A theorem concerning homologies in a compact space.

By

W. T. Puckett, Jr. (Virginia, U. S. A.).

1. Let M be a compact metric space and let M^r be any closed subset of M which contains all the r-th order cyclic elements [1]) of M. This paper proves: *If Z^{r+1} is a complete $(r+1)$-dimensional cycle in M, then there exists a complete cycle Z^{*r+1} in M^r such that $Z^{r+1} \sim Z^{*r+1}$ in M* [2]). The notions of a complete cycle and its associate concepts are used in the sense of Vietoris [3]) and Alexandroff [4]). All relations of complexes are modulo 2, and oriented complexes play no part in this paper.

2. An infinite sequence $Z^r = \{z_n^r\}$ of r-dimensional cycles is called a fundamental sequence [3]) or complete cycle [4]) provided that (i) z_n^r is a δ_n-cycle with δ_n approaching zero, and (ii) for every $\varepsilon > 0$ there exists an N_ε such that $i,j > N_\varepsilon$ implies $z_i^r \underset{\varepsilon}{\sim} z_j^r$. Among the properties of complete cycles we have occasion to use are the following:

(2.1) *Let $Z^{r+1} = \{z_n^{r+1}\}$ be a complete $(r+1)$-dimensional cycle in M such that for some subsequence $\{z_{n_i}^{r+1}\}$ each $z_{n_i}^{r+1}$ is the sum of two complexes, say $z_{n_i}^{r+1} = K_i^{r+1} + Q_i^{r+1}$. Then $Z^r = \{z_i^r\} = \{\dot{K}_i^{r+1}\} = \{\dot{Q}_i^{r+1}\}$* [5]) *is a complete r-dimensional cycle and $Z^r \sim 0$ in M.*

[1]) See G. T. Whyburn, *Cyclic elements of higher order*, Amer. Journ. Math., vol. 56 (1934), pp. 133-146; also *On the structure of continua*, Bull. Amer. Math. Soc., vol. 42 (1936), pp. 57-61.

[2]) The writer is greatly indebted to G. T. Whyburn for his suggestions and criticism during the preparation of this paper, as well as for his formulation of the problem as an extension of a result due to S. Eilenberg. See *Deux théorèmes sur l'homologie dans les espaces compacts*, Fund. Math., vol. 24 (1935), pp. 151-159.

[3]) L. Vietoris, *Über den höheren Zusammenhang kompakter Räume...*, Math. Ann., vol. 97 (1926), pp. 454-472.

[4]) P. Alexandroff, *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, Ann. of Math., ser. 2, vol. 30 (1928), pp. 101-187.

[5]) If K is a homogeneous n-dimensional complex, then $\dot{K}$ denotes the boundary (mod 2) of the complex K.

(2.2) *If $Z^r = \{z_n^r\}$ and $X^r = \{x_n^r\}$ are complete cycles, while $Y^r = \{y_n^r\}$ is a sequence of cycles such that $z_n^r = x_n^r + y_n^r$, then Y^r is a complete cycle.*

(2.3) *If $\{z_n^r\}$ is a sequence of r-dimensional δ_n-cycles $(\delta_n \to 0)$ in M, then there exists a complete r-dimensional cycle $Z^r = \{z_{n_i}^r\}$ contained in $\{z_n^r\}$.* [6])

(2.4) *Let $Z^r = \{z_i^r\}$ be a complete cycle in M and for each integer n let $Z_n^r = \{z_{n_i}^r\}$ be a complete cycle such that $Z^r \sim Z_n^r$ in M. Then there exists a complete cycle $Z'^r = \{z_n'^r\}$ such that $z_n'^r$ is a cycle of Z_n^r and $Z'^r \sim Z^r$ in M.*

Select a sequence of positive numbers $\{\varepsilon_m\}$ such that $\varepsilon_m > \varepsilon_{m+1}$ and $\varepsilon_m \to 0$. Since Z^r is a complete cycle and $Z^r \sim Z_1^r$, there must exist a positive integer N_1 such that $i,j \geqslant N_1$ implies $z_i^r \underset{\varepsilon_1}{\sim} z_j^r$ and $z_i^r \underset{\varepsilon_1}{\sim} z_{1i}^r$. Define $z_1'^r = z_{1N_1}^r$. Likewise there exists an integer $N_2 > N_1$ such that $i,j \geqslant N_2$ implies $z_i^r \underset{\varepsilon_2}{\sim} z_j^r$ and $z_i^r \underset{\varepsilon_2}{\sim} z_{2i}^r$, since Z^r and Z_2^r are homologous complete cycles. Define $z_2'^r = z_{2N_2}^r$. Generally, since Z^r and Z_m^r are homologous complete cycles there exists a positive integer N_m with $N_m > N_{m-1} > \ldots > N_2 > N_1$ such that $i,j \geqslant N_m$ implies $z_i^r \underset{\varepsilon_m}{\sim} z_j^r$ and $z_i^r \underset{\varepsilon_m}{\sim} z_{mi}^r$. Define $z_m'^r = z_{mN_m}^r$, then $Z'^r = \{z_m'^r\}$ is the complete cycle sought.

