ACTA ARITHMETICA
XCII.3 (2000)

Multiple exponential sums with monomials

by

Xiaodong Cao (Beijing) and Wenguang Zhai (Jinan)

1. Introduction. In 1989, Fouvry and Iwaniec [4] used the double large sieve inequality due to Bombieri and Iwaniec [1] to estimate multiple exponential sums with monomials

$$\sum_{m_1\sim M_1}\cdots\sum_{m_k\sim M_k}\varphi_{m_1}\cdots\varphi_{m_k}e\biggl(T\frac{m_1^{\alpha_1}\cdots m_k^{\alpha_k}}{M_1^{\alpha_1}\cdots M_k^{\alpha_k}}\biggr),$$

where $e(\theta):=e^{2\pi i\theta}$, $\alpha_j\in\mathbb{R}$, $\varphi_{m_j}\in\mathbb{C}$. This method is often superior to the classical methods (for comparison, see [5]–[7], [10] for example).

Let A be the A-process of Weyl–van der Corput and D the process of applying the large sieve inequality. Then the method of Fouvry and Iwaniec consists in an application of the AD process, which ends with a spacing problem for the points $t(m,q)$ and gives rise to a lot of applications. This method was sharpened in Liu [8]. Recently, this spacing problem was further improved in Sargos and Wu [9] by an ingenious new idea.

But sometimes if T is very large, one should use the A^2D process (i.e. two times the A-process followed by the D-process), which naturally involves the spacing problem for the points $t(m,q_1,q_2):=t(m+q_2,q_1)-t(m-q_2,q_1)$ for $m\sim M$, $q_1\sim Q_1$ and $q_2\sim Q_2$. Recently, the authors [2] studied the spacing problem of $t(m,q_1,q_2)$ with q_1 fixed by the method of Fouvry and Iwaniec [4], and used the result to study some problems in number theory (see Cao and Zhai [2], [3]). This idea plays a key role in the two papers.

Our aim is to give better results on the spacing of $t(m,q_1,q_2)$. In Section 2, some preliminary lemmas are given. In Section 3, we use the method of Fouvry and Iwaniec to study the spacing of $t(m,q_1,q_2)$ for all m, q_1, q_2, which can be used to deal with the case of q_1 near to q_2 in applications. In Section 4 we use the new idea of Sargos and Wu [9] and the method in

2000 *Mathematics Subject Classification*: Primary 11L07; Secondary 11N25, 11N45.

This work is supported by the National Natural Science Foundation of China (Grant No. 19801021) and Natural Science Foundation of Shandong Province (Grant No. Q98A02210).

Section 3 to study the same spacing problem, which is used in application for q_2 larger than q_1. The spacing problems for other related points are considered in Section 5. In Section 6, some estimates for exponential sums with monomials are given. Applications of these results will be given elsewhere.

Notations. $m \sim M$ means that $M < m \le 2M$; $f \asymp g$ means $f \ll g \ll f$; $\|t\| := \min_{n\in\mathbb{Z}} |n-t|$ and $\psi(t) := t-[t]-1/2$ for real t. We also use ε to denote an arbitrarily small positive constant, ε' to denote an unspecified constant multiple of ε, and ε^* to denote a fixed suitably small positive number. We also use the notations

$$C_\alpha^m := \frac{\alpha(\alpha-1)\dots(\alpha-m+1)}{m!} \quad \text{and} \quad C_\alpha^0 = 1.$$

Throughout the paper, we always set (for $\alpha \ne 0,1,2,3$)

$$t(m,q) := t(m,q;\alpha) = (m+q)^\alpha - (m-q)^\alpha,$$
$$t(m,q_1,q_2) := t(m,q_1,q_2;\alpha) = t(m+q_2,q_1) - t(m-q_2,q_1).$$

2. Some preliminary lemmas. Let $\delta > 0$, $M \ge 1$ and $f \in C[M,2M]$. Define

$$\mathcal{R}(f,\delta) := |\{m \in [M,2M] : \|f(m)\| < \delta\}|, \tag{2.1}$$

which denotes the number of integer points in the δ-neighbourhood of $f(t)$ for $M \le t \le 2M$. In this section all constants implied by "$\ll$", "O" and "$\asymp$" may depend only on α and β.

We set $\mathbf{h} := (n,\widetilde{n},q,\widetilde{q})$ and $\mathbf{H} := (N,Q)$. We write $\mathbf{h} \sim \mathbf{H}$ for $n,\widetilde{n} \sim N$, $q,\widetilde{q} \sim Q$ and define $\mathcal{H} := \{\mathbf{h} : \mathbf{h} \sim \mathbf{H}\}$. The following lemma is Theorem 2 of Sargos and Wu [9].

LEMMA 2.1. *Let α, $\beta \in \mathbb{R}$ with $\alpha\beta \ne 0$ and let $M \ge 1$, $N \ge 1$, $Q \ge 1$ and $\mathcal{L} := \log(2MNQ)$. Let $\mathcal{H}^* \subseteq \mathcal{H}$ and $g_{\mathbf{h}}(t) \in C^2[M,2M]$ with $g'_{\mathbf{h}}(t) \asymp \mu M$, $g''_{\mathbf{h}}(t) \asymp \mu$ for $t \sim M$, $\mathbf{h} \in \mathcal{H}^*$. Put $f_{\mathbf{h}}(t) := v(\mathbf{h})t + g_{\mathbf{h}}(t)$ with $v(\mathbf{h}) := \widetilde{n}^\alpha\widetilde{q}^\beta/(n^\alpha q^\beta)$. Then*

$$\begin{aligned}\sum_{\mathbf{h}\in\mathcal{H}^*} \mathcal{R}(f_{\mathbf{h}},\delta) \ll{}& \delta M|\mathcal{H}^*| + |\mathcal{H}^*| + (\delta NQ|\mathcal{H}^*|\mu^{-1}\mathcal{L})^{1/2}\\ &+ \mu^{1/3}M|\mathcal{H}^*| + (\delta^2N^2Q^2|\mathcal{H}^*|^2\mu^{-1})^{1/3}\\ &+ NQ(\delta M|\mathcal{H}^*|)^{1/2}.\end{aligned} \tag{2.2}$$

LEMMA 2.2. *Let $\alpha(\alpha-1)(\alpha-2)(\alpha-3) \ne 0$, $\alpha \in \mathbb{R}$ and $q_1,q_2 > 0$. Then for $|u| \le 1/(10(q_1+q_2))$, $J \in \mathbb{N}$, we have*

$$\begin{aligned}\sigma(u) = \sigma(u,q_1,q_2) &:= \left(\frac{t(1,q_1u,q_2u;\alpha)}{4\alpha(\alpha-1)q_1q_2u^2}\right)^{1/(\alpha-2)}\\ &= 1 + \sum_{j=1}^J \sigma_j(q_1,q_2)u^{2j} + O_J(((q_1+q_2)u)^{2J+2}),\end{aligned} \tag{2.3}$$

where

$$\sigma_j(q_1,q_2)=\sum_{(SC)}\frac{k_1!}{k_2!k_3!\ldots k_{J+1}!}\cdot\frac{C^{k_1}_{(\alpha-2)^{-1}}}{(2C^2_\alpha)^{k_1}}\times\Big(\prod_{i=2}^{J+1}\Big(C^{2i}_\alpha\sum_{k=1}^{i}C^{2k-1}_{2i}q_1^{2(k-1)}q_2^{2(i-k)}\Big)^{k_i}\Big)$$
$$:=\sum_{k=0}^{j}a_{j,k}q_1^{2(j-k)}q_2^{2k}\qquad(a_{j,k}=a_{j,j-k},a_{j,k}\text{ is real}),\tag{2.4}$$

and where SC *denotes the following conditions*: $k_i\ge0$ $(i=1,\ldots,J+1)$, $k_2+2k_3+\ldots+Jk_{J+1}=j$, $k_2+k_3+\ldots+k_{J+1}=k_1$.

