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On the largest prime factor of integers

by

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1. Introduction. Let $Q(x)$ denote the largest prime factor of

$$\prod_{x<n\le x+x^{\frac{1}{2}+\varepsilon}} n.$$

We are interested in a lower bound of $Q(x)$. On the Riemann Hypothesis, one can show that $Q(x) > x$ holds for sufficiently large x.

In 1973, Jutila [11] showed that, for sufficiently large x, $Q(x) > x^{\varphi}$, where $\varphi = \frac{2}{3} - \varepsilon$. Balog [1], [2] improved it to $\varphi = 0.772$. Balog, Harman and Pintz [3] obtained $\varphi = 0.82$. Heath-Brown [7] got $\varphi = \frac{11}{12} - \varepsilon$. Recently, Heath-Brown and C. Jia [8] showed $\varphi = \frac{17}{18} - \varepsilon$.

In this paper, we use some ideas coming from [5], [6], and [8]–[10] on the sieve method and a delicate application of the estimate of Deshouillers and Iwaniec [4] on the mean value of Dirichlet polynomials and ζ function. Then we can prove the following:

THEOREM. *Let ε be a sufficiently small positive constant. Then, for sufficiently large x, we have*

$$Q(x) > x^{\frac{25}{26}-\varepsilon}.$$

Throughout this paper, we suppose that ε is a sufficiently small positive constant and that $B = B(\varepsilon)$ is a sufficiently large positive constant. We choose ε such that $K = \frac{8}{\varepsilon}\left(\frac{1}{26} + \frac{\varepsilon}{2}\right)$ is an integer. Suppose that x $(> x_0(\varepsilon))$ is sufficiently large,

$$v = x^{\frac{25}{26}-\frac{\varepsilon}{2}}, \quad P = x^{\frac{\varepsilon}{8}}, \quad T_0 = x^{\frac{1}{2}-\frac{\varepsilon}{6}}.$$

Let c, c_1 and c_2 denote positive constants which have different values at different places. $m \sim M$ means that there are positive constants c_1 and c_2 such that $c_1 M < m \le c_2 M$.

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We often use $M(s)$ (M standing here for any capital letter except P and L) to denote a Dirichlet polynomial in the form

$$M(s) = \sum_{m\sim M} \frac{a(m)}{m^s},$$

where $a(m)$ is a sequence of complex numbers with $a(m) = O(1)$. We also use $P(s)$ to denote

$$P(s) = \sum_{P<p\le 2P} \frac{1}{p^s},$$

where p denotes a prime number.

All calculations in this paper can be verified on the PC computer. The paper containing full details is obtainable from the authors.

2. Some preliminary lemmas

LEMMA 1. *Suppose that* $MN = v$ *and that* $M(s)$, $N(s)$ *are Dirichlet polynomials. Let* $b = 1 + 1/\log x$, $T_1 = \log^{2B} x$. *Assume that* $v/x^{\frac12} \ll M \ll x^{\frac12}$. *Then for* $T_1 \le T \le T_0$, *we have*

$$\int_T^{2T} |M(b+it)N(b+it)P^K(b+it)|\,dt \ll \log^{-B} x.$$

This is Lemma 1 of [8].

LEMMA 2. *Suppose that* $MNL = v$ *and that* $M(s)$, $N(s)$ *are Dirichlet polynomials*,

$$L(s) = \sum_{l\sim L} \frac{1}{l^s}.$$

Let $b = 1 + 1/\log x$, $T_2 = \sqrt{L}$. *Assume that* $M \ll v^{\frac{13}{25}}$, $N \ll v^{\frac{6.5}{25}}$. *Then for* $T_2 \le T \le T_0$, *we have*

$$\int_T^{2T} |M(b+it)N(b+it)L(b+it)P^K(b+it)|\,dt \ll \log^{-B} x.$$

This can be proved in the same way as in Lemma 2 of [8].

LEMMA 3. *Suppose that* $MNDL = v$ *and that* $M(s)$, $N(s)$, $D(s)$ *are Dirichlet polynomials*,

$$L(s) = \sum_{l\sim L} \frac{1}{l^s}.$$

Let $b = 1+1/\log x$, $T_2 = \sqrt{L}$. *Assume further that* $M \ll v^{\frac{13}{25}}$ *and that* N, D *lie in one of the following regions*:

(i) $N \ll v^{\frac{2.6}{25}},\ D \ll N^{-\frac{1}{2}} v^{\frac{6.5}{25}}$;
(ii) $v^{\frac{2.6}{25}} \ll N \ll v^{\frac{26}{175}},\ D \ll N^{-\frac{4}{3}} v^{\frac{26}{75}}$;
(iii) $v^{\frac{26}{175}} \ll N \ll v^{\frac{5.2}{25}},\ D \ll N^{-\frac{3}{4}} v^{\frac{6.5}{25}}$;
(iv) $v^{\frac{5.2}{25}} \ll N \ll v^{\frac{6.5}{25}},\ D \ll N^{-2} v^{\frac{13}{25}}$.

Then for $T_2 \le T \le T_0$, *we have*

$$\int_T^{2T} |M(b+it)N(b+it)D(b+it)L(b+it)P^K(b+it)|\,dt \ll \log^{-B} x.$$

Proof. If $v/x^{\frac{1}{2}} \ll M \ll v^{\frac{13}{25}}$, it can be dealt with by Lemma 1. We assume that $M \ll v/x^{\frac{1}{2}}$. Let $M(s)P^K(s) = H(s)$. Then $H = MP^K \ll x^{\frac{1}{2}}$. It suffices to show

$$\begin{aligned} I &= \int_T^{2T} \left| M\left(\frac{1}{2}+it\right) N\left(\frac{1}{2}+it\right) D\left(\frac{1}{2}+it\right) L\left(\frac{1}{2}+it\right) P^K\left(\frac{1}{2}+it\right)\right| dt \\ &= \int_T^{2T} \left| H\left(\frac{1}{2}+it\right) N\left(\frac{1}{2}+it\right) D\left(\frac{1}{2}+it\right) L\left(\frac{1}{2}+it\right)\right| dt \\ &\ll x^{\frac{1}{2}} \log^{-B} x. \end{aligned}$$

Applying Cauchy's inequality and the mean value estimate for Dirichlet polynomials, we obtain

$$\begin{aligned} I &\ll \left(\int_T^{2T} \left| H\left(\frac{1}{2}+it\right)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_T^{2T} \left| N\left(\frac{1}{2}+it\right) D\left(\frac{1}{2}+it\right)\right|^2 \left| L\left(\frac{1}{2}+it\right)\right|^2 dt\right)^{\frac{1}{2}} \\ &\ll x^{\frac{1}{4}} \left(\int_T^{2T} \left| N\left(\frac{1}{2}+it\right) D\left(\frac{1}{2}+it\right)\right|^2 \left| L\left(\frac{1}{2}+it\right)\right|^2 dt\right)^{\frac{1}{2}}. \end{aligned}$$

If $D \ll N$, an application of Theorem 2 of Deshouillers and Iwaniec [4] yields

$$\int_T^{2T} \left| N\left(\frac{1}{2}+it\right) D\left(\frac{1}{2}+it\right)\right|^2 \left| L\left(\frac{1}{2}+it\right)\right|^2 dt \ll T^{\frac{\varepsilon}{4}} (T + T^{\frac{1}{2}} N^{\frac{3}{4}} D + T^{\frac{1}{2}} N D^{\frac{1}{2}} + N^{\frac{7}{4}} D^{\frac{3}{2}}).$$

Thus, when $N,\ D$ lie in one of the following regions:

(a) $N \ll v^{\frac{26}{175}},\ D \ll N$;
(b) $v^{\frac{26}{175}} \ll N \ll v^{\frac{5.2}{25}},\ D \ll N^{-\frac{3}{4}} v^{\frac{6.5}{25}}$;
(c) $v^{\frac{5.2}{25}} \ll N \ll v^{\frac{6.5}{25}},\ D \ll N^{-2} v^{\frac{13}{25}}$,

we have

$$\int_T^{2T} \left|N\left(\frac{1}{2}+it\right)D\left(\frac{1}{2}+it\right)\right|^2 \left|L\left(\frac{1}{2}+it\right)\right|^2 dt \ll T_0^{1+\frac{\varepsilon}{4}}.$$

If $D \gg N$, changing the roles of N and D, we get

$$\int_T^{2T} \left|D\left(\frac{1}{2}+it\right)N\left(\frac{1}{2}+it\right)\right|^2 \left|L\left(\frac{1}{2}+it\right)\right|^2 dt$$
$$\ll T^{\frac{\varepsilon}{4}}(T + T^{\frac{1}{2}}D^{\frac{3}{4}}N + T^{\frac{1}{2}}DN^{\frac{1}{2}} + D^{\frac{7}{4}}N^{\frac{3}{2}}).$$

Thus, when $N,\ D$ lie in one of the following regions:

(d) $N \ll v^{\frac{2.6}{25}},\ N \ll D \ll N^{-\frac{1}{2}}v^{\frac{6.5}{25}}$;
(e) $v^{\frac{2.6}{25}} \ll N \ll v^{\frac{26}{175}},\ N \ll D \ll N^{-\frac{4}{3}}v^{\frac{26}{75}}$,

we have

$$\int_T^{2T} \left|N\left(\frac{1}{2}+it\right)D\left(\frac{1}{2}+it\right)\right|^2 \left|L\left(\frac{1}{2}+it\right)\right|^2 dt \ll T_0^{1+\frac{\varepsilon}{4}}.$$

Combining the regions (a)–(e), we get the regions in Lemma 3. If $N,\ D$ lie in one of these regions, then

$$I \ll x^{\frac{1}{4}}T_0^{\frac{1}{2}+\frac{\varepsilon}{8}} \ll x^{\frac{1}{2}}\log^{-B}x.$$

This completes the proof of Lemma 3.

We define $w(u)$ as the continuous solution of the equations

$$\begin{cases} w(u) = 1/u, & 1 \le u \le 2; \\ (uw(u))' = w(u-1), & 2 < u. \end{cases}$$

In particular, when $2 \le u \le 3$,

$$w(u) = \frac{1+\log(u-1)}{u}.$$

LEMMA 4. *For the function $w(u)$, we have the following bounds*:

(i) $w(u) \le 1/1.763$ *if* $u \ge 2$;
(ii) $w(u) \le 0.5644$ *if* $u \ge 3$;
(iii) $w(u) \le 0.5617$ *if* $u \ge 4$.

See Lemma 5 of [8].

Define

$$g(u) = \begin{cases} 1/u & \text{if } 1 \le u \le 2; \\ (1+\log(u-1))/u & \text{if } 2 < u \le 3; \\ 0.5644 & \text{if } 3 < u \le 4; \\ 0.5617 & \text{if } 4 < u. \end{cases} \tag{1}$$

We see that for $u \ge 1$,

$$w(u) \le g(u). \tag{2}$$

Lemma 5. *Let $\mathcal{E} = \{n : t < n \le 2t\}$ and $z \le t$. Set*

$$P(z) = \prod_{p<z} p, \qquad S(\mathcal{E}, z) = \sum_{\substack{t<n\le 2t\\ (n, P(z))=1}} 1.$$

Then for sufficiently large t and z, we have

$$S(\mathcal{E}, z) = w\left(\frac{\log t}{\log z}\right)\frac{t}{\log z} + O\left(\frac{t}{\log^2 z}\right).$$

See Lemma 6 of [8].

3. Sieve method.

3. Sieve method. Let

$$N(d) = \sum_{\substack{x<dp_1\ldots p_K\le x+x^{\frac12+\varepsilon}\\ P<p_i\le 2P}} 1,$$

$$\mathcal{A} = \{n : 2^{-K}v < n \le 2v,\ n \text{ repeats } N(n) \text{ times}\},$$

$$\mathcal{A}_d = \{a : a \in \mathcal{A},\ d \mid a\},$$

$$P(z) = \prod_{p<z} p, \qquad S(\mathcal{A}, z) = \sum_{\substack{a\in\mathcal{A}\\ (a, P(z))=1}} 1.$$

If we prove that

$$\Phi = \sum_{\substack{x<pp_1\ldots p_K\le x+x^{\frac12+\varepsilon}\\ P<p_i\le 2P\\ 2^{-K}v<p\le 2v}} 1 > 0 \tag{3}$$

then we obtain the assertion of the Theorem.

