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On approximation to real numbers by algebraic numbers

by

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1. Introduction. In this paper we study the problem of solvability of the inequality

$$|\xi-\alpha|<c(\xi,n)H(\alpha)^{-A} \tag{1.1}$$

in algebraic numbers α of degree $\le n$, where $A>0$, ξ is a real number which is not an algebraic number of degree $\le n$, $H(\alpha)$ is the height of α. In 1842 Dirichlet proved that for any real number ξ there exist infinitely many rational numbers p/q such that $|\xi-p/q|<q^{-2}$. In 1961 E. Wirsing [9] proved that (1.1) has infinitely many solutions if $A=n/2+\gamma_n$, where $\lim_{n\to\infty}\gamma_n=2$. Moreover, he conjectured that the inequality (1.1) has infinitely many solutions if $A=n+1-\varepsilon$, where $\varepsilon>0$. Further it has been conjectured [5] that the exponent $n+1-\varepsilon$ can be replaced even by $n+1$. This problem has not been solved except in some special cases. In 1965 V. G. Sprindžuk [6] proved that the Conjecture of Wirsing holds for almost all real numbers. In 1967 H. Davenport and W. Schmidt [3] obtained new results in the theory of linear forms. These enabled them to prove the Conjecture for $n=2$. In 1993 [1] the following improvement of the Theorem of Wirsing was obtained: $A=n/2+\gamma_n'$, where $\lim_{n\to\infty}\gamma_n'=3$. In 1992–1997 a new method was introduced, improving the Theorem of Wirsing for $n\le 10$ ([7, 8]).

In this paper we prove the following

Theorem. *For any real number ξ which is not an algebraic number of degree $\le n$, there exist infinitely many algebraic numbers α of degree $\le n$ such that*

$$|\xi-\alpha|\ll H(\alpha)^{-A}. \tag{1.2}$$

Here and below $3\le n\le 7$, $\ll$ is the Vinogradov symbol, and $A=A(n)$ is the positive root of the quadratic equation

$$(3n-5)X^2-(2n^2+n-9)X-n-3=0. \tag{1.3}$$

The implicit constant in $\ll$ depends on ξ and n only.

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The following table contains the values of A corresponding to Wirsing's Theorem, the Theorem above and the Conjecture:

n	Wirsing, 1961	Theorem	Conj.
3	3.2807764	3.4364917	4
4	3.8228757	4.1009866	5
5	4.3507811	4.7677925	6
6	4.8708287	5.4350702	7
7	5.3860009	6.1024184	8

2. Preliminaries. We can confine ourselves to the range $0 < \xi < 1/4$. We suppose that there exists a real number $0 < \xi < 1/4$ which is not an algebraic number of degree $\le n$, such that

$$\forall c > 0\ \exists \widetilde{H}_0 > 0\ \forall \alpha \in \mathbb{A}_n,\ H(\alpha) > \widetilde{H}_0, \qquad |\xi - \alpha| > cH(\alpha)^{-A}, \tag{2.1}$$

where $\mathbb{A}_n$ denotes the set of algebraic numbers of degree $\le n$. Also, we may assume that $\widetilde{H}_0 > ((2n)!)^{30n} e^{60n^2}$.

By Lemma 1 of [2] we have

$$|\xi - \alpha| \le n \frac{|P(\xi)|}{|P'(\xi)|}, \tag{2.2}$$

where α is the root of the polynomial $P(x)$ closest to ξ. In fact, we get

$$\frac{|P'(\xi)|}{|P(\xi)|} = \left| \sum_{i=1}^{n} \frac{1}{\xi - \alpha_i} \right| \le \sum_{i=1}^{n} \frac{1}{|\xi - \alpha_i|} \le \frac{n}{|\xi - \alpha|},$$

which gives (2.2). Put

$$c_T = 4^{n^2} (n!)^{4n^3} \xi^{-2n^5}.$$

By (2.1) and (2.2) we obtain

$$\exists \widetilde{H}_0 > 0\ \forall Q(x) \in \mathbb{Z}[x],\ \deg Q(x) \le n,\ \lceil Q \rceil > \widetilde{H}_0, \qquad \frac{|Q(\xi)|}{|Q'(\xi)|} > c_T \lceil Q \rceil^{-A}. \tag{2.3}$$

Throughout the paper $\lceil L \rceil$ denotes the height of the polynomial $L(x)$.

3. Auxiliary lemmas

LEMMA 3.1. *Let $L(x) = c_n x^n + \ldots + c_1 x + c_0$ be a polynomial with integer coefficients such that $|L(\xi)| < 1/2$. Then there is an index $j_1 \in \{1, \ldots, n\}$ such that $|c_{j_1}| = \lceil L \rceil$.*

Proof. Assume that $|c_{j_1}| < \lceil L\rceil$ for any $j_1 \in \{1, \ldots, n\}$. Then

$$|L(\xi)| = \Big|\sum_{\nu=0}^{n} c_\nu \xi^\nu\Big| > \Big| -\sum_{\nu=1}^{n} \lceil L\rceil \xi^\nu + \lceil L\rceil \Big| = \lceil L\rceil \Big| -\sum_{\nu=1}^{n} \xi^\nu + 1\Big| > \frac{1}{2}. \ \blacksquare$$

Lemma 3.2. *Let $L(x)$ be a polynomial and j_1 an index as in Lemma 3.1. Suppose $|c_i| \le \xi^{n-1}\lceil L\rceil$ for every $i \in \{1, \ldots, n\}\backslash\{j_1\}$. Then $\lceil L\rceil < \xi^{-n+1}|L'(\xi)|$.*

Proof. We have

$$|L'(\xi)| = \Big|\sum_{\nu=1}^{n} \nu c_\nu \xi^{\nu-1}\Big| = \Big|j_1 c_{j_1}\xi^{j_1-1} + \Big(\sum_{\nu=1}^{n} \nu c_\nu \xi^{\nu-1} - j_1 c_{j_1}\xi^{j_1-1}\Big)\Big|.$$

Since $|j_1 c_{j_1}\xi^{j_1-1}| = j_1\lceil L\rceil\xi^{j_1-1} \ge n\lceil L\rceil\xi^{n-1}$,

$$\begin{aligned}\Big|\sum_{\nu=1}^{n} \nu c_\nu \xi^{\nu-1} - j_1 c_{j_1}\xi^{j_1-1}\Big| &\le \xi^{n-1}\lceil L\rceil\Big(\sum_{\nu=1}^{n} \nu\xi^{\nu-1} - j_1\xi^{j_1-1}\Big)\\ &\le \xi^{n-1}\lceil L\rceil\sum_{\nu=1}^{n-1} \nu\xi^{\nu-1},\end{aligned}$$

and $n - \sum_{\nu=1}^{n-1} \nu\xi^{\nu-1} > 1$, we obtain

$$\begin{aligned}|L'(\xi)| &\ge |j_1 c_{j_1}\xi^{j_1-1}| - \Big|\sum_{\nu=1}^{n} \nu c_\nu \xi^{\nu-1} - j_1 c_{j_1}\xi^{j_1-1}\Big|\\ &\ge n\xi^{n-1}\lceil L\rceil - \xi^{n-1}\lceil L\rceil\sum_{\nu=1}^{n-1} \nu\xi^{\nu-1} = \xi^{n-1}\lceil L\rceil\Big(n - \sum_{\nu=1}^{n-1} \nu\xi^{\nu-1}\Big)\\ &> \xi^{n-1}\lceil L\rceil. \ \blacksquare\end{aligned}$$

Notations. In this section $L^{(k)}(x)$ denotes the kth derivative of a polynomial $L(x)$. However, in Sections 4–7 we will use $\widetilde{Q}_i^{(l)}(x)$ to denote the polynomial with *indices* l and i.

Lemma 3.3. *For any polynomials $F(x)$ and $G(x)$ the following identity is valid*:

$$R(F,G) \equiv \left|\begin{matrix} \frac{F^{(l)}(\xi)}{l!} & \ldots & F'(\xi) & F(\xi) & \\ & \ddots & \ddots & \ddots & \\ & \frac{F^{(l)}(\xi)}{l!} & \ldots & F'(\xi) & F(\xi) \\ \frac{G^{(m)}(\xi)}{m!} & \ldots & G'(\xi) & G(\xi) & \\ & \ddots & \ddots & \ddots & \\ & \frac{G^{(m)}(\xi)}{m!} & \ldots & G'(\xi) & G(\xi) \end{matrix}\right| \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} l \end{matrix}, \tag{3.1}$$

where $R(F,G)$ denotes the resultant of $F(x)$ and $G(x)$, ξ is any real, complex or p-adic number, $\deg F(x)=l$, $\deg G(x)=m$.

Proof. Write

$$F(x)=\sum_{\nu=0}^{l}a_\nu x^\nu=a_l\prod_{\nu=1}^{l}(x-\alpha_\nu),\qquad G(x)=\sum_{\nu=0}^{m}b_\nu x^\nu=b_m\prod_{\nu=1}^{m}(x-\beta_\nu),$$

$$\widetilde{F}(x)=F(x+\xi)=\sum_{\nu=0}^{l}\widetilde{a}_\nu x^\nu,\qquad \widetilde{G}(x)=G(x+\xi)=\sum_{\nu=0}^{m}\widetilde{b}_\nu x^\nu.$$

Denote by $\Delta_{l,m}(A_i,B_j)$ the determinant obtained from (3.1) by replacing $F^{(i)}(\xi)/i!$ and $G^{(j)}(\xi)/j!$ with A_i and B_j, $0\le i\le l$, $0\le j\le m$, respectively. For example, according to the definition of resultant we have $R(F,G)=\Delta_{l,m}(a_i,b_j)$. We now obtain

$$\begin{aligned}R(F,G)&=a_l^m b_m^l\prod_{i,j}(\alpha_i-\beta_j)=a_l^m b_m^l\prod_{i,j}(\alpha_i-\xi-(\beta_j-\xi))=\Delta_{l,m}(\widetilde{a}_i,\widetilde{b}_j)\\&=\Delta_{l,m}\Big(\frac{\widetilde{F}^{(i)}(0)}{i!},\frac{\widetilde{G}^{(j)}(0)}{j!}\Big)=\Delta_{l,m}\Big(\frac{F^{(i)}(\xi)}{i!},\frac{G^{(j)}(\xi)}{j!}\Big).\ \blacksquare\end{aligned}$$

LEMMA 3.4. *Let $F(x),G(x)\in\mathbb{Z}[x]$ be nonzero polynomials with* $\deg F(x)=l\le n$, $\deg G(x)=m\le n$, $lm\ge 2$. *Suppose that $F(x)$ and $G(x)$ have no common root. Then at least one of the following estimates is true*:

$$\begin{aligned}&\text{(I)}\quad 1<c_R\max(|F(\xi)|,|G(\xi)|)^2\max(\lceil F\rceil,\lceil G\rceil)^{m+l-2},\\ \text{(3.2)}\quad&\text{(II)}\quad 1<c_R\max(|F(\xi)|\cdot|F'(\xi)|\cdot|G'(\xi)|,|G(\xi)|\cdot|F'(\xi)|^2)\lceil F\rceil^{m-2}\lceil G\rceil^{l-1},\\&\text{(III)}\quad 1<c_R\max(|G(\xi)|\cdot|F'(\xi)|\cdot|G'(\xi)|,|F(\xi)|\cdot|G'(\xi)|^2)\lceil F\rceil^{m-1}\lceil G\rceil^{l-2},\end{aligned}$$

where $0<\xi<1$ and $c_R=(2n)!((n+1)!)^{2n-2}$.

