The situation is different from IV in the following point. The space $X_s(\omega_4)$ possesses the property (A) since (Σ_1) and (Σ_2) are fulfilled (see [2]). The norms $\| \|_1^*$ and $\| \|_3^*$ being non-equivalent in X_s with $\| \|_4^*$, the corresponding Saks spaces do not possess the property (A).

Especially *in* $X_s(\omega_1)$ *we have* (B_1) *but not* (B_2).

VI. Let us denote by X the space of all bounded continuous functions defined in an open interval (a, b). (The end points need not be finite here). As Y_0 we take the set of all linear functionals of the form

$$\int_{a+}^{b-} x(t)\,dy,$$

where y denotes a function of finite variation in (a, b), continuous from the left and equal to zero at the point $(a+b)/2$. It is easy to see that Y_0 is not identical with Y and possesses the property (T). The space Y_0 is non-separable, since

$$\|y\| = \operatorname*{var}_{(a,b)} y(t) \quad \text{for} \quad y \in Y_0.$$

Let a, b be finite and let us denote by B the set of all $y \in Y_0$ such that $y(t) = 0$ for $t \in (a, a+1/n)+(b-1/n, b)$ and $\operatorname*{var}_{\langle a+1/n,\, b-1/n\rangle} y(t) = 1/n$.

Then

$$\|x\|^* = \sup_{y\in B} |y(x)| = \sup_{n} \sup_{\langle a+1/n,\, b-1/n\rangle} |x(t)|/n.$$

In the case when a, b are infinite we define B and the norm $\| \|^*$ analogically.

It is possible to show that $X_s(\omega)$ is a Saks space fulfilling conditions (Σ_1) and (Σ_2) and that $Y_s(\omega) = Y_0$ (see [1], [2]).

References

[1] J. Musielak and W. Orlicz, *Linear functionals over the space of functions continuous in an open interval*, Studia Math. 15 (1956), p. 216-224.

[2] W. Orlicz, *Linear operations in Saks spaces (I)*, ibidem 11 (1950), p. 237-272.

[3] — *Linear operations in Saks spaces (II)*, ibidem 15 (1955), p. 1-25.

Reçu par la Rédaction le 3. 2. 1956

On the continuity of linear operations in Saks spaces with an application to the theory of summability

by

W. ORLICZ (Poznań)

1. Let X be a linear space and let a B-norm $\| \|$ (*fundamental norm*) and a B- or F-norm $\| \|^*$ (*starred norm*) be defined in X. If the set

$$X_s = \mathop{E}_{x}\{x \in X,\ \|x\| \leqslant 1\}$$

with the distance defined as $d(x, y) = \|x-y\|^*$ is a complete space, it will be called a *Saks space* (with the norm $\| \|^*$, see [2][1])). The following theorem is a generalization of the result given in [3]:

1.1. *Let* $X_1, X_2, \ldots, X_n, \ldots$ *be linear subspaces of the space* X *and et an* F*-norm* $\| \|_n^*$ *be defined in* X_n *for* $n = 1, 2, \ldots$ *Writing*

$$X_0 = \bigcap_{n=1}^{\infty} X_n,$$

we suppose the following conditions to be satisfied:

(a) $X_1 \supset X_2 \supset \ldots \supset X_n \supset \ldots$;

(b) *there exists a linear subspace* $Y_0 \subset X_0$ *such that the set* $\overline{X}_n = Y_0 \cap X_n \cap \overline{X}_s$ *is dense in* $X_n \cap X_s$, *the distance being induced by* $\| \|_n^*$ *for* $n = 1, 2, \ldots$;

(c) *the set* $X_n \cap X_s$ *is a Saks space under the norm* $\| \|_n^*$, *satisfying the condition* (Σ_1)[2]), *for* $n = 1, 2, \ldots$);

(d) *if* $x_i \in X_0$ *and* $\|x_i\|_k^* \to 0$ *for a fixed* k *and* $i \to \infty$ *then* $\|x_i\|_{k'}^* \to 0$ *for every* $k' < k$.

Further suppose that in X_0 *additive operations* U_n *with values in a Fréchet space* Y *are defined, such that*

(α) *for every* $x \in X_0$ *the sequence* $\{U_n(x)\}$ *is convergent;*

(β) *for every fixed positive integer* n, k, $\|x_i\| \leqslant 1$, $x_i \in X_0$ *and* $\|x_i\|_k^* \to 0$ *for* $i \to \infty$ *imply* $U_n(x_i) \to 0$.

---

[1]) The numbers in square brackets refer to the references at the end of this paper.

[2]) Concerning the definition of the condition (Σ_1) see [2], p. 240.

Under these assumptions $\|x_i\| \leqslant 1, x_i \in X_0$, and $\|x_i\|_k^* \to 0$ for $k = 1, 2, \ldots$ implies $U_i(x_i) \to 0$.

