Pour établir le théorème 6 il suffit donc de démontrer l'existence d'un point fixe unique de la transformation

$$\tilde{U}(x,y) = L[U], \tag{6.2}$$

considérée dans l'espace E^* des fonctions $U(x,y)$ de classe C^1 dans le rectangle D et dont la norme est définie par (3.11). Supposons à cet effet que ${}^1U(x,y)\epsilon E^*$, ${}^2U(x,y)\epsilon E^*$. Soit

$${}^1\tilde{U}(x,y) = L[{}^1U], \quad {}^2\tilde{U}(x,y) = L[{}^2U].$$

On démontre, de même que dans le cas du théorème 1, que

$$\begin{aligned} \|{}^2\tilde{U}(x,y) - {}^1\tilde{U}(x,y)\| \\ \leqslant [(K_1+K_2)(1+h)+hL(h+2)]\,\|{}^2U(x,y)-{}^1U(x,y)\|. \end{aligned} \tag{6.3}$$

La relation $L[E^*]\subset E^*$ étant évidente, l'existence d'un point fixe de la transformation (6.2) résulte des inégalités (6.1) et (6.3) en vertu du théorème de Banach. Le théorème 6 se trouve ainsi démontré.

Travaux cités

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On the estimation of Cesàro means of orthonormal series

by J. Meder (Szczecin)

1. A sequence of real functions $\varphi_0(x), \varphi_1(x), \varphi_2(x), \ldots$ defined in the interval $\langle 0,1\rangle$ and such that the $\varphi_n^2(x)$ are integrable in $\langle 0,1\rangle$ is called an orthonormal system if

$$\int_0^1 \varphi_i(x)\varphi_k(x)\,dx = \begin{cases} 0 & \text{for} \quad i\neq k, \\ 1 & \text{for} \quad i=k. \end{cases} \tag{1}$$

Instead of *an orthonormal system* we shall write *an ON-system*. If the system of functions is ON, then the series

$$\sum_{n=0}^{\infty} a_n\varphi_n(x) \tag{2}$$

with real coefficients $a_0, a_1, a_2, \ldots$ will be called an orthonormal series. We shall consider only orthonormal series satisfying the condition

$$\sum_{n=0}^{\infty} a_n^2 < \infty. \tag{3}$$

We do not repeat this assumption in the theorems presented here.

In this paper we shall be concerned with the estimation of Cesàro means of orthonormal series

$$\sigma_n^{(r)}(x) = \frac{\sum_{k=0}^{n} A_{n-k}^r a_k\varphi_k(x)}{A_n^r}, \tag{4}$$

where

$$A_0^r = 1, \quad A_n^r = \frac{(r+1)(r+2)\ldots(r+n)}{n!}. \tag{5}$$

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2. At the beginning we give a new proof of the theorem known as the Kaczmarz-Zygmund lemma, which will find application in the course of our further considerations (see [5] and [6], [13] and [14]).

Lemma of Kaczmarz-Zygmund. *If $r > \frac{1}{2}$, then for every orthonormal series* (2) *the series*

$$\sum_{n=1}^{\infty} n[\sigma_n^{(r)}(x)-\sigma_{n-1}^{(r)}(x)]^2$$

is convergent almost everywhere.

Proof. Formulas (4) and (5) imply

$$\sigma_n^{(r)}(x)-\sigma_{n-1}^{(r)}(x) = \frac{\sum_{k=0}^{n} A_{n-k}^{r} a_k\varphi_k(x)}{A_n^r} - \frac{\sum_{k=0}^{n-1} A_{n-k-1}^{r} a_k\varphi_k(x)}{A_{n-1}^r}$$

$$= \frac{1}{A_n^r}\sum_{k=0}^{n-1} \frac{A_{n-k}^r A_{n-1}^r - A_n^r A_{n-k-1}^r}{A_{n-1}^r} a_k\varphi_k(x) + \frac{a_n\varphi_n(x)}{A_n^r},$$

whence

$$\sigma_n^{(r)}(x)-\sigma_{n-1}^{(r)}(x) = \frac{r}{A_n^r}\sum_{k=1}^{n-1}\frac{kA_{n-k-1}^r}{n(n-k)}a_k\varphi_k(x)+\frac{a_n\varphi_n(x)}{A_n^r}. \tag{6}$$

This together with the orthonormality of the system $\{\varphi_n(x)\}$ implies

$$\int_0^1 [\sigma_n^{(r)}(x)-\sigma_{n-1}^{(r)}(x)]^2dx = \frac{r^2}{(A_n^r)^2}\sum_{k=1}^{n-1}\frac{k^2(A_{n-k-1}^r)^2a_n^2}{n^2(n-k)^2}+\frac{a_n^2}{(A_n^r)^2}.$$

Since

$$A_n^r \simeq n^r/\Gamma(r+1) \quad \text{for} \quad n\to\infty$$

(*e. g.* see [9], p. 42 (2)), then there exist positive constants C_r and D_r such that

$$C_r n^r < A_n^r < D_r n^r \quad (n = 1, 2, \ldots).$$

Therefore we have

$$\int_0^1 [\sigma_n^{(r)}(x)-\sigma_{n-1}^{(r)}(x)]^2dx < A_r\left(\sum_{k=1}^{n-1}\frac{k^2a_k^2}{n^{2+2r}(n-k)^{2-2r}}+\frac{a_n^2}{n^{2r}}\right), \tag{7}$$

where $A_r = \max(r^2D_r^2/C_r^2, 1/C_r^2)$, whence

$$\sum_{n=2}^{\infty}\int_0^1 n[\sigma_n^{(r)}(x)-\sigma_{n-1}^{(r)}(x)]^2dx < A_r\left(\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{k^2a_k^2}{n^{1+2r}(n-k)^{2-2r}}+\sum_{n=2}^{\infty}\frac{a_n^2}{n^{2r-1}}\right).$$

We shall prove that both series on the right side of this inequality are convergent for $r > \frac{1}{2}$. We have

$$\sum_{n=2}^{\infty}\frac{a_n^2}{n^{2r-1}} < \sum_{n=1}^{\infty} a_n^2 < \infty.$$

Further, since for $r > \frac{1}{2}$

$$\sum_{n=k+1}^{\infty}\frac{k^2}{n^{1+2r}(n-k)^{2-2r}} = \sum_{n=k+1}^{2k}\frac{k^2}{n^{1+2r}(n-k)^{2-2r}}+\sum_{n=2k+1}^{\infty}\frac{k^2}{n^{1+2r}(n-k)^{2-2r}}$$

$$< k^{1-2r}\sum_{n=1}^{k} n^{2r-2}+k^2\sum_{n=2k+1}^{\infty}\frac{1}{(n-k)^3}$$

$$= k^{1-2r}O(k^{2r-1})+k^2O\left(\frac{1}{k^2}\right) = O(1),$$

we have

$$\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{k^2a_k^2}{n^{1+2r}(n-k)^{2-2r}} = \sum_{k=1}^{\infty}k^2a_k^2\sum_{n=k+1}^{\infty}\frac{1}{n^{1+2r}(n-k)^{2-2r}} < A\sum_{k=1}^{\infty}a_k^2 < \infty,$$

A being a constant. This concludes the proof of convergence. Consequently we have

$$\sum_{n=2}^{\infty}\int_0^1 n[\sigma_n^{(r)}(x)-\sigma_{n-1}^{(r)}(x)]^2dx < B_r\sum_{n=1}^{\infty}a_n^2 < \infty, \tag{8}$$

where $B_r = A_r(A+1)$. By applying Levy's theorem (see [7], p. 8 [124]) it follows from (8) that the series

$$\sum_{n=2}^{\infty} n[\sigma_n^{(r)}(x)-\sigma_{n-1}^{(r)}(x)]^2$$

is convergent almost everywhere.