In particular

$$\sigma_1=\frac{2C^4_\alpha C^1_{(\alpha-2)^{-1}}}{C^2_\alpha}(q_1^2+q_2^2):=C(q_1^2+q_2^2),\tag{2.5}$$

$$\sigma_2=\frac{4C^2_{(\alpha-2)^{-1}}(C^4_\alpha)^2}{(C^2_\alpha)^2}(q_1^2+q_2^2)^2+\frac{C^1_{(\alpha-2)^{-1}}C^6_\alpha}{C^2_\alpha}(3q_1^4+10q_1^2q_2^2+3q_2^4).\tag{2.6}$$

Proof. For $\mu\ne0$ and $|x|<1/2$ we have

$$(1+x)^\mu=\sum_{m=0}^{\infty}C^m_\mu x^m.\tag{2.7}$$

From (2.7) one easily gets

$$\begin{aligned}t(1,q_1u,q_2u;\alpha)&=\sum_{m=0}^{\infty}C^m_\alpha\{((q_1+q_2)u)^m-(-(q_1-q_2)u)^m\}\\&\quad+\sum_{m=0}^{\infty}C^m_\alpha\{-((q_1-q_2)u)^m+(-(q_1+q_2)u)^m\}\\&=2\sum_{m=1}^{\infty}C^{2m}_\alpha\{(q_1+q_2)^{2m}-(q_1-q_2)^{2m}\}u^{2m}.\end{aligned}\tag{2.8}$$

It is well known that

$$(x_1+\ldots+x_m)^n=\sum_{\substack{k_1+\ldots+k_m=n\\k_i\ge0,\,i=1,\ldots,m}}\frac{n!}{k_1!\ldots k_m!}x_1^{k_1}\ldots x_m^{k_m}.\tag{2.9}$$

It follows from (2.9) that

$$(2.10)\quad (q_1+q_2)^{2m}-(q_1-q_2)^{2m} = \sum_{n=0}^{2m} C_{2m}^n q_1^n q_2^{2m-n} - \sum_{n=0}^{2m} C_{2m}^n q_1^n (-q_2)^{2m-n} = 2\sum_{k=1}^{m} C_{2m}^{2k-1} q_1^{2k-1} q_2^{2m-2k+1}.$$

Combining (2.8) and (2.10) we obtain

$$(2.11)\qquad \sigma(u) = (1 + A(q_1,q_2,u) + O((q_1u+q_2u)^{2J+2}))^{1/(\alpha-2)},$$

where

$$A(q_1,q_2,u) = \sum_{m=2}^{J+1} \frac{C_\alpha^{2m}}{2C_\alpha^2}\Big(\sum_{k=1}^{m} C_{2m}^{2k-1} q_1^{2(k-1)} q_2^{2(m-k)}\Big) u^{2m-2}.$$

Now, using (2.11), (2.7) and the mean-value theorem, we have

$$(2.12)\qquad \sigma(u) = 1 + \sum_{k_1=1}^{J+1} C_{(\alpha-2)^{-1}}^{k_1} A^{k_1}(q_1,q_2,u) + O(((q_1+q_2)u)^{2J+2}).$$

Finally, by (2.12), (2.9) and after a simple calculation, we easily deduce

$$\begin{aligned}\sigma(u) = 1 &+ \sum_{k_1=1}^{J+1} \frac{C_{(\alpha-2)^{-1}}^{k_1}}{(2C_\alpha^2)^{k_1}} \Bigg(\sum_{\substack{k_2+k_3+\ldots+k_{J+1}=k_1 \\ k_i\ge 0,\, i=2,3,\ldots,J+1}} \frac{k_1!}{k_2!k_3!\ldots k_{J+1}!} \\ &\times \Big\{ \prod_{i=2}^{J+1} \Big(C_\alpha^{2i} \sum_{k=1}^{i} C_{2i}^{2k-1} q_1^{2(k-1)} q_2^{2(i-k)}\Big)^{k_i}\Big\}\Big\{\prod_{i=2}^{J+1} u^{2(i-1)k_i}\Big\}\Bigg) \\ &+ O(((q_1+q_2)u)^{2J+2}),\end{aligned}$$

and from the above expression Lemma 2.2 can be proved at once.

Lemma 2.3. *Let $\alpha,\beta\in\mathbb{R}$, $\alpha(\alpha-1)\beta(\alpha\beta-\beta-1)\neq 0$, $r,q>0$ and let $\gamma := 1/(\alpha\beta-\beta-1)$. Then for $|u|\le 1/(10(r+q))$, $J\in\mathbb{N}$,*

$$\begin{aligned}(2.13)\qquad \sigma^*(u) = \sigma^*(u,q,r) &:= \Big(\frac{t(1,qu,ru;\alpha)}{2(\alpha-1)(2\alpha)^\beta q^\beta r u^{\beta+1}}\Big)^\gamma \\ &= 1 + \sum_{j=1}^{J} \sigma_j^*(q,r) u^{2j} + O(((q+r)u)^{2J+2}),\end{aligned}$$

where

$$\begin{aligned}(2.14)\qquad \sigma_1^*(q,r) = {}&\gamma\Big(\frac{4C_\alpha^4}{\alpha(\alpha-1)} + \frac{6C_\beta^2}{\alpha\beta} + \frac{(\alpha-1)C_\beta^3}{\beta}\Big) r^2 \\ &+ \gamma\Big(\frac{4C_\alpha^4}{\alpha(\alpha-1)} + \frac{2C_\beta^2}{\alpha\beta}\Big) q^2,\end{aligned}$$

$$\sigma_j^*(q,r)=\sum_{i=0}^{j} b_{j,i}q^{2i}r^{2(j-i)}, \tag{2.15}$$

where $b_{j,i}=b_{j,i}(\alpha,\beta)$ *is real for all* $i,j\ge 0$.

Proof. The proof is similar to that of Lemma 2.2, so we omit the details. We notice that the degree of u in the Taylor expansion of $\sigma^*(u,q,r)$ is always even since $\sigma^*(-u,q,r)=\sigma^*(u,q,r)$. Moreover, we have $\sigma^*(u,q,-r)=\sigma^*(u,q,r)$.

3. Spacing problem for the points $t(m,q_1,q_2)$ **(I).** This section is devoted to investigating the spacing problem for the points $t(m,q_1,q_2)$ with q_1 "near" to q_2 by the method of Fouvry and Iwaniec [4]. Throughout this and the next section all constants implied by "$\ll$", "O" and "$\asymp$" depend at most on α and ε (or ε^*); we also use the following notations. Let $M\ge 10$, $Q_1\ge 1$, $Q_2\ge 1$, $\eta>0$, $\delta>0$ and $\Delta>0$. We set $T:=M^{\alpha-2}Q_1Q_2$ and $\mathcal{L}:=\log(2MQ_1Q_2)$. Therefore for $m\sim M$, $q_1\sim Q_1$, $q_2\sim Q_2$ and $Q_1+Q_2<M/3$, we have $t(m,q_1,q_2)\asymp T$.

Let $\mathcal{F}(M,Q_1,Q_2,\Delta)$ denote the number of sextuplets $(m,\widetilde{m},q_1,\widetilde{q}_1,q_2,\widetilde{q}_2)$ with $m,\widetilde{m}\sim M$, $q_1,\widetilde{q}_1\sim Q_1$ and $q_2,\widetilde{q}_2\sim Q_2$, satisfying

$$|t(m,q_1,q_2)-t(\widetilde{m},\widetilde{q}_1,\widetilde{q}_2)|\le \Delta T. \tag{3.1}$$

Theorem 1. *Let* $1\le Q_1\le Q_2\le M^{2/3-\varepsilon}$. *Then*

$$\begin{aligned}\mathcal{F}(M,Q_1,Q_2,\Delta)M^{-2\varepsilon}\ll MQ_1Q_2+\Delta(MQ_1Q_2)^2\\ +M^{-2}Q_1^2Q_2^6+Q_1^2Q_2^{8/3}.\end{aligned} \tag{3.2}$$

We set

$$A:=\left(\frac{q_1q_2}{\widetilde{q}_1\widetilde{q}_2}\right)^{1/(\alpha-2)},\qquad B:=A\sigma_1(q_1,q_2)-A^{-1}\sigma_1(\widetilde{q}_1,\widetilde{q}_2),$$

where $\sigma_1(q_1,q_2)$ is defined by (2.5). The following lemma is an analogue of Lemma 4 of Fouvry and Iwaniec [4].

Lemma 3.1. *Let* $D(M,Q_1,Q_2,\Delta)$ *denote the number of couples* $(m,\mathbf{q})$, $\mathbf{q}=(q_1,\widetilde{q}_1,q_2,\widetilde{q}_2)$, *with* $m\sim M$; $q_1,\widetilde{q}_1\sim Q_1$; $q_2,\widetilde{q}_2\sim Q_2$, *satisfying* $\|Am-Bm^{-1}\|\le\Delta$. *If* $1\le Q_1\le Q_2\le M^{3/4-\varepsilon^*}$, *then*

$$M^{-\varepsilon}D(M,Q_1,Q_2,\Delta)\ll MQ_1Q_2+\Delta M(Q_1Q_2)^2+Q_1^2Q_2^{8/3}. \tag{3.3}$$

Proof. Since $D(M,Q_1,Q_2,\Delta)$ is non-decreasing in Δ, we can assume that $(Q_1Q_2)^{-1}<\Delta<1$. First we estimate the number of couples $(m,\mathbf{q})$ with $|B|\le\Delta M$ by a crude argument. In this case we have $\|Am\|\le 2\Delta$, which implies

$$\left|\left(\frac{q_1q_2}{\widetilde{q}_1\widetilde{q}_2}\right)^{1/(\alpha-2)}-\frac{\widetilde{m}}{m}\right|\ll\Delta M^{-1}. \tag{3.4}$$

By Lemma 1 of Fouvry and Iwaniec [4], the number of quadruples $(m,\widetilde{m},n,\widetilde{n})$ with $m,\widetilde{m}\sim M$, $n,\widetilde{n}\sim Q_1Q_2$ such that

$$\left|\left(\frac{n}{\widetilde{n}}\right)^{1/(\alpha-2)}-\frac{\widetilde{m}}{m}\right|\ll \Delta M^{-1}$$

is

$$\ll MQ_1Q_2\mathcal{L}+(\Delta M^{-1})M^2(Q_1Q_2)^2=MQ_1Q_2\mathcal{L}+\Delta M(Q_1Q_2)^2. \tag{3.5}$$

Since $Q_1Q_2\le q_1q_2,\widetilde{q}_1\widetilde{q}_2\le 4Q_1Q_2$, the number of such $(m,\mathbf{q})$ is

$$\ll M(Q_1Q_2)^{1+\varepsilon}+\Delta M(Q_1Q_2)^{2+\varepsilon} \tag{3.6}$$

by a divisor argument.