Proof. Consider the identity of Lemma 3.3. Since the polynomials $F(x),G(x)\in\mathbb{Z}[x]$ have no common root, it follows that

$$|R(F,G)|\ge 1. \tag{3.3}$$

We will obtain an upper bound for the absolute value of the determinant (3.1). Let us expand it with respect to the last column. Obviously, any nonzero term contains the factor $F(\xi)$ or $G(\xi)$. We distinguish two cases.

CASE A. Suppose that some nonzero term contains $F(\xi)^2$, $G(\xi)^2$ or $F(\xi)G(\xi)$. Using the inequality

$$|L^{(k)}(\xi)|<(n+1)!\lceil L\rceil, \tag{3.4}$$

where $\deg L(x)\le n$, we estimate other factors. Hence this term has absolute value at most

$$((n+1)!)^{m+l-2}\max(|F(\xi)|,|G(\xi)|)^2\max(\lceil F\rceil,\lceil G\rceil)^{m+l-2}.$$

Case B. Suppose that some nonzero term contains $F(\xi)$ or $G(\xi)$ together with the other factors $F^{(i)}(\xi)/i!$ or $G^{(j)}(\xi)/j!$ where $1 \le i \le l,\ 1 \le j \le m$. If we expand the determinant (3.1) according to the last three columns, we see that the term considered contains one of the following expressions: $F(\xi)F'(\xi)G'(\xi),\ G(\xi)F'(\xi)^2,\ G(\xi)F'(\xi)G'(\xi)$ or $F(\xi)G'(\xi)^2$. Using (3.4) we conclude that this term has absolute value at most

$$((n+1)!)^{m+l-3}\max(|F(\xi)|\cdot|F'(\xi)|\cdot|G'(\xi)|,|G(\xi)|\cdot|F'(\xi)|^2)\lceil F\rceil^{m-2}\lceil G\rceil^{l-1}$$

or

$$((n+1)!)^{m+l-3}\max(|G(\xi)|\cdot|F'(\xi)|\cdot|G'(\xi)|,|F(\xi)|\cdot|G'(\xi)|^2)\lceil F\rceil^{m-1}\lceil G\rceil^{l-2}.$$

Finally, when expanding the determinant (3.1), we obtain $(l+m)!$ terms. Combining this information with (3.3), we get (3.2)(I)–(III). ∎

The following two lemmas are well known.

Lemma 3.5 (see [4], [5]). *Let $R(x), R_1(x), \ldots, R_\nu(x)$ be polynomials such that $R(x) = R_1(x)\ldots R_\nu(x)$, $\deg R(x) = l$. Then*

$$e^{-l}\lceil R_1\rceil\ldots\lceil R_\nu\rceil \le \lceil R\rceil \le (l+1)^{\nu-1}\lceil R_1\rceil\ldots\lceil R_\nu\rceil. \tag{3.5}$$

Lemma 3.6. *Let $F(x)$ and $G(x)$ be polynomials with integer coefficients of degree $\le l$. Let $F(x)$ be a polynomial irreducible over $\mathbb{Z}$ with $\lceil F\rceil > e^l\lceil G\rceil$. Then $F(x)$ and $G(x)$ have no common root.*

Proof. Assume that $F(x)$ and $G(x)$ have a common root. Then there exists a polynomial $\widetilde{F}(x) \in \mathbb{Z}[x]$, $\widetilde{F}(x) \not\equiv 1$, dividing both $F(x)$ and $G(x)$. Since $F(x)$ is irreducible, we have $\widetilde{F}(x) \equiv F(x)$. Therefore $G(x) = F(x)\widetilde{G}(x)$, where $\widetilde{G}(x) \in \mathbb{Z}[x]$. By (3.5) we have $\lceil G\rceil \ge e^{-l}\lceil F\rceil\lceil \widetilde{G}\rceil \ge e^{-l}\lceil F\rceil$. ∎

Lemma 3.7. *Consider the following system of inequalities*:

$$\begin{cases} |a_{11}x_1 + \ldots + a_{1n}x_n| \le A_1, \\ |a_{21}x_1 + \ldots + a_{2n}x_n| \le A_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ |a_{n1}x_1 + \ldots + a_{nn}x_n| \le A_n, \end{cases} \tag{3.6}$$

where $a_{ij} \in \mathbb{R},\ \ A_i \in \mathbb{R}^+,\ 1 \le i,j \le n$. Suppose that

(I) *for any* $1 \le j \le n,\ \max_{2\le i\le n}(|a_{ij}|) \le B_j, \min_{1\le j\le n-1}(B_j) \ge B_n > 0$;
(II) $\max_{1\le j\le n-1}(|a_{1j}|) \le |a_{1n}|,\ a_{1n} \ne 0$;
(III) $\max_{2\le\nu\le n-1}(A_\nu) \le A_n$;
(IV) $|\Delta| > c_d\,|a_{1n}|B_1\ldots B_{n-1}$, *where Δ is the determinant of the system* (3.6), *and c_d is some positive constant.*

Then for any solution $(\widetilde{x}_1, \ldots, \widetilde{x}_n) \in \mathbb{R}^n$ of the system (3.6) *the following estimates hold*:

$$|\widetilde{x}_l| < \frac{n!}{c_d}B_l^{-1}\max\left(\frac{A_1B_n}{|a_{1n}|}, A_n\right) \qquad (1 \le l \le n). \tag{3.7}$$

P r o o f. Using the Theorem of Cramer, we have

$$|\widetilde{x}_l| = \frac{|\Delta_l|}{|\Delta|} \quad (1 \le l \le n), \tag{3.8}$$

where Δ_l is the determinant obtained from Δ by replacing lth column with $[\theta_1 A_1, \ldots, \theta_n A_n]^{\mathrm{T}}$, $|\theta_\nu| \le 1$, $1 \le \nu \le n$.

When expanding Δ_l with respect to the lth column, we get

$$|\Delta_l| \le n \max(A_1|M_1|, \ldots, A_n|M_n|), \tag{3.9}$$

where M_ν are the minors corresponding to $\theta_\nu A_\nu$ for $1 \le \nu \le n$.

By (I) we have

$$|M_1| \le (n-1)! B_1 \ldots B_n B_l^{-1}. \tag{3.10}$$

Let us show that

$$|M_\nu| \le (n-1)! |a_{1n}| B_1 \ldots B_{n-1} B_l^{-1} \quad (2 \le \nu \le n). \tag{3.11}$$

In fact, by (II) the absolute values of a_{1j} from the first line of the minors M_ν, $2 \le \nu \le n$, are less than or equal to $|a_{1n}|$. On the other hand, by (I) the absolute values of any minors $m_{\nu j}$ of M_ν which correspond to the elements a_{1j} are less than or equal to $(n-2)! B_1 \ldots B_{n-1} B_l^{-1}$. This gives (3.11).

Using (III) and (3.9)–(3.11), we get

$$|\Delta_l| \le n! B_1 \ldots B_{n-1} B_l^{-1} \max(A_1 B_n, A_n |a_{1n}|). \tag{3.12}$$

By substituting the estimate (IV) and (3.12) into (3.8), we obtain (3.7). ■

4. Construction of $\widetilde{Q}_i^{(0)}(x), \ldots, \widetilde{Q}_i^{(n-1)}(x)$. Fix some $h \in \mathbb{N}$, $h > \widetilde{H}_0$. We consider the finite set of polynomials $P(x) \in \mathbb{Z}[x]$ with $\deg P(x) \le n$, $\lceil P \rceil \le h$. Their values at ξ are distinct. Hence we can choose a unique (up to sign) polynomial $\widetilde{P}_0(x) \in \mathbb{Z}[x]$, $\widetilde{P}_0(x) \not\equiv 0$, with minimal absolute value at ξ.

Put

$$c_p = n!\, \xi^{-n^2}.$$

We now increase h until a polynomial $\widetilde{P}_1(x) \in \mathbb{Z}[x]$, $\widetilde{P}_1(x) \not\equiv 0$, of degree $\le n$ with $\lceil \widetilde{P}_1 \rceil \le h$, $|\widetilde{P}_1(\xi)| < c_p^{-1} |\widetilde{P}_0(\xi)|$ appears. If there are several polynomials of this kind, pick one with minimal absolute value at ξ. It is clear that $\widetilde{H}_0 < \lceil \widetilde{P}_1 \rceil$. We increase h again until a polynomial $\widetilde{P}_2(x) \in \mathbb{Z}[x]$ of degree $\le n$ with $\widetilde{H}_0 < \lceil \widetilde{P}_1 \rceil < \lceil \widetilde{P}_2 \rceil \le h$, $|\widetilde{P}_2(\xi)| < c_p^{-1} |\widetilde{P}_1(\xi)|$ appears. By repeating this process, we obtain a sequence of polynomials $\widetilde{P}_i(x) \in \mathbb{Z}[x]$,

$\deg \widetilde{P}_i(x) \le n$, such that

$$(4.1)\quad \begin{aligned} &\text{(i)}\quad 1/2 > |\widetilde{P}_1(\xi)| > c_p|\widetilde{P}_2(\xi)| > \ldots > c_p^{k-1}|\widetilde{P}_k(\xi)| > \ldots, \\ &\text{(ii)}\quad \widetilde{H}_0 < \boxed{\widetilde{P}_1} < \boxed{\widetilde{P}_2} < \ldots < \boxed{\widetilde{P}_k} < \ldots, \\ &\text{(iii)}\quad \forall P(x) \in \mathbb{Z}[x],\ P(x) \not\equiv 0,\ \deg P(x) \le n,\ \boxed{P} < \boxed{\widetilde{P}_{k+1}}, \\ &\qquad\qquad |P(\xi)| \ge c_p^{-1}|\widetilde{P}_k(\xi)|. \end{aligned}$$

For any natural i we set

$$\widetilde{Q}_i^{(0)}(x) = \widetilde{P}_i(x).$$

Write $\widetilde{Q}_i^{(0)}(x) = a_n^{(0)}x^n + \ldots + a_1^{(0)}x + a_0^{(0)}$. By Lemma 3.1 there is an index $j_1 \in \{1, \ldots, n\}$ such that $|a_{j_1}^{(0)}| = \boxed{\widetilde{Q}_i^{(0)}}$.