3. To simplify further formulations we now introduce the following notation. If the sequence

$$|f_n(x)/g_n(x)|$$

is bounded or convergent to zero for $n\to\infty$ almost everywhere in $\langle 0,1\rangle$, then we shall write

$$f_n(x) \doteq O(g_n(x)) \quad \text{or} \quad f_n(x) \doteq o(g_n(x)),$$

respectively. The signs $\doteq$, $\dot{\to}$, etc. mean that the considered equality, convergence, or other relation, is to be assumed almost everywhere in the interval $\langle 0,1\rangle$.

THEOREM 1. *We have for orthonormal series* (2)

(a) $$\sigma_n^{(r)}(x) \doteq o(\sqrt{\log n}) \quad \text{for} \quad r \geqslant \tfrac{1}{2},$$

(b) $$\sigma_n^{(r)}(x) \doteq o(\sqrt{\log^{1+\varepsilon} n}) \quad \text{for} \quad 0 < r < \tfrac{1}{2}$$

for arbitrary $\varepsilon > 0$.

Proof. At first let us consider the case $r > \frac{1}{2}$. Assuming $a_0 = 0$ in (2) we do not restrict the generality of the proof. Then $\sigma_0^{(r)}(x) = 0$ and

$$\sigma_n^{(r)}(x) = \sum_{k=1}^{n} [\sigma_k^{(r)}(x) - \sigma_{k-1}^{(r)}(x)] = \sum_{k=1}^{n} \sqrt{k}[\sigma_k^{(r)}(x) - \sigma_{k-1}^{(r)}(x)] \frac{1}{\sqrt{k}}.$$

Applying the inequality of Buniakowski-Schwarz to the right side of this equality we obtain

$$|\sigma_n^{(r)}(x)| \leqslant \Big(\sum_{k=1}^{n} k[\sigma_k^{(r)}(x) - \sigma_{k-1}^{(r)}(x)]^2\Big)^{1/2} \Big(\sum_{k=1}^{n} \frac{1}{k}\Big)^{1/2}.$$

This inequality implies the existence of a constant C such that

$$\sup_n \frac{|\sigma_n^{(r)}(x)|}{\sqrt{\log(n+3)}} \leqslant C \Big(\sum_{k=1}^{\infty} k[\sigma_k^{(r)}(x) - \sigma_{k-1}^{(r)}(x)]^2\Big)^{1/2}.$$

The lemma of Kaczmarz-Zygmund implies that the series on the right side of this inequality is convergent almost everywhere. Hence

(9) $$\sup_n \frac{|\sigma_n^{(r)}(x)|}{\sqrt{\log(n+3)}} \doteq O(1)$$

for $r > \frac{1}{2}$.

4. We now proceed to prove that for $r > \frac{1}{2}$

(10) $$\sigma_n^{(r)}(x) \doteq o\big(\sqrt{\log(n+3)}\big).$$

The proof is conducted for two distinct cases:

$$1.\ \tfrac{1}{2} < r \leqslant 1 \quad \text{and} \quad 2.\ r > 1.$$

For $\frac{1}{2} < r \leqslant 1$ we apply the following lemma by Bosanquet ([3], p. 484):

If

$$s_n^r = \sum_{\nu=0}^{n} A_{n-\nu}^{r-1} s_\nu,$$

then for each number $0 \leqslant r \leqslant 1$ *and for arbitrary integers* $0 \leqslant m < n$ *the following inequality is satisfied*:

$$\Big|\sum_{\nu=0}^{m} A_{n-\nu}^{r-1} s_\nu\Big| \leqslant \sup_{0 \leqslant \mu \leqslant m} |s_\mu^r|.$$

Applying the method used by G. Alexits (see [1]) and G. Sunouchi (see [9]), we choose a sequence of positive numbers λ_n, increasing to infinity, and such that

(11) $$\sum_{n=1}^{\infty} a_n^2 \lambda_n^2 < \infty.$$

For such a sequence the proofs of existence are provided by the theory of series (*e. g.* see [4], p. 338, 4).

We write

$$\bar{\sigma}_n^{(r)}(x) = \frac{1}{A_n^r} \sum_{k=0}^{n} A_{n-k}^r a_k \lambda_k \varphi_k(x), \qquad \bar{s}_n^{(r)} = \sum_{k=0}^{n} A_{n-k}^r a_k \lambda_k \varphi_k(x).$$

Then

$$\sigma_n^{(r)}(x) = \frac{1}{A_n^r} \sum_{k=0}^{n} A_{n-k}^r a_k \lambda_k \varphi_k(x) \frac{1}{\lambda_k}$$

$$= \frac{1}{\lambda_n A_n^r} \sum_{k=0}^{n} A_{n-k}^r a_k \lambda_k \varphi_k(x) + \frac{1}{A_n^r} \sum_{k=0}^{n-1} \Delta \frac{1}{\lambda_k} \sum_{\nu=0}^{k} A_{n-\nu}^r a_\nu \lambda_\nu \varphi_\nu(x)$$

$$= \frac{1}{\lambda_{n+1}} \bar{\sigma}_n^{(r)}(x) + \frac{1}{A_n^r} \sum_{k=0}^{n} \Delta \frac{1}{\lambda_k} \sum_{\nu=0}^{k} A_{n-\nu}^r a_\nu \lambda_\nu \varphi_\nu(x),$$

and Bosanquet's lemma implies that

$$|\sigma_n^{(r)}(x)| \leqslant \frac{1}{\lambda_{n+1}} |\bar{\sigma}_n^{(r)}(x)| + \frac{1}{A_n^r} \sum_{k=0}^{n} \Delta \frac{1}{\lambda_k} \sup_{0 \leqslant \mu \leqslant k} |\bar{s}_\mu^r(x)|.$$