By Lemma 3 of Fouvry and Iwaniec [4] (see also the proof of Lemma 2 in Liu [8]) we find that for $S=\Delta^{-1}$, the number of $(m,\mathbf{q})$ with $|B|\ge \Delta M$ is

$$\ll \Delta M(Q_1Q_2)^{2+\varepsilon}+E_1+E_2, \tag{3.7}$$

where

$$E_1:=E_1(M,Q_1,Q_2,\Delta)=\Delta\sum_{1\le s\le S}\sum_{\substack{q_1,\widetilde{q}_1\sim Q_1\\ q_2,\widetilde{q}_2\sim Q_2}}\min(M,1/\|As\|), \tag{3.8}$$

$$\begin{aligned}E_2&:=E_2(M,Q_1,Q_2,\Delta)\\ &=\Delta\sum_{1\le s\le S}\sum_{\substack{q_1,\widetilde{q}_1\sim Q_1,\,q_2,\widetilde{q}_2\sim Q_2\\ \|As\|\ll 3s|B|M^{-2}}}\min(M,(|B|sM^{-3})^{-1/2}).\end{aligned} \tag{3.9}$$

We estimate E_1 first. By a simple splitting argument for the interval $[1,\Delta^{-1}]$, we get for some $1\le S_1\le \Delta^{-1}=S$,

$$\begin{aligned}E_1&\ll \Delta\mathcal{L}\sum_{s\sim S_1}\sum_{\substack{q_1,\widetilde{q}_1\sim Q_1,\,q_2,\widetilde{q}_2\sim Q_2\\ \|As\|\le M^{-1}}}\min(M,1/\|As\|)\\ &\quad+\Delta\mathcal{L}\sum_{s\sim S_1}\sum_{\substack{q_1,\widetilde{q}_1\sim Q_1,\,q_2,\widetilde{q}_2\sim Q_2\\ \|As\|> M^{-1}}}\min(M,1/\|As\|)\\ &=\Delta\mathcal{L}T_1+\Delta\mathcal{L}T_2.\end{aligned} \tag{3.10}$$

Applying the same argument as for (3.6), we obtain

$$T_1\ll M\{S_1(Q_1Q_2)^{1+\varepsilon}+(M^{-1}S_1^{-1})S_1^2(Q_1Q_2)^{2+\varepsilon}\}. \tag{3.11}$$

To treat T_2, applying the splitting argument to the interval $[M^{-1},1]$, we get for some $M^{-1}\le\delta\le 1$,

$$T_2\ll\frac{\mathcal{L}}{\delta}\sum_{s\sim S_1}\sum_{\substack{q_1,\widetilde{q}_1\sim Q_1,\,q_2,\widetilde{q}_2\sim Q_2\\ \delta<\|As\|\le 2\delta}}1\ll\frac{\mathcal{L}}{\delta}\sum_{s\sim S_1}\sum_{\substack{q_1,\widetilde{q}_1\sim Q_1,\,q_2,\widetilde{q}_2\sim Q_2\\ \delta<\|As\|\le 2\delta}}1.$$

Similarly to the estimate for T_1 in (3.11), we also have

$$T_2 \ll \delta^{-1}\mathcal{L}\{S_1(Q_1Q_2)^{1+\varepsilon} + (\delta S_1^{-1})S_1^2(Q_1Q_2)^{2+\varepsilon}\}. \tag{3.12}$$

From (3.10)–(3.12) we obtain

$$E_1 \ll M(Q_1Q_2)^{1+\varepsilon}\mathcal{L} + (Q_1Q_2)^{2+\varepsilon}\mathcal{L}. \tag{3.13}$$

To treat E_2, we split the range of summation into subsets defined by $S_1 < s \le 2S_1$ and $R < |B| \le 2R$, where $1 \le S_1 \le \Delta^{-1}$ and $\Delta M \le R \le Q_2^2$. Thus for some S_1 and R,

$$\begin{aligned} E_2 &\ll \mathcal{L}^2\Delta \sum_{s\sim S_1} \sum_{\substack{q_1,\widetilde{q}_1\sim Q_1,\, q_2,\widetilde{q}_2\sim Q_2 \\ \|As\|\ll 3s|B|M^{-2},\, R<|B|\le 2R}} \min(M, (|B|sM^{-3})^{-1/2}) \\ &\ll \mathcal{L}^2\Delta \sum_{s\sim S_1} \sum_{\substack{q_1,\widetilde{q}_1\sim Q_1,\, q_2,\widetilde{q}_2\sim Q_2 \\ \|As\|\ll S_1RM^{-2}}} \min(M, (RS_1M^{-3})^{-1/2}) \\ &:= \mathcal{L}^2\Delta T_3. \end{aligned} \tag{3.14}$$

If $R \le MS_1^{-1}$, similarly to (3.11) we have

$$\Delta T_3 \ll \Delta\{S_1M(Q_1Q_2)^{1+\varepsilon} + S_1(Q_1Q_2)^{2+\varepsilon}\}. \tag{3.15}$$

If $R > MS_1^{-1}$, similarly to (3.12) we also have

$$\begin{aligned} \Delta T_3 &\ll \mathcal{L}\Delta(RS_1M^{-3})^{-1/2} \sum_{s\sim S_1} \sum_{\substack{q_1,\widetilde{q}_1\sim Q_1,\, q_2,\widetilde{q}_2\sim Q_2 \\ \|As\|\ll S_1RM^{-2}}} 1 \\ &\ll \mathcal{L}\Delta(RS_1M^{-3})^{-1/2}\{S_1(Q_1Q_2)^{1+\varepsilon} + (RM^{-2})S_1^2(Q_1Q_2)^{2+\varepsilon}\}. \end{aligned} \tag{3.16}$$

Combining (3.14)–(3.16) we get

$$E_2 \ll M(Q_1Q_2)^{1+\varepsilon}\mathcal{L} + (Q_1Q_2)^{2+\varepsilon}\mathcal{L} + (\Delta M)^{-1/2}(Q_1Q_2)^{2+\varepsilon}Q_2\mathcal{L}. \tag{3.17}$$

Finally, since $D(M, Q_1, Q_2, \Delta)$ is non-decreasing in Δ, we can replace Δ on the right-hand side of (3.17) by $\Delta + M^{-1}Q_2^{2/3}$; this new Δ and the condition $Q_1 \le Q_2 \le M^{3/4-\varepsilon^*}$ assure that $|B| < \Delta M^2$ holds. This completes the proof of Lemma 3.1.

Proof of Theorem 1. Inequality (3.1) implies that

$$|t^{1/(\alpha-2)}(m, q_1, q_2) - t^{1/(\alpha-2)}(\widetilde{m}, \widetilde{q}_1, \widetilde{q}_2)| \le \Delta T^{1/(\alpha-2)}.$$

By (2.3) with $u = 1/m$ and $J = 1$, we deduce that

$$|Am + Bm^{-1} - \widetilde{m}| \ll \Delta M + Q_2^4M^{-3}. \tag{3.18}$$

Clearly (3.18) implies $\|Am + Bm^{-1}\| \ll \Delta M + Q_2^4M^{-3}$, and for given $m, q_1, \widetilde{q}_1, q_2, \widetilde{q}_2$ the number of $\widetilde{m}$ is bounded by $O(1 + \Delta M)$. Applying

Lemma 3.1 with $\Delta = \Delta M + Q_2^4 M^{-3}$ we get

$$\begin{aligned}\mathcal{F}(M, Q_1, Q_2, \Delta) \ll (1 + \Delta M)&\\ &\times (M(Q_1Q_2)^{1+\varepsilon} + \Delta M^2 (Q_1Q_2)^{2+\varepsilon} + M^{-2} Q_1^{2+\varepsilon} Q_2^{6+\varepsilon})\\ &+ (1 + \Delta M) Q_1^{2+\varepsilon} Q_2^{8/3+\varepsilon} \mathcal{L}.\end{aligned}$$

If $\Delta M < 1$ we obtain the bound (3.2), otherwise the trivial estimate yields

$$\mathcal{F}(M, Q_1, Q_2, \Delta) \ll (1 + \Delta M) M (Q_1Q_2)^2 \ll \Delta (M Q_1 Q_2)^2.$$

Now the proof of Theorem 1 is finished.