In the following, we set

$$\mathcal{B} = \{n : v < n \le 2v\}, \tag{4}$$

$$X = x^{\frac12+\varepsilon}\left(\sum_{P<p\le 2P}\frac{1}{p}\right)^K. \tag{5}$$

Buchstab's identity yields

$$\Phi = S(\mathcal{A}, (2v)^{\frac12}) = S(\mathcal{A}, v^{\frac1{25}}) - \sum_{v^{\frac1{25}}<p\le v^{\frac{12}{25}}} S(\mathcal{A}_p, p) - \sum_{v^{\frac{12}{25}}<p\le (2v)^{\frac12}} S(\mathcal{A}_p, p). \tag{6}$$

By the discussion in Lemma 8 of [8] with the application of Lemma 1, we can get the asymptotic formula

$$\sum_{v^{\frac{12}{25}}<p\le(2v)^{\frac12}} S(\mathcal{A}_p,p)=\frac{X}{v}\sum_{v^{\frac{12}{25}}<p\le(2v)^{\frac12}} S(\mathcal{B}_p,p)+O\left(\frac{X}{\log^2 v}\right). \tag{7}$$

The discussion in Lemma 9 of [8] yields the asymptotic formula

$$S(\mathcal{A},v^{\frac{1}{25}})=\frac{X}{v}\cdot S(\mathcal{B},v^{\frac{1}{25}})+O\left(\frac{X}{\log^2 v}\right). \tag{8}$$

Applying Buchstab's identity again, we get

$$\begin{aligned}\sum_{v^{\frac{1}{25}}<p\le v^{\frac{12}{25}}} S(\mathcal{A}_p,p) &= \sum_{v^{\frac{1}{25}}<p\le v^{\frac{12}{25}}} S(\mathcal{A}_p,v^{\frac{1}{25}})-\sum_{v^{\frac{1}{25}}<p\le v^{\frac{12}{25}}}\ \sum_{\substack{v^{\frac{1}{25}}<q<p\\ q<\left(\frac{2v}{p}\right)^{\frac12}}} S(\mathcal{A}_{pq},q).\end{aligned} \tag{9}$$

By the discussion in Lemma 9 of [8], the first sum on the right side in (9) has an asymptotic formula.

We therefore need to deal with the sum

$$\sum_{v^{\frac{1}{25}}<p\le v^{\frac{12}{25}}}\ \sum_{\substack{v^{\frac{1}{25}}<q<p\\ q<\left(\frac{v}{p}\right)^{\frac12}}} S(\mathcal{A}_{pq},q).$$

Removing the sum with $v^{\frac{12}{25}}<pq\le v^{\frac{13}{25}}$ which has an asymptotic formula, we have to consider the sums

$$\Omega_1=\sum_{v^{\frac{1}{25}}<p\le v^{\frac{11}{25}}}\ \sum_{\substack{v^{\frac{1}{25}}<q<p\\ q<\frac{v^{\frac{12}{25}}}{p}}} S(\mathcal{A}_{pq},q), \tag{10}$$

$$\Omega_2=\sum_{v^{\frac{6.5}{25}}<p\le v^{\frac{12}{25}}}\ \sum_{\substack{\frac{v^{\frac{13}{25}}}{p}<q<p\\ q<\left(\frac{v}{p}\right)^{\frac12}}} S(\mathcal{A}_{pq},q). \tag{11}$$

Now we define the deficiency of a sum as defined in Section 3 of [9]. If

$$\Sigma=\sum_{p,\,q} S(\mathcal{A}_{pq},q)\ge\frac{X}{v}(1+O(\varepsilon))\sum_{p,\,q} S(\mathcal{B}_{pq},q)-C\frac{X}{\log v},$$

then we call the constant C the *deficiency* of Σ. Of course any constant greater than C can be used as the deficiency of Σ. If a sum has an asymptotic formula, then its deficiency is 0.

When we write

$$\sum_{p,\,q} S(\mathcal{A}_{pq}, q) \ge 0 = \frac{X}{v}\sum_{p,\,q} S(\mathcal{B}_{pq}, q) - \frac{X}{v}\sum_{p,\,q} S(\mathcal{B}_{pq}, q),$$

by Lemma 5 and the prime number theorem, we have

$$\begin{aligned}\frac{X}{v}\sum_{p,\,q} S(\mathcal{B}_{pq}, q) &= (1+O(\varepsilon))\frac{X}{v}\sum_{p,\,q} w\left(\frac{\log\frac{v}{pq}}{\log q}\right)\frac{v}{pq\log q}\\ &= (1+O(\varepsilon))X\int\frac{dx}{x\log x}\int w\left(\frac{\log\frac{v}{xy}}{\log y}\right)\frac{dy}{y\log^2 y}\\ &= (1+O(\varepsilon))\frac{X}{\log v}\int\frac{dt}{t}\int w\left(\frac{1-t-u}{u}\right)\frac{du}{u^2}.\end{aligned}$$

Hence, the deficiency is

$$\int\frac{dt}{t}\int w\left(\frac{1-t-u}{u}\right)\frac{du}{u^2}.$$

Similarly, we can define the deficiency of the sum in $2n$ $(n \ge 2)$ variables.

4. The deficiency of Ω_1. Applying Buchstab's identity twice, we get

$$\begin{aligned}\Omega_1 &= \sum_{v^{\frac{1}{25}}<p\le v^{\frac{11}{25}}}\ \sum_{\substack{v^{\frac{1}{25}}<q<p\\ q<\frac{v^{\frac{12}{25}}}{p}}} S(\mathcal{A}_{pq}, q)\\ &= \sum_{v^{\frac{1}{25}}<p\le v^{\frac{11}{25}}}\ \sum_{\substack{v^{\frac{1}{25}}<q<p\\ q<\frac{v^{\frac{12}{25}}}{p}}} S(\mathcal{A}_{pq}, v^{\frac{1}{25}})\\ &\quad - \sum_{v^{\frac{1}{25}}<p\le v^{\frac{11}{25}}}\ \sum_{\substack{v^{\frac{1}{25}}<q<p\\ q<\frac{v^{\frac{12}{25}}}{p}}}\ \sum_{\substack{v^{\frac{1}{25}}<r<q\\ r<\left(\frac{2v}{pq}\right)^{\frac{1}{2}}}} S(\mathcal{A}_{pqr}, v^{\frac{1}{25}})\\ &\quad + \sum_{v^{\frac{1}{25}}<p\le v^{\frac{11}{25}}}\ \sum_{\substack{v^{\frac{1}{25}}<q<p\\ q<\frac{v^{\frac{12}{25}}}{p}}}\ \sum_{\substack{v^{\frac{1}{25}}<r<q\\ r<\left(\frac{2v}{pq}\right)^{\frac{1}{2}}}}\ \sum_{\substack{v^{\frac{1}{25}}<s<r\\ s<\left(\frac{2v}{pqr}\right)^{\frac{1}{2}}}} S(\mathcal{A}_{pqrs}, s),\end{aligned}\tag{12}$$

where p, q, r, s denote prime numbers.

Note that $pq < v^{\frac{13}{25}}$, $r < q < v^{\frac{6.5}{25}}$ in the second term in the above formula. By the discussion in Lemma 9 of [8] with the application of Lemma 2, we see that the first two terms in the above formula have asymptotic formulas. Therefore we have to deal with the sum

$$\Lambda = \sum_{v^{\frac{1}{25}} < p \le v^{\frac{11}{25}}} \sum_{\substack{v^{\frac{1}{25}} < q < p \\ q < \frac{v^{\frac{12}{25}}}{p}}} \sum_{\substack{v^{\frac{1}{25}} < r < q \\ r < \left(\frac{v}{pq}\right)^{\frac{1}{2}}}} \sum_{\substack{v^{\frac{1}{25}} < s < r \\ s < \left(\frac{v}{pqr}\right)^{\frac{1}{2}}}} S(\mathcal{A}_{pqrs}, s). \tag{13}$$

We call the above procedure *process I*. In the following, we shall discuss the deficiency of Λ in some cases. Firstly $qrs > v^{\frac{13}{25}}$ is assumed.

I. $qrs > v^{\frac{13}{25}}$. Now we discuss the deficiency of

$$\Gamma = \sum_{v^{\frac{13}{75}} < p \le v^{\frac{23}{75}}} \sum_{\substack{v^{\frac{13}{75}} < q < p \\ q < \frac{v^{\frac{12}{25}}}{p}}} \sum_{\substack{\left(\frac{v^{\frac{13}{25}}}{q}\right)^{\frac{1}{2}} < r < q \\ r < \left(\frac{v}{pq}\right)^{\frac{1}{2}}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{qr} < s < r \\ s < \left(\frac{v}{pqr}\right)^{\frac{1}{2}}}} S(\mathcal{A}_{pqrs}, s).$$

Let $\Gamma \ge 0$. Then the deficiency of Γ is

$$\begin{aligned}
&\int_{\frac{13}{75}}^{\frac{23}{75}} \frac{dt}{t} \int_{\frac{13}{75}}^{\min(t, \frac{12}{25}-t)} \frac{du}{u} \int_{\frac{1}{2}(\frac{13}{25}-u)}^{u} \frac{dr}{r} \\
&\qquad \times \int_{\frac{13}{25}-u-r}^{\min(r, \frac{1}{2}(1-t-u-r))} w\left(\frac{1-t-u-r-s}{s}\right) \frac{ds}{s^2} \\
&\le \int_{\frac{13}{75}}^{\frac{23}{75}} \frac{dt}{t} \int_{\frac{13}{75}}^{\min(t, \frac{12}{25}-t)} \frac{du}{u} \int_{\frac{1}{2}(\frac{13}{25}-u)}^{u} \frac{dr}{r} \\
&\qquad \times \int_{\frac{13}{25}-u-r}^{\min(r, \frac{1}{2}(1-t-u-r))} g\left(\frac{1-t-u-r-s}{s}\right) \frac{ds}{s^2} \\
&\le 0.021170.
\end{aligned}$$

Next we assume $qrs < v^{\frac{13}{25}}$. The sum with $v^{\frac{12}{25}} \le qrs < v^{\frac{13}{25}}$ can be removed, so that we can assume $qrs < v^{\frac{12}{25}}$. As in [9], we use $qrs < v^{\frac{13}{25}} \to qrs < v^{\frac{12}{25}}$ to mean that the sum with $v^{\frac{12}{25}} \le qrs < v^{\frac{13}{25}}$ is removed.

II. $qrs < v^{\frac{12}{25}}$, $prs > v^{\frac{13}{25}}$. The corresponding deficiency is

$$\int_{\frac{5}{25}}^{\frac{11}{25}} \frac{dt}{t} \int_{\frac{1}{2}(\frac{13}{25}-t)}^{\min(t-\frac{1}{25}, \frac{12}{25}-t)} \frac{du}{u} \int_{\frac{1}{2}(\frac{13}{25}-t)}^{u} \frac{dr}{r} \int_{\frac{13}{25}-t-r}^{\min(r, \frac{12}{25}-u-r)} w\left(\frac{1-t-u-r-s}{s}\right) \frac{ds}{s^2},$$

where we used the fact that

$$\min\left(r,\ \frac{12}{25}-u-r\right) \le \frac{r}{2}+\frac{1}{2}\left(\frac{12}{25}-u-r\right) < \frac{1}{2}(1-t-u-r).$$

We discuss several cases.

1. $r < v^{\frac{2.6}{25}}$. The corresponding deficiency is

$$\int_{\frac{7.8}{25}}^{\frac{11}{25}} \frac{dt}{t} \int_{\frac{1}{2}(\frac{13}{25}-t)}^{\frac{12}{25}-t} \frac{du}{u} \int_{\frac{1}{2}(\frac{13}{25}-t)}^{\min(u,\ \frac{2.6}{25})} \frac{dr}{r} \int_{\frac{13}{25}-t-r}^{r} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

Applying Buchstab's identity twice, we get

$$\begin{aligned}
&\sum_{v^{\frac{7.8}{25}}<p\le v^{\frac{11}{25}}} \sum_{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac12}<q<\frac{v^{\frac{12}{25}}}{p}} \sum_{\substack{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac12}<r<q\\ r<v^{\frac{2.6}{25}}}} \sum_{\frac{v^{\frac{13}{25}}}{pr}<s<r} S(\mathcal{A}_{pqrs}, s)\\
&= \sum_{v^{\frac{7.8}{25}}<p\le v^{\frac{11}{25}}} \sum_{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac12}<q<\frac{v^{\frac{12}{25}}}{p}} \sum_{\substack{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac12}<r<q\\ r<v^{\frac{2.6}{25}}}} \sum_{\frac{v^{\frac{13}{25}}}{pr}<s<r} S(\mathcal{A}_{pqrs}, v^{\frac{1}{25}})\\
&\quad - \sum_{v^{\frac{7.8}{25}}<p\le v^{\frac{11}{25}}} \sum_{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac12}<q<\frac{v^{\frac{12}{25}}}{p}} \sum_{\substack{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac12}<r<q\\ r<v^{\frac{2.6}{25}}}} \sum_{\frac{v^{\frac{13}{25}}}{pr}<s<r} \sum_{v^{\frac{1}{25}}<t<s} S(\mathcal{A}_{pqrst}, v^{\frac{1}{25}})\\
&\quad + \sum_{v^{\frac{7.8}{25}}<p\le v^{\frac{11}{25}}} \sum_{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac12}<q<\frac{v^{\frac{12}{25}}}{p}} \sum_{\substack{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac12}<r<q\\ r<v^{\frac{2.6}{25}}}} \sum_{\frac{v^{\frac{13}{25}}}{pr}<s<r} \sum_{v^{\frac{1}{25}}<t<s}\\
&\qquad\qquad \sum_{v^{\frac{1}{25}}<w<t} S(\mathcal{A}_{pqrstw}, w),
\end{aligned}$$

where t and w denote prime numbers.

In the second sum above, let $m = pq,\ n = r,\ d = st$. Note that $st < s^2 < r^2 < r^{-\frac12} v^{\frac{6.5}{25}}$. By the discussion in Lemma 9 of [8] with the application of Lemma 3, we can get an asymptotic formula. We deal with the first sum in the same way.

