d + \ldots + p_5^d - N_2}{\varepsilon_2 \log X}\bigg),$$

$$S(x,y)=\sum_{\lambda X<p<X}(\log p)e(xp^c+yp^d),$$

$$D=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}S^5(x,y)e(-N_1x-N_2y)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy,$$

$$\Omega_1=\{(x,y)\mid \max(|x|/\tau_1,|y|/\tau_2)<1\},$$

$$\Omega_2=\{(x,y)\mid \max(|x|/\tau_1,|y|/\tau_2)\ge 1,\ \max(|x|/K_1,|y|/K_2)\le 1\},$$

$$\Omega_3=\{(x,y)\mid \max(|x|/K_1,|y|/K_2)>1\}.$$

2. A short proof of the Theorem. The Theorem follows if we can show that B tends to infinity as X tends to infinity. By Lemma 3 of Tolev [9], it is sufficient to show that D tends to infinity as X tends to infinity. Write

$$D=D_1+D_2+D_3, \tag{2.1}$$

where

$$D_i=\iint_{\Omega_i}S^5(x,y)e(-N_1x-N_2y)\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy. \tag{2.2}$$

By the same arguments as in Section 4 of Tolev [9], we have

$$D_1\gg \varepsilon_1\varepsilon_2X^{5-c-d}. \tag{2.3}$$

By Lemma 4 of Tolev [9], we have

$$D_3\ll 1. \tag{2.4}$$

So now the Theorem follows from (2.1)–(2.4) and the estimate

$$D_2\ll \varepsilon_1\varepsilon_2X^{5-c-d}(\log X)^{-1}. \tag{2.5}$$

By Lemma 14 of Tolev [9] we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|S^4(x,y)|\varphi_{\varepsilon_1}(x)\varphi_{\varepsilon_2}(y)\,dx\,dy\ll X^2\log^6 X. \tag{2.6}$$

It suffices to prove the following

PROPOSITION. *Uniformly for $(x,y)\in\Omega_2$, we have*

$$S(x,y)\ll X^{11/12}\log^{660}X. \tag{2.7}$$

3. Some preliminary lemmas. In order to prove the Proposition, we need the following lemmas. Lemma 1 is Theorem 2.2 of Min [6]. Lemma 2 is Lemma 2.5 of Graham and Kolesnik [2]. Lemma 3 is contained in Lemma 2.8 of Krätzel [3]. Lemma 4 is well known (see Graham and Kolesnik [2], for example).

Lemma 1. *Suppose $f(x)$ and $g(x)$ are algebraic functions in $[a,b]$ and*

$$|f''(x)|\sim 1/R,\qquad |f'''(x)|\ll 1/(RU),$$
$$|g(x)|\ll G,\qquad |g'(x)|\ll GU_1^{-1},\qquad U,U_1\ge 1.$$

Then

$$\begin{aligned}\sum_{a<n\le b} g(n)e(f(n)) = {} & \sum_{\alpha<u\le\beta} b_u\frac{g(n_u)}{\sqrt{|f''(n_u)|}}e(f(n_u)-un_u+1/8)\\ & + O(G\log(\beta-\alpha+2)+G(b-a+R)(U^{-1}+U_1^{-1}))\\ & + O(G\min(\sqrt{R},1/\langle\alpha\rangle)+G\min(\sqrt{R},1/\langle\beta\rangle)),\end{aligned}$$

where $[\alpha,\beta]$ is the image of $[a,b]$ under the mapping $y=f'(x)$, n_u is the solution of the equation $f'(x)=u$,

$$b_u=\begin{cases}1 & \text{for } \alpha<u<\beta,\\ 1/2 & \text{for } u=\alpha\in\mathbb{Z} \text{ or } u=\beta\in\mathbb{Z},\end{cases}$$

and the function $\langle t\rangle$ is defined as follows:

$$\langle t\rangle=\begin{cases}\|t\| & \text{if } t \text{ is not an integer},\\ \beta-\alpha & \text{otherwise},\end{cases}$$

where $\|t\|=\min_{n\in\mathbb{Z}}\{|t-n|\}$.