4. Spacing problem for the points $t(m, q_1, q_2)$ **(II).** In this section, we shall use the new idea developed by Sargos and Wu [9] and the method in the proof of Theorem 1 to investigate the spacing problem for the points $t(m, q_1, q_2)$ for q_2 "larger" than q_1. Before stating our result, we need some notations. For $q_1 \sim Q_1$, $q_2 \sim Q_2$, we have $(q_1 q_2^{\alpha-1})^{1/(\alpha-2)} \in \mathbf{I}$, where $\mathbf{I} := [c(Q_1 Q_2^{\alpha-1})^{1/(\alpha-2)}, c'(Q_1 Q_2^{\alpha-1})^{1/(\alpha-2)}]$, and c, c' are two suitable positive constants depending on α only. Let $\mathbf{I}_\eta \subseteq \mathbf{I}$ with $|\mathbf{I}_\eta| = \eta (Q_1 Q_2^{\alpha-1})^{1/(\alpha-2)}$. We use $\mathcal{E}(M, Q_1, Q_2, \Delta, \mathbf{I}_\eta)$ to denote the number of sextuplets $(m, \widetilde{m}, q_1, \widetilde{q}_1, q_2, \widetilde{q}_2)$ with $m, \widetilde{m} \sim M$, $q_1, \widetilde{q}_1 \sim Q_1$, $q_2, \widetilde{q}_2 \sim Q_2$, such that

$$\begin{gathered}|t(m, q_1, q_2) - t(\widetilde{m}, \widetilde{q}_1, \widetilde{q}_2)| \le \Delta T,\\ (q_1 q_2^{\alpha-1})^{1/(\alpha-2)} \in \mathbf{I}_\eta, \qquad (\widetilde{q}_1 \widetilde{q}_2^{\alpha-1})^{1/(\alpha-2)} \in \mathbf{I}_\eta.\end{gathered} \tag{4.1}$$

THEOREM 2. *If* $Q_1 \ge 1, Q_1 M^\varepsilon \le Q_2 \le M^{1-\varepsilon}$ *and* $Q_1 Q_2 \le M^{3/2-\varepsilon}$, *then there exists an*

$$\eta = \eta(M, Q_1, Q_2) \in \left[\max\left(\frac{Q_1^2}{Q_2^2}, \frac{3\mathcal{L}}{Q_1Q_2}\right), \frac{c'}{c} - 1\right]$$

such that

$$\begin{aligned}\eta^{-1} M^{-\varepsilon} \sum_{0 \le k \le K} &\mathcal{E}(M, Q_1, Q_2, \Delta, \mathbf{I}_{\eta,k})\\ &\ll M Q_1 Q_2 + \Delta M^2 Q_1^2 Q_2^2 + (M Q_1^7 Q_2^9)^{1/4} + M^{-2} Q_1^4 Q_2^4\\ &\quad + (\Delta M^4 Q_1^{15} Q_2^{17})^{1/8} + (\Delta M^4 Q_1^3 Q_2)^{1/2} + (Q_1^{13} Q_2^{15})^{1/6}\\ &\quad + (\Delta M^2 Q_1^8 Q_2^{10})^{1/4} + (M^{-1} Q_1^5 Q_2^6)^{1/2},\end{aligned} \tag{4.2}$$

where $\mathbf{I}_{\eta,k} := [a_k, (1+\eta) a_k]$, $a_k := (1+\eta)^k c (Q_1 Q_2^{\alpha-1})^{1/(\alpha-2)}$, *and* $K := [\log(c'/c)/\eta]$.

Define

$$\mathbf{q} := (q_1, \widetilde{q}_1, q_2, \widetilde{q}_2), \qquad v(\mathbf{q}) := \left(\frac{q_1 q_2}{\widetilde{q}_1 \widetilde{q}_2}\right)^{1/(\alpha-2)},$$
$$\mathbf{Q} := \{\mathbf{q} : q_1, \widetilde{q}_1 \sim Q_1;\ q_2, \widetilde{q}_2 \sim Q_2\},$$

$$v_1(\mathbf{q}) := v(\mathbf{q})\sigma_1(q_1, q_2) - v^{-1}(\mathbf{q})\sigma_1(\widetilde{q}_1, \widetilde{q}_2),$$

where $\sigma_1(q_1, q_2)$ is given by (2.5), and we define $\psi_J := \psi_J(m, \mathbf{q})$ by the recurrence relation

$$\psi_1 := v_1(\mathbf{q})m^{-1}, \tag{4.3}$$

$$\psi_J = \sum_{j=1}^{J} \{v(\mathbf{q})\sigma_j(q_1, q_2)m^{-2j+1} - \sigma_j(\widetilde{q}_1, \widetilde{q}_2)(v(\mathbf{q})m + \psi_{J-1})^{-2j+1}\} \tag{4.4}$$

for $J \geq 2$.

Let $\mathcal{E}_0(M, Q_1, Q_2, \delta, \mathbf{I}_\eta)$ be the number of couples $(m, \mathbf{q})$ with $m \sim M, \mathbf{q} \sim \mathbf{Q}$ such that

$$\begin{gathered} \|v(\mathbf{q})m + \psi_J(m, \mathbf{q})\| \leq \delta, \\ (q_1 q_2^{\alpha-1})^{1/(\alpha-2)} \in \mathbf{I}_\eta, \quad (\widetilde{q}_1 \widetilde{q}_2^{\alpha-1})^{1/(\alpha-2)} \in \mathbf{I}_\eta. \end{gathered} \tag{4.5}$$

LEMMA 4.1. *If* $1 \leq Q_1 \leq Q_2 \leq \varepsilon^* M$, *then for any* $J \geq 1$,

$$\begin{aligned} \mathcal{E}(M, Q_1, Q_2, \Delta, \mathbf{I}_\eta) &\ll (1 + \Delta M + M^{-2J-1} Q_2^{2J+2}) \\ &\quad \times \mathcal{E}_0(M, Q_1, Q_2, \Delta M + M^{-2J-1} Q_2^{2J+2}, \mathbf{I}_\eta). \end{aligned} \tag{4.6}$$

Proof. Obviously it is sufficient to prove

$$\widetilde{m} = v(\mathbf{q})m + \psi_J + O_J(\Delta M + M^{-2J-1} Q_2^{2J+2}). \tag{4.7}$$

We observe that

$$\left\{ \frac{t(m, q_1, q_2)}{4\alpha(\alpha - 1)} \right\}^{1/(\alpha-2)} = (q_1 q_2)^{1/(\alpha-2)} m\sigma(m^{-1}, q_1, q_2).$$

The inequality in (4.1) is equivalent to

$$|t^{1/(\alpha-2)}(m, q_1, q_2) - t^{1/(\alpha-2)}(\widetilde{m}, \widetilde{q}_1, \widetilde{q}_2)| \ll \Delta T^{1/(\alpha-2)},$$

which implies

$$|v(\mathbf{q})m\sigma(m^{-1}, q_1, q_2) - \widetilde{m}\sigma(\widetilde{m}^{-1}, \widetilde{q}_1, \widetilde{q}_2)| \ll \Delta M.$$

Applying Lemma 2.2, we can get for every $J \geq 0$,

$$\begin{aligned} \Big| v(\mathbf{q})m + \sum_{j=1}^{J} \{v(\mathbf{q})\sigma_j(q_1, q_2)m^{-2j+1} - \sigma_j(\widetilde{q}_1, \widetilde{q}_2)\widetilde{m}^{-2j+1}\} - \widetilde{m} \Big| & \\ \ll \Delta M + M^{-2J-1} Q_2^{2J+2}. & \end{aligned} \tag{4.8}$$

For the choice of $J = 0$, one has $\widetilde{m} = v(\mathbf{q})m + O(\Delta M + M^{-1} Q_2^2)$. Taking $J = 1$ in (4.8) and replacing $\widetilde{m}$ by $v(\mathbf{q})m + O(\Delta M + M^{-1} Q_2^2)$, we get the first approximation

$$\widetilde{m} = v(\mathbf{q})m + \psi_1 + O(\Delta M + M^{-3} Q_2^4), \tag{4.9}$$

namely, (4.7) holds for $J=1$. Now we suppose that (4.7) is true for $J-1$. Thus we can replace $\widetilde{m}^{-2j+1}$ by $\{v(\mathbf{q})m+\psi_{J-1}+O(\Delta M+M^{-2J+1}Q_2^{2J})\}^{2j-1}$ in (4.8); we then easily deduce that (4.7) is also true for J. This finishes the proof of Lemma 4.1.

Now we divide the set $\mathbf{Q}$ into two sets $\mathbf{Q}_1$ and $\mathbf{Q}_2$. All $\mathbf{q}$ satisfying

$$|v^2(\mathbf{q})q_2^2-\widetilde{q}_2^{\,2}|\ge 2|v^2(\mathbf{q})q_1^2-\widetilde{q}_1^{\,2}|,\qquad \mathbf{q}\in\mathbf{Q}, \tag{4.10}$$

form the set $\mathbf{Q}_1$. All other $\mathbf{q}$ form $\mathbf{Q}_2$.