We multiply this inequality by $1/\sqrt{\log(n+3)}$ and introduce this fraction under the summation sign. Since $\mu \leqslant k$ implies $A_\mu^r \leqslant A_k^r$ and

$$\frac{1}{A_k^r \sqrt{\log(k+3)}} \sup_{0 \leqslant \mu \leqslant k} |\bar{s}_\mu^r(x)| \leqslant \sup_{0 \leqslant \mu \leqslant k} \frac{|\bar{\sigma}_\mu^{(r)}(x)|}{\sqrt{\log(\mu+3)}} \leqslant \sup_n \frac{|\bar{\sigma}_n^{(r)}(x)|}{\sqrt{\log(n+3)}},$$

we have

$$\frac{|\sigma_n^{(r)}(x)|}{\sqrt{\log(n+3)}} \leqslant \sup_n \frac{|\bar{\sigma}_n^{(r)}(x)|}{\sqrt{\log(n+3)}} \Big[\frac{1}{\lambda_{n+1}} + \frac{1}{A_n^r} \sum_{k=0}^{n} A_k^r \Delta \frac{1}{\lambda_k}\Big].$$

According to (11) the orthonormal series $\sum_{n=0}^{\infty} a_n\lambda_n\varphi_n(x)$ satisfies the assumptions of theorem 1. Thus, (9) yields

$$\sup_n \frac{|\bar{\sigma}_n^{(r)}(x)|}{\sqrt{\log(n+3)}} \doteq O(1).$$

Since the sequence $\{\lambda_n\}$ of positive numbers increases to infinity, the series

$$\sum_{k=0}^{\infty} \Delta\frac{1}{\lambda_k}$$

is absolutely convergent. Thence Kronecker's theorem ([8], p. 980) implies

$$\frac{1}{A_n^r}\sum_{k=0}^{n} A_k^r \Delta\frac{1}{\lambda_k} = o(1).$$

Applying the above two relations to the last inequality we obtain

$$\sigma_n^{(r)}(x) \doteq o\left(\sqrt{\log(n+3)}\right) \tag{12}$$

for $\frac{1}{2} < r \leqslant 1$.

Now suppose that $r > 1$. We choose such numbers ϱ and p that $p > \frac{1}{2}$, $\frac{1}{2} < \varrho < 1$ and $r = \varrho + p$ (see [11], p. 43, 3.13). Then

$$\sigma_n^{(r)}(x) = \sigma_n^{(\varrho+p)}(x) = \frac{1}{A_n^{\varrho+p}}\sum_{k=0}^{n} A_{n-k}^{p-1}A_k^{\varrho}\sigma_k^{(\varrho)}(x),$$

whence

$$\frac{|\sigma_n^{(r)}(x)|}{\sqrt{\log(n+3)}} \leqslant \frac{1}{A_n^{\varrho+p}}\sum_{k=0}^{n} A_{n-k}^{p-1}A_k^{\varrho}\frac{\sigma_k^{(\varrho)}(x)}{\sqrt{\log(k+3)}}.$$

It is easy to see that the matrix

$$\left\{\frac{A_{n-k}^{p-1}A_k^{\varrho}}{A_n^{\varrho+p}}\right\}$$

is a Toeplitz matrix (compare [10]). Thus, the last inequality and (12) give for $r > 1$

$$\sigma_n^{(r)}(x) \doteq o\left(\sqrt{\log(n+3)}\right). \tag{13}$$

5. Now let us consider the case $r = \frac{1}{2}$. Assuming $r = \alpha > 1$ in (6), according to the equality

$$A_{n-k}^{\alpha} = \frac{\alpha A_{n-k-1}^{\alpha}}{n-k},$$

we obtain

$$\sigma_n^{(\alpha)}(x) - \sigma_{n-1}^{(\alpha)}(x) = \frac{1}{nA_n^{\alpha}}\sum_{k=1}^{n} kA_{n-k}^{\alpha-1}a_k\varphi_k(x).$$

On the other hand we have

$$\sigma_n^{(\alpha-1)}(x) - \sigma_n^{(\alpha)}(x) = \frac{1}{\alpha A_n^{\alpha}}\sum_{k=1}^{n} kA_{n-k}^{\alpha-1}a_k\varphi_k(x)$$

(see [7], p. 187). The last two equalities yield, on changing the index n into 2^n,

$$\sigma_{2^n}^{(\alpha-1)}(x) = \frac{2^n}{\alpha}\left[\sigma_{2^n}^{(\alpha)}(x) - \sigma_{2^n-1}^{(\alpha)}(x)\right] + \sigma_{2^n}^{(\alpha)}(x). \tag{14}$$

We shall prove that for $\alpha > 1$

$$2^n\left[\sigma_{2^n}^{(\alpha)}(x) - \sigma_{2^n-1}^{(\alpha)}(x)\right] \doteq o(1).$$

Multiplying both sides of inequality (7) by n^2 and taking $r = \alpha > 1$, we may write

$$\int_0^1 n^2\left[\sigma_n^{(\alpha)}(x) - \sigma_{n-1}^{(\alpha)}(x)\right]^2dx < \frac{A_\alpha}{n^{2\alpha}}\sum_{k=1}^{n-1} k^2a_k^2(n-k)^{2\alpha-2} + \frac{A_\alpha a_n^2}{n^{2\alpha-2}}$$

$$< \frac{A_\alpha}{n^2}\sum_{k=1}^{n} k^2a_k^2 + A_\alpha a_n^2,$$

which, on replacing the index n by 2^n and on the summation from 1 to ∞, yields

$$\sum_{n=1}^{\infty}\int_0^1 4^n\left[\sigma_{2^n}^{(\alpha)}(x) - \sigma_{2^n-1}^{(\alpha)}(x)\right]^2dx < A_\alpha\left(\sum_{n=1}^{\infty}\frac{1}{4^n}\sum_{k=1}^{2^n} k^2a_k^2 + \sum_{n=1}^{\infty} a_{2^n}^2\right) < \infty$$

(see [7], p. 161). Thus we have $2^n[\sigma_{2^n}^{(\alpha)}(x) - \sigma_{2^n-1}^{(\alpha)}(x)] \doteq o(1)$. This and (14) imply

$$\sigma_{2^n}^{(\alpha-1)}(x) \doteq o(1) + \sigma_{2^n}^{(\alpha)}(x).$$

Taking in this equality $\alpha = \frac{3}{2}$ and applying (13) with $r = \alpha = \frac{3}{2}$ we may write

$$\sigma_{2^n}^{(1/2)}(x) \doteq o(\sqrt{n}). \tag{15}$$

6. Now we prove that

$$\sigma_n^{(1/2)}(x) \doteq o\left(\sqrt{\log n}\right).$$

We first investigate the convergence of the series

$$\sum_{k=3}^{\infty} k\left[\frac{\sigma_k^{(1/2)}(x)}{\sqrt{\log k}}-\frac{\sigma_{k-1}^{(1/2)}(x)}{\sqrt{\log(k-1)}}\right]^2. \tag{16}$$