From the definition of $z_m'^r$ it follows that $z_m'^r \underset{\varepsilon_m}{\sim} z_{N_m}^r$, whence $z_m'^r$ is an ε_m-cycle and ε_m approaches zero because of choice of ε_m. Now $Z^{*r} = \{z_{N_m}^r\}$ is a subsequence of Z^r, whence Z^{*r} is a complete cycle homologous to Z^r. [7]) Moreover, from the definition of N_m and the fact $\varepsilon_m > \varepsilon_{m+1}$ it follows that for $i,j > k$

$$\text{(a)} \quad z_{N_i}^r \underset{\varepsilon_k}{\sim} z_{N_j}^r \qquad \text{and} \qquad \text{(b)} \quad z_i'^r \underset{\varepsilon_k}{\sim} z_{N_i}^r.$$

Whence $z_i'^r \underset{\varepsilon_k}{\sim} z_{N_i}^r \underset{\varepsilon_k}{\sim} z_{N_j}^r \underset{\varepsilon_k}{\sim} z_j'^r$ or $z_i'^r \underset{\varepsilon_k}{\sim} z_j'^r$ for $i,j > k$. But $\varepsilon_k \to 0$. Thus for any $\varepsilon > 0$ there exists a k such that $i,j > k$ implies $z_i'^r \underset{\varepsilon}{\sim} z_j'^r$. Therefore Z'^r is a complete cycle. Moreover, it follows from (b) and the fact ε_k approaches zero that $Z'^r \sim Z^r$, which proves the assertion.

[6]) See Vietoris, loc. cit., p. 468; also Alexandroff, *Dimensionstheorie*, Math. Ann., vol. 106 (1932), p. 180.

[7]) See Alexandroff, *Gestalt und Lage*, loc. cit., p. 181.

3. A closed set of points which carries no essential [8]) complete r-dimensional cycle is called a T_r-set or simply a T_r. A non-degenerate subset K of M is an E_r-set [9]), or merely an E_r, provided K is not separated by any T_r-set in K and K is saturated in M relative to this property. The E_r-sets contained in M are the r-th order cyclic elements of M. In case M is locally connected, then its E_0-sets are its non-degenerate cyclic elements [10]). Let M^r denote any closed subset of M which contains all the r-th order cyclic elements of M, then the principal theorem of this paper is:

(3.1) *If Z^{r+1} is a complete $(r+1)$-dimensional cycle in M, then there exists a complete cycle Z^{*r+1} in M^r such that Z^{*r+1} is homologous to Z^{r+1} in M.*

In case $Z^{r+1}=\{z_n^{r+1}\}$ is a non-essential complete cycle the theorem is trivial. Hence it is assumed Z^{r+1} is an essential complete cycle. Moreover, it may be assumed z_n^{r+1} is a δ_n-cycle where $\delta_n > \delta_{n+1}$. The proof depends on the following

Lemma. *For $\varepsilon > 0$ there exists a complete cycle Z_ε^{r+1} contained in $V_\varepsilon(M^r)$ [11]) such that Z_ε^{r+1} is homologous to Z^{r+1} in M.*

Now in order to prove the theorem select a sequence of positive numbers $\{\varepsilon_i\}$ approaching zero. Then by the lemma there exists a complete cycle $Z_{\varepsilon_i}^{r+1}$ in $V_{\varepsilon_i}(M^r)$ for each i such that $Z_{\varepsilon_i}^{r+1} \sim Z^{r+1}$ in M. But by (2.4) there exists a complete cycle $Z'^{r+1}=\{z_i'^{r+1}\}$, where $z_i'^{r+1}$ is contained in $Z_{\varepsilon_i}^{r+1}$, such that $Z'^{r+1} \sim Z^{r+1}$. Now an infinitely small modification [12]) (use ε_i) of Z'^{r+1} gives the desired Z^{*r+1} which is carried by M^r and is homologous to Z^{r+1} in M.

---

[8]) A Z^r is essential if it has at least one carrier in which it is not homologous to zero.

[9]) See Whyburn, *Cyclic elements of higher order*, loc. cit. This paper is hereafter referred to as W^1.