Let us define the starred norm by the formula

$$(\dagger) \qquad \|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x\|_n^*}{1+\|x\|_n^*}, \qquad x \in X_0.$$

Write

$$X_{0s} = \underset{x}{E}\{x \in X_0, \|x\| \leqslant 1\}.$$

Then from (c) it follows that X_{0s} is a Saks space with this norm.

If $x_i \in X_0$, $\|x_i\| \leqslant 1$ for $i = 0, 1, 2, \ldots$, let $\|x_i - x_0\|_k^* \to 0$ as $i \to \infty$ (with fixed k), then in virtue of (β) and the fact that $(x_i - x_0)/2 \in X_{0s}$ it follows that $U_n(x_i) \to U_n(x_0)$. Since, by the condition (α), the sequence $U_n(x)$ converges in X_{0s}, there exists an $x_0 \in X_{0s}$ such that the operations U_n are equicontinuous at x_0. Given an $\varepsilon > 0$ let us choose a positive number ϱ in such a manner that for every pair of elements x', x'' belonging to the sphere

$$K = \underset{x}{E}\{x \in X_0, \|x\| \leqslant 1, \|x - x_0\|^* < \varrho\}$$

the inequalities

$$\|U_n(x') - U_n(x'')\| < \varepsilon \quad \text{for} \quad n = 1, 2, \ldots$$

are satisfied. Further let us choose a positive integer m sufficiently large and a positive number ϱ' sufficiently small to have

$$K_m = \underset{x}{E}\{x \in X_0, \|x\| \leqslant 1, \|x - x_0\|_m^* < \varrho'\} \subset K.$$

This is possible by (d) and (†). By (c), there exists a number $\delta > 0$ such that $\|x\|^* < \delta$, $\|x\| \leqslant 1$ and $x \in X_0 \subset X_m$ imply the possibility of a representation $x = x' - x''$ with

$$x', x'' \in \underset{x}{E}\{x \in X_m, \|x\| \leqslant 1, \|x - x_0\|_m^* < \varrho'\}.$$

According to the condition (b) there exist sequences $x_i', x_i'' \in X_{0s}$ convergent with respect to the norm $\| \|_m^*$ to x', x'' respectively. Since one can suppose, of course, that $x_i', x_i'' \in K_m$, it follows that $\|U_n(x_i') - U_n(x_i'')\| < \varepsilon$ for $i, n = 1, 2, \ldots$ Moreover $[U_n(x_i') - U_n(x_i'')]/2 = U_n[(x_i' - x_i'')/2]$ gives $\|U_n(x)\| < \varepsilon$ for $n = 1, 2, \ldots$

Thus we have proved the equicontinuity of the operations U_n in the space X_{0s} at 0. Hence (see [2], p. 265) and from the definition of the norm $\| \|^*$ follows the statement of our theorem.

1.2. *Let $X_0, X_1, X_2, \ldots, X_n, \ldots$ have the same meaning as in 1.1 and let us suppose that the conditions (a)-(d) of 1.1 are satisfied. Further let X_{0s} denote the same Saks space as in 1.1 with the norm (†).*

Suppose that U is an additive operation in X_0 to a Banach space Y with the following properties:

(a) *the range Y_0 of the operation U is separable;*

(b) *there exists a fundamental set H_0 of linear functionals over Y such that for every $\eta \in H_0$ the functional $\eta(U(x))$ is continuous* ([2], p. 267) *in the sense that $x_i \in X_{0s}$ for $i = 0, 1, 2, \ldots$, and the existence of a positive integer k such that $\|x_i - x_0\|_k^* \to 0$ for $i \to \infty$ implies $\eta(U(x_i)) \to \eta(U(x_0))$.*

Then the operation U is (X_{0s}, Y)-continuous [3].

It is sufficient to prove the continuity of the operation U at 0 ([2], p. 265). Supposing that $x_i \in X_{0s}$, $\|x_i\|^* \to 0$, let us choose a functional $\eta_i \in H_0$ such that $\eta_i(U(x_i)) \geqslant c\|U(x_i)\|$ for $i = 1, 2, \ldots$ Here c denotes a positive constant occurring in the definition of the fundamental set of functionals. The condition (a) implies the existence of a subsequence η_{p_j} of the sequence η_i, convergent in the whole of Y_0. Since the functionals $U_j(x) = \eta_{p_j}(U(x))$ satisfy the assumptions (α), (β) of 1.1, then $\eta_{p_j}(U(x_{p_j})) \to 0$, whence $\|U(x_{p_j})\| \to 0$. Since analogical arguments hold for an arbitrary subsequence of the sequence x_i, it follows that $U(x_i) \to 0$ [4].

1.3. *Let $X_0, X_1, X_2, \ldots, X_n, \ldots$ have the same meaning as in 1.1 and let us suppose that the conditions (a)-(d) of 1.1 are satisfied. Let X_{0s} denote the same Saks space as in 1.1 with the norm (†) and let ξ_n be additive functionals in X_0 satisfying the hypothesis 1.1 (β), where $\xi_n = U_n$ (this implies the continuity of ξ_n in X_{0s}). Suppose that*

(α′) *for every $x \in X_0$, the sequence $\{\xi_n(x)\}$ is bounded;*

(β′) *the sequence $\{\xi_n(x)\}$ is convergent to 0 (is convergent) in a set dense in X_{0s}.*

Then the set of sequences $\{\xi_n(x)\}$, $x \in X_{0s}$, is either non separable in the space T_b or convergent to 0 (convergent for every $x \in X_0$) [5].