The inequality

$$\left[\frac{a_k}{b_k}-\frac{a_{k-1}}{b_{k-1}}\right]^2 \leqslant 2\,\frac{b_{k-1}^2(a_k-a_{k-1})^2+a_{k-1}^2(b_{k-1}-b_k)^2}{b_k^2 b_{k-1}^2}$$

yields

$$\sum_{k=3}^{\infty} k\left[\frac{\sigma_k^{(1/2)}(x)}{\sqrt{\log k}}-\frac{\sigma_{k-1}^{(1/2)}(x)}{\sqrt{\log(k-1)}}\right]^2 \leqslant 2\sum_{k=3}^{\infty}\frac{k[\sigma_k^{(1/2)}(x)-\sigma_{k-1}^{(1/2)}(x)]^2}{\log k}+ \tag{17}$$
$$+2\sum_{k=3}^{\infty}\frac{k(\sqrt{\log(k-1)}-\sqrt{\log k})}{\log k\log(k-1)}[\sigma_{k-1}^{(1/2)}(x)]^2.$$

Assuming $r=\frac{1}{2}$ in the inequality (7), we obtain for $n\geqslant 3$,

$$\int_0^1 \frac{n[\sigma_n^{(1/2)}(x)-\sigma_{n-1}^{(1/2)}(x)]^2}{\log n}\,dx < \frac{A_{1/2}}{n^2\log n}\sum_{k=1}^{n-1}\frac{k^2a_k^2}{n-k}+A_{1/2}a_n^2,$$

whence

$$\sum_{n=3}^{\infty}\int_0^1 \frac{n[\sigma_n^{(1/2)}(x)-\sigma_{n-1}^{(1/2)}(x)]^2}{\log n}\,dx < A_{1/2}\Big(\sum_{n=3}^{\infty}\frac{1}{n^2\log n}\sum_{k=1}^{n-1}\frac{k^2a_k^2}{n-k}+\sum_{n=3}^{\infty}a_n^2\Big)$$
$$< A_{1/2}\Big(\sum_{k=3}^{\infty}k^2a_k^2\sum_{n=k+1}^{\infty}\frac{1}{n^2(n-k)\log n}+\sum_{n=3}^{\infty}a_n^2\Big)$$
$$< A_{1/2}\Big(\sum_{k=3}^{\infty}\frac{k^2a_k^2}{k^2\log k}\sum_{n=k+1}^{2k}\frac{1}{n-k}+\sum_{k=3}^{\infty}k^2a_k^2\sum_{n=2k+1}^{\infty}\frac{1}{n^2(n-k)\log n}+\sum_{n=3}^{\infty}a_n^2\Big)<\infty.$$

According to Levy's theorem ([7], p. 8), the first series on the right side of inequality (17) is convergent almost everywhere. Since

$$\sum_{k=3}^{\infty}\frac{k(\sqrt{\log(k-1)}-\sqrt{\log k})^2}{\log k\log(k-1)}\int_0^1[\sigma_{k-1}^{(1/2)}(x)]^2\,dx$$
$$=\sum_{k=3}^{\infty}\int_0^1 k[\sigma_{k-1}^{(1/2)}(x)]^2\Big(\Delta\frac{1}{\sqrt{\log(k-1)}}\Big)^2dx$$
$$=\sum_{k=3}^{\infty}k\,O\Big(\frac{1}{k^2\log^3k}\Big)\int_0^1[\sigma_{k-1}^{(1/2)}(x)]^2\,dx\leqslant\sum_{k=3}^{\infty}O\Big(\frac{1}{k\log^3k}\Big)\sum_{p=3}^{\infty}a_p^2<\infty,$$

the second series is also convergent almost everywhere. Hence, the series (16) is convergent almost everywhere.

Now we choose such an integer ν that $2^n<\nu<2^{n+1}$. Applying logarithms of base 2 we may write

$$\frac{\sigma_\nu^{(1/2)}(x)}{\sqrt{\log\nu}}-\frac{\sigma_{2^n}^{(1/2)}(x)}{\sqrt{n}}=\sum_{k=2^n+1}^{\nu}\sqrt{k}\left[\frac{\sigma_k^{(1/2)}(x)}{\sqrt{\log k}}-\frac{\sigma_{k-1}^{(1/2)}(x)}{\sqrt{\log(k-1)}}\right]\frac{1}{\sqrt{k}}.$$

Squaring both sides of the last inequality and applying the inequality of Buniakowski-Schwarz, we have for $2^n<\nu<2^{n+1}$

$$\left[\frac{\sigma_\nu^{(1/2)}(x)}{\sqrt{\log\nu}}-\frac{\sigma_{2^n}^{(1/2)}(x)}{\sqrt{n}}\right]^2\leqslant\sum_{k=2^n+1}^{2^{n+1}}k\left[\frac{\sigma_k^{(1/2)}(x)}{\sqrt{\log k}}-\frac{\sigma_{k-1}^{(1/2)}(x)}{\sqrt{\log(k-1)}}\right]^2\sum_{k=2^n+1}^{2^{n+1}}\frac{1}{k}.$$

Since (16) is convergent almost everywhere, the first factor on the right side of the last inequality is independent of ν and converges to zero almost everywhere for $n\to\infty$. It is easy to observe that the second factor is bounded.

Let us denote by $\varepsilon_n(x)$ the expression on the right side of the last inequality. We may then write

$$\lim_{n\to\infty}\varepsilon_n(x)=0 \quad\text{for}\quad x\in E,\quad |E|=1.$$

On the other hand, (15) yields

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\,\sigma_{2^n}^{(1/2)}(x)=0 \quad\text{for}\quad x\in F,\quad |F|=1.$$

Hence for $x\in M=EF$ and for arbitrary $\varepsilon>0$ there exists such a positive integer N_ε that for all indices $n,m>N_\varepsilon$, $2^n<\nu<2^{n+1}$, $2^m<\nu'<2^{m+1}$, the inequalities

$$\left|\frac{1}{\sqrt{n}}\sigma_{2^n}^{(1/2)}(x)-\frac{1}{\sqrt{m}}\sigma_{2^m}^{(1/2)}(x)\right|<\varepsilon,\qquad \left|\frac{1}{\sqrt{\log\nu}}\sigma_\nu^{(1/2)}(x)-\frac{1}{\sqrt{n}}\sigma_{2^n}^{(1/2)}(x)\right|<\varepsilon,$$
$$\left|\frac{1}{\sqrt{\log\nu'}}\sigma_{\nu'}^{(1/2)}(x)-\frac{1}{\sqrt{m}}\sigma_{2^m}^{(1/2)}(x)\right|<\varepsilon$$

hold. Hence for $\nu,\nu'>2^{N_\varepsilon}$ we have

$$\left|\frac{1}{\sqrt{\log\nu}}\sigma_\nu^{(1/2)}(x)-\frac{1}{\sqrt{\log\nu'}}\sigma_{\nu'}^{(1/2)}(x)\right|\leqslant\left|\frac{1}{\sqrt{\log\nu}}\sigma_\nu^{(1/2)}(x)-\frac{1}{\sqrt{n}}\sigma_{2^n}^{(1/2)}(x)\right|+$$
$$+\left|\frac{1}{\sqrt{n}}\sigma_{2^n}^{(1/2)}(x)-\frac{1}{\sqrt{m}}\sigma_{2^m}^{(1/2)}(x)\right|+\left|\frac{1}{\sqrt{m}}\sigma_{2^m}^{(1/2)}(x)-\frac{1}{\sqrt{\log\nu'}}\sigma_{\nu'}^{(1/2)}(x)\right|<3\varepsilon.$$