The corresponding deficiency of the third sum is

$$\begin{aligned}
&\int_{\frac{7.8}{25}}^{\frac{11}{25}} \frac{dt}{t} \int_{\frac{1}{2}(\frac{13}{25}-t)}^{\frac{12}{25}-t} \frac{du}{u} \int_{\frac{1}{2}(\frac{13}{25}-t)}^{\min(u,\ \frac{2.6}{25})} \frac{dr}{r} \int_{\frac{13}{25}-t-r}^{r} \frac{ds}{s}\\
&\qquad\qquad \times \int_{\frac{1}{25}}^{s} \frac{dy}{y} \int_{\frac{1}{25}}^{y} w\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2}.
\end{aligned}$$

1) $ptw > v^{\frac{13}{25}}$. The deficiency is

$$\int_{\frac{7.8}{25}}^{\frac{11}{25}} \frac{dt}{t} \int_{\frac{1}{2}(\frac{13}{25}-t)}^{\frac{12}{25}-t} \frac{du}{u} \int_{\frac{1}{2}(\frac{13}{25}-t)}^{\min(u,\, \frac{2.6}{25})} \frac{dr}{r} \int_{\frac{1}{2}(\frac{13}{25}-t)}^{r} \frac{ds}{s} \times \int_{\frac{1}{2}(\frac{13}{25}-t)}^{s} \frac{dy}{y} \int_{\frac{13}{25}-t-y}^{y} g\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2} \le 0.000108.$$

We now discuss the remaining cases which are similar to case 1), and write the deficiencies in brackets.

2) $ptw < v^{\frac{12}{25}},\ psw > v^{\frac{13}{25}}$ (0.000006). 3) $psw < v^{\frac{12}{25}},\ prw > v^{\frac{13}{25}}$ (0.000010). 4) $prw < v^{\frac{12}{25}},\ prt > v^{\frac{13}{25}}$ (0.000033). 5) $prt < v^{\frac{12}{25}},\ pqt > v^{\frac{13}{25}}$ (0.000013). 6) $pqt < v^{\frac{12}{25}}$ (0.000012).

Therefore the total deficiency in 1. is 0.000182.

2. $v^{\frac{2.6}{25}} < r \le v^{\frac{26}{175}}$. The corresponding deficiency is

$$\int_{\frac{39}{175}}^{\frac{9.4}{25}} \frac{dt}{t} \int_{\max(\frac{1}{2}(\frac{13}{25}-t),\, \frac{2.6}{25})}^{\min(t-\frac{1}{25},\, \frac{12}{25}-t)} \frac{du}{u} \int_{\max(\frac{1}{2}(\frac{13}{25}-t),\, \frac{2.6}{25})}^{\min(u,\, \frac{26}{175})} \frac{dr}{r} \times \int_{\frac{13}{25}-t-r}^{\min(r,\, \frac{12}{25}-u-r)} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

1) $s < r^{-\frac{2}{3}} v^{\frac{13}{75}}$. The corresponding deficiency is

$$\int_{\frac{52}{175}}^{\frac{9.4}{25}} \frac{dt}{t} \int_{\max(\frac{2.6}{25},\, \frac{26}{25}-3t)}^{\frac{12}{25}-t} \frac{du}{u} \int_{\max(\frac{2.6}{25},\, \frac{26}{25}-3t)}^{\min(u,\, \frac{26}{175})} \frac{dr}{r} \times \int_{\frac{13}{25}-t-r}^{\frac{13}{75}-\frac{2}{3}r} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

We use the discussion in 1. Let $m = pq,\ n = r,\ d = st$. Note that $st < s^2 < r^{-\frac{4}{3}} v^{\frac{26}{75}}$. We have to deal with a sum whose deficiency is

$$\int_{\frac{52}{175}}^{\frac{9.4}{25}} \frac{dt}{t} \int_{\max(\frac{2.6}{25},\, \frac{26}{25}-3t)}^{\frac{12}{25}-t} \frac{du}{u} \int_{\max(\frac{2.6}{25},\, \frac{26}{25}-3t)}^{\min(u,\, \frac{26}{175})} \frac{dr}{r} \int_{\frac{13}{25}-t-r}^{\frac{13}{75}-\frac{2}{3}r} \frac{ds}{s} \times \int_{\frac{1}{25}}^{s} \frac{dy}{y} \int_{\frac{1}{25}}^{y} w\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2}.$$

We discuss the following cases.

i) $ptw > v^{\frac{13}{25}}$ (0.000029). ii) $ptw < v^{\frac{12}{25}}$, $psw > v^{\frac{13}{25}}$ (0.000006). iii) $psw < v^{\frac{12}{25}}$, $prw > v^{\frac{13}{25}}$ (0.000175). iv) $prw < v^{\frac{12}{25}}$, $prt > v^{\frac{13}{25}}$ (0.000041). v) $prt < v^{\frac{12}{25}}$, $pqt > v^{\frac{13}{25}}$ (0.000015). vi) $pqt < v^{\frac{12}{25}}$ (0.000013).

Hence, the total deficiency in 1) is 0.000279.

2) $s > r^{-\frac{2}{3}} v^{\frac{13}{75}}$. The corresponding deficiency is

$$\int_{\frac{39}{175}}^{\frac{9.4}{25}} \frac{dt}{t} \int_{\max(\frac{1}{2}(\frac{13}{25}-t),\, \frac{2.6}{25})}^{\min(t-\frac{1}{25},\, \frac{12}{25}-t)} \frac{du}{u} \int_{\max(\frac{1}{2}(\frac{13}{25}-t),\, \frac{2.6}{25})}^{\min(u,\, \frac{26}{175})} \frac{dr}{r} \times \int_{\max(\frac{13}{25}-t-r,\, \frac{13}{75}-\frac{2}{3}r)}^{\min(r,\, \frac{12}{25}-u-r)} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

We have to estimate the deficiency of the sum

$$\Sigma = \sum_{v^{\frac{39}{175}} < p \le v^{\frac{9.4}{25}}} \sum_{\substack{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac{1}{2}} < q < \frac{p}{v^{\frac{1}{25}}} \\ v^{\frac{2.6}{25}} < q < \frac{v^{\frac{12}{25}}}{p}}} \sum_{\substack{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac{1}{2}} < r < q \\ v^{\frac{2.6}{25}} < r < v^{\frac{26}{175}}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{pr} < s < r \\ \frac{v^{\frac{13}{75}}}{r^{\frac{2}{3}}} < s < \frac{v^{\frac{12}{25}}}{qr}}} S(\mathcal{A}_{pqrs}, s). \tag{14}$$

We shall employ Buchstab's identity in the following way:

$$S(\mathcal{A}_{pqrs}, s) = S\left(\mathcal{A}_{pqrs}, \left(\frac{2v}{pqrs}\right)^{\frac{1}{2}}\right) + \sum_{s \le k < \max\left(s,\, \left(\frac{2v}{pqrs}\right)^{\frac{1}{2}}\right)} S(\mathcal{A}_{pqrsk}, k), \tag{15}$$

providing $pqrs^2 < v$.

The first term on the right side of (15) counts prime numbers and the second term counts almost-prime numbers. If $pqrs^3 > 2v$, then the second term is 0.

An application of the decomposition in (15) yields

$$\Sigma = \sum_{v^{\frac{39}{175}} < p \le v^{\frac{9.4}{25}}} \sum_{\substack{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac{1}{2}} < q < \frac{p}{v^{\frac{1}{25}}} \\ v^{\frac{2.6}{25}} < q < \frac{v^{\frac{12}{25}}}{p}}} \sum_{\substack{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac{1}{2}} < r < q \\ v^{\frac{2.6}{25}} < r < v^{\frac{26}{175}}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{pr} < s < r \\ \frac{v^{\frac{13}{75}}}{r^{\frac{2}{3}}} < s < \frac{v^{\frac{12}{25}}}{qr}}} S\left(\mathcal{A}_{pqrs}, \left(\frac{2v}{pqrs}\right)^{\frac{1}{2}}\right) \tag{16}$$

$$+\sum_{v^{\frac{39}{175}}<p\le v^{\frac{9.4}{25}}}\ \sum_{\substack{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac12}<q<\frac{p}{v^{\frac{1}{25}}}\\ v^{\frac{2.6}{25}}<q<\frac{v^{\frac{12}{25}}}{p}}}\ \sum_{\substack{\left(\frac{v^{\frac{13}{25}}}{p}\right)^{\frac12}<r<q\\ v^{\frac{2.6}{25}}<r<v^{\frac{26}{175}}}}$$

$$\sum_{\substack{\frac{v^{\frac{13}{25}}}{pr}<s<r\\ \frac{v^{\frac{13}{75}}}{r^{\frac23}}<s<\frac{v^{\frac{12}{25}}}{qr}\\ s<\left(\frac{2v}{pqr}\right)^{\frac13}}}\ \sum_{s\le k<\left(\frac{2v}{pqrs}\right)^{\frac12}} S(\mathcal{A}_{pqrsk},k)$$

$$=\Sigma_1+\Sigma_2.$$

We call the above procedure *process II*. We call Σ_1 the *prime term* and Σ_2 the *almost-prime term*.

i) The deficiency of the prime term is

$$\int_{\frac{39}{175}}^{\frac{9.4}{25}}\frac{dt}{t}\int_{\max(\frac12(\frac{13}{25}-t),\,\frac{2.6}{25})}^{\min(t-\frac{1}{25},\,\frac{12}{25}-t)}\frac{du}{u}\int_{\max(\frac12(\frac{13}{25}-t),\,\frac{2.6}{25})}^{\min(u,\,\frac{26}{175})}\frac{dr}{r}$$
$$\times\int_{\max(\frac{13}{25}-t-r,\,\frac{13}{75}-\frac23 r)}^{\min(r,\,\frac{12}{25}-u-r)}\frac{ds}{s(1-t-u-r-s)}$$
$$\le 0.019316.$$

ii) The deficiency of the almost-prime term is

$$\int_{\frac{39}{175}}^{\frac{9.4}{25}}\frac{dt}{t}\int_{\max(\frac12(\frac{13}{25}-t),\,\frac{2.6}{25})}^{\min(t-\frac{1}{25},\,\frac{12}{25}-t)}\frac{du}{u}\int_{\max(\frac12(\frac{13}{25}-t),\,\frac{2.6}{25})}^{\min(u,\,\frac{26}{175})}\frac{dr}{r}\int_{\max(\frac{13}{25}-t-r,\,\frac{13}{75}-\frac23 r)}^{\min(r,\,\frac{12}{25}-u-r,\,\frac13(1-t-u-r))}\frac{ds}{s}$$
$$\times\int_{s}^{\frac12(1-t-u-r-s)} w\left(\frac{1-t-u-r-s-k}{k}\right)\frac{dk}{k^2}.$$

a) $kps>v^{\frac{13}{25}}$. The corresponding deficiency is

$$\int_{\frac{39}{175}}^{\frac{9.4}{25}}\frac{dt}{t}\int_{\max(\frac12(\frac{13}{25}-t),\,\frac{2.6}{25})}^{\min(t-\frac{1}{25},\,\frac{12}{25}-t)}\frac{du}{u}\int_{\max(\frac12(\frac{13}{25}-t),\,\frac{2.6}{25})}^{\min(u,\,\frac{26}{175},\,\frac{5.5}{25}+\frac{t}{2}-u)}\frac{dr}{r}$$

$$\times \int\limits_{\max(\frac{13}{25}-t-r,\, \frac{13}{75}-\frac{2}{3}r,\, \frac{1}{25}-t+u+r)}^{\min(r,\, \frac{1}{3}(1-t-u-r))} \frac{ds}{s}$$

$$\times \int\limits_{\max(s,\, \frac{13}{25}-t-s)}^{\frac{1}{2}(1-t-u-r-s)} w\left(\frac{1-t-u-r-s-k}{k}\right)\frac{dk}{k^2},$$

where we used the fact that $\frac{1}{3}(1-t-u-r) \le \frac{12}{25}-u-r$.

α) $kqrs > v^{\frac{13}{25}}$ (0.003549). β) $kqrs < v^{\frac{12}{25}}$ (0.001012).

b) $kps < v^{\frac{12}{25}}$ (0.000156).

Thus the total deficiency in 2) is 0.024033 and that in 2. is 0.024312.

3. $r > v^{\frac{26}{175}}$. The corresponding deficiency is

$$\int\limits_{\frac{5}{25}}^{\frac{58}{175}} \frac{dt}{t} \int\limits_{\max(\frac{1}{2}(\frac{13}{25}-t),\, \frac{26}{175})}^{\min(t-\frac{1}{25},\, \frac{12}{25}-t)} \frac{du}{u} \int\limits_{\max(\frac{1}{2}(\frac{13}{25}-t),\, \frac{26}{175})}^{u} \frac{dr}{r}$$

$$\times \int\limits_{\frac{13}{25}-t-r}^{\min(r,\, \frac{12}{25}-u-r)} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

1) The deficiency of the prime term is 0.017964.

2) The deficiency of the almost-prime term is

$$\int\limits_{\frac{5}{25}}^{\frac{58}{175}} \frac{dt}{t} \int\limits_{\max(\frac{1}{2}(\frac{13}{25}-t),\, \frac{26}{175})}^{\min(t-\frac{1}{25},\, \frac{12}{25}-t)} \frac{du}{u} \int\limits_{\max(\frac{1}{2}(\frac{13}{25}-t),\, \frac{26}{175})}^{u} \frac{dr}{r}$$

$$\times \int\limits_{\frac{13}{25}-t-r}^{\min(r,\, \frac{12}{25}-u-r,\, \frac{1}{3}(1-t-u-r))} \frac{ds}{s}$$

$$\times \int\limits_{s}^{\frac{1}{2}(1-t-u-r-s)} w\left(\frac{1-t-u-r-s-k}{k}\right)\frac{dk}{k^2}.$$

a) $kps > v^{\frac{13}{25}}$ (0.000960). b) $kps < v^{\frac{12}{25}}$. There are two subcases: α) $kqrs > v^{\frac{13}{25}}$ (0.003065); β) $kqrs < v^{\frac{12}{25}}$ (0.000757).