We successively construct nonzero polynomials $\widetilde{Q}_i^{(0)}(x), \ldots, \widetilde{Q}_i^{(n-1)}(x)$ in $\mathbb{Z}[x]$ of degree $\le n$ and distinct integers $j_1, \ldots, j_n$ from $\{1, \ldots, n\}$. We write $\widetilde{Q}_i^{(l)}(x) = a_n^{(l)}x^n + \ldots + a_1^{(l)}x + a_0^{(l)},\ 0 \le l \le n-1$. The polynomials $\widetilde{Q}_i^{(l)}(x)$ and the numbers j_{l+1} (which we call the *indices of the* $\widetilde{Q}_i$*-system*) will have the following properties:

(1_l) $\quad |\widetilde{Q}_i^{(l)}(\xi)| < c_p^{-1}|\widetilde{P}_{i-1}(\xi)|$,

(2_l) $\quad |a_{j_\mu}^{(l)}| \le c_p^{-1}\boxed{\widetilde{Q}_i^{(\mu-1)}} \quad (\mu = 1, \ldots, l)$,

(3_l) $\quad |a_{j_{l+1}}^{(l)}| > \xi^{n-1}\boxed{\widetilde{Q}_i^{(l)}}$

(if $l = 0$, we have (1_l), (3_l) only). Moreover, if for some $0 \le l \le n-1$ any nonzero polynomial $Q(x) = a_nx^n + \ldots + a_1x + a_0 \in \mathbb{Z}[x]$ satisfies

$$|Q(\xi)| < c_p^{-1}|\widetilde{P}_{i-1}(\xi)|,$$

$$|a_{j_\mu}| \le c_p^{-1}\boxed{\widetilde{Q}_i^{(\mu-1)}} \qquad (\mu = 1, \ldots, l)$$

(if $l = 0$, we have $|Q(\xi)| < c_p^{-1}|\widetilde{P}_{i-1}(\xi)|$ only), then $\boxed{Q} \ge \boxed{\widetilde{Q}_i^{(l)}}$. In other words, $\widetilde{Q}_i^{(l)}(x)$ has minimum height among nonzero polynomials in $\mathbb{Z}[x]$ with $(1_l), (2_l)$. We call this the *minimality property* of $\widetilde{Q}_i^{(l)}(x),\ 0 \le l \le n-1$.

The pair $(\widetilde{Q}_i^{(0)}(x), j_1)$ has the desired properties. Suppose $0 \le t < n-1$, and $(\widetilde{Q}_i^{(0)}(x), j_1), \ldots, (\widetilde{Q}_i^{(t)}(x), j_{t+1})$ have been constructed so that $(1_l), (2_l), (3_l)$ with $l = 0, \ldots, t$ and the minimality property hold, and $j_1, \ldots$ $\ldots, j_{t+1}$ are distinct integers in $\{1, \ldots, n\}$. By Minkowski's Theorem on linear forms there is a nonzero polynomial $Q(x) = a_nx^n + \ldots + a_1x + a_0 \in \mathbb{Z}[x]$

having

$$
\begin{aligned}
&|Q(\xi)| < c_p^{-1}|\widetilde{P}_{i-1}(\xi)|,\\
(4.2)\qquad &|a_{j_\mu}| \le c_p^{-1}\left\lceil\widetilde{Q}_i^{(\mu-1)}\right\rceil \qquad (\mu = 1,\dots,t+1),\\
&|a_{k_\eta}| \le \Big(c_p^{-t-2}|\widetilde{P}_{i-1}(\xi)|\prod_{\nu=0}^{t}\left\lceil\widetilde{Q}_i^{(\nu)}\right\rceil\Big)^{-1/(n-t-1)} \qquad (\eta=1,\dots,n-t-1),
\end{aligned}
$$

where $\{k_1,\dots,k_{n-t-1}\} = \{1,\dots,n\}\setminus\{j_1,\dots,j_{t+1}\}$.

If there are several polynomials of this kind, pick one whose height is minimal. We denote it by $\widetilde{Q}_i^{(t+1)}(x)$. By Lemma 3.1, there is an index j in $\{1,\dots,n\}$ such that $|a_j^{(t+1)}| = \left\lceil\widetilde{Q}_i^{(t+1)}\right\rceil$. On the other hand, by the minimality property of $\widetilde{Q}_i^{(l)}(x)$ we have $\left\lceil\widetilde{Q}_i^{(l)}\right\rceil \le \left\lceil\widetilde{Q}_i^{(t+1)}\right\rceil$ for any $0\le l\le t$. Hence $|a_{j_\mu}^{(t+1)}| < \left\lceil\widetilde{Q}_i^{(\mu-1)}\right\rceil \le \left\lceil\widetilde{Q}_i^{(t+1)}\right\rceil$ for $\mu = 1,\dots,t+1$. Therefore j is distinct from $j_1,\dots,j_{t+1}$. We set $j_{t+2} = j$. Then $(1_{t+1}),(2_{t+1}),(3_{t+1})$, and the minimality property hold for $\widetilde{Q}_i^{(t+1)}(x)$. In Section 5 we will slightly modify the construction of the polynomials $Q_i^{(0)}(x),\dots,Q_i^{(n-1)}(x)$ (see (5.7) and Remark 5.8). Therefore we use the inequality $|a_{j_{l+1}}^{(l)}| > \xi^{n-1}\left\lceil\widetilde{Q}_i^{(l)}\right\rceil$ instead of $|a_{j_{l+1}}^{(l)}| = \left\lceil\widetilde{Q}_i^{(l)}\right\rceil$, $0\le l\le n-1$.

In this way $(\widetilde{Q}_i^{(0)}(x),j_1),\dots,(\widetilde{Q}_i^{(n-1)}(x),j_n)$ can be constructed. Clearly

$$
(4.3)\qquad \left\lceil\widetilde{Q}_i^{(0)}\right\rceil \le \left\lceil\widetilde{Q}_i^{(1)}\right\rceil \le \dots \le \left\lceil\widetilde{Q}_i^{(n-1)}\right\rceil.
$$

5. Properties of $\widetilde{Q}_i^{(0)}(x),\dots,\widetilde{Q}_i^{(n-1)}(x)$. Using Lemma 3.1, the last two inequalities from (4.2), and (4.3), we deduce

$$
(5.1)\qquad \left\lceil\widetilde{Q}_i^{(l)}\right\rceil \le c_p^{(l+1)/(n-l)}\Big(|\widetilde{P}_{i-1}(\xi)|\prod_{\nu=0}^{l-1}\left\lceil\widetilde{Q}_i^{(\nu)}\right\rceil\Big)^{-1/(n-l)} \qquad (1\le l\le n-1).
$$

Applying (4.3) to (5.1) with $l = n-1$, we get

$$
(5.2)\qquad \left\lceil\widetilde{Q}_i^{(n-1)}\right\rceil \le c_p^{n}|\widetilde{P}_{i-1}(\xi)|^{-1}\Big(\prod_{\nu=0}^{n-2}\left\lceil\widetilde{Q}_i^{(\nu)}\right\rceil\Big)^{-1} \le c_p^{n}|\widetilde{P}_{i-1}(\xi)|^{-1}\left\lceil\widetilde{P}_i\right\rceil^{-n+1}.
$$

Similarly, (4.3) and (5.1) imply that

$$
\begin{aligned}
(5.3)\qquad \left\lceil\widetilde{Q}_i^{(l)}\right\rceil \le \left\lceil\widetilde{Q}_i^{(n-2)}\right\rceil &\le c_p^{(n-1)/2}|\widetilde{P}_{i-1}(\xi)|^{-1/2}\Big(\prod_{\nu=0}^{n-3}\left\lceil\widetilde{Q}_i^{(\nu)}\right\rceil\Big)^{-1/2}\\
&\le c_p^{(n-1)/2}|\widetilde{P}_{i-1}(\xi)|^{-1/2}\left\lceil\widetilde{P}_i\right\rceil^{1-n/2} \qquad (0\le l\le n-2).
\end{aligned}
$$

LEMMA 5.1. *Let i be any natural number > 1. Suppose that for some $0 \le l \le n-1$ the polynomial $\widetilde{Q}_i^{(l)}(x)$ satisfies the conditions of Lemma* 3.2. *Then*

$$\lceil \widetilde{Q}_i^{(l)} \rceil^{-1} < (c_T c_p\, \xi^{n-1})^{-1/(A-1)} |\widetilde{P}_{i-1}(\xi)|^{1/(A-1)}. \tag{5.4}$$