Thus there exists

$$\lim_{n\to\infty}\frac{1}{\sqrt{\log n}}\,\sigma_n^{(1/2)}(x)$$

almost everywhere and (15) implies

$$\lim_{n\to\infty}\frac{1}{\sqrt{\log n}}\,\sigma_n^{(1/2)}(x)\doteq 0.$$

7. Now we shall prove theorem 1 (b). First we prove that for $0<r<\frac{1}{2}$ and $\varepsilon>0$ the series

$$\sum_{n=3}^{\infty}\frac{\left[\sigma_n^{\left(\frac{-1+r}{2}\right)}(x)-\sigma_n^{\left(\frac{1+r}{2}\right)}(x)\right]^2}{n\log^{1+\varepsilon}n} \tag{18}$$

is convergent almost everywhere.

Taking $\alpha=(1+r)/2$ in the equality

$$\sigma_n^{(\alpha-1)}(x)-\sigma_n^{(\alpha)}(x)=\frac{1}{\alpha A_n^{\alpha}}\sum_{k=1}^{n}kA_{n-k}^{\alpha-1}a_k\varphi_k(x)$$

we obtain

$$\sigma_n^{\left(\frac{-1+r}{2}\right)}(x)-\sigma_n^{\left(\frac{1+r}{2}\right)}(x)=\frac{2}{(1+r)A_n^{\frac{1+r}{2}}}\sum_{k=1}^{n}kA_{n-k}^{\frac{-1+r}{2}}a_k\varphi_k(x).$$

If we square both sides of this equality and then divide it by $n\log^{1+\varepsilon}n$, the orthonormality of the system $\{\varphi_n(x)\}$ enables us to write

$$\sum_{n=3}^{\infty}\int_0^1\frac{\left[\sigma_n^{\left(\frac{-1+r}{2}\right)}(x)-\sigma_n^{\left(\frac{1+r}{2}\right)}(x)\right]^2}{n\log^{1+\varepsilon}n}\,dx$$

$$=\frac{4}{(r+1)^2}\sum_{n=3}^{\infty}\frac{1}{n\left(A_n^{\frac{1+r}{2}}\right)^2\log^{1+\varepsilon}n}\sum_{k=1}^{n}k^2a_k^2\left(A_{n-k}^{\frac{-1+r}{2}}\right)^2.$$

By applying the properties of the symbol A_n^r and the last equality, we obtain

$$\sum_{n=3}^{\infty}\int_0^1\frac{\left[\sigma_n^{\left(\frac{-1+r}{2}\right)}(x)-\sigma_n^{\left(\frac{1+r}{2}\right)}(x)\right]^2}{n\log^{1+\varepsilon}n}\,dx<A_r\sum_{n=3}^{\infty}\frac{1}{n^{2+r}\log^{1+\varepsilon}n}\sum_{k=1}^{n}k^2a_k^2\frac{(n-k+1)^r}{n-k+1}$$

$$<A_r\sum_{n=3}^{\infty}\frac{1}{n^2\log^{1+\varepsilon}n}\sum_{k=1}^{n}\frac{k^2a_k^2}{n-k+1},$$

A_r being a suitable constant. If in the last expression we change the order of summation, we obtain

$$\sum_{n=3}^{\infty}\int_0^1\frac{\left[\sigma_n^{\left(\frac{-1+r}{2}\right)}(x)-\sigma_n^{\left(\frac{1+r}{2}\right)}(x)\right]^2}{n\log^{1+\varepsilon}n}\,dx<A_r\sum_{k=3}^{\infty}k^2a_k^2\sum_{n=k}^{\infty}\frac{1}{n^2(n-k+1)\log^{1+\varepsilon}n}$$

$$<3A_r\sum_{n=3}^{\infty}\frac{1}{n\log^{1+\varepsilon}n}\sum_{k=3}^{\infty}a_k^2<\infty.$$

This implies by Levy's theorem the convergence almost everywhere of the series (18). Applying Kronecker's theorem to this series, we obtain

$$\frac{1}{n\log^{1+\varepsilon}n}\sum_{k=0}^{n}\left[\sigma_k^{\left(\frac{-1+r}{2}\right)}(x)-\sigma_k^{\left(\frac{1+r}{2}\right)}(x)\right]^2\doteq o(1). \tag{19}$$

Analogical considerations show that

$$\frac{1}{n\log^{1+\varepsilon}n}\sum_{k=0}^{n}\left[\sigma_k^{\left(\frac{1+r}{2}\right)}(x)\right]^2\doteq o(1); \tag{20}$$

it may be sufficient to remark that

$$\sum_{n=3}^{\infty}\int_0^1\frac{\left[\sigma_n^{\left(\frac{1+r}{2}\right)}(x)\right]^2}{n\log^{1+\varepsilon}n}\,dx=\sum_{n=3}^{\infty}\frac{1}{n\left(A_n^{\frac{1+r}{2}}\right)^2\log^{1+\varepsilon}n}\sum_{k=1}^{n}\left(A_{n-k}^{\frac{1+r}{2}}\right)^2a_k^2$$

$$<\sum_{n=3}^{\infty}\frac{1}{n\log^{1+\varepsilon}n}\sum_{k=0}^{\infty}a_k^2<\infty.$$

Using the identity

$$\sigma_k^{\left(\frac{-1+r}{2}\right)}(x)=\left[\sigma_k^{\left(\frac{-1+r}{2}\right)}(x)-\sigma_k^{\left(\frac{1+r}{2}\right)}(x)\right]+\sigma_k^{\left(\frac{1+r}{2}\right)}(x),$$

we can write

$$\sum_{k=0}^{n}\left[\sigma_k^{\left(\frac{-1+r}{2}\right)}(x)\right]^2\leqslant 2\sum_{k=0}^{n}\left[\sigma_k^{\left(\frac{-1+r}{2}\right)}(x)-\sigma_k^{\left(\frac{1+r}{2}\right)}(x)\right]^2+2\sum_{k=0}^{n}\left[\sigma_k^{\left(\frac{1+r}{2}\right)}(x)\right]^2.$$