[10]) See Whyburn, *Concerning the structure of a continuous curve*, Amer. Journ. Math., vol. 50 (1928), pp. 167-194; also C. Kuratowski and G. T. Whyburn, *Sur les éléments cycliques et leurs applications*, Fund. Math., vol. 16 (1930), pp. 305-331.

[11]) The symbol $V_\varepsilon(M^r)$ denotes the set of all points in M at a distance less than ε from M^r.

[12]) See Alexandroff, *Gestalt und Lage*, loc. cit., p. 181.

Proof of lemma. For each pair of integers n and i define (a) $M_n^r = V_{2\delta_n}(M^r)$, (b) ${}_nM^r = V_{\delta_n}(M^r)$, (c) Q_{ni}^{r+1} to be the complex of all simplices of z_{n+i}^{r+1} with at least one vertex in $(M-\overline{M_n^r})$, and (d) K_{ni}^{r+1} to be the complement of Q_{ni}^{r+1} in z_{n+i}^{r+1}. Then (a′) Q_{ni}^{r+1} is carried by $(M-{}_nM^r)$, (b′) K_{ni}^{r+1} is carried by $\overline{M_n^r}$, and (c′) $z_{ni}^r = \dot{Q}_{ni}^{r+1} = \dot{K}_{ni}^{r+1}$ is carried by $(\overline{M_n^r}-{}_nM^r)$. Thus by (2.1) $Z_n^r=\{z_{ni}^r\}$ is a complete cycle, and by (c′) it is carried by the closed set $(\overline{M_n^r}-{}_nM^r)$. Now $\overline{M_n^r}\cdot(M-{}_nM^r) = (\overline{M_n^r}-{}_nM^r)$ carries Z_n^r, and Z_n^r is homologous to zero in both $\overline{M_n^r}$ and $(M-{}_nM^r)$, because of (a′), (b′) and (c′). But $(M-{}_nM^r)$ is a closed set and $\overline{M_n^r}$ is a K_r-set [13]) since it contains M^r which carries all the r-th order cyclic elements of M. Thus it follows from a known theorem [14]) that Z_n^r is homologous to zero in $(\overline{M_n^r}-{}_nM^r)$. Therefore, there exists a sequence of $(r+1)$-dimensional complexes $\{R_{ni}^{r+1}\}$ contained in $(\overline{M_n^r}-{}_nM^r)$ such that R_{ni}^{r+1} is a δ_{ni}-complex with $\lim\limits_{i\to\infty}\delta_{ni}=0$ and $\dot{R}_{ni}^{r+1} = z_{ni}^r = \dot{K}_{ni}^{r+1}$. Now z_{n+i}^{r+1} and consequently K_{ni}^{r+1} must have a vertex in ${}_nM^r$ for sufficiently large i, since any carrier of an essential Z^{r+1} must contain a subcontinuum which is contained in some E_r. [15]) Thus for large i $z_{ni}'^{r+1} = K_{ni}^{r+1} + R_{ni}^{r+1}$ is a non-vacuous δ'_{ni}-cycle contained in $\overline{M_n^r}$, where $\delta'_{ni} = \max(\delta_i, \delta_{ni})$ approaches zero as i increases. Whence from (2.3) it follows that there exists for each n a subsequence of $\{z_{ni}'^{r+1}\}$ which is a complete cycle. Consider the cycles of the subsequence so renamed that this complete cycle is $Z_n'^{r+1} = \{z_{ni}'^{r+1}\}$. Define $z_{ni}''^{r+1} = Q_{ni}^{r+1} + R_{ni}^{r+1}$, then $\{z_{ni}''^{r+1}\}$, the corresponding subsequences of Z^{r+1} and of $Z_n'^{r+1}$ satisfy the conditions of (2.2). Whence $Z_n''^{r+1} = \{z_{ni}''^{r+1}\}$ is a complete cycle. Moreover, $Z_n''^{r+1}$ is contained in $(M-{}_nM^r)$ by construction. Thus, since $(M-{}_nM^r)$ contains no E_r-set, it follows [15]) that $Z_n''^{r+1}$ is a non-essential complete cycle, i. e. $Z_n''^{r+1} \sim 0$ in any carrier.