Lemma 4.2. *Let $1\le Q_1\le Q_2\le \varepsilon^*M$. Then for $t\sim M$ and $\mathbf{q}\in\mathbf{Q}_1$, we have*

$$\frac{\partial^i\psi_J(t,\mathbf{q})}{\partial t^i}\asymp|\omega(\mathbf{q})-1|Q_2^2M^{-i-1}\qquad (i=0,1,2), \tag{4.11}$$

where

$$\omega(\mathbf{q})=\left(\frac{q_1q_2^{\alpha-1}}{\widetilde{q}_1\widetilde{q}_2^{\,\alpha-1}}\right)^{1/(\alpha-2)}.$$

Proof. Since

$$\psi_1(t,\mathbf{q})=C\{(v^2(\mathbf{q})q_1^2-\widetilde{q}_1^{\,2})+(v^2(\mathbf{q})q_2^2-\widetilde{q}_2^{\,2})\}\frac{v^{-1}(\mathbf{q})}{t},$$

by (4.10) we get

$$\begin{aligned}\psi_1(t,\mathbf{q})&\asymp|v^2(\mathbf{q})q_2^2-\widetilde{q}_2^{\,2}|M^{-1}\asymp Q_2\left|\left(\frac{q_1q_2}{\widetilde{q}_1\widetilde{q}_2}\right)^{1/(\alpha-2)}q_2-\widetilde{q}_2\right|M^{-1}\\&\asymp Q_2^2|\omega(\mathbf{q})-1|M^{-1}.\end{aligned}$$

Now, (4.11) is true for $J=1$. Suppose that it holds for $J-1$. We write

$$\begin{aligned}\psi_J&=v(\mathbf{q})m\sum_{j=1}^{J}\left\{\frac{\sigma_j(q_1,q_2)}{m^{2j}}-\frac{\sigma_j(\widetilde{q}_1,\widetilde{q}_2)}{(v(\mathbf{q})m+\psi_{J-1})^{2j}}\right\}\\&\quad-\psi_{J-1}\sum_{j=1}^{J}\frac{\sigma_j(\widetilde{q}_1,\widetilde{q}_2)}{(v(\mathbf{q})m+\psi_{J-1})^{2j}}.\end{aligned} \tag{4.12}$$

Using (2.4), the induction hypothesis and $Q_1\le Q_2\le\varepsilon^*M$, it is easy to see that the last term in (4.12) is

$$\ll(|\omega(\mathbf{q})-1|Q_2^2M^{-1})\left(\frac{Q_2}{M}\right)^2\ll{\varepsilon^*}^2|\omega(\mathbf{q})-1|Q_2^2M^{-1}.$$

Similarly by (2.4) we get again

$$\sum_{j=1}^{J}\frac{\sigma_j(q_1,q_2)}{m^{2j}}-\frac{\sigma_j(\widetilde{q}_1,\widetilde{q}_2)}{(v(\mathbf{q})m+\psi_{J-1})^{2j}} \tag{4.13}$$
$$\begin{aligned}
&= C\biggl\{\frac{q_1^2+q_2^2}{m^2}-\frac{\widetilde{q}_1^{\,2}+\widetilde{q}_2^{\,2}}{(v(\mathbf{q})m+\psi_{J-1})^2}\biggr\}\\
&\quad+\sum_{j=2}^{J}\biggl\{\frac{(\sum_{i=0}^{j}a_{j,i}q_1^{2(j-i)}q_2^{2i})}{m^{2j}}-\frac{(\sum_{i=0}^{j}a_{j,i}\widetilde{q}_1^{\,2(j-i)}\widetilde{q}_2^{\,2i})}{(v(\mathbf{q})m+\psi_{J-1})^{2j}}\biggr\}\\
&= C\frac{(v^2(\mathbf{q})q_2^2-\widetilde{q}_2^{\,2})+(v^2(\mathbf{q})q_1^2-\widetilde{q}_1^{\,2})}{(v(\mathbf{q})m+\psi_{J-1})^2}\\
&\quad+C\frac{(2v(\mathbf{q})m+\psi_{J-1})(q_1^2+q_2^2)\frac{\psi_{J-1}}{m^2}}{(v(\mathbf{q})m+\psi_{J-1})^2}\\
&\quad+\sum_{j=2}^{J}\sum_{i=0}^{j}a_{j,i}\biggl\{\frac{q_1^{2(j-i)}q_2^{2i}}{m^{2j}}-\frac{\widetilde{q}_1^{\,2(j-i)}\widetilde{q}_2^{\,2i}}{(v(\mathbf{q})m+\psi_{J-1})^{2j}}\biggr\}.
\end{aligned}$$

Via (4.10) we have

$$(v^2(\mathbf{q})q_2^2-\widetilde{q}_2^{\,2})+(v^2(\mathbf{q})q_1^2-\widetilde{q}_1^{\,2})\asymp Q_2^2|\omega(\mathbf{q})-1|. \tag{4.14}$$

By the induction hypothesis we have

$$\begin{aligned}
&(2v(\mathbf{q})m+\psi_{J-1})(q_1^2+q_2^2)\frac{\psi_{J-1}}{m^2}\\
&\qquad\ll\frac{MQ_2^2(|\omega(\mathbf{q})-1|Q_2^2M^{-1})}{M^2}\ll\varepsilon^{*2}Q_2^2|\omega(\mathbf{q})-1|.
\end{aligned} \tag{4.15}$$

Now we prove

$$\sum_{j=2}^{J}\sum_{i=0}^{j}a_{j,i}\biggl\{\frac{q_1^{2(j-i)}q_2^{2i}}{m^{2j}}-\frac{\widetilde{q}_1^{\,2(j-i)}\widetilde{q}_2^{\,2i}}{(v(\mathbf{q})m+\psi_{J-1})^{2j}}\biggr\}\ll\frac{|\omega(\mathbf{q})-1|Q_2^4}{M^4}. \tag{4.16}$$

For any $1\le j\le J$, by the induction hypothesis we have

$$\begin{aligned}
&\frac{1}{(v(\mathbf{q})m)^{2j}}-\frac{1}{(v(\mathbf{q})m+\psi_{J-1})^{2j}}\\
&\qquad=\frac{(v(\mathbf{q})m+\psi_{J-1})^{2j}-(v(\mathbf{q})m)^{2j}}{(v(\mathbf{q})m)^{2j}(v(\mathbf{q})m+\psi_{J-1})^{2j}}\\
&\qquad\ll jM^{-4j}\int\limits_{v(\mathbf{q})m}^{v(\mathbf{q})m+\psi_{J-1}}u^{2j-1}\,du\ll jM^{-4j}M^{2j-1}|\psi_{J-1}|\\
&\qquad\ll jM^{-4j}M^{2j-1}|\omega(\mathbf{q})-1|\frac{Q_2^2}{M}\ll\frac{|\omega(\mathbf{q})-1|}{M^{2j}}.
\end{aligned}$$

For $1 \le i \le j$, we have

$$\begin{aligned} v^{2i}(\mathbf{q})q_1^{2i} - \widetilde{q}_1^{2i} &= i \int\limits_{\widetilde{q}_1^{\,2}}^{v^2(\mathbf{q})q_1^2} u^{i-1}\,du \\ &\ll iQ_1^{2i-2}|v^2(\mathbf{q})q_1^2 - \widetilde{q}_1^{\,2}| \ll iQ_1^{2i-2}|v^2(\mathbf{q})q_2^2 - \widetilde{q}_2^{\,2}| \\ &\ll iQ_1^{2i-2}Q_2^2|\omega(\mathbf{q}) - 1| \ll iQ_2^{2i}|\omega(\mathbf{q}) - 1| \end{aligned}$$

and

$$\begin{aligned} v^{2i}(\mathbf{q})q_2^{2i} - \widetilde{q}_2^{2i} &= i \int\limits_{\widetilde{q}_2^{\,2}}^{v^2(\mathbf{q})q_2^2} u^{i-1}\,du \\ &\ll iQ_2^{2i-2}|v^2(\mathbf{q})q_2^2 - \widetilde{q}_2^{\,2}| \ll iQ_2^{2i}|\omega(\mathbf{q}) - 1|. \end{aligned}$$

So for $0 \le i \le j$, we have

$$\begin{aligned} v^{2i}(\mathbf{q})&q_1^{2(j-i)}q_2^{2i} - \widetilde{q}_1^{\,2(j-i)}\widetilde{q}_2^{\,2i} \\ &= v^{2i}(\mathbf{q})q_1^{2(j-i)}q_2^{2i} - (v(\mathbf{q})q_1)^{2(j-i)}\widetilde{q}_2^{\,2i} \\ &\quad + (v(\mathbf{q})q_1)^{2(j-i)}\widetilde{q}_2^{\,2i} - \widetilde{q}_1^{\,2(j-i)}\widetilde{q}_2^{\,2i} \\ &= (v(\mathbf{q})q_1)^{2(j-i)}(v^{2i}(\mathbf{q})q_2^{2i} - \widetilde{q}_2^{\,2i}) \\ &\quad + \widetilde{q}_2^{\,2i}((v(\mathbf{q})q_1)^{2(j-i)} - \widetilde{q}_1^{\,2(j-i)}) \\ &\ll (Q_1^{2(j-i)}Q_2^{2i} + Q_2^{2j})|\omega(\mathbf{q}) - 1| \\ &\ll Q_2^{2j}|\omega(\mathbf{q}) - 1|. \end{aligned}$$