This and formulas (19) and (20) imply

$$\sum_{k=0}^{n}\left[\sigma_k^{\left(\frac{-1+r}{2}\right)}(x)\right]^2\doteq o(n\log^{1+\varepsilon}n). \tag{21}$$

Since

$$s_n^{\alpha+\beta+1}(x) = \sum_{k=0}^{n} A_{n-k}^{\beta} s_k^{\alpha}(x),$$

(see [11], p. 42), we have

$$s_n^r(x) = s_n^{\frac{-1+r}{2}+\frac{-1+r}{2}+1}(x) = \sum_{k=0}^{n} A_{n-k}^{\frac{-1+r}{2}} s_k^{\frac{-1+r}{2}}(x) = \sum_{k=0}^{n} A_{n-k}^{\frac{-1+r}{2}} A_k^{\frac{-1+r}{2}} \sigma_k^{\left(\frac{-1+r}{2}\right)}(x).$$

Applying the inequality of Buniakowski-Schwarz and (21) to the last equality, we obtain

$$|s_n^r(x)| \leqslant \left[\sum_{k=0}^{n} \left(A_{n-k}^{\frac{-1+r}{2}}\right)^2 \left(A_k^{\frac{-1+r}{2}}\right)^2 \sum_{k=0}^{n} \left|\sigma_k^{\left(\frac{-1+r}{2}\right)}(x)\right|^2\right]^{1/2}$$

$$= O\left[\sum_{k=0}^{n} A_{n-k}^{-1+r} A_k^{-1+r} \sum_{k=0}^{n} \left[\sigma_k^{\left(\frac{-1+r}{2}\right)}(x)\right]^2\right]^{1/2} = O\left[A_n^{2r-1} \sum_{k=0}^{n} \left[\sigma_k^{\left(\frac{-1+r}{2}\right)}(x)\right]^2\right]^{1/2}$$

$$\doteq o(\sqrt{n\log^{1+\varepsilon} n})\, O(\sqrt{n^{2r-1}}) \doteq o(n^r \sqrt{\log^{1+\varepsilon} n}).$$

Hence $\sigma_n^{(r)}(x) \doteq o(\sqrt{\log^{1+\varepsilon} n})$ for $0 < r < \frac{1}{2}$, which concludes the proof.

Remark. Theorem 1(b) still holds if in place of $\log^{1+\varepsilon} n$ we write $\log n (\log\log n) \ldots \underbrace{(\log\log\ldots\log n)}_{p} \underbrace{(\log\log\ldots\log n)^{1+\varepsilon}}_{p+1}$, p an integer $\geqslant 2$.

THEOREM 2. *Let us assume that for a certain* $0 < \varepsilon \leqslant 1$

$$\sum_{n=1}^{\infty} a_n^2 \log^{2-\varepsilon} n < \infty \tag{22}$$

and let $f(x)$ *be the function with an integrable square given by the theorem of Riesz-Fischer having the expansion* $\sum_{n=0}^{\infty} a_n \varphi_n(x)$. *Then*

$$\sigma_n(x) \doteq f(x) + o\left(\frac{1}{\sqrt{\log^{1-\varepsilon} n}}\right) \quad \textit{for} \quad n \to \infty,$$

$\sigma_n(x)$ *being the first Cesàro means of the expansion of the function* $f(x)$.

Proof. In this proof we apply logarithms of base 2. First we shall prove that

$$\sqrt{n^{1-\varepsilon}}[\sigma_{2^n}(x) - f(x)] \doteq 0 \quad \text{for} \quad n \to \infty. \tag{23}$$

Taking in the known formula

$$\sigma_n(x) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k \varphi_k(x) \tag{24}$$

the index 2^n in place of n, and indicating by $s_n(x)$ the n-th partial sum of the series (1), we obtain the equality

$$\sigma_{2^n}(x) = s_{2^n}(x) - \frac{1}{2^n+1} \sum_{k=0}^{2^n} k a_k \varphi_k(x). \tag{25}$$

Subtracting $f(x)$ from both sides of this equality, squaring it, integrating from 0 to 1, and using

$$\int_0^1 f^2(x)\,dx = \sum_{n=0}^{\infty} a_n^2 \quad \text{and} \quad \int_0^1 [f(x) - s_n(x)]^2 dx = \sum_{k=n+1}^{\infty} a_k^2 \to 0,$$

we obtain

$$\int_0^1 [\sigma_{2^n}(x) - f(x)]^2 dx = \sum_{k=2^n+1}^{\infty} a_k^2 + \frac{1}{(2^n+1)^2} \sum_{k=1}^{2^n} k^2 a_k^2.$$

Hence

$$\sum_{n=2}^{\infty} n^{1-\varepsilon} \int_0^1 [\sigma_{2^n}(x) - f(x)]^2 dx < \sum_{n=2}^{\infty} n^{1-\varepsilon} \sum_{k=2^n+1}^{\infty} a_k^2 + \sum_{n=2}^{\infty} \frac{n^{1-\varepsilon}}{4^n} \sum_{k=1}^{2^n} k^2 a_k^2 = \Sigma_1 + \Sigma_2,$$

where Σ_1 stands for the former and Σ_2 for the latter of the last two series.

We shall prove that the series Σ_1 and Σ_2 converge. At first we investigate the series

$$\sum_{n=1}^{\infty} r_{2^n}, \quad \text{where} \quad r_{2^n} = \sum_{k=2^n+1}^{\infty} a_k^2 \log^{1-\varepsilon} k.$$

Applying Abel's transformation we may write

$$\sum_{n=1}^{\infty} r_{2^n} = \sum_{n=1}^{\infty} (r_{2^n} - r_{2^{n+1}})\, n + \lim_{n\to\infty} n r_{2^n}.$$

Since

$$n r_{2^n} = n \sum_{k=2^n+1}^{\infty} a_k^2 \log^{1-\varepsilon} k < \sum_{k=2^n+1}^{\infty} a_k^2 \log^{2-\varepsilon} k,$$

(22) implies

$$\lim_{n\to\infty} n r_{2^n} = 0.$$

Thus we have

$$\sum_{n=1}^{\infty} r_{2^n} = \sum_{n=1}^{\infty} (r_{2^n} - r_{2^{n+1}})\, n.$$

On the other hand

$$\sum_{n=1}^{\infty}(r_{2^n}-r_{2^{n+1}})n = \sum_{n=1}^{\infty} n \sum_{k=2^n+1}^{2^{n+1}} a_k^2\log^{1-\varepsilon}k = \sum_{n=1}^{\infty} n \sum_{k=2^n+1}^{2^{n+1}} \frac{a_k^2\log^{2-\varepsilon}k}{\log k}$$

$$< \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{2^{n+1}} a_k^2\log^{2-\varepsilon}k < \sum_{k=2}^{\infty} a_k^2\log^{2-\varepsilon}k < \infty.$$