Therefore the total deficiency in 3. is 0.022746 and that in II is 0.047240.

III. $prs < v^{\frac{12}{25}}$, $pqs > v^{\frac{13}{25}}$. The corresponding deficiency is

$$\int\limits_{\frac{14}{75}}^{\frac{10}{25}} \frac{dt}{t} \int\limits_{\frac{1}{2}(\frac{14}{25}-t)}^{\min(t,\, \frac{12}{25}-t)} \frac{du}{u} \int\limits_{\frac{13}{25}-t-u}^{u-\frac{1}{25}} \frac{dr}{r} \int\limits_{\frac{13}{25}-t-u}^{\min(r,\, \frac{12}{25}-t-r)} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

1. $q < v^{\frac{26}{175}}$. Applying process I again and writing $m = prs,\ n = q$ and $d = t$, we have to deal with a sum whose deficiency is

$$\int\limits_{\frac{46}{175}}^{\frac{10}{25}} \frac{dt}{t} \int\limits_{\frac{1}{2}(\frac{14}{25}-t)}^{\min(\frac{12}{25}-t,\, \frac{26}{175})} \frac{du}{u} \int\limits_{\frac{13}{25}-t-u}^{u-\frac{1}{25}} \frac{dr}{r} \int\limits_{\frac{13}{25}-t-u}^{\min(r,\, \frac{12}{25}-t-r)} \frac{ds}{s} \times \int\limits_{\frac{1}{25}}^{s} \frac{dy}{y} \int\limits_{\frac{1}{25}}^{y} w\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2}.$$

1) $pqw > v^{\frac{13}{25}}$ (0.000334). 2) $pqw < v^{\frac{12}{25}},\ pqt > v^{\frac{13}{25}}$ (0.000006). 3) $pqt < v^{\frac{12}{25}}$ (0.000013).

Hence, the total deficiency in 1. is 0.000353.

2. $v^{\frac{26}{175}} < q < v^{\frac{5.2}{25}}$.

1) $s < q^{-\frac{3}{4}} v^{\frac{6.5}{25}}$. Applying process I again and writing $m = prs,\ n = q$ and $d = t$, we have to deal with a sum whose deficiency is

$$\int\limits_{\frac{5.2}{25}}^{\frac{58}{175}} \frac{dt}{t} \int\limits_{\max(\frac{1}{2}(\frac{14}{25}-t),\, \frac{26}{175},\, \frac{26}{25}-4t)}^{\min(\frac{12}{25}-t,\, \frac{5.2}{25})} \frac{du}{u} \int\limits_{\frac{13}{25}-t-u}^{u-\frac{1}{25}} \frac{dr}{r} \times \int\limits_{\frac{13}{25}-t-u}^{\min(r,\, \frac{12}{25}-t-r,\, \frac{6.5}{25}-\frac{3}{4}u)} \frac{ds}{s} \int\limits_{\frac{1}{25}}^{s} \frac{dy}{y} \times \int\limits_{\frac{1}{25}}^{\min(y,\, \frac{1}{2}(1-t-u-r-s-y))} w\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2},$$

where we used the fact that $s < \frac{12}{25} - t - r$ and $s < \frac{6.5}{25} - \frac{3}{4}u \Rightarrow t+u+r+3s = (t+r+s)+(u+2s) < 1 \Rightarrow s < \frac{1}{2}(1-t-u-r-s)$.

i) $pqw > v^{\frac{13}{25}}$: a) $pstw > v^{\frac{13}{25}}$ (0.000324); b) $pstw < v^{\frac{12}{25}},\ prtw > v^{\frac{13}{25}}$ (0.000108); c) $prtw < v^{\frac{12}{25}},\ prsw > v^{\frac{13}{25}}$ (0.000010); d) $prsw < v^{\frac{12}{25}},\ prst > v^{\frac{13}{25}}$ (0.000006); e) $prst < v^{\frac{12}{25}}$ (0.000363). ii) $pqw < v^{\frac{12}{25}}, pqt > v^{\frac{13}{25}}$ (0.000300). iii) $pqt < v^{\frac{12}{25}}, pqtw > v^{\frac{13}{25}}$ (0.000322). iv) $pqtw < v^{\frac{12}{25}}$ (0.000006).

Hence, the total deficiency in 1) is 0.001439.

2) $s > q^{-\frac{3}{4}} v^{\frac{6.5}{25}}$ (0.002038).

Thus the total deficiency in 2. is 0.003477.

3. $v^{\frac{5.2}{25}} < q\ (< v^{\frac{6.5}{25}})$.

1) $s < q^{-2} v^{\frac{13}{25}}$. Applying process I again and writing $m = prs,\ n = q$ and $d = t$, we have to deal with a sum whose deficiency is

$$\int_{\frac{5.2}{25}}^{\frac{6.8}{25}} \frac{dt}{t} \int_{\frac{5.2}{25}}^{\min(t,\,\frac{12}{25}-t)} \frac{du}{u} \int_{\frac{13}{25}-t-u}^{u-\frac{1}{25}} \frac{dr}{r}$$
$$\times \int_{\frac{13}{25}-t-u}^{\min(r,\,\frac{12}{25}-t-r,\,\frac{13}{25}-2u)} \frac{ds}{s} \int_{\frac{1}{25}}^{s} \frac{dy}{y}$$
$$\times \int_{\frac{1}{25}}^{\min(y,\,\frac{1}{2}(1-t-u-r-s-y))} w\Big(\frac{1-t-u-r-s-y-z}{z}\Big)\frac{dz}{z^2}.$$

i) $pqw > v^{\frac{13}{25}}$ (0.001662). ii) $pqw < v^{\frac{12}{25}},\ pqt > v^{\frac{13}{25}}$ (0.000020). iii) $pqt < v^{\frac{12}{25}},\ pqtw > v^{\frac{13}{25}}$ (0.000019). iv) $pqtw < v^{\frac{12}{25}}$ (0).

Therefore the total deficiency in 1) is 0.001701.

2) $s > q^{-2}v^{\frac{13}{25}}$.

i) The deficiency of the prime term is 0.009678.

ii) The deficiency of the almost-prime term is

$$\int_{\frac{5.2}{25}}^{\frac{6.8}{25}} \frac{dt}{t} \int_{\frac{5.2}{25}}^{\min(t,\,\frac{12}{25}-t)} \frac{du}{u} \int_{\frac{13}{25}-2u}^{2u-t-\frac{1}{25}} \frac{dr}{r} \int_{\frac{13}{25}-2u}^{\min(r,\,\frac{12}{25}-t-r)} \frac{ds}{s}$$
$$\times \int_{s}^{\frac{1}{2}(1-t-u-r-s)} w\Big(\frac{1-t-u-r-s-k}{k}\Big)\frac{dk}{k^2},$$

where we used the fact that $\min\big(r,\,\frac{12}{25}-t-r\big) \le \frac{r}{3}+\frac{2}{3}\big(\frac{12}{25}-t-r\big) < \frac{1}{3}(1-t-u-r)$.

a) $kps > v^{\frac{13}{25}}$ (0.000006). b) $kps < v^{\frac{12}{25}},\ kpr > v^{\frac{13}{25}}$ (0.000653). c) $kpr < v^{\frac{12}{25}}$: α) $kqrs > v^{\frac{13}{25}}$ (0.002882); β) $kqrs < v^{\frac{12}{25}}$ (0.001823).

Hence, the total deficiency in 2) is 0.015042 and that in 3. is 0.016743. Therefore the total deficiency in III is 0.020573.

IV. $pqs < v^{\frac{12}{25}},\ pqr > v^{\frac{13}{25}}$. The corresponding deficiency is

$$\int_{\frac{13}{75}}^{\frac{9}{25}} \frac{dt}{t} \int_{\frac{1}{2}(\frac{13}{25}-t)}^{\min(t,\,\frac{11}{25}-t)} \frac{du}{u} \int_{\frac{13}{25}-t-u}^{u} \frac{dr}{r}$$
$$\times \int_{\frac{1}{25}}^{\frac{12}{25}-t-u} w\Big(\frac{1-t-u-r-s}{s}\Big)\frac{ds}{s^2}.$$

1. $r < v^{\frac{26}{175}}$. Applying process I again and writing $m = pqs,\ n = r$ and $d = t$, we have to deal with a sum whose deficiency is

$$\int_{\frac{32.5}{175}}^{\frac{9}{25}} \frac{dt}{t} \int_{\max(\frac{1}{2}(\frac{13}{25}-t),\, \frac{65}{175}-t)}^{\min(t,\, \frac{11}{25}-t)} \frac{du}{u} \int_{\frac{13}{25}-t-u}^{\min(u,\, \frac{26}{175})} \frac{dr}{r} \int_{\frac{1}{25}}^{\frac{12}{25}-t-u} \frac{ds}{s}$$
$$\times \int_{\frac{1}{25}}^{s} \frac{dy}{y} \int_{\frac{1}{25}}^{\min(y,\, \frac{1}{2}(1-t-u-r-s-y))} w\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2}.$$

1) $prtw > v^{\frac{13}{25}}$ (0.000164). 2) $prtw < v^{\frac{12}{25}},\ prsw > v^{\frac{13}{25}}$ (0.000008). 3) $prsw < v^{\frac{12}{25}},\ prst > v^{\frac{13}{25}}$ (0.000009). 4) $prst < v^{\frac{12}{25}},\ prstw > v^{\frac{13}{25}}$ (0.000519). 5) $prstw < v^{\frac{12}{25}}$ (0.000269).

Hence, the total deficiency in 1. is 0.000969.

2. $v^{\frac{26}{175}} < r < v^{\frac{5.2}{25}}$.

1) $s < r^{-\frac{3}{4}} v^{\frac{6.5}{25}}$. Applying process I again and writing $m = pqs,\ n = r$ and $d = t$, we have to deal with a sum whose deficiency is

$$\int_{\frac{13}{75}}^{\frac{51}{175}} \frac{dt}{t} \int_{\max(\frac{1}{2}(\frac{13}{25}-t),\, \frac{26}{175})}^{\min(t,\, \frac{11}{25}-t)} \frac{du}{u} \int_{\max(\frac{13}{25}-t-u,\, \frac{26}{175})}^{\min(u,\, \frac{5.2}{25})} \frac{dr}{r}$$
$$\times \int_{\frac{1}{25}}^{\min(\frac{12}{25}-t-u,\, \frac{6.5}{25}-\frac{3}{4}r)} \frac{ds}{s} \int_{\frac{1}{25}}^{s} \frac{dy}{y}$$
$$\times \int_{\frac{1}{25}}^{\min(y,\, \frac{1}{2}(1-t-u-r-s-y))} w\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2}.$$

i) $prtw > v^{\frac{13}{25}}$ (0.000387). ii) $prtw < v^{\frac{12}{25}},\ prsw > v^{\frac{13}{25}}$ (0.000052). iii) $prsw < v^{\frac{12}{25}},\ prst > v^{\frac{13}{25}}$ (0.000009). iv) $prst < v^{\frac{12}{25}},\ prstw > v^{\frac{13}{25}}$ (0.000293). v) $prstw < v^{\frac{12}{25}}$ (0.000015).

Hence, the total deficiency in 1) is 0.000756.

2) $s > r^{-\frac{3}{4}} v^{\frac{6.5}{25}}$ (0.000006).

Thus the total deficiency in 2. is 0.000762.

3. $r > v^{\frac{5.2}{25}}$ (0.000108).

Therefore the total deficiency in IV is 0.001839.

V. $pqr < v^{\frac{12}{25}},\ pqrs > v^{\frac{13}{25}}$. The corresponding deficiency is

$$\int_{\frac{13}{100}}^{\frac{10}{25}} \frac{dt}{t} \int_{\frac{1}{3}(\frac{13}{25}-t)}^{\min(t,\, \frac{11}{25}-t)} \frac{du}{u} \int_{\frac{1}{2}(\frac{13}{25}-t-u)}^{\min(u,\, \frac{12}{25}-t-u)} \frac{dr}{r} \int_{\frac{13}{25}-t-u-r}^{r} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

1. $s < v^{\frac{26}{175}}$. Applying process I again and writing $m = pqr,\ n = s$ and $d = t$, we have to deal with a sum whose deficiency is

$$\int_{\frac{13}{100}}^{\frac{10}{25}} \frac{dt}{t} \int_{\frac{1}{3}(\frac{13}{25}-t)}^{\min(t,\,\frac{11}{25}-t)} \frac{du}{u} \int_{\frac{1}{2}(\frac{13}{25}-t-u)}^{\min(u,\,\frac{12}{25}-t-u)} \frac{dr}{r}$$
$$\times \int_{\frac{13}{25}-t-u-r}^{\min(r,\,\frac{26}{175})} \frac{ds}{s} \int_{\frac{1}{25}}^{s} \frac{dy}{y} \int_{\frac{1}{25}}^{\min(y,\,\frac{1}{2}(1-t-u-r-s-y))} w\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2}.$$

1) $prtw > v^{\frac{13}{25}}$ (0.000700). 2) $prtw < v^{\frac{12}{25}},\ pqtw > v^{\frac{13}{25}}$ (0.000849). 3) $pqtw < v^{\frac{12}{25}},\ pqsw > v^{\frac{13}{25}}$ (0.000515). 4) $pqsw < v^{\frac{12}{25}},\ pqst > v^{\frac{13}{25}}$: i) $pqrw > v^{\frac{13}{25}}$ (0.000017); ii) $pqrw < v^{\frac{12}{25}}$ (0.000374). 5) $pqst < v^{\frac{12}{25}},\ pqrt > v^{\frac{13}{25}}$: i) $qrstw > v^{\frac{13}{25}}$ (0.000065); ii) $qrstw < v^{\frac{12}{25}},\ prstw > v^{\frac{13}{25}}$ (0.000148); iii) $prstw < v^{\frac{12}{25}}$ (0.000910). 6) $pqrt < v^{\frac{12}{25}}$: i) $pqstw > v^{\frac{13}{25}}$ (0.000193); ii) $pqstw < v^{\frac{12}{25}},\ pqrtw > v^{\frac{13}{25}}$ (0.000008); iii) $pqrtw < v^{\frac{12}{25}}$ (0.000006).