Proof. By Lemma 3.2 we obtain $\lceil \widetilde{Q}_i^{(l)} \rceil < \xi^{-n+1} |\widetilde{Q}_i^{(l)\prime}(\xi)|$. On the other hand, $\lceil \widetilde{Q}_i^{(l)} \rceil > \widetilde{H}_0$. Therefore by (2.3) and (1_l) we get

$$c_T \lceil \widetilde{Q}_i^{(l)} \rceil^{-A} < \frac{|\widetilde{Q}_i^{(l)}(\xi)|}{|\widetilde{Q}_i^{(l)\prime}(\xi)|} < \xi^{-n+1} \frac{|\widetilde{Q}_i^{(l)}(\xi)|}{\lceil \widetilde{Q}_i^{(l)} \rceil} < c_p^{-1} \xi^{-n+1} |\widetilde{P}_{i-1}(\xi)| \lceil \widetilde{Q}_i^{(l)} \rceil^{-1},$$

hence

$$\lceil \widetilde{Q}_i^{(l)} \rceil^{-A+1} < c_T^{-1} c_p^{-1} \xi^{-n+1} |\widetilde{P}_{i-1}(\xi)|,$$

and the result follows. ■

Define

$$c_M = \min_{\substack{P(x) \in \mathbb{Z}[x],\, P(x) \not\equiv 0 \\ \deg P(x) \le n,\, \lceil P \rceil \le e^n \lceil \widetilde{P}_1 \rceil}} (|P(\xi)|), \tag{5.5}$$

$$H_0 = c_M^{-30n} c_R^{15n} e^{60n^2} \lceil \widetilde{P}_1 \rceil^n. \tag{5.6}$$

By (4.1)(ii) there exists an index $k_0 \in \mathbb{N}$ such that $\lceil \widetilde{P}_{k_0} \rceil \le H_0 < \lceil \widetilde{P}_{k_0+1} \rceil$. From now on

$$Q_i^{(l)}(x) = \widetilde{Q}_{k_0+i}^{(l)}(x) \quad \text{for any } i \in \mathbb{N} \text{ and } l = 0, \ldots, n-1. \tag{5.7}$$

In particular,

$$P_i(x) = \widetilde{P}_{k_0+i}(x) \quad \text{for any } i \in \mathbb{N}.$$

LEMMA 5.2. *For any natural $i > 1$ we have*

$$\begin{aligned} &\text{(I)} \quad |\widetilde{P}_{i-1}(\xi)| < \lceil \widetilde{P}_i \rceil^{-(n-1)(A-1)/(A-2)}, \\ &\text{(II)} \quad \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil < c_p^{-n} |P_{i-1}(\xi)|^{-(A-2)/(A-1)}. \end{aligned} \tag{5.8}$$

Proof. It follows from (2_l) with $l = n-1$ and (4.3) that the polynomials $\widetilde{Q}_i^{(n-1)}(x)$ satisfy the conditions of Lemma 3.2. Substituting (5.2) into (5.4), we get

$$\left(c_p^n |\widetilde{P}_{i-1}(\xi)|^{-1} \lceil \widetilde{P}_i \rceil^{-n+1}\right)^{-1} < (c_T c_p \xi^{n-1})^{-1/(A-1)} |\widetilde{P}_{i-1}(\xi)|^{1/(A-1)},$$

hence

$$|\widetilde{P}_{i-1}(\xi)|^{(A-2)/(A-1)} < c_p^n (c_T c_p \xi^{n-1})^{-1/(A-1)} \lceil \widetilde{P}_i \rceil^{-n+1},$$

and so, by the definitions of c_T and c_p, we obtain

$$|\widetilde{P}_{i-1}(\xi)|^{(A-2)/(A-1)} < \lceil \widetilde{P}_i \rceil^{-n+1},$$

which gives (5.8)(I).

Similarly, substituting (5.1) with $l = n-1$ into (5.4) and keeping (5.7) in mind, we deduce

$$\Big(c_p^n |P_{i-1}(\xi)|^{-1}\Big(\prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil\Big)^{-1}\Big)^{-1} < (c_T c_p \xi^{n-1})^{-1/(A-1)} |P_{i-1}(\xi)|^{1/(A-1)},$$

hence

$$\prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil < c_p^n (c_T c_p \xi^{n-1})^{-1/(A-1)} |P_{i-1}(\xi)|^{-(A-2)/(A-1)}.$$

Using the definitions of c_T and c_p, we get (5.8)(II). ■

Corollary 5.3. *For any natural $i > 1$ we have*

$$\text{(5.9)}\qquad \begin{aligned} &\text{(I)} \quad |P_{i-1}(\xi)| < \lceil P_i \rceil^{-(n-1)(A-1)/(A-2)},\\ &\text{(II)} \quad |P_{i-1}(\xi)| < \lceil P_i \rceil^{-n}. \end{aligned}$$

Proof. The inequality (5.9)(I) immediately follows from (5.7) and (5.8)(I). To obtain (5.9)(II) we must use (5.9)(I) and the inequality $A < n+1$:

$$|P_{i-1}(\xi)| < \lceil P_i \rceil^{-(n-1)(A-1)/(A-2)} < \lceil P_i \rceil^{-(n-1)(n+1-1)/(n+1-2)} = \lceil P_i \rceil^{-n}. \ ■$$

Lemma 5.4. *For any $i \in \mathbb{N}$ the polynomials $P_i(x)$ are irreducible over $\mathbb{Z}$ and have degree n.*

Proof. Assume that $P_i(x) = P_{i_1}(x) \ldots P_{i_\gamma}(x)$, $1 \le \gamma \le n$, where $P_{i_1}(x), \ldots, P_{i_\gamma}(x)$ are irreducible over $\mathbb{Z}$, have degree $< n$ and integer coefficients. Let the heights of $P_{i_1}(x), \ldots, P_{i_\lambda}(x)$ be greater than $e^n \lceil \widetilde{P}_1 \rceil$ and the heights of the others be at most $e^n \lceil \widetilde{P}_1 \rceil$. It is obvious that $\lambda \le n$. We now show that $\lambda \ge 1$. In fact, assume that the heights of $P_{i_1}(x), \ldots, P_{i_\gamma}(x)$ do not exceed $e^n \lceil \widetilde{P}_1 \rceil$. Then by (3.5) we get

$$\lceil P_i \rceil \le (n+1)^{n-1} \lceil P_{i_1} \rceil \ldots \lceil P_{i_\gamma} \rceil \le (n+1)^{n-1} \big(e^n \lceil \widetilde{P}_1 \rceil\big)^n,$$

hence $\lceil P_i \rceil \le (n+1)^{n-1} e^{n^2} \lceil \widetilde{P}_1 \rceil^n$. On the other hand, (5.6) and (5.7) yield

$$\text{(5.10)}\qquad \lceil P_i \rceil > c_M^{-30n} c_R^{15n} e^{60n^2} \lceil \widetilde{P}_1 \rceil^n$$

for any $i \in \mathbb{N}$. This gives a contradiction.

We now prove that there exists an index $1 \le j_0 \le \lambda$ such that

$$(5.11) \qquad |P_{i_{j_0}}(\xi)| < c_R^{-1/2} \lceil P_{i_{j_0}} \rceil^{-(n-1)(A-1)/(A-2)+1/30}.$$

Assume the contrary. Then by (5.9)(I), the definition of c_M, (3.5), and (5.10) we have

$$\begin{aligned}
&\lceil P_{i+1} \rceil^{-(n-1)(A-1)/(A-2)} \\
&\quad > |P_i(\xi)| = \prod_{\nu=1}^{\gamma} |P_{i_\nu}(\xi)| \ge c_M^{\gamma-\lambda} \prod_{\nu=1}^{\lambda} |P_{i_\nu}(\xi)| \\
&\quad \ge c_M^{\gamma-\lambda} c_R^{-\lambda/2} \Big(\prod_{\nu=1}^{\lambda} \lceil P_{i_\nu} \rceil \Big)^{-(n-1)(A-1)/(A-2)+1/30} \\
&\quad > c_M^n c_R^{-n/2} (e^n \lceil P_i \rceil)^{-(n-1)(A-1)/(A-2)+1/30} \\
&\quad = c_M^n c_R^{-n/2} e^{-n(n-1)(A-1)/(A-2)+n/30} \lceil P_i \rceil^{1/30} \lceil P_i \rceil^{-(n-1)(A-1)/(A-2)} \\
&\quad > \lceil P_i \rceil^{-(n-1)(A-1)/(A-2)},
\end{aligned}$$

which is impossible.

Since $1 \le j_0 \le \lambda$, we have $\lceil P_{i_{j_0}} \rceil > e^n \lceil \widetilde{P}_1 \rceil$. Therefore there exists an index $k \in \mathbb{N}$ such that

$$(5.12) \qquad e^n \lceil \widetilde{P}_k \rceil < \lceil P_{i_{j_0}} \rceil \le e^n \lceil \widetilde{P}_{k+1} \rceil.$$

Combining (5.8)(I) with (5.12), then using the inequality $\lceil P_{i_{j_0}} \rceil > \widetilde{H}_0 > c_R^{15} e^{60n^2}$, we obtain

$$\begin{aligned}
(5.13) \quad |\widetilde{P}_k(\xi)| &< \lceil \widetilde{P}_{k+1} \rceil^{-(n-1)(A-1)/(A-2)} \le (e^{-n} \lceil P_{i_{j_0}} \rceil)^{-(n-1)(A-1)/(A-2)} \\
&= e^{n(n-1)(A-1)/(A-2)} \lceil P_{i_{j_0}} \rceil^{-1/30} \lceil P_{i_{j_0}} \rceil^{-(n-1)(A-1)/(A-2)+1/30} \\
&< c_R^{-1/2} \lceil P_{i_{j_0}} \rceil^{-(n-1)(A-1)/(A-2)+1/30}.
\end{aligned}$$

Since $\lceil P_{i_{j_0}} \rceil > e^n \lceil \widetilde{P}_k \rceil$ and $P_{i_{j_0}}(x)$ is irreducible over $\mathbb{Z}$, by Lemma 3.6 the polynomials $\widetilde{P}_k(x)$ and $P_{i_{j_0}}(x)$ have no common root. Moreover, $\deg \widetilde{P}_k(x) \ge 2$ and $\deg P_{i_{j_0}}(x) \ge 2$, since otherwise we get

$$\frac{|\widetilde{P}_k(\xi)|}{|\widetilde{P}_k'(\xi)|} = \frac{|\widetilde{P}_k(\xi)|}{\lceil \widetilde{P}_k \rceil} < \lceil \widetilde{P}_k \rceil^{-(n-1)(A-1)/(A-2)+1/30-1},$$

and a simple calculation shows that

$$-(n-1)\frac{A-1}{A-2} - \frac{29}{30} < -A,$$

hence

$$\frac{|\widetilde{P}_k(\xi)|}{|\widetilde{P}'_k(\xi)|} < \lceil \widetilde{P}_k \rceil^{-A},$$

which contradicts (2.3). The same holds for $P_{i_{j_0}}(x)$. Thus, we can apply (3.2) to $\widetilde{P}_k(x)$ and $P_{i_{j_0}}(x)$.