Thus

$$\sum_{n=1}^{\infty} r_{2^n} < \infty.$$

To prove the convergence of the series Σ_1 it suffices to observe that

$$\Sigma_1 = \sum_{n=2}^{\infty} n^{1-\varepsilon} \sum_{k=2^n+1}^{\infty} a_k^2 = \sum_{n=2}^{\infty} n^{1-\varepsilon} \sum_{k=2^n+1}^{\infty} \frac{a_k^2\log^{1-\varepsilon}k}{\log^{1-\varepsilon}k}$$

$$< \sum_{n=2}^{\infty} \sum_{k=2^n+1}^{\infty} a_k^2\log^{1-\varepsilon}k = \sum_{n=2}^{\infty} r_{2^n} < \infty.$$

Now we prove the convergence of the series Σ_2. Let us observe that

$$s_m = \sum_{n=2}^{m} \frac{n^{1-\varepsilon}}{4^n} \sum_{k=2}^{2^n} k^2 a_k^2 = \sum_{n=2}^{m} \sum_{p=n}^{m} \frac{p^{1-\varepsilon}}{4^p} \sum_{k=2^{n-1}+1}^{2^n} k^2 a_k^2$$

$$< \sum_{n=2}^{m} \frac{1}{n^{\varepsilon}} \sum_{p=n}^{m} \frac{p}{4^p} \sum_{k=2^{n-1}+1}^{2^n} k^2 a_k^2 < \sum_{n=2}^{m} \sum_{p=n}^{m} \frac{p}{4^p} \sum_{k=2^{n-1}+1}^{2^n} k^2 a_k^2 \log^{-\varepsilon}k.$$

Since the sequence $\{p/4^p\}$ decreases, we have

$$s_m < \sum_{n=2}^{m} \sum_{p=n}^{m} \frac{p}{4^p} \sum_{k=2^{n-1}+1}^{2^n} k^2 a_k^2 \log^{-\varepsilon}k < \sum_{n=2}^{m} \int_{n-1}^{\infty} \frac{x\,dx}{4^x} \sum_{k=2^{n-1}+1}^{2^n} k^2 a_k^2 \log^{-\varepsilon}k.$$

Since $n < 3\log(2^{n-1}+1)$ and since for $n \geqslant 2$ we have

$$\int_{n-1}^{\infty} \frac{x\,dx}{4^x} = \frac{n-1}{4^{n-1}\log 4} + \frac{1}{4^{n-1}\log^2 4} < \frac{4n}{4^n},$$

it follows that

$$s_m < 4\sum_{n=2}^{m} \frac{n}{4^n} \sum_{k=2^{n-1}+1}^{2^n} k^2 a_k^2 \log^{-\varepsilon}k < 12 \sum_{n=2}^{m} \sum_{k=2^{n-1}+1}^{2^n} a_k^2 \log^{1-\varepsilon}k$$

$$< 12 \sum_{n=2}^{\infty} \sum_{k=2^{n-1}+1}^{2^n} a_k^2 \log^{1-\varepsilon}k = 12 \sum_{k=3}^{\infty} a_k^2 \log^{1-\varepsilon}k < \infty.$$

Hence, the sequence $\{s_m\}$ is bounded. However, it is increasing, whence there exists

$$\lim_{m\to\infty} s_m = \sum_{n=2}^{\infty} \frac{n^{1-\varepsilon}}{4^n} \sum_{k=2}^{2^n} k^2 a_k^2 < \infty;$$

thus

$$\Sigma_2 = a_1^2 \sum_{n=2}^{\infty} \frac{n^{1-\varepsilon}}{4^n} + \lim_{m\to\infty} s_m < \infty.$$

This proves the convergence almost everywhere of the series

$$\sum_{n=2}^{\infty} n^{1-\varepsilon}[\sigma_{2^n}(x)-f(x)]^2,$$

which implies (23).

8. Now we prove that the series

$$\sum_{k=4}^{\infty} k\left\{\sqrt{\log^{1-\varepsilon}k}\,[\sigma_k(x)-f(x)]-\sqrt{\log^{1-\varepsilon}(k-1)}\,[\sigma_{k-1}(x)-f(x)]\right\}^2 \tag{26}$$

is convergent almost everywhere. To simplify the notation we write

$$\Phi_n(x) = \sqrt{\log^{1-\varepsilon}n}\,[\sigma_n(x)-f(x)] \quad \text{for} \quad n = 2, 3, \ldots$$

This and formula (23) yield after simple transformations

$$\Phi_k(x)-\Phi_{k-1}(x) = \sqrt{\log^{1-\varepsilon}k} \sum_{i=0}^{k}\left(1-\frac{i}{k+1}\right)a_i\varphi_i(x) -$$

$$-\sqrt{\log^{1-\varepsilon}(k-1)} \sum_{i=0}^{k-1}\left(1-\frac{i}{k}\right)a_i\varphi_i(x) + f(x)\,\Delta\left(\sqrt{\log^{1-\varepsilon}(k-1)}\right).$$

Hence

$$\Phi_k(x)-\Phi_{k-1}(x) = -\sum_{i=0}^{k-1} a_i\varphi_i(x)\,\Delta\left(\sqrt{\log^{1-\varepsilon}(k-1)}\right) + \tag{27}$$

$$+\sum_{i=0}^{k-1} i a_i\varphi_i(x)\,\Delta\left(\frac{\sqrt{\log^{1-\varepsilon}(k-1)}}{k}\right) +$$

$$+\frac{\sqrt{\log^{1-\varepsilon}k}}{k+1}\,a_k\varphi_k(x) + f(x)\,\Delta\left(\sqrt{\log^{1-\varepsilon}(k-1)}\right).$$

However, it is easy to see that

$$\Delta\left(\sqrt{\log^{1-\varepsilon}(k-1)}\right) = O\left(\frac{1}{k\sqrt{\log^{1+\varepsilon}k}}\right); \quad \Delta\left(\frac{\sqrt{\log^{1-\varepsilon}(k-1)}}{k}\right) = O\left(\frac{\sqrt{\log^{1-\varepsilon}k}}{k^2}\right).$$