For each n, the complete cycle $Z_n'^{r+1}$ is homologous to Z^{r+1} in M. Let $Z_n^{r+1} = \{z_{n+i}^{r+1}\}$, then, since Z_n^{r+1} is a subsequence of Z^{r+1}, these two cycles are homologous in any carrier. But $Z_n^{r+1} = Z_n'^{r+1} + Z_n''^{r+1}$, whence $Z_n^{r+1} + Z_n'^{r+1} = Z_n''^{r+1} \sim 0$, since $Z_n''^{r+1}$ is non-essential. Thus

---

[13]) A closed subset K of M having the property that $K \cdot E_r \neq T_r$ implies E_r is contained in K is called a K_r-set. See W^1, p. 139.

[14]) See W^1 (3.1), p. 140.

[15]) See W^1 (1.4), p. 134.

$Z_n'^{r+1}$ is homologous to $Z_n'^{r+1}$ in M, and therefore $Z_n'^{r+1}$ is homologous to Z^{r+1} in M for every n.

Thus to establish the lemma for any $\varepsilon>0$, one takes n so great that $\varepsilon>2\delta_n$ and puts $Z_\varepsilon^{r+1}=Z_n'^{r+1}$.

(3.11) ***Corollary.*** *The $(r+1)$-dimensional Betti group (mod 2)* [16]) *of M^r is isomorphic with the corresponding group of M.*

4. Let one replace M^r in the above theorem by a closed subset M^* of M such that dimension of $(M-M^*)\leqslant r$, then one obtains a known result, which is the consequence of a theorem proved by Eilenberg [17]). It will be shown that M^r *is contained in* M^*, *while the converse is not true.* Thus for the case of complete cycles (mod 2) M^r is a more essential kernel of the set M. If $M^r=0$ the assertion is trivial. However, it is well to point out that in this case M can contain no essential Z^{r+1}. [15]) If M contains an E_r, it suffices to show that a point p belonging to the minimum M^r implies that there exists a sequence of points $\{p_i\}$ contained in M^r such that for each i the dimension of M^r at p_i is greater than or equal to $(r+1)$, and that $\overline{\Sigma p_i}$ contains p. Select a sequence of positive numbers $\{\varepsilon_i\}$ approaching zero and take ε_i-neighborhoods, $V_{\varepsilon_i}(p)$, of p. Then by definition of M^r there exists a point p_i contained in $V_{\varepsilon_i}(p)$ for each i, such that p_i belongs to some E_r which in turn is contained in M^r. By a known theorem [18]) the dimension of E_r at p_i is greater than or equal to $(r+1)$. Hence the dimension of M^r at p_i is not less than $(r+1)$.

That M^r need not contain M^* is shown by the example of a space consisting of a 2-sphere and of a tangent square.

The University of Virginia.

[16]) This group is called the r-th connectivity group by Vietoris, loc. cit.

[17]) S. Eilenberg, *Deux théorèmes sur l'homologie dans les espaces compacts*, loc. cit.

[18]) See W^1 (1.7), p. 135.

Eine Normalitätsbedingung für Familien von Potentialfunktionen.

Von

A. Tulajkov (Moskau).

Es sei E eine Famille reeller Potentialfunktionen $U(P)$, die wir uns in einem offenen (eventuell unbeschränkten) zusammenhängenden Gebiet D des n-dimensionalen Euklidischen Raumes definiert und regulär denken. Man nennt E *normal* in D, wenn eine beliebige Folge von Funktionen aus E eine Teilfolge enthält, welche auf jedem beschränkten abgeschlossenen $\overline{D}_1\subset D$ entweder im üblichen Sinne gleichmässig konvergiert oder im ganzen $\overline{D}_1$ nach $+\infty$ bzw. nach $-\infty$ gleichmässig strebt.

Die erste Normalitätsbedingung für solche Funktionenfamilien bei beliebiger Dimensionszahl wurde von Privaloff [1]) gegeben. Weitere Bedingungen findet man bei Montel [2]). Da seine letzte Abhandlung auch notwendige und hinreichende Bedingungen enthält, gestattet sich Verfasser folgenden Satz mitzuteilen, obwohl der Beweis fast trivialerweise aus Montelschen Ausführungen folgt:

Satz. *Die Famille E in dem Gebiet D* (unter den oben angegebenen Bedingungen) *ist dann und nur dann normal, wenn für jedes beschränkte, zusammenhängende, abgeschlossene Teilgebiet $\overline{D}_1$ von D es eine Zahl $M=M(\overline{D}_1)>0$ gibt, derart dass jede Funktion $U\in E$ mindestens einen Wert $a=a(U)$ mit $|a|<M$ auf $\overline{D}_1$ nicht annimmt.*

[1]) I. Privaloff, *Sur les fonctions harmoniques*, Rec. Math. Moscou 32 (1925), 464-469.

[2]) P. Montel, *Familles de fonctions harmoniques*, Fund. Math. 25 (1935), 388-407.