Combining the above estimates we get, for any $0 \le i \le j$,

$$\begin{aligned} \frac{q_1^{2(j-i)}q_2^{2i}}{m^{2j}} &- \frac{\widetilde{q}_1^{\,2(j-i)}\widetilde{q}_2^{\,2i}}{(v(\mathbf{q})m + \psi_{J-1})^{2j}} \\ &= \frac{q_1^{2(j-i)}q_2^{2i}}{m^{2j}} - \frac{\widetilde{q}_1^{\,2(j-i)}\widetilde{q}_2^{\,2i}}{m^{2j}} \\ &\quad + \widetilde{q}_1^{\,2(j-i)}\widetilde{q}_2^{\,2i}\left(\frac{1}{(v(\mathbf{q})m)^{2j}} - \frac{1}{(v(\mathbf{q})m + \psi_{J-1})^{2j}}\right) \\ &= m^{-2j}(v^{2i}(\mathbf{q})q_1^{2(j-i)}q_2^{2i} - \widetilde{q}_1^{\,2(j-i)}\widetilde{q}_2^{\,2i}) \\ &\quad + \widetilde{q}_1^{\,2(j-i)}\widetilde{q}_2^{\,2i}\left(\frac{1}{(v(\mathbf{q})m)^{2j}} - \frac{1}{(v(\mathbf{q})m + \psi_{J-1})^{2j}}\right) \\ &\ll M^{-2j}Q_2^{2j}|\omega(\mathbf{q}) - 1|. \end{aligned}$$

Hence (4.16) follows upon summing over $j \ge 2, 0 \le i \le j$.

Combining (4.12)–(4.16) we conclude that the first relation of (4.11) (namely, $i = 0$) holds for J. The other two can be shown similarly.

The following lemma is a key to the proof of Theorem 2. To prove the lemma, we shall use Lemma 2.1 and the idea in the proof of Theorem 1.

LEMMA 4.3. *If* $Q_1Q_2 \le \varepsilon^* M^{3/2}$ *and* $1 \le Q_1 \le Q_2 \le \varepsilon^* M$, *then there is an*

$$\eta = \eta(M, Q_1, Q_2) \in \left[\max\left(\frac{Q_1^2}{Q_2^2}, \frac{3\mathcal{L}}{Q_1Q_2}\right), \frac{c'}{c} - 1\right]$$

such that

$$\begin{aligned}(4.17)\qquad \eta^{-1}M^{-\varepsilon} \sum_{0\le k\le K} &\mathcal{E}_0(M, Q_1, Q_2, \delta, \mathbf{I}_{\eta,k})\\ &\ll MQ_1Q_2 + \delta MQ_1^2Q_2^2 + (MQ_1^7Q_2^9)^{1/4} + M^{-2}Q_1^4Q_2^4\\ &\quad + (\delta M^3Q_1^{15}Q_2^{17})^{1/8} + (\delta M^3Q_1^3Q_2)^{1/2} + (Q_1^{13}Q_2^{15})^{1/6}\\ &\quad + (\delta MQ_1^8Q_2^{10})^{1/4} + (M^{-1}Q_1^5Q_2^6)^{1/2}.\end{aligned}$$

Proof. We put $f_{\mathbf{q}}(t) := v(\mathbf{q})t + \psi_J(t, \mathbf{q})$ and use S_η to denote the quantity to be estimated in Lemma 4.3. Let $\mathcal{E}_0^{(1)}(M, Q_1, Q_2, \delta, \mathbf{I}_{\eta,k})$ denote the number of couples $(m, \mathbf{q})$ with $m \sim M$, $\mathbf{q} \in \mathbf{Q}_1$ such that (4.5) holds, and $\mathcal{E}_0^{(2)}(M, Q_1, Q_2, \delta, \mathbf{I}_{\eta,k})$ be the number of couples $(m, \mathbf{q})$ with $m \sim M$, $\mathbf{q} \in \mathbf{Q}_2$ such that (4.5) holds. Clearly we have

$$\begin{aligned}(4.18)\qquad &\mathcal{E}_0(M, Q_1, Q_2, \delta, \mathbf{I}_{\eta,k})\\ &\qquad = \mathcal{E}_0^{(1)}(M, Q_1, Q_2, \delta, \mathbf{I}_{\eta,k}) + \mathcal{E}_0^{(2)}(M, Q_1, Q_2, \delta, \mathbf{I}_{\eta,k}).\end{aligned}$$

We estimate $\eta^{-1}\sum_{0\le k\le K}\mathcal{E}_0^{(1)}(M, Q_1, Q_2, \delta, \mathbf{I}_{\eta,k})$ first. Define

$$\mathcal{H}_\eta^{(1)} := \{\mathbf{q} \in \mathbf{Q}_1 : |\omega(\mathbf{q}) - 1| \le \eta\}, \quad \mathcal{H}_{\eta,l}^{(1)} := \{\mathbf{q} \in \mathbf{Q}_1 : \eta_l/2 < |\omega(\mathbf{q}) - 1| \le \eta_l\}$$

with $\eta_l := \eta/2^l$ $(0 \le l \le L := [\log(\eta Q_1Q_2/\mathcal{L})/\log 2])$. Noticing that the last two conditions of (4.5) imply $|\omega(\mathbf{q}) - 1| \le \eta$, we can write

$$\begin{aligned}(4.19)\qquad \eta^{-1}\sum_{0\le k\le K} &\mathcal{E}_0^{(1)}(M, Q_1, Q_2, \delta, \mathbf{I}_{\eta,k})\\ &\ll \eta^{-1}\sum_{\mathbf{q}\in\mathcal{H}_{\eta,L+1}^{(1)}} \mathcal{R}(f_{\mathbf{q}}, \delta) + \eta^{-1}\sum_{0\le l\le L}\sum_{\mathbf{q}\in\mathcal{H}_{\eta,l}^{(1)}} \mathcal{R}(f_{\mathbf{q}}, \delta).\end{aligned}$$

For $\mathbf{q} \in \mathcal{H}_{\eta,L+1}^{(1)}$, we have $\mathcal{R}(f_{\mathbf{q}}, \delta) \ll M$ trivially, which implies that the first term on the right-hand side of (4.19) is

$$(4.20)\qquad \ll \eta^{-1}M\{Q_1Q_2\mathcal{L} + \eta_{L+1}(Q_1^2Q_2^2)\} \ll MQ_1Q_2\mathcal{L}\eta^{-1}$$

in view of Lemma 3.2 in [9].

When $\mathbf{q} \in \mathcal{H}_{\eta,l}^{(1)}$ $(0 \le l \le L)$, Lemma 4.2 shows that the function $f_{\mathbf{q}}(t)$ satisfies the condition of Lemma 2.1 with $\mu = \eta_l Q_2^2/M^3$. Hence the second term on the right-hand side of (4.19) is

$$\begin{aligned} (4.21)\qquad &\ll \eta^{-1} \sum_{0\le l\le L} \{\delta M|\mathcal{H}_{\eta,l}^{(1)}| + (\eta_l Q_2^2)^{1/3}|\mathcal{H}_{\eta,l}^{(1)}| + |\mathcal{H}_{\eta,l}^{(1)}|\} \\ &\quad + \eta^{-1} \sum_{0\le l\le L} (\delta M^3 Q_1 |\mathcal{H}_{\eta,l}^{(1)}| \mathcal{L} \eta_l^{-1} Q_2^{-1})^{1/2} \\ &\quad + \eta^{-1} \sum_{0\le l\le L} (\delta^2 M^3 Q_1^2 |\mathcal{H}_{\eta,l}^{(1)}|^2 \eta_l^{-1})^{1/3} \\ &\quad + \eta^{-1} \sum_{0\le l\le L} Q_1 Q_2 (\delta M |\mathcal{H}_{\eta,l}^{(1)}|)^{1/2}. \end{aligned}$$

When $0 \le l \le L$, Lemma 3.2 of [9] implies that

$$|\mathcal{H}_{\eta,l}^{(1)}| \ll Q_1 Q_2 \mathcal{L} + \eta_l (Q_1 Q_2)^2 \ll \eta_l (Q_1 Q_2)^2,$$

so we replace $|\mathcal{H}_{\eta,l}^{(1)}|$ by the estimate $\eta_l (Q_1 Q_2)^2$ in (4.21). Combining (4.19) and (4.20), a simple calculation shows that

$$\begin{aligned} (4.22)\qquad \eta^{-1} &\sum_{0\le k\le K} \mathcal{E}_0^{(1)}(M, Q_1, Q_2, \delta, \mathbf{I}_{\eta,k}) \\ &\ll M Q_1 Q_2 \mathcal{L} \eta^{-1} + \delta M Q_1^2 Q_2^2 + Q_1^2 Q_2^2 + Q_1^2 Q_2^{8/3} \eta^{1/3} \\ &\quad + (\delta M^3 Q_1^3 Q_2)^{1/2} \mathcal{L}^2 \eta^{-1} + (\delta^2 M^3 Q_1^6 Q_2^4)^{1/3} \eta^{-2/3} \\ &\quad + (\delta M Q_1^4 Q_2^4)^{1/2} \eta^{-1/2}. \end{aligned}$$