Thus the total deficiency in 1. is 0.003785.

2. $s > v^{\frac{26}{175}}$ (0.000027).

Therefore the total deficiency in V is 0.003812.

VI. $pqrs < v^{\frac{12}{25}}$. The corresponding deficiency is

$$\int_{\frac{1}{25}}^{\frac{9}{25}} \frac{dt}{t} \int_{\frac{1}{25}}^{\min(t,\,\frac{10}{25}-t)} \frac{du}{u} \int_{\frac{1}{25}}^{\min(u,\,\frac{11}{25}-t-u)} \frac{dr}{r} \times \int_{\frac{1}{25}}^{\min(r,\,\frac{12}{25}-t-u-r)} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

Applying process I again, we have to deal with a sum whose deficiency is

$$\int_{\frac{1}{25}}^{\frac{9}{25}} \frac{dt}{t} \int_{\frac{1}{25}}^{\min(t,\,\frac{10}{25}-t)} \frac{du}{u} \int_{\frac{1}{25}}^{\min(u,\,\frac{11}{25}-t-u)} \frac{dr}{r}$$
$$\times \int_{\frac{1}{25}}^{\min(r,\,\frac{12}{25}-t-u-r)} \frac{ds}{s} \int_{\frac{1}{25}}^{s} \frac{dy}{y} \int_{\frac{1}{25}}^{y} w\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2},$$

where we used the fact that $s \le \min\left(r,\ \frac{12}{25}-t-u-r\right) \le \frac{r}{2}+\frac{1}{2}\left(\frac{12}{25}-t-u-r\right) = \frac{1}{2}\left(\frac{12}{25}-t-u\right) < \frac{1}{4}(1-t-u-r) \Rightarrow y < \frac{1}{2}(1-t-u-r-s-y)$.

1. $prstw > v^{\frac{13}{25}}$ (0.000453). 2. $prstw < v^{\frac{12}{25}},\ pqstw > v^{\frac{13}{25}}$ (0.000324). 3. $pqstw < v^{\frac{12}{25}},\ pqrtw > v^{\frac{13}{25}}$ (0.000303). 4. $pqrtw < v^{\frac{12}{25}},\ pqrsw > v^{\frac{13}{25}}$ (0.000153). 5. $pqrsw < v^{\frac{12}{25}},\ pqrst > v^{\frac{13}{25}}$ (0.000046).

6. $pqrst < v^{\frac{12}{25}}$, $pqrstw > v^{\frac{13}{25}}$. Applying process I once more, we have the deficiency

$$\int_{\frac{13}{150}}^{\frac{8}{25}} \frac{dt}{t} \int_{\frac{1}{5}(\frac{13}{25}-t)}^{\min(t,\,\frac{9}{25}-t)} \frac{du}{u} \int_{\frac{1}{4}(\frac{13}{25}-t-u)}^{\min(u,\,\frac{10}{25}-t-u)} \frac{dr}{r}$$
$$\times \int_{\frac{1}{3}(\frac{13}{25}-t-u-r)}^{\min(r,\,\frac{11}{25}-t-u-r)} \frac{ds}{s} \int_{\frac{1}{2}(\frac{13}{25}-t-u-r-s)}^{\min(s,\,\frac{12}{25}-t-u-r-s)} \frac{dy}{y} \int_{\frac{13}{25}-t-u-r-s-y}^{y} \frac{dz}{z} \int_{\frac{1}{25}}^{z} \frac{d\alpha}{\alpha}$$
$$\times \int_{\frac{1}{25}}^{\alpha} w\left(\frac{1-t-u-r-s-y-z-\alpha-\beta}{\beta}\right) \frac{d\beta}{\beta^2}$$
$$\le \frac{1}{1.763} \int_{\frac{13}{150}}^{\frac{8}{25}} \frac{dt}{t} \int_{\frac{1}{5}(\frac{13}{25}-t)}^{\min(t,\,\frac{9}{25}-t)} \frac{du}{u} \int_{\frac{1}{4}(\frac{13}{25}-t-u)}^{\min(u,\,\frac{10}{25}-t-u)} \frac{dr}{r}$$
$$\times \int_{\frac{1}{3}(\frac{13}{25}-t-u-r)}^{\min(r,\,\frac{11}{25}-t-u-r)} \frac{ds}{s} \int_{\frac{1}{2}(\frac{13}{25}-t-u-r-s)}^{\min(s,\,\frac{12}{25}-t-u-r-s)} \frac{dy}{y}$$
$$\times \int_{\frac{13}{25}-t-u-r-s-y}^{y} \left(25\log(25z) - 25 + \frac{1}{z}\right) \frac{dz}{z}$$
$$\le 0.000359,$$

where we used the fact that $\beta \le \frac{1}{3}(1-t-u-r-s-y-z-\alpha)$ and Lemma 4.

7. $pqrstw < v^{\frac{12}{25}}$. Applying process I twice, we have the deficiency

$$0.5617 \int_{\frac{1}{25}}^{\frac{7}{25}} \frac{dt}{t} \int_{\frac{1}{25}}^{\min(t,\,\frac{8}{25}-t)} \frac{du}{u} \int_{\frac{1}{25}}^{\min(u,\,\frac{9}{25}-t-u)} \frac{dr}{r}$$
$$\times \int_{\frac{1}{25}}^{\min(r,\,\frac{10}{25}-t-u-r)} \frac{ds}{s} \int_{\frac{1}{25}}^{\min(s,\,\frac{11}{25}-t-u-r-s)} \frac{dy}{y}$$
$$\times \int_{\frac{1}{25}}^{\min(y,\,\frac{12}{25}-t-u-r-s-y)} \left(25\log(25z) - 25 + \frac{1}{z}\right) \frac{dz}{z}$$
$$\le 0.000495,$$

where we used the fact that $\beta \le \frac{1}{5}(1-t-u-r-s-y-z-\alpha)$ and Lemma 4.

Therefore the total deficiency in VI is 0.002133. Summing up the above estimates, we conclude that the total deficiency of Ω_1 is 0.096767.

5. The deficiency of Ω_2. Now we consider the sum

$$\Omega_2 = \sum_{v^{\frac{6.5}{25}} < p \le v^{\frac{12}{25}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{p} < q < p \\ q < \left(\frac{v}{p}\right)^{\frac{1}{2}}}} S(\mathcal{A}_{pq}, q).$$

The corresponding deficiency is

$$\int_{\frac{6.5}{25}}^{\frac{12}{25}} \frac{dt}{t} \int_{\frac{13}{25}-t}^{\min(t, \frac{1}{2}(1-t))} w\left(\frac{1-t-u}{u}\right) \frac{du}{u^2}.$$

We discuss several cases.

I. $q < v^{\frac{2.6}{25}}$. The corresponding deficiency is

$$\int_{\frac{10.4}{25}}^{\frac{12}{25}} \frac{dt}{t} \int_{\frac{13}{25}-t}^{\frac{2.6}{25}} w\left(\frac{1-t-u}{u}\right) \frac{du}{u^2}.$$

Let $m = p,\ n = q$ and $d = r$. Applying process I with Lemma 3, we have to deal with a sum whose deficiency is

$$\int_{\frac{10.4}{25}}^{\frac{12}{25}} \frac{dt}{t} \int_{\frac{13}{25}-t}^{\frac{2.6}{25}} \frac{du}{u} \int_{\frac{1}{25}}^{u} \frac{dr}{r} \int_{\frac{1}{25}}^{r} w\left(\frac{1-t-u-r-s}{s}\right) \frac{ds}{s^2}.$$

1. $ps > v^{\frac{13}{25}}$.

1) $s < q^{-\frac{1}{4}} r^{-\frac{1}{2}} v^{\frac{6.5}{50}}$. Applying process I again and writing $m = p,\ n = q$ and $d = rst$, we have to deal with a sum whose deficiency is

$$\int_{\frac{78}{175}}^{\frac{12}{25}} \frac{dt}{t} \int_{\frac{13}{25}-t}^{\min(\frac{2.6}{25}, 6t-\frac{65}{25})} \frac{du}{u} \int_{\frac{13}{25}-t}^{\min(u, 2t-\frac{u}{2}-\frac{19.5}{25})} \frac{dr}{r} \int_{\frac{13}{25}-t}^{\min(r, \frac{6.5}{50}-\frac{u}{4}-\frac{r}{2})} \frac{ds}{s}$$
$$\times \int_{\frac{1}{25}}^{s} \frac{dy}{y} \int_{\frac{1}{25}}^{\min(y, \frac{1}{2}(1-t-u-r-s-y))} w\left(\frac{1-t-u-r-s-y-z}{z}\right) \frac{dz}{z^2}.$$

i) $pw > v^{\frac{13}{25}}$ (0.000508). ii) $pw < v^{\frac{12}{25}}$ (0).

2) $s > q^{-\frac{1}{4}} r^{-\frac{1}{2}} v^{\frac{6.5}{50}}$ (0.007963).

Thus the total deficiency in 1. is 0.008471.

2. $ps < v^{\frac{12}{25}},\ pr > v^{\frac{13}{25}}$ (0.000996). 3. $pr < v^{\frac{12}{25}},\ prs > v^{\frac{13}{25}}$ (0.000831). 4. $prs < v^{\frac{12}{25}}$ (0).

Therefore the total deficiency in I is 0.010298.

II. $v^{\frac{2.6}{25}} < q < v^{\frac{26}{175}}$. The corresponding deficiency is

$$\int_{\frac{65}{175}}^{\frac{12}{25}} \frac{dt}{t} \int_{\max(\frac{13}{25}-t,\, \frac{2.6}{25})}^{\frac{26}{175}} w\Big(\frac{1-t-u}{u}\Big)\frac{du}{u^2}.$$

Let $m = p,\ n = q$ and $d = r$. Applying process I with Lemma 3, we have to deal with a sum whose deficiency is

$$\int_{\frac{65}{175}}^{\frac{12}{25}} \frac{dt}{t} \int_{\max(\frac{13}{25}-t,\, \frac{2.6}{25})}^{\frac{26}{175}} \frac{du}{u} \int_{\frac{1}{25}}^{u} \frac{dr}{r} \int_{\frac{1}{25}}^{\min(r,\, \frac{1}{2}(1-t-u-r))} w\Big(\frac{1-t-u-r-s}{s}\Big)\frac{ds}{s^2}.$$

1. $ps > v^{\frac{13}{25}}$.

1) $s < q^{-\frac{2}{3}}r^{-\frac{1}{2}}v^{\frac{13}{75}}$. Let $m = p,\ n = q$ and $d = rst$. Applying process I with Lemma 3, we have to deal with a sum the deficiency of which is

$$\int_{\frac{33.8}{75}}^{\frac{12}{25}} \frac{dt}{t} \int_{\frac{2.6}{25}}^{\min(\frac{26}{175},\, \frac{9}{4}t-\frac{91}{100})} \frac{du}{u} \int_{\frac{13}{25}-t}^{\min(u,\, 2t-\frac{4}{3}u-\frac{52}{75})} \frac{dr}{r}$$
$$\times \int_{\frac{13}{25}-t}^{\min(r,\, \frac{13}{75}-\frac{2}{3}u-\frac{r}{2})} \frac{ds}{s} \int_{\frac{1}{25}}^{s} \frac{dy}{y}$$
$$\times \int_{\frac{1}{25}}^{\min(y,\, \frac{1}{2}(1-t-u-r-s-y))} g\Big(\frac{1-t-u-r-s-y-z}{z}\Big)\frac{dz}{z^2}$$
$$\le 0.000486.$$

2) $s > q^{-\frac{2}{3}}r^{-\frac{1}{2}}v^{\frac{13}{75}}$ (0.071032).

Hence, the total deficiency in 1. is 0.071518.

2. $ps < v^{\frac{12}{25}},\ pr > v^{\frac{13}{25}}$.