If we square both sides of equality (27) and multiply it by k, the integration from 0 to 1 and summation from $k=4$ to ∞ together with the last equalities yield

$$\sum_{k=4}^{\infty}\int_0^1 k[\Phi_k(x)-\Phi_{k-1}(x)]^2dx = \sum_{k=4}^{\infty} O\left(\frac{1}{k\log^{1+\varepsilon}k}\right)\sum_{i=0}^{k-1}a_i^2 +$$
$$+\sum_{k=4}^{\infty} O\left(\frac{\log^{1-\varepsilon}k}{k^3}\right)\sum_{i=1}^{k-1}i^2a_i^2 + \sum_{k=4}^{\infty} O\left(\frac{\log^{1-\varepsilon}k}{k}\right)a_k^2 +$$
$$+\sum_{k=4}^{\infty} O\left(\frac{1}{k\log^3 k}\right)a_k^2 + \sum_{k=4}^{\infty} O\left(\frac{1}{k\log^{1+\varepsilon}k}\right)\sum_{i=0}^{\infty}a_i^2.$$

This and condition (3) imply

$$\sum_{k=4}^{\infty}\int_0^1 k[\Phi_k(x)-\Phi_{k-1}(x)]^2dx = O(1) + \sum_{k=4}^{\infty} O\left(\frac{\log^{1-\varepsilon}k}{k^3}\right)\sum_{i=4}^{k-1}i^2a_i^2. \tag{28}$$

We now prove that the series on the right side of the last equality is convergent. In the first place we observe that

$$\sum_{k=4}^{m}\frac{\log^{1-\varepsilon}k}{k^3}\sum_{i=4}^{k-1}i^2a_i^2 = \sum_{i=4}^{m}i^2a_i^2\sum_{k=i+1}^{m}\frac{\log^{1-\varepsilon}k}{k^3} < \sum_{i=4}^{m}i^2a_i^2\log^{-\varepsilon}i\int_i^{\infty}\frac{\log x}{x^3}\,dx.$$

However, for $i \geqslant 4$,

$$\int_i^{\infty}\frac{\log x}{x^3}\,dx = \log e\left(\frac{\log i}{2i^2}+\frac{1}{4i^2}\right) < \frac{2\log i}{i^2},$$

whence

$$\sum_{k=4}^{m}\frac{\log^{1-\varepsilon}k}{k^3}\sum_{i=4}^{k}i^2a_i^2 < 2\sum_{i=4}^{m}a_i^2\log^{1-\varepsilon}i.$$

Hence and from (22) follows the convergence of the series under consideration. This, together with formula (28) and the definition of $\Phi_n(x)$ implies the convergence almost everywhere of the series (26).

9. Let ν indicate an integer satisfying inequalities $2^n < \nu < 2^{n+1}$. With the above definition of $\Phi_n(x)$ we may write

$$\Phi_\nu(x)-\Phi_{2^n}(x) = \sum_{k=2^n+1}^{\nu}\sqrt{k}\,[\Phi_k(x)-\Phi_{k-1}(x)]\frac{1}{\sqrt{k}}.$$

Squaring both sides of this equality and applying the inequality of Buniakowski-Schwarz to the right side, we obtain

$$|\Phi_\nu(x)-\Phi_{2^n}(x)|^2 \leqslant \sum_{k=2^n+1}^{\nu}k[\Phi_k(x)-\Phi_{k-1}(x)]^2\sum_{k=2^n+1}^{\nu}\frac{1}{k}.$$

Since $2^n < \nu < 2^{n+1}$, we have

$$|\Phi_\nu(x)-\Phi_{2^n}(x)|^2 < \sum_{k=2^n+1}^{2^{n+1}}k[\Phi_k(x)-\Phi_{k-1}(x)]^2\sum_{k=2^n+1}^{2^{n+1}}\frac{1}{k}.$$

From the definition of the function $\Phi_k(x)$ it follows that the series

$$\sum_{k=3}^{\infty}k[\Phi_k(x)-\Phi_{k-1}(x)]^2$$

is identical with the series (25), and thus is convergent almost everywhere. Hence

$$\sum_{k=2^n+1}^{2^{n+1}}k[\Phi_k(x)-\Phi_{k-1}(x)]^2 \doteq o(1) \quad \text{for} \quad n\to\infty.$$

Since

$$\sum_{k=2^n+1}^{2^{n+1}}(1/k) = O(1) \quad \text{for} \quad n\to\infty,$$

we have

$$|\Phi_\nu(x)-\Phi_{2^n}(x)| < \varepsilon_n(x), \tag{29}$$

where $\varepsilon_n(x)\to 0$ for $n\to\infty$.

Considerations analogical to those contained in the proof of theorem 1(a) allow us to conclude from (29) that

$$\lim_{n\to\infty}\Phi_n(x) \doteq 0.$$

This and the definition of the function $\Phi_n(x)$ imply

$$\sigma_n(x) \doteq f(x)+o\left(\frac{1}{\sqrt{\log^{1-\varepsilon}n}}\right) \quad \text{for} \quad n\to\infty,$$

which concludes the proof.

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A note on some properties of the functions $\varphi(n)$, $\sigma(n)$ and $\theta(n)$

by A. Schinzel (Warszawa) and Y. Wang (Peking)

§ 1. Introduction. A. Schinzel has proved in [4] that for every sequence a of h positive numbers $a_1, a_2, \ldots, a_h$ and $\varepsilon > 0$ there exist natural numbers n and n' such that

$$\left|\frac{\varphi(n+i)}{\varphi(n+i-1)} - a_i\right| < \varepsilon, \qquad \left|\frac{\sigma(n'+i)}{\sigma(n'+i-1)} - a_i\right| < \varepsilon \qquad (i = 1, 2, \ldots, h)\,(^1).$$

Professor Hua Loo-Keng has pointed out that by Brun's method we can prove the existence of positive constants $c = c(a, \varepsilon)$ and $X_0 = X_0(a, \varepsilon)$ such that the number of numbers n satisfying the first of these inequalities in the interval $1 \leqslant n \leqslant X$ is greater than

$$cX/\log^{h+1}X \quad \text{for} \quad X > X_0.$$

In the present paper we give the proof of this theorem, of an analogous theorem on the function $\sigma(n)$ and of a theorem on the function $\theta(n)$(2) which is weaker but gives a positive solution of the problem put forward in paper [2] of A. Schinzel and comprises the theorem from paper [3] of A. Schinzel.

The question whether a theorem analogous to the theorems on functions φ and σ is true for the function θ remains open.

§ 2. An auxiliary theorem. Let

$$A_0 = h!\,q_1 \ldots q_s q_{01} \ldots q_{0t_0}, \qquad A_i = q_{i1} \ldots q_{it_i} \quad (1 \leqslant i \leqslant h)$$

be positive integers, where $q_1, q_2, \ldots, q_s$ are all the prime numbers in the interval $0 < x \leqslant 10(h+1)$ and q_{ij} $(0 \leqslant i \leqslant h,\ 1 \leqslant j \leqslant t_i)$ are primes greater than $10(h+1)$ such that $A_0, A_1, \ldots, A_h > 1$ are relatively prime in pairs.

(1) $\varphi(n)$ denotes the Euler function, $\sigma(n)$ — the sum of divisors of number n.

(2) $\theta(n)$ denotes the number of divisors of n.