Lemma 2. *Suppose $z(n)$ is any complex number and $1\le Q\le N$. Then*

$$\Big|\sum_{N<n\le CN} z(n)\Big|^2\ll\frac{N}{Q}\sum_{0\le q\le Q}\left(1-\frac{q}{Q}\right)\mathrm{Re}\sum_{N<n\le CN-q} z(n)\overline{z(n+q)}.$$

Lemma 3. *Suppose $f(x)\ll P$ and $f'(x)\gg\varDelta$ for $x\sim N$. Then*

$$\sum_{n\sim N}\min\left(D,\frac{1}{\|f(n)\|}\right)\ll(P+1)(D+\varDelta^{-1})\log(2+\varDelta^{-1}).$$

Lemma 4. *Suppose $5<A<B\le 2A$ and $f''(x)$ is continuous on $[A,B]$. If $0<c_1\lambda_1\le|f'(x)|\le c_2\lambda_1\le 1/2$, then*

$$\sum_{A<n\le B} e(f(n))\ll\lambda_1^{-1}.$$

If $0<c_3\lambda_2\le|f''(x)|\le c_4\lambda_2$, then

$$\sum_{A<n\le B} e(f(n))\ll A\lambda_2^{1/2}+\lambda_2^{-1/2}.$$

Now we prove the following two lemmas, which are important in the proof of the Proposition. Let

$$S=S(M,a,b,\gamma_1,\gamma_2)=\sum_{M<m\le M_1} e(am^{\gamma_1}+bm^{\gamma_2}),$$

where M and M_1 are positive numbers such that $5 \le M < M_1 \le 2M$, a and b are real numbers such that $ab \ne 0$, and γ_1 and γ_2 are real numbers such that $1 < \gamma_1, \gamma_2 < 2, \gamma_1 \ne \gamma_2$. Let $R = |a|M^{\gamma_1} + |b|M^{\gamma_2}$.

LEMMA 5. *If* $RM^{-1} \le 1/8$, *then*

$$S \ll MR^{-1/2}.$$

P r o o f. Suppose $R > 100$; otherwise Lemma 5 is trivial. Let

$$f(m) = am^{\gamma_1} + bm^{\gamma_2}.$$

Then

$$f'(m) = \gamma_1 am^{\gamma_1 - 1} + \gamma_2 bm^{\gamma_2 - 1}.$$

If $ab > 0$, then $R/M \le |f'(m)| \le 4R/M \le 1/2$, hence the assertion follows from Lemma 4.

Now suppose $ab < 0$. Let

$$I = \{t \in [M, M_1] \mid |f'(t)| \le R^{1/2}M^{-1}\},$$
$$J = \{t \in [M, M_1] \mid |f'(t)| > R^{1/2}M^{-1}\}.$$

By the definition we see that if $m \in J$, then

$$R^{1/2}/M \le |f'(m)| \le 4R/M \le 1/2;$$

thus by Lemma 4,

$$\sum_{m \in J} e(f(m)) \ll MR^{-1/2}. \tag{3.1}$$

We only need to estimate $|I|$. If $t \in I$, then

$$\gamma_1 at^{\gamma_1} = -\gamma_2 bt^{\gamma_2} + O(R^{1/2}) = -\gamma_2 bt^{\gamma_2}(1 + O(R^{-1/2})),$$
$$t^{\gamma_1 - \gamma_2} = \frac{-\gamma_2 b}{\gamma_1 a}(1 + O(R^{-1/2})),$$

which implies that

$$\begin{aligned} t &= \left(\frac{-\gamma_2 b}{\gamma_1 a}\right)^{1/(\gamma_1 - \gamma_2)} (1 + O(R^{-1/2}))^{1/(\gamma_1 - \gamma_2)} \\ &= \left(\frac{-\gamma_2 b}{\gamma_1 a}\right)^{1/(\gamma_1 - \gamma_2)} (1 + O(R^{-1/2})) \\ &= \left(\frac{-\gamma_2 b}{\gamma_1 a}\right)^{1/(\gamma_1 - \gamma_2)} + O(MR^{-1/2}). \end{aligned} \tag{3.2}$$

So

$$|I| \ll MR^{-1/2}. \tag{3.3}$$

Now the conclusion follows from (3.1) and (3.3).

LEMMA 6. *If $M \ll R \ll M^2$, then*

$$S \ll R^{1/2} + MR^{-1/3}.$$

Proof. We have

$$f''(m) = \gamma_1(\gamma_1 - 1)am^{\gamma_1 - 2} + \gamma_2(\gamma_2 - 1)bm^{\gamma_2 - 2}.$$

If $ab > 0$, then $|f''(m)| \sim RM^{-2}$, and by Lemma 4 we get $S \ll R^{1/2} + MR^{-1/2}$.

Now suppose $ab < 0$. Let $\Delta_0 = R^{2/3}M^{-2}$. Define

$$I_0 = \{t \in [M, M_1] \mid |f''(t)| \le \Delta_0\},$$

$$I_j = \{t \in [M, M_1] \mid 2^{j-1}\Delta_0 < |f''(t)| \le 2^j\Delta_0 \le 2R/M^2\},$$

$$1 \le j \le \frac{\log\left(\frac{2R}{M^2\Delta_0}\right)}{\log 2} = J_0.$$

If I_0 is not empty, then by the same argument as in Lemma 5 we get $|I_0| \ll MR^{-1/3}$. Thus Lemma 4 yields

$$\begin{aligned}
\sum_{M<m\le M_1} e(f(m)) &= \sum_{m\in I_0} e(f(m)) + \sum_{1\le j\le J_0} \sum_{m\in I_j} e(f(m)) \\
&\ll MR^{-1/3} + \sum_{1\le j\le J_0} \{M(2^j\Delta_0)^{1/2} + (2^j\Delta_0)^{-1/2}\} \\
&\ll MR^{-1/3} + R^{1/2}.
\end{aligned} \tag{3.4}$$

This completes the proof.