(a) Substituting (5.11) and (5.13) into (3.2)(I), then using (5.12), we deduce

$$\begin{aligned}1 &< c_R \max(|\widetilde{P}_k(\xi)|, |P_{i_{j_0}}(\xi)|)^2 \max\left(\lceil \widetilde{P}_k \rceil, \lceil P_{i_{j_0}} \rceil\right)^{2n-3} \\ &< c_R c_R^{-1} \lceil P_{i_{j_0}} \rceil^{-2(n-1)(A-1)/(A-2)+1/15} \lceil P_{i_{j_0}} \rceil^{2n-3} \\ &= \lceil P_{i_{j_0}} \rceil^{-2(n-1)(A-1)/(A-2)+2n-44/15}.\end{aligned}$$

Here we have used the inequalities $\deg \widetilde{P}_k(x) \le n$, $\deg P_{i_{j_0}}(x) \le n-1$. It is easy to verify that

$$-2(n-1)\frac{A-1}{A-2} + 2n - \frac{44}{15} < 0 \quad \text{for } n = 3, \ldots, 7,$$

and we obtain a contradiction.

Since $\min(\lceil \widetilde{P}_k \rceil, \lceil P_{i_{j_0}} \rceil) > \widetilde{H}_0$, we can apply (2.3) to the polynomials $\widetilde{P}_k(x)$ and $P_{i_{j_0}}(x)$.

(b) Applying (2.3) to (3.2)(II)–(III), then using (5.11)–(5.13) and the definitions of c_T and c_R, we have

$$\begin{aligned}1 &< c_R c_T^{-2} \max(|\widetilde{P}_k(\xi)|, |P_{i_{j_0}}(\xi)|)^3 \max\left(\lceil \widetilde{P}_k \rceil, \lceil P_{i_{j_0}} \rceil\right)^{2A} \max\left(\lceil \widetilde{P}_k \rceil, \lceil P_{i_{j_0}} \rceil\right)^{2n-4} \\ &< c_R c_R^{-3/2} c_T^{-2} \lceil P_{i_{j_0}} \rceil^{-3(n-1)(A-1)/(A-2)+1/10} \lceil P_{i_{j_0}} \rceil^{2A+2n-4} \\ &< \lceil P_{i_{j_0}} \rceil^{-3(n-1)(A-1)/(A-2)+2A+2n-39/10}.\end{aligned}$$

Since

$$-3(n-1)\frac{A-1}{A-2} + 2A + 2n - \frac{39}{10} < 0 \quad \text{for } n = 3, \ldots, 7,$$

we come to a contradiction again. This completes the proof. ∎

LEMMA 5.5. *For any natural $i > 1$ we have*

$$|P_{i-1}(\xi)|^{-1} < \lceil P_i \rceil^{(2A+n-2)/3} \lceil P_{i-1} \rceil^{(n-1)/3}. \tag{5.14}$$

P r o o f. By Lemma 5.4 the polynomials $P_{i-1}(x)$ and $P_i(x)$ are irreducible over $\mathbb{Z}$ and have degree n. Therefore they have no common root. Moreover, $\deg P_{i-1}(x) \ge 2$ and $\deg P_i(x) \ge 2$, since otherwise by (5.9)(II) we get

$$\frac{|P_i(\xi)|}{|P'_i(\xi)|} = \frac{|P_i(\xi)|}{\lceil P_i \rceil} < \lceil P_i \rceil^{-n-1},$$

which contradicts (2.3). The same holds for $P_{i-1}(x)$. Thus, we can apply (3.2) to $P_{i-1}(x)$ and $P_i(x)$.

(a) Substituting (5.9)(II) into (3.2)(I) and using (4.1)(ii), we obtain

$$\begin{aligned}1 &< c_R \max(|P_{i-1}(\xi)|,\ |P_i(\xi)|)^2 \max(\lceil P_{i-1}\rceil,\ \lceil P_i\rceil)^{2n-2}\\ &< c_R \lceil P_i\rceil^{-2n} \lceil P_i\rceil^{2n-2} = c_R \lceil P_i\rceil^{-2},\end{aligned}$$

hence $\lceil P_i\rceil^2 < c_R$. This gives a contradiction with (5.10).

Since $\min(\lceil P_{i-1}\rceil,\ \lceil P_i\rceil) > \widetilde{H}_0$, we can apply (2.3) to the polynomials $P_{i-1}(x)$ and $P_i(x)$.

(b) Applying (2.3) to (3.2)(II), then using (4.1)(i), (4.1)(ii), and the definitions of c_T and c_R, we deduce

$$\begin{aligned}1 &< c_R \max(|P_{i-1}(\xi)|\cdot|P'_{i-1}(\xi)|\cdot|P'_i(\xi)|, |P_i(\xi)|\cdot|P'_{i-1}(\xi)|^2)\lceil P_{i-1}\rceil^{n-2}\lceil P_i\rceil^{n-1}\\ &< c_R c_T^{-2}\,|P_{i-1}(\xi)|^3\lceil P_{i-1}\rceil^{A}\lceil P_i\rceil^{A}\lceil P_{i-1}\rceil^{n-2}\lceil P_i\rceil^{n-1}\\ &= c_R c_T^{-2}\,|P_{i-1}(\xi)|^3\lceil P_i\rceil^{A+n-1}\lceil P_{i-1}\rceil^{A+n-2}\\ &< |P_{i-1}(\xi)|^3\lceil P_i\rceil^{2A+n-2}\lceil P_{i-1}\rceil^{n-1}.\end{aligned}$$

(c) Similarly, by (2.3), (3.2)(III), (4.1)(i), (4.1)(ii), and the definitions of c_T and c_R, we have

$$\begin{aligned}1 &< c_R \max(|P_i(\xi)|\cdot|P'_{i-1}(\xi)|\cdot|P'_i(\xi)|, |P_{i-1}(\xi)|\cdot|P'_i(\xi)|^2)\lceil P_{i-1}\rceil^{n-1}\lceil P_i\rceil^{n-2}\\ &< c_R c_T^{-2}\,|P_{i-1}(\xi)|^3\lceil P_i\rceil^{2A}\lceil P_{i-1}\rceil^{n-1}\lceil P_i\rceil^{n-2}\\ &= c_R c_T^{-2}\,|P_{i-1}(\xi)|^3\lceil P_i\rceil^{2A+n-2}\lceil P_{i-1}\rceil^{n-1}\\ &< |P_{i-1}(\xi)|^3\lceil P_i\rceil^{2A+n-2}\lceil P_{i-1}\rceil^{n-1}.\end{aligned}$$

It is easy to see that either one of the above two inequalities gives (5.14). ∎

LEMMA 5.6. *For any natural $i > 1$ we have*

$$\prod_{\nu=0}^{n-2}\lceil Q_i^{(\nu)}\rceil < c_p^{-n}|P_{i-1}(\xi)|^{-1/2}\lceil P_i\rceil^{-1+n/2}. \tag{5.15}$$

Proof. From (5.8)(II) we deduce

$$\begin{aligned}\prod_{\nu=0}^{n-2}\lceil Q_i^{(\nu)}\rceil &< c_p^{-n}|P_{i-1}(\xi)|^{-(A-2)/(A-1)}\\ &\equiv c_p^{-n}|P_{i-1}(\xi)|^{-1/2}\lceil P_i\rceil^{-1+n/2}|P_{i-1}(\xi)|^{-(A-3)/(2(A-1))}\lceil P_i\rceil^{1-n/2}.\end{aligned} \tag{5.16}$$

We now prove that

$$|P_{i-1}(\xi)|^{-(A-3)/(2(A-1))}\lceil P_i\rceil^{1-n/2} < 1. \tag{5.17}$$

If the result were false, we should have

$$|P_{i-1}(\xi)| \le \lceil P_i \rceil^{-(n-2)(A-1)/(A-3)}.$$

Substituting this into (5.14), we get

$$\begin{aligned} 1 &< |P_{i-1}(\xi)|\,\lceil P_i \rceil^{(2A+n-2)/3}\lceil P_{i-1} \rceil^{(n-1)/3} \\ &\le \lceil P_i \rceil^{-(n-2)(A-1)/(A-3)+(2A+n-2)/3}\lceil P_{i-1} \rceil^{(n-1)/3} \\ &< \lceil P_i \rceil^{-(n-2)(A-1)/(A-3)+(2A+n-2)/3+(n-1)/3}. \end{aligned}$$

A simple calculation shows that

$$-\frac{(n-2)(A-1)}{A-3} + \frac{2A+2n-3}{3} < 0 \quad \text{for } n = 3, \ldots, 7,$$

and we obtain a contradiction. This gives (5.17). Finally, (5.16) and (5.17) imply (5.15). ∎

Lemma 5.7. *Let i be any natural number > 1. Then for any $0 \le l \le n-2$ there exist at least two indices $\{k_1, k_2\} \subset \{1, \ldots, n\}$ such that*

$$|a_{k_\nu}^{(l)}| > \xi^{n-1}\lceil Q_i^{(l)} \rceil \quad (\nu = 1, 2).$$

Proof. By Lemma 3.1 for any $0 \le l \le n-1$ there exists an index $k_1 \in \{1, \ldots, n\}$ such that $|a_{k_1}^{(l)}| = \lceil Q_i^{(l)} \rceil$.