1) $s < q^{-\frac{4}{3}}r^{-1}v^{\frac{26}{75}}$. Let $m = ps,\ n = q$ and $d = rt$. Applying process I with Lemma 3, we have to deal with a sum whose deficiency is

$$\int_{\frac{68}{175}}^{\frac{11}{25}} \frac{dt}{t} \int_{\max(\frac{13}{25}-t,\, \frac{2.6}{25})}^{\min(\frac{26}{175},\, \frac{3}{4}t-\frac{4}{25})} \frac{du}{u} \int_{\frac{13}{25}-t}^{\min(u,\, \frac{23}{75}-\frac{4}{3}u)} \frac{dr}{r}$$
$$\times \int_{\frac{1}{25}}^{\min(\frac{12}{25}-t,\, \frac{26}{75}-\frac{4}{3}u-r)} \frac{ds}{s} \int_{\frac{1}{25}}^{\min(s,\, \frac{1}{2}(1-t-u-r-s))} \frac{dy}{y}$$

$$\times \int_{\frac{1}{25}}^{\min(y,\,\frac12(1-t-u-r-s-y))} g\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2}$$
$$\le 0.000290.$$

2) $s > q^{-\frac43} r^{-1} v^{\frac{26}{75}}$.

i) The deficiency of the prime term is 0.005291.

ii) The deficiency of the almost-prime term is

$$\int_{\frac{65}{175}}^{\frac{11}{25}} \frac{dt}{t} \int_{\max(\frac{13}{25}-t,\,\frac37(t-\frac{10}{75}))}^{\frac{26}{175}} \frac{du}{u} \int_{\max(\frac{13}{25}-t,\,t-\frac43u-\frac{10}{75})}^{u} \frac{dr}{r}$$
$$\times \int_{\max(\frac{1}{25},\,\frac{26}{75}-\frac43u-r)}^{\frac{12}{25}-t} \frac{ds}{s}$$
$$\times \int_{s}^{\frac12(1-t-u-r-s)} w\left(\frac{1-t-u-r-s-k}{k}\right)\frac{dk}{k^2}.$$

a) $kp > v^{\frac{13}{25}}$ (0.001783). b) $kp < v^{\frac{12}{25}},\ kps > v^{\frac{13}{25}}$ (0.001370). c) $kps < v^{\frac{12}{25}}$ (0.000068).

Thus the total deficiency in 2. is 0.008802.

3. $pr < v^{\frac{12}{25}},\ prs > v^{\frac{13}{25}}$.

1) $s < q^{-\frac23} v^{\frac{13}{75}}$. Let $m = pr,\ n = q$ and $d = st$. Applying process I with Lemma 3, we have to deal with a sum whose deficiency is

$$\int_{\frac{65}{175}}^{\frac{11}{25}} \frac{dt}{t} \int_{\max(\frac{13}{25}-t,\,\frac{2.6}{25})}^{\frac{26}{175}} \frac{du}{u} \int_{\frac12(\frac{13}{25}-t)}^{\frac{12}{25}-t} \frac{dr}{r} \int_{\frac{13}{25}-t-r}^{\min(r,\,\frac{13}{75}-\frac23u)} \frac{ds}{s}$$
$$\times \int_{\frac{1}{25}}^{s} \frac{dy}{y} \int_{\frac{1}{25}}^{y} w\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2}.$$

i) $ptw > v^{\frac{13}{25}}$ (0.000088). ii) $ptw < v^{\frac{12}{25}},\ psw > v^{\frac{13}{25}}$ (0.000006). iii) $psw < v^{\frac{12}{25}},\ prw > v^{\frac{13}{25}}$ (0.000011). iv) $prw < v^{\frac{12}{25}}$ (0).

2) $s > q^{-\frac23} v^{\frac{13}{75}}$ (0.000571).

Therefore the total deficiency in 3. is 0.000676 and that in II is 0.080996.

III. $v^{\frac{26}{175}} < q < v^{\frac{5.2}{25}}$. Now we consider the deficiency of the sum

$$\Sigma = \sum_{v^{\frac{7.8}{25}} < p \le v^{\frac{12}{25}}} \ \sum_{\substack{\frac{v^{\frac{13}{25}}}{p} < q < v^{\frac{5.2}{25}} \\ v^{\frac{26}{175}} < q}} S(\mathcal{A}_{pq}, q).$$

We write

$$\Sigma = \Psi_1 + \Psi_2, \tag{17}$$

where

$$\Psi_1 = \sum_{v^{\frac{7.8}{25}} < p \le v^{\frac{11.5}{25}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{p} < q < v^{\frac{5.2}{25}} \\ v^{\frac{26}{175}} < q < \frac{v^{\frac{9.55}{25}}}{p^{\frac{1}{2}}}}} S(\mathcal{A}_{pq}, q),$$

$$\Psi_2 = \sum_{v^{\frac{8.7}{25}} < p \le v^{\frac{11.5}{25}}} \sum_{\frac{v^{\frac{9.55}{25}}}{p^{\frac{1}{2}}} < q < v^{\frac{5.2}{25}}} S(\mathcal{A}_{pq}, q) + \sum_{v^{\frac{11.5}{25}} < p \le v^{\frac{12}{25}}} \sum_{v^{\frac{26}{175}} < q < v^{\frac{5.2}{25}}} S(\mathcal{A}_{pq}, q).$$

The deficiency of Ψ_2 is

$$\int\limits_{\frac{8.7}{25}}^{\frac{11.5}{25}} \frac{dt}{t} \int\limits_{\frac{9.55}{25}-\frac{t}{2}}^{\frac{5.2}{25}} g\left(\frac{1-t-u}{u}\right)\frac{du}{u^2} + \int\limits_{\frac{11.5}{25}}^{\frac{12}{25}} \frac{dt}{t} \int\limits_{\frac{26}{175}}^{\frac{5.2}{25}} g\left(\frac{1-t-u}{u}\right)\frac{du}{u^2} \le 0.157504. \tag{18}$$

Next we shall discuss the deficiency of Ψ_1. Applying Buchstab's identity twice, we have

$$\begin{aligned}\Psi_1 = & \sum_{v^{\frac{7.8}{25}} < p \le v^{\frac{11.5}{25}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{p} < q < v^{\frac{5.2}{25}} \\ v^{\frac{26}{175}} < q < \frac{v^{\frac{9.55}{25}}}{p^{\frac{1}{2}}}}} S(\mathcal{A}_{pq}, v^{\frac{1}{25}}) \\ & - \sum_{v^{\frac{7.8}{25}} < p \le v^{\frac{11.5}{25}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{p} < q < v^{\frac{5.2}{25}} \\ v^{\frac{26}{175}} < q < \frac{v^{\frac{9.55}{25}}}{p^{\frac{1}{2}}}}} \sum_{v^{\frac{1}{25}} < r < q} S(\mathcal{A}_{pqr}, v^{\frac{1}{25}}) \\ & + \sum_{v^{\frac{7.8}{25}} < p \le v^{\frac{11.5}{25}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{p} < q < v^{\frac{5.2}{25}} \\ v^{\frac{26}{175}} < q < \frac{v^{\frac{9.55}{25}}}{p^{\frac{1}{2}}}}} \sum_{v^{\frac{1}{25}} < r < q} \sum_{\substack{v^{\frac{1}{25}} < s < r \\ s < \left(\frac{2v}{pqr}\right)^{\frac{1}{2}}}} S(\mathcal{A}_{pqrs}, s).\end{aligned} \tag{19}$$

The first sum has an asymptotic formula. For the second sum

$$\sum_{v^{\frac{7.8}{25}} < p \le v^{\frac{11.5}{25}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{p} < q < v^{\frac{5.2}{25}} \\ v^{\frac{26}{175}} < q < \frac{v^{\frac{9.55}{25}}}{p^{\frac{1}{2}}}}} \sum_{v^{\frac{1}{25}} < r < q} S(\mathcal{A}_{pqr}, v^{\frac{1}{25}}),$$

if $pr \le v^{\frac{13}{25}}$, then we have an asymptotic formula. If $r \le q^{-\frac{3}{4}} v^{\frac{6.5}{25}}$, then writing $m = p$, $n = q$ and $d = r$, we also have an asymptotic formula. Hence, we assume that $pr > v^{\frac{13}{25}}$, $r > q^{-\frac{3}{4}} v^{\frac{6.5}{25}}$ and have to deal with the sum

$$\Lambda = \sum_{v^{\frac{7.8}{25}} < p \le v^{\frac{11.5}{25}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{p} < q < v^{\frac{5.2}{25}} \\ v^{\frac{26}{175}} < q < \frac{v^{\frac{9.55}{25}}}{p^{\frac{1}{2}}}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{p} < r < q \\ \frac{v^{\frac{6.5}{25}}}{q^{\frac{3}{4}}} < r}} S(\mathcal{A}_{pqr}, v^{\frac{1}{25}}).$$

Now we use the discussion in [6]. Applying Buchstab's identity to the p-summation in Λ, we have

$$\begin{aligned} \Lambda = & \sum_{\substack{v^{\frac{7.8}{25}} < n \le v^{\frac{11.5}{25}} \\ (n, P(v^{\frac{1}{25}}))=1}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{n} < q < v^{\frac{5.2}{25}} \\ v^{\frac{26}{175}} < q < \frac{v^{\frac{9.55}{25}}}{n^{\frac{1}{2}}}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{n} < r < q \\ \frac{v^{\frac{6.5}{25}}}{q^{\frac{3}{4}}} < r}} S(\mathcal{A}_{nqr}, v^{\frac{1}{25}}) \\ & - \sum_{\substack{v^{\frac{7.8}{25}} < z\nu \le v^{\frac{11.5}{25}} \\ v^{\frac{1}{25}} \le z < \nu \\ (\nu, P(z))=1}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{z\nu} < q < v^{\frac{5.2}{25}} \\ v^{\frac{26}{175}} < q < \frac{v^{\frac{9.55}{25}}}{(z\nu)^{\frac{1}{2}}}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{z\nu} < r < q \\ \frac{v^{\frac{6.5}{25}}}{q^{\frac{3}{4}}} < r}} S(\mathcal{A}_{z\nu qr}, v^{\frac{1}{25}}), \end{aligned}$$

where n, ν denote integers and z denotes a prime number.

We write

$$\begin{aligned} & \sum_{\substack{v^{\frac{7.8}{25}} < n \le v^{\frac{11.5}{25}} \\ (n, P(v^{\frac{1}{25}}))=1}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{n} < q < v^{\frac{5.2}{25}} \\ v^{\frac{26}{175}} < q < \frac{v^{\frac{9.55}{25}}}{n^{\frac{1}{2}}}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{n} < r < q \\ \frac{v^{\frac{6.5}{25}}}{q^{\frac{3}{4}}} < r}} S(\mathcal{A}_{nqr}, v^{\frac{1}{25}}) \\ & = \#\{nqrl \in \mathcal{A} : v^{\frac{7.8}{25}} < n \le v^{\frac{11.5}{25}},\ (n, P(v^{\frac{1}{25}})) = 1,\ v^{\frac{13}{25}}/n < q < v^{\frac{5.2}{25}}, \\ & \qquad v^{\frac{26}{175}} < q < v^{\frac{9.55}{25}}/n^{\frac{1}{2}}, v^{\frac{13}{25}}/n < r < q, v^{\frac{6.5}{25}}/q^{\frac{3}{4}} < r,\ (l,\ P(v^{\frac{1}{25}})) = 1\}. \end{aligned}$$

Note that $v^{\frac{13}{25}} < nr \Rightarrow ql \ll v^{\frac{12}{25}}$ and that $r < q < v^{\frac{6.5}{25}}$. By the discussion in Lemma 9 of [8] with the application of Lemma 2, we can get an asymptotic formula. The idea of transferring the application of the sieve method from the l-summation to the n-summation comes from [6].

Now we shall deal with the deficiencies of the following two sums:

$$\Gamma_1 = \sum_{\substack{v^{\frac{7.8}{25}} < z\nu \le v^{\frac{11.5}{25}} \\ v^{\frac{1}{25}} \le z < \nu \\ (\nu, P(z))=1}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{z\nu} < q < v^{\frac{5.2}{25}} \\ v^{\frac{26}{175}} < q < \frac{v^{\frac{9.55}{25}}}{(z\nu)^{\frac{1}{2}}}}} \sum_{\substack{\frac{v^{\frac{13}{25}}}{z\nu} < r < q \\ \frac{v^{\frac{6.5}{25}}}{q^{\frac{3}{4}}} < r}} S(\mathcal{A}_{z\nu qr}, v^{\frac{1}{25}}), \tag{20}$$

$$\Gamma_2=\sum_{v^{\frac{7.8}{25}}<p\le v^{\frac{11.5}{25}}}\ \sum_{\substack{\frac{v^{\frac{13}{25}}}{p}<q<v^{\frac{5.2}{25}}\\ v^{\frac{26}{175}}<q<\frac{v^{\frac{9.55}{25}}}{p^{\frac12}}}}\ \sum_{v^{\frac{1}{25}}<r<q}\ \sum_{\substack{v^{\frac{1}{25}}<s<r\\ s<\left(\frac{v}{pqr}\right)^{\frac12}}} S(\mathcal{A}_{pqrs},s). \tag{21}$$

By Lemma 5, the deficiency of Γ_1 is

$$\int\limits_{\frac{7.8}{25}}^{\frac{11.5}{25}} dt \int\limits_{\max(\frac{13}{25}-t,\,\frac{26}{175})}^{\min(\frac{5.2}{25},\,\frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int\limits_{\frac{1}{25}}^{\frac{t}{2}} w\left(\frac{t-z}{z}\right)\frac{dz}{z^2}$$
$$\times \int\limits_{\max(\frac{13}{25}-t,\,\frac{6.5}{25}-\frac34 u)}^{u} 25w(25(1-t-u-r))\frac{dr}{r}$$

and that of Γ_2 is

$$\int\limits_{\frac{7.8}{25}}^{\frac{11.5}{25}} \frac{dt}{t} \int\limits_{\max(\frac{13}{25}-t,\,\frac{26}{175})}^{\min(\frac{5.2}{25},\,\frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int\limits_{\frac{1}{25}}^{u} \frac{dr}{r}$$
$$\times \int\limits_{\frac{1}{25}}^{\min(r,\,\frac12(1-t-u-r))} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

We refer to [6] and [10]. Firstly we consider the deficiency of Γ_1 in some cases.