4. Proof of the Proposition. In this section we shall estimate $S(x, y)$ for $(x, y) \in \Omega_2$. Suppose $1 < d < c < 25/24$ and fix $(x, y) \in \Omega_2$. Let $R = |x|X^c + |y|X^d$. Obviously, $X^{3/4-\eta} \ll R \ll X^{25/24}\log^{-300} X$. Without loss of generality, we suppose $xy \ne 0$. For the case $x = 0$ or $y = 0$, previous methods yield better results (see [1, 5]).

LEMMA 7. *Suppose $a(m)$ are complex numbers such that*

$$\sum_{m\sim M} |a(m)|^2 \ll M\log^{2A} M, \qquad A > 0.$$

Then for $M \ll \min(X^{2/3}, X^{19/12}R^{-1})$, $MN \sim X$, we have

$$S_{\mathrm{I}} = \sum_{m\sim M} a(m) \sum_{n\sim N} e(x(mn)^c + y(mn)^d) \ll X^{11/12}\log^{A+1} X. \tag{4.1}$$

Proof. If $M \ll X^{11/12}R^{-1/2}$, then by Lemma 6 we get

$$S_{\mathrm{I}} \ll M(R^{1/2} + NR^{-1/3})\log^A X \ll X^{11/12}\log^A X. \tag{4.2}$$

From now on we always suppose $M \gg X^{11/12}R^{-1/2}$. Let $Q = [X^{1/6}]$. By Cauchy's inequality and Lemma 2 we have

$$(4.3)\qquad |S_{\mathrm{I}}|^2 \ll \sum_{m\sim M}|a(m)|^2 \sum_{m\sim M}\Big|\sum_{n\sim N} e(x(mn)^c + y(mn)^d)\Big|^2$$
$$\ll X^2Q^{-1}\log^{2A} X + XQ^{-1}\log^{2A} X\sum_{q=1}^{Q}|E_q|,$$

where

$$E_q = \sum_{m\sim M}\sum_{N<n\le 2N-q} e(xm^c\Delta(n,q;c) + ym^d\Delta(n,q;d)),$$
$$\Delta(n,q;t) = (n+q)^t - n^t.$$

Now the problem is reduced to showing that

$$(4.4)\qquad \sum_{q=1}^{Q}|E_q| \ll X\log^2 X.$$

For each fixed $1\le q\le Q$, let

$$f(m,n) = xm^c\Delta(n,q;c) + ym^d\Delta(n,q;d).$$

We first consider several simple cases.

CASE 0: *A special case.* For constants $a,b>0$, let $N(a,b)$ denote the solution of the inequality

$$(4.5)\qquad |ax(mn)^c + by(mn)^d| \le \frac{R}{Q^{1/2}\log X},\qquad m\sim M,\ n\sim N.$$

Suppose $0<\sigma<1$ is a positive constant small enough. Then we can prove that uniformly for $a,b\in[\sigma,1/\sigma]$, we have

$$(4.6)\qquad N(a,b)\ll_\sigma X^{11/12}.$$

If $xy>0$, then $N(a,b)=0$; so suppose $xy<0$. If (m,n) satisfies the inequality (4.5), then

$$ax(mn)^c = -by(mn)^d + O\bigg(\frac{R}{Q^{1/2}\log X}\bigg)$$
$$= -by(mn)^d(1+O(Q^{-1/2}\log^{-1} X)),$$

which implies that

$$\begin{aligned} mn &= \left(\frac{-by}{ax}\right)^{1/(c-d)} (1 + O(Q^{-1/2}\log^{-1} X))^{1/(c-d)} \\ &= \left(\frac{-by}{ax}\right)^{1/(c-d)} (1 + O(Q^{-1/2}\log^{-1} X)) \\ &= \left(\frac{-by}{ax}\right)^{1/(c-d)} + O(XQ^{-1/2}\log^{-1} X). \end{aligned}$$

Thus (4.5) follows from a divisor argument. Why we study this case will be explained later.