Now we estimate $\eta^{-1} \sum_{0\le k\le K} \mathcal{E}_0^{(2)}(M, Q_1, Q_2, \delta, \mathbf{I}_{\eta,k})$ by the technique in the proof of Theorem 1. Let $\mathcal{E}_0^*(M, Q_1, Q_2, \delta)$ be the number of couples $(m, \mathbf{q})$ with $m \sim M$, $\mathbf{q} \in \mathbf{Q}_2$ such that the first condition of (4.5) holds. We can obtain

$$\sum_{0\le k\le K} \mathcal{E}_0^{(2)}(M, Q_1, Q_2, \delta, \mathbf{I}_{\eta,k}) \le \mathcal{E}_0^*(M, Q_1, Q_2, \delta).$$

From the first condition of (4.5), we get

$$|v^2(\mathbf{q}) q_2^2 - \widetilde{q}_2^{\,2}| < 2|v^2(\mathbf{q}) q_1^2 - \widetilde{q}_1^{\,2}|.$$

So applying similar arguments to those for the estimate (4.11) in Lemma 4.2 for $i = 0$, we get

$$(4.23)\qquad \|v(\mathbf{q}) m + v_1(\mathbf{q}) m^{-1}\| \ll \delta + Q_1^2 Q_2^2 M^{-3}.$$

Let $\mathcal{B}(M, Q_1, Q_2, \delta)$ denote the number of couples $(m, \mathbf{q})$ with $m \sim M$,

$\mathbf{q} \in \mathbf{Q}_2$ such that

$$\|v(\mathbf{q})m + v_1(\mathbf{q})m^{-1}\| \ll \delta + M^{-1}Q_1^{2/3} + Q_1^2Q_2^2M^{-3}. \tag{4.24}$$

It is easily seen that $\mathcal{E}_0^*(M, Q_1, Q_2, \delta) \le \mathcal{B}(M, Q_1, Q_2, \delta)$. Now, using the same argument as in Lemma 3.1 and noticing $B = v_1(\mathbf{q}) \ll Q_1^2$, one obtains

$$\begin{aligned} M^{-\varepsilon}\mathcal{B}(M, Q_1, Q_2, \delta) \ll{}& MQ_1Q_2 + \delta M(Q_1Q_2)^2 \\ &+ Q_1^{8/3}Q_2^2 + M^{-2}(Q_1Q_2)^4. \end{aligned} \tag{4.25}$$

Combining (4.18), (4.22) and (4.25), we get

$$\begin{aligned} M^{-\varepsilon}S_\eta \ll{}& MQ_1Q_2\eta^{-1} + \delta MQ_1^2Q_2^2\eta^{-1} + Q_1^2Q_2^2 \\ &+ Q_1^2Q_2^{8/3}\eta^{1/3} + Q_1^{8/3}Q_2^2\eta^{-1} + (\delta M^3Q_1^3Q_2)^{1/2}\eta^{-1} \\ &+ (\delta^2M^3Q_1^6Q_2^4)^{1/3}\eta^{-2/3} + (\delta MQ_1^4Q_2^4)^{1/2}\eta^{-1/2} \\ &+ M^{-2}(Q_1Q_2)^4\eta^{-1}. \end{aligned} \tag{4.26}$$

Now applying Lemma 2.4 of [9] to optimize the parameter $\eta \in [\max(Q_1^2/Q_2^2, 3\mathcal{L}/(Q_1Q_2)), c'/c - 1]$, one has

$$\begin{aligned} M^{-\varepsilon}S_\eta \ll{}& MQ_1Q_2 + \delta MQ_1^2Q_2^2 + Q_1^2Q_2^2 \\ &+ (MQ_1^7Q_2^9)^{1/4} + M^{-2}(Q_1Q_2)^4 + (\delta M^3Q_1^{15}Q_2^{17})^{1/8} \\ &+ (\delta^2M^3Q_1^{18}Q_2^{20})^{1/9} + (\delta MQ_1^{10}Q_2^{12})^{1/5} \\ &+ (\delta M^3Q_1^3Q_2)^{1/2} + (\delta^2M^3Q_1^6Q_2^4)^{1/3} \\ &+ (\delta MQ_1^4Q_2^4)^{1/2} + (Q_1^5Q_2^7)^{1/3} + Q_1^{8/3}Q_2^2 \\ &+ (Q_1^{13}Q_2^{15})^{1/6} + (\delta MQ_1^8Q_2^{10})^{1/4} + (M^{-1}Q_1^5Q_2^6)^{1/2} \\ :={}& T_1 + T_2 + \ldots + T_{16}. \end{aligned} \tag{4.27}$$

It is easy to check that

$$T_3, T_{12}, T_{13} \ll T_{14}, \qquad T_{11} \le (T_2T_3)^{1/2},$$
$$T_{10} \le T_1^{1/3}T_2^{2/3}, \qquad T_8 \le T_{15}^{4/5}T_{14}^{1/5}, \qquad T_7 \le T_{15}^{8/9}T_1^{1/9},$$

and this completes the proof of Lemma 4.3.

Proof of Theorem 2. Now Theorem 2 is a simple consequence of Lemmas 4.1 and 4.3.

5. Spacing problem for the points $t^\beta(m+q_2, q_1) - t^\beta(m-q_2, q_1)$. The transformation formula (B-process) is important in the theory of exponential sums. When estimating an exponential sum of the type

$$\sum_{m\sim M}\sum_{n\sim N} a_mb_ne(Am^cn^d),$$

sometimes we have to use a B-process (over the variable n) between two A-processes (over m). In this case we have to consider the spacing problem for the points

$$t_\beta(m,q_1,q_2) := t_\beta(m,q_1,q_2;\alpha) = t^\beta(m+q_2,q_1) - t^\beta(m-q_2,q_1), \tag{5.1}$$

where α and β are real numbers such that

$$\beta(\alpha\beta-\beta-1)u(\alpha,\beta)v(\alpha,\beta)\neq 0,$$

$$u(\alpha,\beta) = \frac{(\alpha-2)(\alpha-3)}{6} + \frac{\beta-1}{\alpha},$$

$$v(\alpha,\beta) = u(\alpha,\beta) + \frac{(\alpha-1)(\beta-1)(\beta-2)}{6} + \frac{2(\beta-1)}{\alpha}.$$

Let $M\ge 10$, $Q_1\ge 1$, $Q_2\ge 1$, $\eta>0$, and $\Delta>0$. We set $T := M^{\alpha\beta-\beta-1}Q_1^\beta Q_2$ and $\mathcal{L} := \log(2MQ_1Q_2)$. Hence for $m\sim M$, $q_1\sim Q_1$, $q_2\sim Q_2$, and $Q_1+Q_2<M/3$ we have $t_\beta(m,q_1,q_2)\asymp T$.

Let $\mathcal{B}_1(M,Q_1,\Delta,\eta)$ denote the number of quadruples $(m,\widetilde{m},q_1,\widetilde{q}_1)$ with $m,\widetilde{m}\sim M$, $Q_1\le q_1,\widetilde{q}_1\le(1+\eta)Q_1$, and q_2 fixed, $q_2\in[Q_2,2Q_2]$, such that

$$|t_\beta(m,q_1,q_2)-t_\beta(\widetilde{m},\widetilde{q}_1,q_2)|\le \Delta T.$$

Theorem 3. *If $Q_2\ge 1, Q_1\ge C_1(\alpha,\beta)Q_2$, $Q_1\le M^{1-\varepsilon}$, where $C_1(\alpha,\beta)$ is a computable constant depending on α and β only, then there exists an $\eta=\eta(M,Q_1)\in[1/\sqrt{Q_1},1]$ such that*

$$\eta^{-1}\sum_{0\le k\le K}\mathcal{B}_1(M,Q_{1,k},\Delta,\eta) \ll \{MQ_1+\Delta(MQ_1)^2+(MQ_1^9)^{1/4}\}\mathcal{L}^4, \tag{5.2}$$

where $Q_{1,k} := (1+\eta)^kQ_1$ and $K := [(\log 2)/\eta]$.

Let $\mathcal{B}_2(M,Q_2,\Delta,\eta)$ denote the number of quadruples $(m,\widetilde{m},q_2,\widetilde{q}_2)$ with $m,\widetilde{m}\sim M$, $Q_2\le q_2,\widetilde{q}_2\le(1+\eta)Q_2$, and q_1 fixed, $q_1\in[Q_1,2Q_1]$, such that

$$|t_\beta(m,q_1,q_2)-t_\beta(\widetilde{m},q_1,\widetilde{q}_2)|\le \Delta T.$$

Theorem 4. *If $Q_1\ge 1$, $Q_2\ge C_2(\alpha,\beta)Q_1$, $Q_2\le M^{1-\varepsilon}$, where $C_2(\alpha,\beta)$ is a computable constant depending on α and β only, then there exists an $\eta=\eta(M,Q_2)\in[1/\sqrt{Q_2},1]$ such that*

$$\eta^{-1}\sum_{0\le k\le K}\mathcal{B}_2(M,Q_{2,k},\Delta,\eta) \ll \{MQ_2+\Delta(MQ_2)^2+(MQ_2^9)^{1/4}\}\mathcal{L}^4, \tag{5.3}$$

where $Q_{2,k} := (1+\eta)^kQ_2$ and $K := [(\log 2)/\eta]$.