1. $r\nu > v^{\frac{13}{25}}$. The corresponding deficiency is

$$\int\limits_{\frac{8.9}{25}}^{\frac{11.5}{25}} dt \int\limits_{\max(\frac{14}{25}-t,\,\frac{26}{175})}^{\frac{9.55}{25}-\frac{t}{2}} \frac{du}{u} \int\limits_{\frac{1}{25}}^{t+u-\frac{13}{25}} w\left(\frac{t-z}{z}\right)\frac{dz}{z^2}$$
$$\times \int\limits_{\max(\frac{13}{25}-t+z,\,\frac{6.5}{25}-\frac34 u)}^{u} 25w(25(1-t-u-r))\frac{dr}{r}.$$

We use Buchstab's identity

$$S(\mathcal{A}_{z\nu qr},v^{\frac{1}{25}})=S(\mathcal{A}_{z\nu qr},r)+\sum_{\substack{v^{\frac{1}{25}}\le s<r\\ s<\left(\frac{2v}{z\nu qr}\right)^{\frac12}}} S(\mathcal{A}_{z\nu qrs},s)$$

to produce two sums

$$\Lambda_1=\sum_z\sum_\nu\sum_q\sum_r S(\mathcal{A}_{z\nu qr},r),$$

$$\Lambda_2 = \sum_z \sum_\nu \sum_q \sum_r \sum_{\substack{v^{\frac{1}{25}} \le s < r \\ s < \left(\frac{2v}{z\nu qr}\right)^{\frac12}}} S(\mathcal{A}_{z\nu qrs}, s).$$

1) The deficiency of Λ_1 is

$$\int_{\frac{8.9}{25}}^{\frac{11.5}{25}} dt \int_{\max(\frac{14}{25}-t,\, \frac{26}{175})}^{\frac{9.55}{25}-\frac{t}{2}} \frac{du}{u} \int_{\frac{1}{25}}^{t+u-\frac{13}{25}} w\left(\frac{t-z}{z}\right)\frac{dz}{z^2}$$
$$\times \int_{\max(\frac{13}{25}-t+z,\, \frac{6.5}{25}-\frac34 u)}^{u} w\left(\frac{1-t-u-r}{r}\right)\frac{dr}{r^2}.$$

Applying process II to the ν-summation, we have to consider the prime term and the almost-prime term. For convenience, we call them the z-prime term and the z-almost-prime term respectively.

i) The deficiency of the z-prime term is

$$\int_{\frac{8.9}{25}}^{\frac{11.5}{25}} dt \int_{\max(\frac{14}{25}-t,\, \frac{26}{175})}^{\frac{9.55}{25}-\frac{t}{2}} \frac{du}{u} \int_{\frac{1}{25}}^{t+u-\frac{13}{25}} \frac{dz}{z(t-z)}$$
$$\times \int_{\max(\frac{13}{25}-t+z,\, \frac{6.5}{25}-\frac34 u)}^{u} g\left(\frac{1-t-u-r}{r}\right)\frac{dr}{r^2} \le 0.002064.$$

ii) The deficiency of the z-almost-prime term. Now ν is replaced by $e\beta$, where $z < e < \beta$, $(\beta, P(e)) = 1$, e is a prime number and β is an integer. The corresponding deficiency is

$$\int_{\frac{8.9}{25}}^{\frac{11.5}{25}} dt \int_{\max(\frac{14}{25}-t,\, \frac{26}{175})}^{\frac{9.55}{25}-\frac{t}{2}} \frac{du}{u} \int_{\frac{1}{25}}^{t+u-\frac{13}{25}} \frac{dz}{z}$$
$$\times \int_{z}^{\frac12(t-z)} w\left(\frac{t-z-e}{e}\right)\frac{de}{e^2} \int_{\max(\frac{13}{25}-t+z,\, \frac{6.5}{25}-\frac34 u)}^{u} w\left(\frac{1-t-u-r}{r}\right)\frac{dr}{r^2}.$$

a) $qre > v^{\frac{13}{25}}$ (0.000029). b) $qre < v^{\frac{12}{25}}$, $qrze > v^{\frac{13}{25}}$ (0.000229). c) $qrze < v^{\frac{12}{25}}$: α) $r\beta > v^{\frac{13}{25}}$ (0.000006); β) $r\beta < v^{\frac{12}{25}}$, $rz\beta > v^{\frac{13}{25}}$ (0.000644); γ) $rz\beta < v^{\frac{12}{25}}$ (0.000463).

Therefore the total deficiency of Λ_1 is 0.003435.

2) The deficiency of Λ_2 is

$$\int_{\frac{8.9}{25}}^{\frac{11.5}{25}} dt \int_{\max(\frac{14}{25}-t,\,\frac{26}{175})}^{\frac{9.55}{25}-\frac{t}{2}} \frac{du}{u} \int_{\frac{1}{25}}^{t+u-\frac{13}{25}} w\left(\frac{t-z}{z}\right)\frac{dz}{z^2}$$
$$\times \int_{\max(\frac{13}{25}-t+z,\,\frac{6.5}{25}-\frac{3}{4}u)}^{u} \frac{dr}{r} \int_{\frac{1}{25}}^{\min(r,\,\frac{1}{2}(1-t-u-r))} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

i) $s\nu > v^{\frac{13}{25}}$ (0.000102). ii) $s\nu < v^{\frac{12}{25}},\ sz\nu > v^{\frac{13}{25}}$ (0.001319).

iii) $sz\nu < v^{\frac{12}{25}}$. The corresponding deficiency is

$$\int_{\frac{8.9}{25}}^{\frac{11}{25}} dt \int_{\max(\frac{14}{25}-t,\,\frac{26}{175})}^{\frac{9.55}{25}-\frac{t}{2}} \frac{du}{u} \int_{\frac{1}{25}}^{t+u-\frac{13}{25}} w\left(\frac{t-z}{z}\right)\frac{dz}{z^2}$$
$$\times \int_{\max(\frac{13}{25}-t+z,\,\frac{6.5}{25}-\frac{3}{4}u)}^{u} \frac{dr}{r} \int_{\frac{1}{25}}^{\min(\frac{1}{2}(1-t-u-r),\,\frac{12}{25}-t)} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

a) The deficiency of the z-prime term is 0.001807.

b) The deficiency of the z-almost-prime term is

$$\int_{\frac{8.9}{25}}^{\frac{11}{25}} dt \int_{\max(\frac{14}{25}-t,\,\frac{26}{175})}^{\frac{9.55}{25}-\frac{t}{2}} \frac{du}{u} \int_{\frac{1}{25}}^{t+u-\frac{13}{25}} \frac{dz}{z} \int_{z}^{\frac{1}{2}(t-z)} w\left(\frac{t-z-e}{e}\right)\frac{de}{e^2}$$
$$\times \int_{\max(\frac{13}{25}-t+z,\,\frac{6.5}{25}-\frac{3}{4}u)}^{u} \frac{dr}{r} \int_{\frac{1}{25}}^{\min(\frac{1}{2}(1-t-u-r),\,\frac{12}{25}-t)} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

α) $r\beta > v^{\frac{13}{25}}$ (0.000006). β) $r\beta < v^{\frac{12}{25}},\ rz\beta > v^{\frac{13}{25}}$ (0.000471). γ) $rz\beta < v^{\frac{12}{25}},\ rsz\beta > v^{\frac{13}{25}}$ (0.000396). δ) $rsz\beta < v^{\frac{12}{25}},\ qsz\beta > v^{\frac{13}{25}}$ (0.000006). λ) $qsz\beta < v^{\frac{12}{25}}$ (0.000611).

Therefore the total deficiency of Λ_2 is 0.004718 and that in 1. is 0.008153.

2. $r\nu < v^{\frac{12}{25}},\ q\nu > v^{\frac{13}{25}}$. The corresponding deficiency is

$$\int_{\frac{8.9}{25}}^{\frac{73.7}{175}} dt \int_{\max(\frac{14}{25}-t,\,\frac{30}{175})}^{\frac{9.55}{25}-\frac{t}{2}} \frac{du}{u} \int_{\max(\frac{1}{25},\,t-\frac{3}{4}u-\frac{5.5}{25})}^{t+u-\frac{13}{25}} w\left(\frac{t-z}{z}\right)\frac{dz}{z^2}$$
$$\times \int_{\max(\frac{13}{25}-t,\,\frac{6.5}{25}-\frac{3}{4}u)}^{\frac{12}{25}-t+z} 25w(25(1-t-u-r))\,\frac{dr}{r}.$$

Note that $z < q^{-\frac{3}{4}} v^{\frac{6.5}{25}}$. Let $m = r\nu,\ n = q$ and $d = z$. Then we have an asymptotic formula. Hence, the deficiency is 0.

3. $q\nu < v^{\frac{12}{25}}$. The corresponding deficiency is

$$\int_{\frac{7.8}{25}}^{\frac{11.5}{25}} dt \int_{\max(\frac{13}{25}-t,\, \frac{26}{175})}^{\min(\frac{5.2}{25},\, \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int_{t+u-\frac{12}{25}}^{\frac{t}{2}} w\left(\frac{t-z}{z}\right)\frac{dz}{z^2}$$
$$\times \int_{\max(\frac{13}{25}-t,\, \frac{6.5}{25}-\frac{3}{4}u)}^{u} 25w(25(1-t-u-r))\,\frac{dr}{r}.$$

1) $qrz > v^{\frac{13}{25}}$.

i) The deficiency of Λ_1 is

$$\int_{\frac{7.8}{25}}^{\frac{11.5}{25}} dt \int_{\max(\frac{13}{25}-t,\, \frac{26}{175},\, \frac{6.5}{25}-\frac{t}{4})}^{\min(\frac{5.2}{25},\, \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int_{\frac{13}{25}-2u}^{\frac{t}{2}} g\left(\frac{t-z}{z}\right)\frac{dz}{z^2}$$
$$\times \int_{\max(\frac{13}{25}-t,\, \frac{6.5}{25}-\frac{3}{4}u,\, \frac{13}{25}-u-z)}^{u} g\left(\frac{1-t-u-r}{r}\right)\frac{dr}{r^2}$$
$$\le 0.004218.$$

ii) The deficiency of Λ_2 is

$$\int_{\frac{7.8}{25}}^{\frac{11.5}{25}} dt \int_{\max(\frac{13}{25}-t,\, \frac{26}{175},\, \frac{6.5}{25}-\frac{t}{4})}^{\min(\frac{5.2}{25},\, \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int_{\frac{13}{25}-2u}^{\frac{t}{2}} w\left(\frac{t-z}{z}\right)\frac{dz}{z^2}$$
$$\times \int_{\max(\frac{13}{25}-t,\, \frac{6.5}{25}-\frac{3}{4}u,\, \frac{13}{25}-u-z)}^{u} \frac{dr}{r} \int_{\frac{1}{25}}^{\min(r,\, \frac{1}{2}(1-t-u-r))} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

a) $sz\nu > v^{\frac{13}{25}}$ (0.000236).

b) $sz\nu < v^{\frac{12}{25}}$.

α) $qrs > v^{\frac{13}{25}}$ (0.000146). β) $qrs < v^{\frac{12}{25}}$: β_1) $qs\nu > v^{\frac{13}{25}}$ (0.000006); β_2) $qs\nu < v^{\frac{12}{25}}$ (0.007355).

Therefore the total deficiency in 1) is 0.011961.