Case 1: $|\partial f/\partial m| \le 500^{-1}$. It is obvious that

$$\begin{aligned} |xm^c \varDelta(n,q;c)| &\sim q|x|m^c n^{c-1} \sim q|x|X^c N^{-1}, \\ |ym^d \varDelta(n,q;d)| &\sim q|y|m^d n^{d-1} \sim q|y|X^d N^{-1}, \end{aligned}$$

thus

$$|xm^c \varDelta(n,q;c)| + |ym^d \varDelta(n,q;d)| \sim qRN^{-1}.$$

We use Lemma 5 to estimate the sum over m and get

$$E_q \ll NM(qRN^{-1})^{-1/2} \ll MN^{3/2}q^{-1/2}R^{-1/2}.$$

Summing over q we find that (4.4) holds if noticing $M \gg X^{11/12}R^{-1/2}$ and $R \ll X^{25/24}$.

Case 2: $|\partial f/\partial n| \le 500^{-1}$. For fixed m, we estimate the sum over n. Since

$$\begin{aligned} \partial f/\partial n &= cxm^c \varDelta(n,q;c-1) + dym^d \varDelta(n,q;d-1), \\ \varDelta(n,q;c-1) &= (c-1)qn^{c-2} + O(q^2N^{c-3}), \\ \varDelta(n,q;d-1) &= (d-1)qn^{d-2} + O(q^2N^{d-3}), \end{aligned}$$

we get

$$\partial f/\partial n = c(c-1)xqm^c n^{c-2} + d(d-1)yqm^d n^{d-2} + O(q^2RN^{-3}).$$

If $xy > 0$, then

$$c_1 qRN^{-2} < |\partial f/\partial n| \le c_2 qRN^{-2} < 1/2$$

for some constants $c_1, c_2 > 0$. Thus by Lemma 4 we get

$$E_q \ll MN^2q^{-1}R^{-1}.$$

Now suppose $xy < 0$, $0 < \delta = o(qRN^{-2})$ is a parameter to be determined. Define

$$\begin{aligned} I &= \{t \in [N, 2N-q] \mid |\partial f/\partial t| \le \delta\}, \\ J &= \{t \in [N, 2N-q] \mid |\partial f/\partial t| > \delta\}. \end{aligned}$$

If $n \in I$, then we have

$$\begin{aligned} c(c-1)xqm^c n^{c-2} &= -d(d-1)yqm^d n^{d-2} + O(\delta + q^2 R N^{-3}) \\ &= -d(d-1)yqm^d n^{d-2}(1 + O(\delta N^2 (qR)^{-1} + qN^{-1})), \end{aligned}$$

which gives

$$\begin{aligned} n &= \left(\frac{-d(d-1)ym^d}{c(c-1)xm^c}\right)^{1/(c-d)} (1 + O(\delta N^2 (qR)^{-1} + qN^{-1}))^{1/(c-d)} \\ &= \left(\frac{-d(d-1)ym^d}{c(c-1)xm^c}\right)^{1/(c-d)} (1 + O(\delta N^2 (qR)^{-1} + qN^{-1})) \\ &= \left(\frac{-d(d-1)ym^d}{c(c-1)xm^c}\right)^{1/(c-d)} + (q + \delta N^3 q^{-1} R^{-1}). \end{aligned}$$

Thus

$$|I| \ll q + \delta N^3 q^{-1} R^{-1}. \tag{4.7}$$

By Lemma 4 we get

$$\sum_{n \in J,\, |\partial f/\partial n| \le 500^{-1}} e(f(m,n)) \ll \delta^{-1}. \tag{4.8}$$

Thus we get

$$\sum_{n \sim N,\, |\partial f/\partial n| \le 500^{-1}} e(f(m,n)) \ll q + N^{3/2}(qR)^{-1/2}, \tag{4.9}$$

by choosing $\delta = (qR)^{1/2} N^{-3/2}$.

Combining the above, we get

$$\sum_{\substack{(m,n) \\ |\partial f/\partial n| \le 500^{-1}}} e(f(m,n)) \ll Mq + MN^{3/2}(qR)^{-1/2} + MN^2 (qR)^{-1}. \tag{4.10}$$

Summing over q we find

$$\begin{aligned} \sum_q \sum_{\substack{(m,n) \\ |\partial f/\partial n| \le 500^{-1}}} & e(f(m,n)) \\ &\ll MQ^2 + MN^{3/2} Q^{1/2} R^{-1/2} + MN^2 R^{-1} \log Q \ll X \log X, \end{aligned} \tag{4.11}$$

if we recall $X^{11/12} R^{-1/2} \ll M \ll X^{2/3}$.

Case 3: *For some i and j, $2 \le i + j \le 3$,*

$$\left|\frac{\partial^{i+j} f}{\partial m^i \partial n^j}\right| \le \frac{qR \log X}{QM^i N^{j+1}}. \tag{$*$}$$

Let $c(\gamma,0)=1$, $c(\gamma,n)=\gamma(\gamma-1)\dots(\gamma-n+1)$ for $n\neq 0$. Then

$$\frac{\partial^{i+j}f}{\partial m^i\partial n^j}=c(c,i)c(c,j)xm^{c-i}\Delta(n,q;c-j)\\ +c(d,i)c(d,j)ym^{d-i}\Delta(n,q;d-j).$$

Since $c(c,i)c(c,j)$ and $c(d,i)c(d,j)$ always have the same sign, we may suppose $xy<0$; otherwise there is no (m,n) satisfying $(*)$.