Fix some $0 \le l \le n-2$ and suppose that $|a_k^{(l)}| \le \xi^{n-1}\lceil Q_i^{(l)} \rceil$ for all $k \in \{1, \ldots, n\} \setminus \{k_1\}$. Then the polynomial $Q_i^{(l)}(x)$ satisfies the conditions of Lemma 3.2. Therefore we can apply Lemma 5.1 to $Q_i^{(l)}(x)$. Substituting (5.3) into (5.4) and keeping (5.7) in mind, we obtain

$$(c_p^{(n-1)/2}|P_{i-1}(\xi)|^{-1/2}\lceil P_i \rceil^{1-n/2})^{-1} < (c_T c_p \xi^{n-1})^{-1/(A-1)}|P_{i-1}(\xi)|^{1/(A-1)}.$$

This inequality can be written as

$$|P_{i-1}(\xi)|^{(A-3)/(2(A-1))}\lceil P_i \rceil^{-1+n/2} < (c_T c_p \xi^{n-1})^{-1/(A-1)} c_p^{(n-1)/2},$$

and so, by the definitions of c_T and c_p, we get

$$|P_{i-1}(\xi)|^{(A-3)/(2(A-1))}\lceil P_i \rceil^{-1+n/2} < 1,$$

which contradicts (5.17). ∎

Remark 5.8. We now can slightly modify the construction of the polynomials $Q_i^{(0)}(x), \ldots, Q_i^{(n-1)}(x)$. By Lemma 5.7 there are at least two indices $\{k_1, k_2\} \subset \{1, \ldots, n\}$ such that

$$|a_{k_\nu}^{(0)}| > \xi^{n-1}\lceil Q_i^{(0)} \rceil \quad (\nu = 1, 2).$$

We may suppose that $k_1 \in \{1, \ldots, n-1\}$ and set $j_1 = k_1$. We now construct $(Q_i^{(1)}(x), j_2), \ldots, (Q_i^{(n-1)}(x), j_n)$ with this (possibly new) value of j_1. Again there are at least two indices $\{k_1, k_2\} \subset \{1, \ldots, n\}$ with

$$|a_{k_\nu}^{(1)}| > \xi^{n-1} \boxed{Q_i^{(1)}} \quad (\nu = 1, 2).$$

Since $|a_{j_1}^{(1)}| \le c_p^{-1} \boxed{Q_i^{(0)}} < \xi^{n-1} \boxed{Q_i^{(1)}}$, these indices are distinct from j_1. So, we can pick $j_2 \in \{1, \ldots, n-1\} \setminus \{j_1\}$, etc. In this way we can arrange $j_1, \ldots, j_{n-1}$ so that $\{j_1, \ldots, j_{n-1}\} = \{1, \ldots, n-1\}$. Below, we assume this is true.

6. Three statements. The following results are of great importance for this paper.

STATEMENT 6.1. *Let i be any natural number > 1. Write*

$$P_{i-1}(x) = b_n x^n + \ldots + b_1 x + b_0.$$

Then the polynomials $P_{i-1}(x)$, $Q_i^{(0)}(x), \ldots, Q_i^{(n-2)}(x)$ are linearly independent and also

$$|\Delta| = \left\| \begin{matrix} a_{j_1}^{(n-2)} & \ldots & a_{j_{n-1}}^{(n-2)} & Q_i^{(n-2)}(\xi) \\ \ldots & \ldots & \ldots & \ldots \\ a_{j_1}^{(0)} & \ldots & a_{j_{n-1}}^{(0)} & Q_i^{(0)}(\xi) \\ b_{j_1} & \ldots & b_{j_{n-1}} & P_{i-1}(\xi) \end{matrix} \right\| > \xi^{n^2} |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \boxed{Q_i^{(\nu)}}, \tag{6.1}$$

where $j_1, \ldots, j_{n-1}$ are the indices of the Q_i-system.

Proof. From this moment on, we will take into account the notation (5.7) when using the formulas from Section 4. By (2_l) with $1 \le l \le n-2$ and (4.3) we have

$$|a_{j_\mu}^{(l)}| \le c_p^{-1} \boxed{Q_i^{(\mu-1)}} \le c_p^{-1} \boxed{Q_i^{(l)}} \quad (1 \le \mu \le l),$$

hence

$$\left\| \begin{matrix} a_{j_1}^{(n-2)} & \ldots & a_{j_{n-1}}^{(n-2)} \\ \ldots & \ldots & \ldots \\ a_{j_1}^{(0)} & \ldots & a_{j_{n-1}}^{(0)} \end{matrix} \right\| \ge \prod_{\nu=0}^{n-2} |a_{j_{\nu+1}}^{(\nu)}| - \frac{(n-1)!}{c_p} \prod_{\nu=0}^{n-2} \boxed{Q_i^{(\nu)}}.$$

Applying (3_l) with $l = 0, \ldots, n-2$ to $\prod_{\nu=0}^{n-2} |a_{j_{\nu+1}}^{(\nu)}|$, we obtain

$$\begin{aligned} \left\| \begin{matrix} a_{j_1}^{(n-2)} & \ldots & a_{j_{n-1}}^{(n-2)} \\ \ldots & \ldots & \ldots \\ a_{j_1}^{(0)} & \ldots & a_{j_{n-1}}^{(0)} \end{matrix} \right\| &\ge \xi^{(n-1)^2} \prod_{\nu=0}^{n-2} \boxed{Q_i^{(\nu)}} - \frac{(n-1)!}{c_p} \prod_{\nu=0}^{n-2} \boxed{Q_i^{(\nu)}} \\ &= \left(\xi^{(n-1)^2} - \frac{(n-1)!}{c_p} \right) \prod_{\nu=0}^{n-2} \boxed{Q_i^{(\nu)}}. \end{aligned} \tag{6.2}$$

On the other hand, by (4.1)(ii) and (4.3) the absolute values of other minors from the first $n-1$ columns of the determinant Δ are less than or equal to $(n-1)!\prod_{\nu=0}^{n-2}\lceil Q_i^{(\nu)}\rceil$. Hence by (1_l) with $l=0,\dots,n-2$, (6.2) and the definition of c_p, we get

$$\begin{aligned}|\Delta| &> \Big(\xi^{(n-1)^2}-\frac{(n-1)!}{c_p}\Big)|P_{i-1}(\xi)|\prod_{\nu=0}^{n-2}\lceil Q_i^{(\nu)}\rceil\\ &\quad-(n-1)!\Big(\sum_{\nu=0}^{n-2}|Q_i^{(\nu)}(\xi)|\Big)\prod_{\nu=0}^{n-2}\lceil Q_i^{(\nu)}\rceil\\ &> \Big(\xi^{(n-1)^2}-\frac{(n-1)!}{c_p}\Big)|P_{i-1}(\xi)|\prod_{\nu=0}^{n-2}\lceil Q_i^{(\nu)}\rceil\\ &\quad-\frac{(n-1)!(n-1)}{c_p}|P_{i-1}(\xi)|\prod_{\nu=0}^{n-2}\lceil Q_i^{(\nu)}\rceil\\ &> \xi^{n^2}\,|P_{i-1}(\xi)|\prod_{\nu=0}^{n-2}\lceil Q_i^{(\nu)}\rceil.\end{aligned}$$

This gives (6.1). Finally, since $|\Delta|>0$, the polynomials $P_{i-1}(x)$, $Q_i^{(0)}(x),\dots$ $\dots,Q_i^{(n-2)}(x)$ are linearly independent. ■

STATEMENT 6.2. *Let i and τ be natural numbers such that*

$$\lceil P_{i-1}\rceil\le c_h\lceil P_\tau\rceil,\qquad 1\le\tau\le i-1,\ i>1,\tag{6.3}$$

where

$$c_h=4\,(n!)^2\,c_p^{2n}.$$

Let also $L(x)$ be a nonzero polynomial satisfying

$$|L(\xi)|<|P_{i-1}(\xi)|^{1/2}\lceil P_i\rceil^{-1+n/2}\lceil P_\tau\rceil^{-n+1},\tag{6.4}$$

$$|L'(\xi)|<|P_{i-1}(\xi)|^{1-A/2}\lceil P_i\rceil^{(n-2)(1-A/2)}\lceil P_\tau\rceil^{-n+2},\tag{6.5}$$

$$\lceil L\rceil<\xi^{-n+1}|L'(\xi)|.\tag{6.6}$$

Then

$$\frac{|L(\xi)|}{|L'(\xi)|}<(c_h\,\xi^{-1})^{(n-1)A}\lceil L\rceil^{-A}.\tag{6.7}$$

Proof. By (6.4), (6.3), (5.9)(II), and (5.14) we get

$$\begin{aligned}|L(\xi)| &< |P_{i-1}(\xi)|^{1/2}\lceil P_i\rceil^{-1+n/2}\lceil P_\tau\rceil^{-n+1}\\ &\le c_h^{n-1}|P_{i-1}(\xi)|^{1/2}\lceil P_i\rceil^{-1+n/2}\lceil P_{i-1}\rceil^{-n+1}\\ &= c_h^{n-1}\,|P_{i-1}(\xi)|^{1/2+\alpha_1-\alpha_2}|P_{i-1}(\xi)|^{-\alpha_1}|P_{i-1}(\xi)|^{\alpha_2}\\ &\quad\times\lceil P_i\rceil^{-1+n/2}\lceil P_{i-1}\rceil^{-n+1}\\ &< c_h^{n-1}|P_{i-1}(\xi)|^{1/2+\alpha_1-\alpha_2}\lceil P_i\rceil^{(2A+n-2)\alpha_1/3}\lceil P_{i-1}\rceil^{(n-1)\alpha_1/3}\\ &\quad\times\lceil P_i\rceil^{-n\alpha_2}\lceil P_i\rceil^{-1+n/2}\lceil P_{i-1}\rceil^{-n+1}\end{aligned}\tag{6.8}$$

$$= c_h^{n-1}|P_{i-1}(\xi)|^{1/2+\alpha_1-\alpha_2}\lceil P_i\rceil^{(2A+n-2)\alpha_1/3-n\alpha_2-1+n/2} \times \lceil P_{i-1}\rceil^{(n-1)\alpha_1/3-n+1},$$

where α_1 and α_2 are any nonnegative constants. Put

$$(6.9)\qquad \begin{aligned} \alpha_1 &= \frac{3(n-2)(A-1)}{n-1}+3,\\ \alpha_2 &= \frac{7}{2}+\frac{3(n-2)(A-1)}{n-1}-\left(\frac{A}{2}-1\right)(A-1). \end{aligned}$$

It is easy to verify that for $n=3,\dots,7$ the constants α_1 and α_2 are positive. By (6.9) we have

$$(6.10)\qquad \frac{1}{2}+\alpha_1-\alpha_2=\left(\frac{A}{2}-1\right)(A-1),\qquad \frac{n-1}{3}\alpha_1-n+1=(n-2)(A-1),$$

and

$$\begin{aligned} &\frac{2A+n-2}{3}\alpha_1-n\alpha_2-1+\frac{n}{2}\\ &=\frac{n^2A^2+3nA^2-7n^2A+7nA-8A^2+12A+2n^2-8n-2}{2(n-1)}\\ &\equiv\frac{2((3n-5)A^2-(2n^2+n-9)A-n-3)+(n-1)(n-2)(A-2)(A-1)}{2(n-1)}, \end{aligned}$$

hence by (1.3) we obtain

$$(6.11)\qquad \begin{aligned} &\frac{2A+n-2}{3}\alpha_1-n\alpha_2-1+\frac{n}{2}\\ &=\frac{(n-1)(n-2)(A-2)(A-1)}{2(n-1)}=(n-2)\left(\frac{A}{2}-1\right)(A-1). \end{aligned}$$