2) $qrz < v^{\frac{12}{25}}$. If $\nu \le v^{\frac{6.5}{25}}$, writing $m = qrz$ and $n = \nu$, we have an asymptotic formula. Hence, we assume $\nu > v^{\frac{6.5}{25}}$. The corresponding deficiency is

$$\int_{\frac{7.8}{25}}^{\frac{11.5}{25}} dt \int_{\max(\frac{13}{25}-t,\, \frac{26}{175})}^{\min(\frac{5.2}{25},\, \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int_{t+u-\frac{12}{25}}^{\min(t-\frac{6.5}{25},\, \frac{5.5}{25}-\frac{u}{4})} w\left(\frac{t-z}{z}\right)\frac{dz}{z^2}$$
$$\times \int_{\max(\frac{13}{25}-t,\, \frac{6.5}{25}-\frac{3}{4}u)}^{\min(u,\, \frac{12}{25}-u-z)} 25w(25(1-t-u-r))\frac{dr}{r}.$$

i) $r \le v^{\frac{26}{175}}$. The corresponding deficiency is

$$\int_{\frac{65}{175}}^{\frac{11.5}{25}} dt \int_{\frac{26}{175}}^{\min(\frac{5.2}{25},\, \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int_{t+u-\frac{12}{25}}^{\min(t-\frac{6.5}{25},\, \frac{5.5}{25}-\frac{u}{4})} w\left(\frac{t-z}{z}\right)\frac{dz}{z^2} \times \int_{\max(\frac{13}{25}-t,\, \frac{6.5}{25}-\frac{3}{4}u)}^{\min(\frac{12}{25}-u-z,\, \frac{26}{175})} 25w(25(1-t-u-r))\,\frac{dr}{r}.$$

Note that $v^{\frac{2.6}{25}} < r$. If $z < r^{-\frac{4}{3}} v^{\frac{26}{75}}$ $(r < z^{-\frac{3}{4}} v^{\frac{6.5}{25}})$, writing $m = q\nu,\ n = r$ and $d = z$, we have an asymptotic formula. Hence, we assume $r \ge z^{-\frac{3}{4}} v^{\frac{6.5}{25}}$. The deficiency is

$$\int_{\frac{71.5}{175}}^{\frac{11.5}{25}} dt \int_{\frac{26}{175}}^{\frac{9.55}{25}-\frac{t}{2}} \frac{du}{u} \int_{\frac{26}{175}}^{\min(t-\frac{6.5}{25},\, \frac{5.5}{25}-\frac{u}{4})} g\left(\frac{t-z}{z}\right)\frac{dz}{z^2} \times \int_{\max(\frac{13}{25}-t,\, \frac{6.5}{25}-\frac{3}{4}u,\, \frac{6.5}{25}-\frac{3}{4}z)}^{\min(\frac{12}{25}-u-z,\, \frac{26}{175})} 25g(25(1-t-u-r))\frac{dr}{r}$$

$$\le 0.000754.$$

ii) $v^{\frac{26}{175}} < r$ $(< v^{\frac{5.2}{25}})$. The corresponding deficiency is

$$\int_{\frac{7.8}{25}}^{\frac{11.5}{25}} dt \int_{\max(\frac{13}{25}-t,\, \frac{26}{175})}^{\min(\frac{5.2}{25},\, \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int_{t+u-\frac{12}{25}}^{\min(t-\frac{6.5}{25},\, \frac{58}{175}-u)} w\left(\frac{t-z}{z}\right)\frac{dz}{z^2} \times \int_{\max(\frac{13}{25}-t,\, \frac{26}{175})}^{\min(u,\, \frac{12}{25}-u-z)} 25w(25(1-t-u-r))\frac{dr}{r}.$$

If $z < r^{-\frac{3}{4}} v^{\frac{6.5}{25}}$ $(r < z^{-\frac{4}{3}} v^{\frac{26}{75}})$, writing $m = q\nu,\ n = r$ and $d = z$, we have an asymptotic formula. Hence, we assume $r \ge z^{-\frac{4}{3}} v^{\frac{26}{75}}$. The deficiency is

$$\int_{\frac{7.8}{25}}^{\frac{11.5}{25}} dt \int_{\max(\frac{13}{25}-t,\, \frac{26}{175})}^{\min(\frac{5.2}{25},\, \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int_{t+u-\frac{12}{25}}^{\min(t-\frac{6.5}{25},\, \frac{58}{175}-u)} g\left(\frac{t-z}{z}\right)\frac{dz}{z^2} \times \int_{\max(\frac{13}{25}-t,\, \frac{26}{175},\, \frac{26}{75}-\frac{4}{3}z)}^{\max(\max(\frac{13}{25}-t,\, \frac{26}{175},\, \frac{26}{75}-\frac{4}{3}z),\, \min(u,\, \frac{12}{25}-u-z))} 25g(25(1-t-u-r))\frac{dr}{r}$$

$$\le 0.001733.$$

Thus the deficiency in 2) is 0.002487 and that in 3. is 0.014448. Therefore the deficiency of Γ_1 is 0.022601.

Next we consider the deficiency of Γ_2.

1. $ps > v^{\frac{13}{25}}$. The deficiency is

$$\int_{\frac{9}{25}}^{\frac{11.5}{25}} \frac{dt}{t} \int_{\max(\frac{13}{25}-t,\, \frac{26}{175})}^{\min(\frac{9.55}{25}-\frac{t}{2},\, 2t-\frac{14}{25})} \frac{du}{u} \int_{\frac{13}{25}-t}^{\min(u,\, t-u-\frac{1}{25})} \frac{dr}{r} \times \int_{\frac{13}{25}-t}^{\min(r,\, \frac{1}{2}(1-t-u-r))} g\left(\frac{1-t-u-r-s}{s}\right) \frac{ds}{s^2}$$
$$\leq 0.013694.$$

2. $ps < v^{\frac{12}{25}},\ pr > v^{\frac{13}{25}}$. The corresponding deficiency is

$$\int_{\frac{7.8}{25}}^{\frac{11}{25}} \frac{dt}{t} \int_{\max(\frac{13}{25}-t,\, \frac{26}{175})}^{\min(\frac{5.2}{25},\, \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int_{\frac{13}{25}-t}^{u} \frac{dr}{r} \times \int_{\frac{1}{25}}^{\min(\frac{1}{2}(1-t-u-r),\, \frac{12}{25}-t)} w\left(\frac{1-t-u-r-s}{s}\right) \frac{ds}{s^2}.$$

1) $qrs > v^{\frac{13}{25}}$ (0.000236).

2) $qrs < v^{\frac{12}{25}}$.

i) The deficiency of the prime term is 0.036198.

ii) The deficiency of the almost-prime term is

$$\int_{\frac{7.8}{25}}^{\frac{11}{25}} \frac{dt}{t} \int_{\max(\frac{13}{25}-t,\, \frac{26}{175})}^{\min(\frac{5.2}{25},\, \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int_{\frac{13}{25}-t}^{u} \frac{dr}{r} \int_{\frac{1}{25}}^{\min(\frac{12}{25}-t,\, \frac{12}{25}-u-r,\, \frac{1}{3}(1-t-u-r))} \frac{ds}{s} \times \int_{s}^{\frac{1}{2}(1-t-u-r-s)} w\left(\frac{1-t-u-r-s-k}{k}\right) \frac{dk}{k^2}.$$

a) $kp > v^{\frac{13}{25}}$ (0.002569). b) $kp < v^{\frac{12}{25}},\ kps > v^{\frac{13}{25}}$: α) $kqrs > v^{\frac{13}{25}}$ (0.000774); β) $kqrs < v^{\frac{12}{25}}$ (0.004275). c) $kps < v^{\frac{12}{25}}$: α) $kqrs > v^{\frac{13}{25}}$ (0.000402); β) $kqrs < v^{\frac{12}{25}}$ (0.003268).

Therefore the total deficiency in 2. is 0.047722.

3. $pr < v^{\frac{12}{25}},\ prs > v^{\frac{13}{25}}$. The corresponding deficiency is

$$\int_{\frac{7.8}{25}}^{\frac{11}{25}} \frac{dt}{t} \int_{\max(\frac{13}{25}-t,\, \frac{26}{175})}^{\min(\frac{5.2}{25},\, \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int_{\frac{1}{2}(\frac{13}{25}-t)}^{\frac{12}{25}-t} \frac{dr}{r} \int_{\frac{13}{25}-t-r}^{\min(r,\, \frac{1}{2}(1-t-u-r))} w\left(\frac{1-t-u-r-s}{s}\right) \frac{ds}{s^2}.$$

1) $s < q^{-\frac{3}{8}} v^{\frac{6.5}{50}}$. Let $m = pr,\ n = q$ and $d = st$. Applying process I with Lemma 3, we have to deal with a sum whose deficiency is

$$\int\limits_{\frac{7.8}{25}}^{\frac{11}{25}} \frac{dt}{t} \int\limits_{\max(\frac{13}{25}-t,\ \frac{26}{175})}^{\min(\frac{5.2}{25},\ \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int\limits_{\max(\frac{1}{2}(\frac{13}{25}-t),\ \frac{19.5}{50}-t+\frac{3}{8}u)}^{\frac{12}{25}-t} \frac{dr}{r} \times \int\limits_{\frac{13}{25}-t-r}^{\min(r,\ \frac{6.5}{50}-\frac{3}{8}u)} \frac{ds}{s} \int\limits_{\frac{1}{25}}^{s} \frac{dy}{y} \int\limits_{\frac{1}{25}}^{y} w\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2}.$$

i) $prw > v^{\frac{13}{25}}$ (0.000643). ii) $prw < v^{\frac{12}{25}}$ (0).

Hence, the total deficiency in 1) is 0.000643.

2) $s > q^{-\frac{3}{8}} v^{\frac{6.5}{50}}$.

i) The deficiency of the prime term is 0.021656.

ii) The deficiency of the almost-prime term is

$$\int\limits_{\frac{7.8}{25}}^{\frac{11}{25}} \frac{dt}{t} \int\limits_{\max(\frac{13}{25}-t,\ \frac{26}{175})}^{\min(\frac{5.2}{25},\ \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int\limits_{\frac{1}{2}(\frac{13}{25}-t)}^{\frac{12}{25}-t} \frac{dr}{r} \times \int\limits_{\max(\frac{13}{25}-t-r,\ \frac{6.5}{50}-\frac{3}{8}u)}^{\max(\max(\frac{13}{25}-t-r,\ \frac{6.5}{50}-\frac{3}{8}u),\ \min(r,\ \frac{1}{3}(1-t-u-r)))} \frac{ds}{s} \times \int\limits_{s}^{\frac{1}{2}(1-t-u-r-s)} w\left(\frac{1-t-u-r-s-k}{k}\right)\frac{dk}{k^2}.$$

a) $kp > v^{\frac{13}{25}}$ (0.000349). b) $kp < v^{\frac{12}{25}},\ kps > v^{\frac{13}{25}}$: α) $kqrs > v^{\frac{13}{25}}$ (0.001240); β) $kqrs < v^{\frac{12}{25}}$ (0.004339). c) $kps < v^{\frac{12}{25}}$ (0.000786).

Therefore the total deficiency in 2) is 0.028370 and that in 3. is 0.029013.

4. $prs < v^{\frac{12}{25}}$. The corresponding deficiency is

$$\int\limits_{\frac{7.8}{25}}^{\frac{10}{25}} \frac{dt}{t} \int\limits_{\max(\frac{13}{25}-t,\ \frac{26}{175})}^{\min(\frac{5.2}{25},\ \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int\limits_{\frac{1}{25}}^{\frac{11}{25}-t} \frac{dr}{r} \int\limits_{\frac{1}{25}}^{\min(r,\ \frac{12}{25}-t-r)} w\left(\frac{1-t-u-r-s}{s}\right)\frac{ds}{s^2}.$$

Let $m = prs,\ n = q$ and $d = t$. Note that $t < s < \min(r,\ v^{\frac{12}{25}}/pr) \le (v^{\frac{12}{25}}/p)^{\frac{1}{2}} < v^{\frac{6.5}{25}} q^{-\frac{3}{4}}$. Applying process I with Lemma 3, we have to deal with a sum whose deficiency is

$$\int_{\frac{7.8}{25}}^{\frac{10}{25}} \frac{dt}{t} \int_{\max(\frac{13}{25}-t,\, \frac{26}{175})}^{\min(\frac{5.2}{25},\, \frac{9.55}{25}-\frac{t}{2})} \frac{du}{u} \int_{\frac{1}{25}}^{\frac{11}{25}-t} \frac{dr}{r} \int_{\frac{1}{25}}^{\min(r,\, \frac{12}{25}-t-r)} \frac{ds}{s} \int_{\frac{1}{25}}^{s} \frac{dy}{y}$$
$$\times \int_{\frac{1}{25}}^{\min(y,\, \frac{1}{2}(1-t-u-r-s-y))} w\left(\frac{1-t-u-r-s-y-z}{z}\right)\frac{dz}{z^2}.$$

1) $prsw > v^{\frac{13}{25}}$ (0.000373). 2) $prsw < v^{\frac{12}{25}}$ (0.000063).

Thus the total deficiency in 4. is 0.000436. Hence, the deficiency of Γ_2 is 0.090865. The deficiency of Ψ_1 is 0.113466 and that of Ψ_2 is 0.157504. Therefore the total deficiency in III is 0.270970.

IV. $q > v^{\frac{5.2}{25}}$. The deficiency is

$$\int_{\frac{6.5}{25}}^{\frac{12}{25}} \frac{dt}{t} \int_{\max(\frac{13}{25}-t,\, \frac{5.2}{25})}^{\min(t,\, \frac{1}{2}(1-t))} g\left(\frac{1-t-u}{u}\right)\frac{du}{u^2} \le 0.529493.$$

Therefore the deficiency of Ω_2 is 0.891757.

6. Proof of the Theorem. The above discussion yields

$$\begin{aligned}
\Phi &\ge \frac{X}{v}\cdot\Big(S(\mathcal{B}, v^{\frac{1}{25}}) - \sum_{v^{\frac{1}{25}} < p \le v^{\frac{12}{25}}} S(\mathcal{B}_p, p) - \sum_{v^{\frac{12}{25}} < p \le (2v)^{\frac{1}{2}}} S(\mathcal{B}_p, p)\Big) \\
&\quad - 0.096767\cdot\frac{X}{\log v} - 0.891757\cdot\frac{X}{\log v} \\
&= \frac{X}{v}\cdot S(\mathcal{B}, (2v)^{\frac{1}{2}}) - 0.988524\cdot\frac{X}{\log v} \\
&= (1+O(\varepsilon))\cdot\frac{X}{\log v} - 0.988524\cdot\frac{X}{\log v} \\
&\ge 0.011475\cdot\frac{X}{\log v}.
\end{aligned}$$

Thus (3) holds and so the proof of the Theorem is complete.

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