Let us suppose that the set of the sequences $\{\xi_n(x)\}$, $x \in X_{0s}$, is contained in the separable, closed, linear subspace $\overline{T}_b \subset T_b$. Define the operation U on X_{0s} to $\overline{T}_b$ by the formula $U(x) = \{\xi_n(x)\}$. Since the set H_0 of linear functionals (over the space T_b) of the form

---

[3] This means: $\|x_i - x_0\|^* \to 0$, $x_i, x_0 \in X_{0s}$, implies $U(x_i) \to U(x_0)$.

[4] The arguments used in this proof are known.

[5] T_0 and T_b denote the space of sequences convergent to 0 and that of bounded sequences respectively, with usual norms.

$$\eta(y) = \sum_{n=1}^{\infty} c_n t_n, \quad \text{where} \quad \sum_{n=1}^{\infty} |c_n| \leqslant 1, \ y = \{t_n\} \epsilon T_b,$$

and almost all c's vanish, is a fundamental set, 1.1 (β) implies that H_0 satisfies the hypothesis 1.2 (b). It follows by 1.2 that the operation U is (X_{0s}, T_b)-continuous. Since, by (β'), $\xi_n(x)$ is convergent to 0 (convergent) in a set dense in X_{0s}, this is also true for every $x \epsilon X_{0s}$.

2. Now we give an application of 1.3 to the theory of linear methods of summability. In the sequel we use the notation and definition introduced in [1].

THEOREM. *Suppose the linear methods of summability $A^1, A^2, \ldots$ to be permanent for null-sequences. Let X_0 denote the set af all bounded sequences summable to 0 by all the methods A^n simultaneously. Let B be an arbitrary method of summability permanent for null-sequences. Then the set of all sequences of transforms $\{B_i(x)\}$, $x \epsilon X_0$, is either non-separable in T_b or convergent to 0 for every $x \epsilon X_0$.*

Put $X = T_b$ and

$$\|x\| = \sup_n |t_n|.$$

Let us denote by X_n for $n = 1, 2, \ldots$ the set of all bounded sequences summable to 0 by the methods $A^1, A^2, \ldots, A^n$, and by C^n the method of summability corresponding to the matrix (c_{in}) arising by the juxtaposition of all rows of the methods $A^1, A^2, \ldots, A^n$. Obviously $X_n = C_0^{n*} \cap T_b$. We define in X_n the norm

$$\|x\|_n^* = \sup_n |C_n(x)| + \sum_{i=1}^{\infty} \frac{1}{2^i} |t_i|.$$

According to the result of [2], the condition 1.1 (c) is satisfied. The fulfilment of the conditions 1.1(a) and (d) follows immediately from the definitions of the method C^n and of the starred norm. From lemma 2.2 ([1]) it follows that the set Y_0 of all bounded sequences with almost all elements equal to 0 satisfies the hypothesis 1.1(b). Further let us observe that

$$X_0 = \bigcap_{n=1}^{\infty} X_n$$

and that the transforms $B_n(x)$ fulfil the hypothesis 1.1 (β) (for $U_n(x) = B_n(x)$). If we write in 1.3 $\xi_n = B_n$ then the classical conditions of permanence for null-sequences imply the fulfilment of the condition 1.3 (α'). Finally, since $B_n(x) \to 0$ for $x \epsilon Y_0$, the condition 1.3 (β') is also satisfied.

Under the same hypothesis about the methods $A^1, A^2, \ldots, A^n, \ldots$ and with the same meaning of X_0 as above we get the following corollary:

A. *Let B be a permanent method such that $B_n(x) \to B(x)$ for $x \epsilon X_0$, then $B(x) = 0$ for every $x \epsilon X_0$* (see [1], theorem 1′).

Our hypothesis ensures that the set of all sequences $\{B_i(x)\}$, $x \epsilon X_0$, is separable.

B. *If there exists a bounded divergent sequence summable to 0 by the methods A^n, then the set of all bounded divergent sequences, A^n-summable to 0, is non separable in space T_b* [6].

For the proof one can take as B the identical method.

Bibliography

[1] S. Mazur and W. Orlicz, *On linear methods of summability*, Studia Mathem. 14 (1955), p. 129-160.

[2] W. Orlicz, *Linear operations in Saks spaces (I)*, ibidem 11 (1950), p. 237-272.

[3] — *Linear operations in Saks spaces (II)*, ibidem 15 (1955), p. 1-25.

[6] See [1], where this theorem is proved in the special case when $A^1 = A^2 = \ldots = A^n = \ldots$.

Reçu par la Rédaction le 10. 2. 1956