Remark. Since the proofs of Theorems 3 and 4 are very similar to that of Theorem 3 in Sargos and Wu [9], we omit them.

When $Q_1\asymp Q_2$, we give the following theorem, which is an analogue of Theorem 1.

THEOREM 5. *If* $Q_1 \geq 1$, $Q_2 \geq 1$, $Q_1 \ll Q_2 \ll Q_1 \ll M^{2/3}$, *then*

$$\mathcal{B}_1(M, Q_1, \Delta, 1) \ll \{MQ_1 + \Delta(MQ_1)^2 + M^{-2}Q_1^6\}\mathcal{L}^4. \tag{5.4}$$

6. Estimation for exponential sums with monomials. In this section we give two estimates of the general three-dimensional exponential sum

$$S_I(M, M_1, M_2) := \sum_{m\sim M} \sum_{m_1\sim M_1} \sum_{m_2\sim M_2} a(m)b(m_1, m_2)e(Am^\alpha m_1^\beta m_2^\gamma).$$

This sum was studied by Fouvry and Iwaniec ([4], Theorem 3) and by Liu ([8], Theorem 3), and then sharpened by Sargos and Wu ([9], Theorem 7).

Throughout this section we use the notation $\mathcal{L} := \log 2MM_1M_2$.

THEOREM 6. *Suppose that* $\alpha, \beta, \gamma, A, M, M_1, M_2 \in \mathbb{R}$, $\alpha(\alpha-1)(\alpha-2) \times (\alpha-3)\gamma(\gamma-1) \neq 0$, $A \neq 0$, $M, M_1, M_2 \geq 1$. *Let* $|a(m)| \leq 1, |b(m_1, m_2)| \leq 1$, $F = |A|M^\alpha M_1^\beta M_2^\gamma$, $F \gg M$. *Then*

$$\begin{aligned}(6.1)\quad & S_I(M, M_1, M_2)M^{-\varepsilon} \\ &\ll (FM^5M_1^7M_2^7)^{1/8} + (M^8M_1^7M_2^7)^{1/8} + (F^4M^{43}M_1^{54}M_2^{54})^{1/58} \\ &\quad + (F^7M^{82}M_1^{100}M_2^{100})^{1/108} + (F^3M^{37}M_1^{46}M_2^{46})^{1/49} \\ &\quad + (F^3M^{46}M_1^{54}M_2^{54})^{1/58} + (F^{29}M^{250}M_1^{294}M_2^{294})^{1/336} \\ &\quad + (F^{25}M^{230}M_1^{266}M_2^{266})^{1/304} + (F^{25}M^{262}M_1^{294}M_2^{294})^{1/336} \\ &\quad + (F^{-1}M^{181}M_1^{188}M_2^{188})^{1/200} + (F^{-1}M^{334}M_1^{344}M_2^{344})^{1/368} \\ &\quad + (F^{-4}M^{190}M_1^{188}M_2^{188})^{1/200} + (M^5M_1^6M_2^6)^{1/6} \\ &\quad + (F^{-1}M^9M_1^8M_2^8)^{1/8}.\end{aligned}$$

Proof. We only give a sketch of proof; the details are similar to Theorem 7 of Sargos and Wu [9].

If $F \ll M^2$, Theorem 6 follows from Theorem 3 of Fouvry and Iwaniec [4]. Now we suppose $F \gg M^2$. By Cauchy's inequality and Lemma 2.1 of Sargos and Wu [9], for some $10 \leq Q_1 \leq M^{1/3}$ we get

$$|S_I|^2\mathcal{L}^{-1} \ll (MM_1M_2)^2Q_1^{-1} + MM_1M_2Q_1^{-1}\Sigma$$

with

$$\Sigma := \sum_{q_1\sim Q_1^*} \sum_{m\sim M} \sum_{m_1\sim M_1,\, m_2\sim M_2} c(m, q_1)e(At(m, q_1)m_1^\beta m_2^\gamma)$$

for some $1 \ll Q_1^* \ll Q_1$, where $|c(m, q_1)| \leq 1$.

Take $Q_2 = Q_1^{*2}$. By Cauchy's inequality and Lemma 2.1 of Sargos and Wu [9] again, we get

$$\Sigma^2\mathcal{L}^{-1} \ll (MM_1M_2Q_1^*)^2Q_2^{-1} + MM_1M_2Q_1^*Q_2^{-1}\Sigma_1$$

with

$$\Sigma_1 := \sum_{\substack{q_1\sim Q_1^*,\, q_2\sim Q_2^*\\ m\sim M}} \sum_{\substack{m_1\sim M_1\\ m_2\sim M_2}} c(m,q_1,q_2)e(At(m,q_1,q_2)m_1^\beta m_2^\gamma)$$

for some $1 \ll Q_2^* \ll Q_2$, where $|c(m,q_1,q_2)| \le 1$.

Now we use the D-process to Σ_1. For the spacing of $m_1^\beta m_2^\gamma$, we use Lemma 1 of Fouvry and Iwaniec [4]. For the spacing of $t(m,q_1,q_2)$, we use Theorems 1 and 2. If $Q_2^* \ge Q_1^* M^\varepsilon$ or $Q_1^* \ge Q_2^* M^\varepsilon$, we use Theorem 2; if $Q_1^* M^{-\varepsilon} \le Q_2^* \le Q_1^* M^\varepsilon$, we use Theorem 1. Finally, using Lemma 2.4 of [9] to choose a best Q_1 finishes the proof of Theorem 6.

When one of the coefficients is smooth, combining with B-process we may often obtain a better estimate.

Let

$$S_{II}(M,M_1,M_2) := \sum_{m\sim M}\sum_{m_1\sim M_1}\sum_{m_2\sim M_2} a(m_1)b(m_2)e(Am^\alpha m_1^\beta m_2^\gamma).$$

THEOREM 7. *Let* $M,M_1,M_2 \ge 1, A > 0,\ \alpha\beta/(\alpha-1) \notin \{0,1,2,\ldots\}$, $a(m_1) \ll 1$ *and* $b(m_2) \ll 1$. *Then*

$$\begin{aligned}(6.2)\quad & S_{II}(M,M_1,M_2)M^{-\varepsilon}\\ &\ll (F^2M^3M_1^7M_2^7)^{1/8} + (F^4M_1^7M_2^7)^{1/8} + (F^{18}M^{15}M_1^{54}M_2^{54})^{1/58}\\ &\quad + (F^{35}M^{26}M_1^{100}M_2^{100})^{1/108} + (F^{31}M^{24}M_1^{92}M_2^{92})^{1/98}\\ &\quad + (F^{10}M^6M_1^{27}M_2^{27})^{1/29} + (F^{111}M^{86}M_1^{294}M_2^{294})^{1/336}\\ &\quad + (F^{103}M^{74}M_1^{266}M_2^{266})^{1/304} + (F^{119}M^{74}M_1^{294}M_2^{294})^{1/336}\\ &\quad + (F^{80}M^{19}M_1^{188}M_2^{188})^{1/200} + (F^{149}M^{34}M_1^{344}M_2^{344})^{1/368}\\ &\quad + (F^{43}M^5M_1^{94}M_2^{94})^{1/100} + (F^2MM_1^6M_2^6)^{1/6}\\ &\quad + (F^4M^{-1}M_1^8M_2^8)^{1/8} + F^{-1/2}MNH.\end{aligned}$$

Proof. Using the B-process to m and then using Theorem 6, we get the assertion.

Acknowledgements. The authors thank the referee for his valuable suggestions.

References

[1] E. Bombieri and H. Iwaniec, *On the order of* $\zeta(\frac12+it)$, Ann. Scuola Norm. Sup. Pisa 13 (1986), 449–472.

[2] X.-D. Cao and W.-G. Zhai, *On the distribution of* p^α *modulo one*, J. Théor. Nombres Bordeaux, to appear.

[3] —, —, *On the distribution of square-full integers*, submitted.

[4] E. Fouvry and H. Iwaniec, *Exponential sums with monomials*, J. Number Theory 33 (1989), 311–333.
[5] S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge Univ. Press, Cambridge, 1991.
[6] M. N. Huxley and N. Watt, *Exponential sums with a parameter*, Proc. London Math. Soc. 59 (1989), 233–252.
[7] E. Krätzel, *Lattice Points*, Deutscher Verlag Wiss., Berlin, 1988.
[8] H.-Q. Liu, *On the number of abelian groups of a given order* (*supplement*), Acta Arith. 64 (1993), 285–296.
[9] P. Sargos and J. Wu, *Multiple exponential sums with monomials and their applications in number theory*, Acta Math. Hungar. 88 (2000), to appear.
[10] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Clarendon Press, Oxford, 1951.

Beijing Institute of Petro-Chemical Technology
Daxing, Beijing, 102600
P.R. China
E-mail: biptiao@info.iuol.cn.net

Department of Mathematics
Shandong Normal University
Jinan, Shandong, 250014
P.R. China
E-mail: wgzhai@jn-public.sd.cninfo.net

Received on 12.8.1998
and in revised form on 30.8.1999 (3444)