If (m,n) satisfies $(*)$, then

$$\begin{aligned}&c(c,i)c(c,j)xm^{c-i}\Delta(n,q;c-j)\\ &\qquad=-c(d,i)c(d,j)ym^{d-i}\Delta(n,q;d-j)+O\Big(\frac{qR\log X}{QM^iN^{j+1}}\Big)\\ &\qquad=-c(d,i)c(d,j)ym^{d-i}\Delta(n,q;d-j)\Big(1+O\Big(\frac{\log X}{Q}\Big)\Big),\end{aligned}$$

which implies that

$$\begin{aligned}m&=\Big(\frac{-c(d,i)c(d,j)y\Delta(n,q;d-j)}{c(c,i)c(c,j)x\Delta(n,q;c-j)}\Big)^{1/(c-d)}\Big(1+O\Big(\frac{\log X}{Q}\Big)\Big)^{1/(c-d)}\\ &=\Big(\frac{-c(d,i)c(d,j)y\Delta(n,q;d-j)}{c(c,i)c(c,j)x\Delta(n,q;c-j)}\Big)^{1/(c-d)}\Big(1+O\Big(\frac{\log X}{Q}\Big)\Big)\\ &=\Big(\frac{-c(d,i)c(d,j)y\Delta(n,q;d-j)}{c(c,i)c(c,j)x\Delta(n,q;c-j)}\Big)^{1/(c-d)}+O\Big(\frac{M\log X}{Q}\Big).\end{aligned}$$

Thus

$$\sum_{(m,n),\,(*)}e(f(m,n))\ll\frac{X\log X}{Q}$$

and

$$\sum_q\sum_{(m,n),\,(*)}e(f(m,n))\ll X\log X. \tag{4.12}$$

Now we turn to the most difficult part. We suppose that none of the conditions from Cases 0 to 3 holds. Without loss of generality, we suppose $\partial f/\partial n>0$. For any fixed $0\le j\le(\log 10Q)/\log 2$, let I_j denote the subinterval of $[N,2N-q]$ in which

$$\frac{2^jqR}{QN^3}<\Big|\frac{\partial^2f}{\partial n^2}\Big|\le\frac{2^{j+1}qR}{QN^3}.$$

We suppose $I_j=[A_j,B_j]$, say; A_j and B_j may depend on m, but this does not affect our final result.

By Lemma 1 we get

$$\sum_{n\in I_j} e(f(m,n)) = e(1/8)\sum_{v_1(m)<v\le v_2(m)} \frac{b_v e(s(m,v))}{\sqrt{|G(m,v)|}} + O(R(m,q,j)), \tag{4.13}$$

where

$$\begin{aligned}
&f_n(m,g(m,v)) = v,\\
&s(m,v) = f(m,g(m,v)) - vg(m,v),\\
&G(m,v) = f_{nn}(m,g(m,v)),\\
&R(m,q,j) = \log X + \frac{QN^2}{2^j qR} + \min\left(\frac{Q^{1/2}N^{3/2}}{2^{j/2}q^{1/2}R^{1/2}}, \frac{1}{\|v_1(m)\|}\right)\\
&\qquad\qquad + \min\left(\frac{Q^{1/2}N^{3/2}}{2^{j/2}q^{1/2}R^{1/2}}, \frac{1}{\|v_2(m)\|}\right),\\
&\frac{qR}{QN^2} \ll v_1(m), v_2(m) \ll \frac{qR}{N^2}.
\end{aligned}$$

Since

$$\begin{gathered}
qRN^{-2} \gg 1,\\
v_1'(m) = \frac{\partial^2 f}{\partial n\partial m}(m,B_j) \gg qRQ^{-1}M^{-1}N^{-2},\\
v_2'(m) = \frac{\partial^2 f}{\partial n\partial m}(m,A_j) \gg qRQ^{-1}M^{-1}N^{-2},
\end{gathered}$$

by Lemma 3 we get

$$\begin{aligned}
&\sum_{1\le q\le Q}\sum_{j\ge 0}\sum_m R(m,q,j) \qquad (4.14)\\
&\ll \sum_{1\le q\le Q}\sum_{j\ge 0}\left(M\log X + \frac{QMN^2}{2^j qR} + \frac{qR}{N^2}\cdot\frac{Q^{1/2}N^{3/2}}{2^{j/2}q^{1/2}R^{1/2}} + \frac{qR}{N^2}\cdot\frac{QMN^2}{qR}\right)\\
&\ll MQ^2\log^2 X + QMN^2R^{-1}\log X + Q^2R^{1/2}N^{-1/2}\\
&\ll X\log^2 X.
\end{aligned}$$

Let $v_1 = \min v_1(m), v_2 = \max v_2(m)$. Then

$$\sum_{M<m\le 2M}\sum_{v_1(m)<v\le v_2(m)} \frac{b_v e(s(m,v))}{\sqrt{|G(m,v)|}} \ll \sum_{v_1\le v\le v_2}\left|\sum_{m\in I_v}\frac{e(s(m,v))}{\sqrt{|G(m,v)|}}\right|, \tag{4.15}$$

where I_v is a subinterval of $[M,2M]$.