Finally, (6.8), (6.10), and (6.11) imply that

$$(6.12)\qquad \begin{aligned} |L(\xi)| < c_h^{n-1}\,&|P_{i-1}(\xi)|^{(A/2-1)(A-1)}\\ &\times\lceil P_i\rceil^{(n-2)(A/2-1)(A-1)}\lceil P_{i-1}\rceil^{(n-2)(A-1)}. \end{aligned}$$

On the other hand, if we raise both sides of (6.5) to the power $-A+1$ and apply (6.3), we get

$$\begin{aligned} |L'(\xi)|^{-A+1} &> |P_{i-1}(\xi)|^{(A/2-1)(A-1)}\lceil P_i\rceil^{(n-2)(A/2-1)(A-1)}\lceil P_\tau\rceil^{(n-2)(A-1)}\\ &\ge c_h^{-(n-2)(A-1)}|P_{i-1}(\xi)|^{(A/2-1)(A-1)}\\ &\quad\times\lceil P_i\rceil^{(n-2)(A/2-1)(A-1)}\lceil P_{i-1}\rceil^{(n-2)(A-1)}. \end{aligned}$$

Combining this with (6.12), we find that $|L(\xi)| < c_h^{(n-1)A}\,|L'(\xi)|^{-A+1}$. We now divide both sides of this inequality by $|L'(\xi)|$ and apply (6.6):

$$\frac{|L(\xi)|}{|L'(\xi)|} < c_h^{(n-1)A}|L'(\xi)|^{-A} < c_h^{(n-1)A}\xi^{-(n-1)A}\,\overline{|L|}^{\,-A},$$

which gives (6.7). ■

STATEMENT 6.3. *Let i and τ be as in Statement 6.2. Let also $A_1, \ldots, A_n$ be positive numbers such that*

$$\prod_{\nu=1}^{n} A_\nu \geq n!|P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \overline{|Q_i^{(\nu)}|}, \tag{6.13}$$

$$A_1 \leq c_p^n |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \overline{|Q_i^{(\nu)}|}\,\overline{|P_\tau|}^{\,-n+1}, \tag{6.14}$$

$$\overline{|\widetilde{P}_1|} \leq A_2 \leq \ldots \leq A_n \leq c_p^{(n-3)/2}|P_{i-1}(\xi)|^{-1/2}\,\overline{|P_i|}^{\,-1+n/2}\,\overline{|P_\tau|}^{\,-n+2}. \tag{6.15}$$

Then there exists a nonzero polynomial $L(x) = c_n x^n + \ldots + c_1 x + c_0$ with integer coefficients which satisfies

$$|L(\xi)| < A_1, \tag{6.16}$$

$$|c_{k_\nu}| \leq A_{\nu+1} \quad (1 \leq \nu \leq n-1), \tag{6.17}$$

$$\overline{|L|} < \xi^{-n+1} A_n, \tag{6.18}$$

where $\{k_1, \ldots, k_{n-1}\} = \{1, \ldots, n-1\}$.

Proof. First we note that by (5.9)(II), (6.3), and (4.1)(ii) we obtain

$$\overline{|\widetilde{P}_1|} \leq |P_{i-1}(\xi)|^{-1/2}\,\overline{|P_i|}^{\,-1+n/2}\,\overline{|P_\tau|}^{\,-n+2},$$

so (6.15) is correct.

Consider the following system of inequalities:

$$\begin{cases} \Big|\sum_{\nu=0}^{n-2} Q_i^{(\nu)}(\xi)x_\nu + P_{i-1}(\xi)x_{n-1}\Big| < A_1, \\ \Big|\sum_{\nu=0}^{n-2} a_{k_1}^{(\nu)}x_\nu + b_{k_1}x_{n-1}\Big| \leq A_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Big|\sum_{\nu=0}^{n-2} a_{k_{n-1}}^{(\nu)}x_\nu + b_{k_{n-1}}x_{n-1}\Big| \leq A_n. \end{cases} \tag{6.19}$$

We now prove that

$$|\Delta| = \left\|\begin{matrix} Q_i^{(0)}(\xi) & \ldots & Q_i^{(n-2)}(\xi) & P_{i-1}(\xi) \\ a_{k_1}^{(0)} & \ldots & a_{k_1}^{(n-2)} & b_{k_1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k_{n-1}}^{(0)} & \ldots & a_{k_{n-1}}^{(n-2)} & b_{k_{n-1}} \end{matrix}\right\| \leq n!|P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \overline{|Q_i^{(\nu)}|}. \tag{6.20}$$

In fact, it follows from (1_l) with $l = 0, \ldots, n-2$ that the entries of the first line of the determinant Δ are at most $|P_{i-1}(\xi)|$ in absolute value. On the other hand, (4.1)(ii) and (4.3) imply that any minor of the other $n-1$ lines has absolute value at most $(n-1)! \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil$. This gives (6.20).

Thanks to (6.13), (6.20), and Minkowski's Theorem on linear forms there exists a nonzero integer solution $(\widetilde{x}_0, \ldots, \widetilde{x}_{n-1}) \in \mathbb{Z}^n$ of (6.19). Using Remark 5.8, we have $\{k_1, \ldots, k_{n-1}\} = \{j_1, \ldots, j_{n-1}\}$, where $j_1, \ldots, j_{n-1}$ are the indices of the Q_i-system. Therefore we can apply (6.1) to the determinant Δ. It follows from (1_l) with $l = 0, \ldots, n-2$, (4.1)(ii), (4.3), (6.1), and (6.15) that the system (6.19) satisfies the conditions of Lemma 3.7. By this lemma and the definition of c_p we have

$$
\begin{aligned}
&|\widetilde{x}_\nu| \le c_p \max\left(\frac{A_1 \lceil P_{i-1} \rceil}{|P_{i-1}(\xi)| \lceil Q_i^{(\nu)} \rceil}, \frac{A_n}{\lceil Q_i^{(\nu)} \rceil}\right) \qquad (\nu = 0, \ldots, n-2),\\
&|\widetilde{x}_{n-1}| \le c_p \max\left(\frac{A_1}{|P_{i-1}(\xi)|}, \frac{A_n}{\lceil P_{i-1} \rceil}\right).
\end{aligned}
\tag{6.21}
$$

Put

$$
L(x) = \sum_{\nu=0}^{n-2} Q_i^{(\nu)}(x) \widetilde{x}_\nu + P_{i-1}(x) \widetilde{x}_{n-1} = c_n x^n + \ldots + c_1 x + c_0. \tag{6.22}
$$

The polynomials $Q_i^{(0)}(x), \ldots, Q_i^{(n-2)}(x)$ and $P_{i-1}(x)$ have integer coefficients and by Statement 6.1 are linearly independent. On the other hand, the solution $(\widetilde{x}_0, \ldots, \widetilde{x}_{n-1})$ is nonzero and integer. Hence the polynomial $L(x)$ is nonzero and has integer coefficients as well.

From (6.19) and (6.22) we deduce (6.16) and (6.17). Let us prove (6.18). We first obtain an upper bound for $|L(\xi)|$ and $|L'(\xi)|$.

Applying (6.14) to (6.16) and using (5.15), we find that

$$
|L(\xi)| < |P_{i-1}(\xi)|^{1/2} \lceil P_i \rceil^{-1+n/2} \lceil P_\tau \rceil^{-n+1}. \tag{6.23}
$$

Using (6.22), (2.3), (6.21), (1_l) with $l = 0, \ldots, n-2$, (4.1)(ii), and (4.3), we get

$$
\begin{aligned}
|L'(\xi)| &\le \sum_{\nu=0}^{n-2} |Q_i^{(\nu)\prime}(\xi)| \cdot |\widetilde{x}_\nu| + |P_{i-1}'(\xi)| \cdot |\widetilde{x}_{n-1}|\\
&\le c_T^{-1} c_p \Bigg(\sum_{\nu=0}^{n-2} |Q_i^{(\nu)}(\xi)| \lceil Q_i^{(\nu)} \rceil^A \max\left(\frac{A_1 \lceil P_{i-1} \rceil}{|P_{i-1}(\xi)| \lceil Q_i^{(\nu)} \rceil}, \frac{A_n}{\lceil Q_i^{(\nu)} \rceil}\right)\\
&\qquad + |P_{i-1}(\xi)| \lceil P_{i-1} \rceil^A \max\left(\frac{A_1}{|P_{i-1}(\xi)|}, \frac{A_n}{\lceil P_{i-1} \rceil}\right)\Bigg)
\end{aligned}
\tag{6.24}
$$

$$< c_T^{-1} c_p |P_{i-1}(\xi)| \Big(\sum_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil^{A-1} + \lceil P_{i-1} \rceil^{A-1} \Big)$$
$$\times \max \left(\frac{A_1 \lceil P_{i-1} \rceil}{|P_{i-1}(\xi)|}, A_n \right)$$
$$< n c_T^{-1} c_p |P_{i-1}(\xi)| \lceil Q_i^{(n-2)} \rceil^{A-1} \max \left(\frac{A_1 \lceil P_{i-1} \rceil}{|P_{i-1}(\xi)|}, A_n \right).$$

By (6.14), (5.15), and (6.3) we have

$$\begin{aligned}
\frac{A_1 \lceil P_{i-1} \rceil}{|P_{i-1}(\xi)|} &\le c_p^n \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil \lceil P_\tau \rceil^{-n+1} \lceil P_{i-1} \rceil \\
&< |P_{i-1}(\xi)|^{-1/2} \lceil P_i \rceil^{-1+n/2} \lceil P_\tau \rceil^{-n+1} \lceil P_{i-1} \rceil \\
&\le c_h |P_{i-1}(\xi)|^{-1/2} \lceil P_i \rceil^{-1+n/2} \lceil P_\tau \rceil^{-n+2}.
\end{aligned} \tag{6.25}$$