Now the problem is reduced to estimating the sum over m. We first prove that $|G(m,v)|^{-1/2}$ is monotonic. Let $g = g(m,v)$. Differentiating the equation $f_n(m, g(m, v)) = v$ over m we get

$$g_m(m, v) = -\frac{f_{nm}(m, g)}{f_{nn}(m, g)}. \tag{4.16}$$

Thus

$$G_m(m, v) = f_{mnn} + f_{nnn} g_m = \frac{f_{nnm} f_{nn} - f_{nnn} f_{nm}}{f_{nn}}. \tag{4.17}$$

We only need to consider $f_{nnm} f_{nn} - f_{nnn} f_{nm}$, since f_{nn} always has the same sign. Here we remark that we actually consider subintervals of $[M, 2M]$ such that f_{nn} is always positive or negative. This is so for other derivatives.

We now compute the corresponding derivatives. We have

$$\begin{aligned} f_{nm} &= c^2 x m^{c-1} \Delta(g, q; c-1) + d^2 y m^{d-1} \Delta(g, q; d-1) \\ &= c^2(c-1) x q m^{c-1} g^{c-2} + d^2(d-1) y q m^{d-1} g^{d-2} + O\left(\frac{q^2 R}{M N^3}\right). \end{aligned}$$

Since $|f_{nm}| > (qR \log X)/(QMN^2)$, we have

$$f_{nm} = (c^2(c-1) x q m^{c-1} g^{c-2} + d^2(d-1) y q m^{d-1} g^{d-2}) \left(1 + O\left(\frac{Q^2}{N \log X}\right)\right).$$

Similarly,

$$\begin{aligned} f_{nn} &= (c(c-1)(c-2) x q m^c g^{c-3} + d(d-1)(d-2) y q m^d g^{d-3}) \\ &\quad \times \left(1 + O\left(\frac{Q^2}{N \log X}\right)\right), \\ f_{nnm} &= (c^2(c-1)(c-2) x q m^{c-1} g^{c-3} + d^2(d-1)(d-2) y q m^{d-1} g^{d-3}) \\ &\quad \times \left(1 + O\left(\frac{Q^2}{N \log X}\right)\right), \\ f_{nnn} &= (D(c) x q m^c g^{c-4} + D(d) y q m^d g^{d-4}) \left(1 + O\left(\frac{Q^2}{N \log X}\right)\right), \end{aligned}$$

where $D(\gamma) = \gamma(\gamma-1)(\gamma-2)(\gamma-3)$.

For simplicity, we write $s = x m^c g^c$, $t = y m^d g^d$. Then we get

$$\begin{aligned} & f_{nn} f_{nnm} - f_{nm} f_{nnn} \\ &\qquad = m^{-1} g^{-6} (A s^2 + 2B st + C t^2) \left(1 + O\left(\frac{Q^2}{N \log X}\right)\right), \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} A &= c^3 (c-2)^2 (c-2) < 0, \\ B &= c(c-1) d(d-1)(3cd - c^2 - d^2 - c - d) < 0, \\ C &= d^3 (d-2)^2 (d-2) < 0. \end{aligned}$$

We only need to show that

$$As^2 + 2Bst + Ct^2 \neq 0. \tag{4.19}$$

If $xy > 0$, (4.19) is obvious. Now suppose $xy < 0$. It is easy to show that

$$B^2 - AC = c^2(c-1)^2 d^2(d-1)^2(c-d)^2(2c+2d+1+c^2+d^2-4cd) > 0.$$

Thus there exist constants a_1, a_2, b_1, b_2 such that

$$As^2 + 2Bst + Ct^2 = (a_1 s + b_1 t)(a_2 s + b_2 t).$$

Since $A < 0, B < 0, C < 0$, it can be easily seen that $a_1 b_1 > 0, a_2 b_2 > 0$. Now we recall that s and t do not satisfy the condition of Case 0. Taking $\sigma = \frac{1}{2}\min(|a_1|, |a_2|, |b_1|^{-1}, |b_2|^{-1})$ in Case 0, we obtain

$$|a_1 s + b_1 t| > \frac{R}{Q^{1/2}\log X}, \qquad |a_2 s + b_2 t| > \frac{R}{Q^{1/2}\log X}.$$

Thus

$$|As^2 + 2Bst + Ct^2| \geq \frac{R^2}{Q\log^2 X}.$$

This is the reason why we consider Case 0.

By the above discussion we know that $|G(m,v)|$ is monotonic in m. So is $|G(m,v)|^{-1/2}$.

Now we compute $s_{mm}(m,v)$. We have

$$\begin{aligned} s_m(m,v) &= f_m(m,g) + f_n(m,g)g_m - vg_m = f_m(m,g), \\ s_{mm}(m,v) &= f_{mm}(m,g) + f_{mn}(m,g)g_m = (f_{mm}f_{nn} - f_{mn}^2)/f_{nn}. \end{aligned} \tag{4.20}$$

Similar to G_m, we have

$$f_{mm}f_{nn} - f_{mn}^2 = -\frac{2q^2}{m^2 n^4}(A_1 s^2 + B_1 st + C_1 t^2)\left(1 + O\left(\frac{Q^2}{N\log X}\right)\right),$$

where $A_1 = c^3(c-1)^2$, $B_1 = c(c-1)d(d-1)(c+d)$, $C_1 = d^3(d-1)^2$, $B_1^2 - 4A_1C_1 > 0$. Now if $xy > 0$, we immediately get

$$|f_{mm}f_{nn} - f_{mn}^2| \gg \frac{q^2R^2}{M^2N^4};$$

if $xy < 0$, then similar to G_m, we have

$$|A_1 s^2 + B_1 st + C_1 t^2| \gg \frac{R^2}{Q\log^2 X},$$

which implies

$$|f_{mm}f_{nn} - f_{mn}^2| \gg \frac{q^2R^2}{QM^2N^4\log^2 X}.$$

Combining the above, we get

$$|s_{mm}| \gg \frac{qR}{QM^2N\log^2 X}. \tag{4.21}$$

On the other hand, we trivially have

$$|s_{mm}| \ll |f_{mm}| + |f_{mn}g_m| \ll \frac{qR}{M^2N} + \frac{qR}{N^2M}\cdot\frac{N}{M} \ll \frac{qR}{M^2N}. \tag{4.22}$$

Now let

$$I_{v,l} = \left\{ m \in I_v \;\middle|\; \frac{2^l qR}{QM^2N\log^2 X} < |s_{mm}| \le \frac{2^{l+1}qR}{QM^2N\log^2 X} \right\},$$
$$0 \le l \le \log(Q\log X)/\log 2.$$

Then by partial summation and Lemma 4 we get

$$\begin{aligned} &\sum_{q=1}^{Q}\sum_{j\ge 0}\sum_{v=v_1}^{v_2}\Bigg|\sum_{m\in I_v}\frac{e(s(m,v))}{\sqrt{|G(m,v)|}}\Bigg| \\ &\ll \sum_{q=1}^{Q}\sum_{j\ge 0}\sum_{v=v_1}^{v_2}\sum_{l\ge 0}\Bigg|\sum_{m\in I_{v,l}}\frac{e(s(m,v))}{\sqrt{|G(m,v)|}}\Bigg| \\ &\ll \sum_{q=1}^{Q}\sum_{j\ge 0}\sum_{v=v_1}^{v_2}\sum_{l\ge 0}\left(\frac{QN^3}{qR}\right)^{1/2} \\ &\quad\times\left(M\left(\frac{2^l qR}{QM^2N\log^2 X}\right)^{1/2} + \left(\frac{QM^2N\log^2 X}{2^l qR}\right)^{1/2}\right) \\ &\ll \sum_{q=1}^{Q}\sum_{j\ge 0}\sum_{v=v_1}^{v_2}\left(\frac{QN^3}{qR}\right)^{1/2}\left(\frac{(qR)^{1/2}}{N^{1/2}} + \frac{M(QN\log^2 X)^{1/2}}{(qR)^{1/2}}\right) \\ &\ll \sum_{q=1}^{Q}\sum_{j\ge 0}\frac{qR}{N^2}\left(\frac{QN^3}{qR}\right)^{1/2}\left(\frac{(qR)^{1/2}}{N^{1/2}} + \frac{M(QN\log^2 X)^{1/2}}{(qR)^{1/2}}\right) \\ &\ll Q^{5/2}RN^{-1}\log^2 X + MQ^2\log^2 X \\ &\ll X\log^2 X, \end{aligned} \tag{4.23}$$

if we recall the condition $M \ll \min(x^{2/3}, x^{19/12}R^{-1})$. This completes the proof of Lemma 7.