Substituting (6.15) and (6.25) into (6.24), then using (5.3) and the definitions of c_T, c_p and c_h, we obtain

$$\begin{aligned}
|L'(\xi)| &< n c_T^{-1} c_p c_h |P_{i-1}(\xi)|^{1/2} \lceil Q_i^{(n-2)} \rceil^{A-1} \lceil P_i \rceil^{-1+n/2} \lceil P_\tau \rceil^{-n+2} \\
&< n c_T^{-1} c_p c_h c_p^{(n-1)(A-1)/2} |P_{i-1}(\xi)|^{1/2} |P_{i-1}(\xi)|^{-(A-1)/2} \\
&\quad \times \lceil P_i \rceil^{(1-n/2)(A-1)} \lceil P_i \rceil^{-1+n/2} \lceil P_\tau \rceil^{-n+2} \\
&< |P_{i-1}(\xi)|^{1-A/2} \lceil P_i \rceil^{(n-2)(1-A/2)} \lceil P_\tau \rceil^{-n+2}.
\end{aligned} \tag{6.26}$$

Now we can complete the proof of (6.18). Assume that $\lceil L \rceil \ge \xi^{-n+1} A_n$. Hence by (6.15) and (6.17) we have $|c_{k_\nu}| \le A_{\nu+1} \le A_n \le \xi^{n-1} \lceil L \rceil$, $\nu = 1, \ldots, n-1$. Therefore $L(x)$ satisfies the conditions of Lemma 3.2. Thus $\lceil L \rceil < \xi^{-n+1} |L'(\xi)|$. Hence by (6.23) and (6.26) the polynomial $L(x)$ satisfies the conditions of Statement 6.2. It follows that

$$\frac{|L(\xi)|}{|L'(\xi)|} < (c_h \xi^{-1})^{(n-1)A} \lceil L \rceil^{-A}.$$

Since $\lceil L \rceil \ge \xi^{-n+1} A_n > \lceil \widetilde{P}_1 \rceil$ and $c_T > (c_h \xi^{-1})^{(n-1)A}$, we obtain a contradiction with (2.3). Hence $\lceil L \rceil < \xi^{-n+1} A_n$. ∎

Corollary 6.4. *For any natural $i > 2$ we have*

$$|P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_\tau^{(\nu)} \rceil < n!\, c_p^n |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil, \tag{6.27}$$

where $P_{\tau-1}(x)$ is the polynomial from (4.1) *with $\lceil P_{i-1} \rceil \le c_h \lceil P_\tau \rceil$, $1 < \tau \le i-1$.*

P r o o f. Suppose that

$$|P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_\tau^{(\nu)} \rceil \geq n!\, c_p^n |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil \tag{6.28}$$

for some natural $i > 2$. Put

$$A_1 = \min\Big(c_p^{-1}|P_{\tau-1}(\xi)|, c_p^n |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil \lceil P_\tau \rceil^{-n+1}\Big), \tag{6.29}$$

$$A_\nu = c_p^{-1} \lceil Q_\tau^{(\nu-2)} \rceil \qquad (2 \leq \nu \leq n). \tag{6.30}$$

We now prove that $A_1, \ldots, A_n$ satisfy the conditions of Statement 6.3. In fact, if $A_1 = c_p^{-1}|P_{\tau-1}(\xi)|$, then by (6.28)–(6.30) we get

$$\begin{aligned}\prod_{\nu=1}^{n} A_\nu &= c_p^{-n} |P_{\tau-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_\tau^{(\nu)} \rceil \geq c_p^{-n} n!\, c_p^n |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil \\ &= n! |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil.\end{aligned}$$

Similarly, if $A_1 = c_p^n |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil \lceil P_\tau \rceil^{-n+1}$, then by (6.30), (4.3), and the definition of c_p,

$$\prod_{\nu=1}^{n} A_\nu = c_p |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil \lceil P_\tau \rceil^{-n+1} \prod_{\nu=0}^{n-2} \lceil Q_\tau^{(\nu)} \rceil > n! |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil.$$

By (6.29) we have

$$A_1 \leq c_p^n |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil \lceil P_\tau \rceil^{-n+1}.$$

Finally, by (6.30) and (5.3) we obtain

$$\begin{aligned}A_n = c_p^{-1} \lceil Q_\tau^{(n-2)} \rceil &\leq c_p^{(n-3)/2} |P_{\tau-1}(\xi)|^{-1/2} \lceil P_\tau \rceil^{1-n/2} \\ &= c_p^{(n-3)/2} |P_{i-1}(\xi)|^{-1/2} \frac{|P_{\tau-1}(\xi)|^{-1/2}}{|P_{i-1}(\xi)|^{-1/2}} \\ &\quad \times \lceil P_i \rceil^{-1+n/2} \frac{\lceil P_\tau \rceil^{-1+n/2}}{\lceil P_i \rceil^{-1+n/2}} \lceil P_\tau \rceil^{-n+2}.\end{aligned} \tag{6.31}$$

Since $\tau \leq i-1$, from (6.31), (4.1)(i), and (4.1)(ii) we deduce

$$A_n < c_p^{(n-3)/2} |P_{i-1}(\xi)|^{-1/2} \lceil P_i \rceil^{-1+n/2} \lceil P_\tau \rceil^{-n+2},$$

hence by (6.30) and (4.3) we get

$$\lceil \widetilde{P}_1 \rceil \leq A_2 \leq \ldots \leq A_n \leq c_p^{(n-3)/2} |P_{i-1}(\xi)|^{-1/2} \lceil P_i \rceil^{-1+n/2} \lceil P_\tau \rceil^{-n+2}.$$

Thus, $A_1, \ldots, A_n$ satisfy the conditions of Statement 6.3. Hence there exists a nonzero polynomial $L(x) = c_n x^n + \ldots + c_1 x + c_0$ with integer coefficients which satisfies

$$\begin{gathered}
|L(\xi)| < A_1 \leq c_p^{-1} |P_{\tau-1}(\xi)|, \\
|c_{j_\nu}| \leq c_p^{-1} \left\lceil Q_\tau^{(\nu-1)} \right\rceil \qquad (\nu = 1, \ldots, n-1), \\
\lceil L \rceil < \xi^{-n+1} A_n = \xi^{-n+1} c_p^{-1} \left\lceil Q_\tau^{(n-2)} \right\rceil < \left\lceil Q_\tau^{(n-2)} \right\rceil \leq \left\lceil Q_\tau^{(n-1)} \right\rceil,
\end{gathered}$$

where $j_1, \ldots, j_{n-1}$ are the indices of the Q_τ-system. We obtain a contradiction with the minimality property of $Q_\tau^{(n-1)}(x)$. This contradiction proves Corollary 6.4. ■

7. Proof of the Theorem. We consider a sequence of natural numbers $1 = m_1 < m_2 < \ldots$ such that

$$\lceil P_{m_{k+1}} \rceil \leq \max(c_h \lceil P_{m_k} \rceil, \lceil P_{m_k+1} \rceil) < \lceil P_{m_{k+1}+1} \rceil.$$

We have

$$c_h \lceil P_{m_{k-1}+1} \rceil \leq c_h \lceil P_{m_k} \rceil \leq \max(c_h \lceil P_{m_k} \rceil, \lceil P_{m_k+1} \rceil) < \lceil P_{m_{k+1}+1} \rceil,$$

hence

$$\lceil P_{m_{k+1}+1} \rceil^{-1} < c_h^{-1} \lceil P_{m_{k-1}+1} \rceil^{-1},$$

for any natural $k > 1$. If we multiply these inequalities together for all $1 < k \leq l$, we obtain

$$\lceil P_{m_{l+1}+1} \rceil^{-1} < c_h^{-l/2} \lceil P_2 \rceil^{-1}, \tag{7.1}$$

where l is even. It follows from Corollary 6.4 that for any $k \in \mathbb{N}$,

$$|P_{m_k}(\xi)| \prod_{\nu=0}^{n-2} \left\lceil Q_{m_k+1}^{(\nu)} \right\rceil < n! c_p^n |P_{m_{k+1}}(\xi)| \prod_{\nu=0}^{n-2} \left\lceil Q_{m_{k+1}+1}^{(\nu)} \right\rceil.$$

If we multiply these inequalities together for all $1 \leq k \leq l$, we obtain

$$|P_1(\xi)| \prod_{\nu=0}^{n-2} \left\lceil Q_2^{(\nu)} \right\rceil < (n! c_p^n)^l |P_{m_{l+1}}(\xi)| \prod_{\nu=0}^{n-2} \left\lceil Q_{m_{l+1}+1}^{(\nu)} \right\rceil,$$

for any $l \in \mathbb{N}$. Hence

$$|P_1(\xi)| < (n! c_p^n)^l |P_{m_{l+1}}(\xi)| \prod_{\nu=0}^{n-2} \left\lceil Q_{m_{l+1}+1}^{(\nu)} \right\rceil. \tag{7.2}$$

Let l be even. We substitute (5.15) into (7.2) and apply first (5.9)(II), then

(7.1) and the definition of c_h:

$$|P_1(\xi)| < (n!c_p^n)^l |P_{m_{l+1}}(\xi)|^{1/2} \lceil P_{m_{l+1}+1} \rceil^{-1+n/2} < (n!c_p^n)^l \lceil P_{m_{l+1}+1} \rceil^{-1}$$
$$< (n!c_p^n)^l \, c_h^{-l/2} \lceil P_2 \rceil^{-1} < (n!c_p^n)^l \, c_h^{-l/2} = \left(\frac{1}{2}\right)^l.$$

Letting $l \to \infty$ we come to a contradiction with the boundedness of $|P_1(\xi)|$. Thus, the assumption

$$\exists \widetilde{H}_0 > 0 \ \forall Q(x) \in \mathbb{Z}[x], \ \deg Q(x) \le n, \ \lceil Q \rceil > \widetilde{H}_0,$$
$$\frac{|Q(\xi)|}{|Q'(\xi)|} > c_T \lceil Q \rceil^{-A},$$

cannot be true. So neither can (2.1). Hence for any real number $0 < \xi < 1/4$ which is not an algebraic number of degree $\le n$, we have

$$\exists c > 0 \ \forall \widetilde{H}_0 > 0 \ \exists \alpha \in \mathbb{A}_n, \ H(\alpha) > \widetilde{H}_0, \quad |\xi - \alpha| \le cH(\alpha)^{-A},$$

and this completes the proof of the Theorem.

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