LEMMA 8. *Suppose a_m and b_n are complex numbers such that*

$$\sum_{m\sim M}|a_m|^2 \ll M\log^{2A} M, \qquad \sum_{n\sim N}|b_n|^2 \ll N\log^{2A} N, \qquad A>0, B>0.$$

Then for $X^{1/6} \ll N \ll \min(X^{3/2}R^{-1}, RX^{-1/3})$, we have

$$S_{\mathrm{II}} = \sum_{m\sim M}\sum_{n\sim N} a_m b_n e(x(mn)^c + y(mn)^d) \ll X^{11/12}\log^{A+B+1} X. \tag{4.24}$$

P r o o f. Take $Q = [X^{1/6} \log^{-1} X] = o(N)$. Then by Cauchy's inequality and Lemma 2 again we get

$$(4.25)\qquad |S_{\mathrm{II}}|^2 \ll \frac{X^2 \log^{2A+2B} X}{Q} + \frac{X \log^{2A} X}{Q} \sum_{q=1}^{Q} \sum_{n} |b_n b_{n+q}| \Big| \sum_{m\sim M} e(f(m,n)) \Big|,$$

where $f(m,n)$ is defined as in the proof of Lemma 7.

By Lemma 6 we get

$$(4.26)\qquad \sum_{m\sim M} e(f(m,n)) \ll q^{1/2} R^{1/2} N^{-1/2} + M N^{1/3} q^{-1/3} R^{-1/3}.$$

Notice that for fixed q, we have

$$(4.27)\qquad \sum_{n} |b_n b_{n+q}| \ll \sum_{n} |b_n|^2 + \sum_{n} |b_{n+q}|^2 \ll N \log^{2B} N.$$

The conclusion follows from the above three estimates.

Now we prove our Proposition. Let

$$D = \min(X^{2/3}, X^{19/12} R^{-1}), \qquad E = \min(X^{3/2} R^{-1}, R X^{-1/3}), \qquad F = X^{1/6}.$$

Then it is easy to check that under our assumptions we have

$$DE > X, \qquad X/D > (2X)^{1/13}, \qquad F^2 < E.$$

Using Heath-Brown's identity ($k = 13$) we know that $S(x,y)$ can be written as $O(\log^{26} X)$ exponential sums of the form

$$T = \sum_{n_1\sim N_1} \cdots \sum_{n_{26}\sim N_{26}} a_1(n_1)\ldots a_{26}(n_{26}) e(x(n_1\ldots n_{26})^c + y(n_1\ldots n_{26})^d),$$

where

$$\begin{aligned}
&N_i < n_i \le 2N_i \ (i = 1,\ldots,26), \qquad X \ll N_1 \ldots N_{26} \ll X,\\
&N_i \le (2X)^{1/13} \ (i = 14,\ldots,26),\\
&a_1(n_1) = \log n_1, \qquad a_i(n_i) = 1 \ (i = 2,\ldots,13),\\
&a_i(n_i) = \mu(n_i) \ (i = 14,\ldots,26).
\end{aligned}$$

Some n_i may only take value 1. It suffices to show that for each T we have

$$(4.28)\qquad T \ll X^{11/12} \log^{630} X.$$

We consider three cases.

Case 1: *There is an N_j such that $N_j \ge X/D$.* Since $X/D > X^{1/13}$, it follows that $1 \le j \le 13$. Without loss of generality, suppose $j = 1$. Let $m = n_2 n_3 \ldots n_{26}$, $a_m = \sum_{m=n_2n_3\ldots n_{26}} \mu(n_{14})\ldots\mu(n_{26}) \ll d_{25}(m)$, $n = n_1$.

Then T is a sum of type I. By partial summation, Lemma 7 and a divisor argument we get

$$T \ll X^{11/12} \log^{630} X.$$

CASE 2: *There is an N_j such that $F \le N_j < X/D \le E$.* In this case we take $n = n_j$, $m = \prod_{i \ne j} n_i$. Then T forms a sum of type II and (4.28) follows from Lemma 8.

CASE 3: $N_j < F$ $(j = 1, \ldots, 26)$. Without loss of generality, we suppose $N_1 \ge \ldots \ge N_{26}$. Let $1 \le l \le 26$ be an integer such that

$$N_1 \ldots N_{l-1} \le F, \qquad N_1 \ldots N_l > F.$$

It is easy to check that $3 \le l \le 23$. We have

$$F < N_1 \ldots N_l = (N_1 \ldots N_{l-1}) N_l < F^2 < E.$$

Let $n = n_1 \ldots n_l$, $m = n_{l+1} \ldots n_{26}$, $a_n = \prod_{i=1}^{l} a_i(n_i)$, $b_m = \prod_{i=l+1}^{26} a_i(n_i)$. Then T forms a sum of type II and (4.28) follows from Lemma 8.

Now the Proposition follows from the above three cases.

Acknowledgements. Prof. Tolev kindly sent his papers to the author and the author is very grateful to him.

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Received on 11.8.1998 (3445)