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Differential inequalities in linear spaces

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In this paper we discuss several forms of the generalized mean value theorem (see [1], [5] and [7]). In the second part of this work it is shown how the classical theorems concerning differential inequalities may be generalized by means of the notion of the cone.

§ 1. I. Suppose E to be a real topological vector space. Its elements are denoted by $x, y, z, \ldots$ The functions of the real variable t, with values lying in E are denoted by $x(t), y(t), z(t)$. By $\bar{A}, A^0$ where $A \subset E$ we mean the closure and the interior of A respectively. FA denotes the boundary of A. The class of all linear (additive and homogeneous) and continuous functionals defined on E is designated by $\bar{E}$. We consider closed hyperplanes, *i. e.*, the sets given by the equations $\xi x = a$ where $\xi \in \bar{E}$, a real — in symbols $H(\xi, a)$. A closed (open) right half-space is defined as follows:

$$H^+(\xi, a) = \mathop{E}_{x}\{\xi x \geqslant a\} \qquad \big(H_+(\xi, a) = \mathop{E}_{x}\{\xi x > a\}\big).$$

A convex body is a closed convex subset of E, possessing interior points. We have the following lemmas:

Lemma 1 (1). *For every point x of the boundary of the convex body A there exists such a $\xi \in \bar{E}$ that $A \subset H^+(\xi, \xi x)$.*

Lemma 2 (1). *A convex body A is the common part of all closed right half-spaces of the form $H^+(\xi, \xi x)$ where $x \in FA$ and $A \subset H^+(\xi, \xi x)$.*

We now prove

Lemma 3. *The interior of the convex body A is the common part of all open right half-spaces of the form $H_+(\xi, \xi x)$ where $x \in FA$ and $A \subset H^+(\xi, \xi x)$.*

Proof. Suppose that for every $\xi \in \bar{E}$ and every $x \in FA$ such that $A \subset H^+(\xi, \xi x)$ we have the relation $x_0 \in H_+(\xi, \xi x)$. From lemma 2 we get $x_0 \in A$. If $x_0 \notin A^0$, then by lemma 1 there is such a $\xi_0 \in \bar{E}$ that $A \subset H^+(\xi_0, \xi_0 x_0)$ and $x_0 \in H(\xi_0, \xi_0 x_0)$ — this contradicts the fact that x_0 is supposed to belong in particular to $H_+(\xi_0, \xi_0 x_0)$.

(1) These lemmas are essentially due to S. Mazur — see [3]. See also [2], p. 72, prop. 3.

II. The function $x(t)$ defined in the interval Δ is called weakly continuous if for every $\xi \in \bar{E}$ the real valued function $\xi x(t)$ is continuous in Δ.

Applying lemma 2 and lemma 3 in the same way as it has been done in [5] we may prove the following theorems:

THEOREM 1. *Suppose A to be a convex body lying in a linear topological space E. We assume that $x(t)$ is weakly continuous in Δ. Suppose that for every $\xi \in \bar{E}$ there exists an at most denumerable set $Z(\xi) \subset \Delta$ such that for every $t \in \Delta - Z(\xi)$ there are a sequence of reals $\tau_n \to 0+$ and a sequence $y_n \in A$ such that*

$$\lim_{n\to\infty} \xi \left\{ \frac{x(t+\tau_n)-x(t)}{\tau_n} - y_n \right\} = 0.$$

Under the assumptions given above we have

$$\frac{x(t_1)-x(t_2)}{t_1-t_2} \in A \quad \text{for} \quad t_1 \neq t_2,\ t_1, t_2 \in \Delta.$$

THEOREM 2. *Suppose that V is an open and convex subset of E. We assume that $x(t)$ is weakly continuous in the interval Δ. Suppose that for every $\xi \in \bar{E}$ there exists an at most denumerable set $Z(\xi) \subset \Delta$ such that for every $t \in \Delta - Z(\xi)$ there are a sequence of reals $\tau_n \to 0+$ and an element $z \in V$ such that*

$$\lim_{n\to\infty} \xi \left\{ \frac{x(t+\tau_n)-x(t)}{\tau_n} \right\} = \xi z.$$

Under our assumptions we have

$$\frac{x(t_1)-x(t_2)}{t_1-t_2} \in V \quad \text{for} \quad t_1 \neq t_2,\ t_1, t_2 \in \Delta.$$

Similar theorems may be proved for functions weakly ACG (see [5], th. 2).

§ 2. I. Suppose E to be a real topological vector space. We introduce the following definition:

DEFINITION. A set S is a *cone in E* if it is closed, non-void and the following conditions hold:

(1) if $x \in S$, $y \in S$, then $x+y \in S$,

(2) if $\lambda \geqslant 0$ and $x \in S$, then $\lambda x \in S$.

Given a cone S one can introduce the relation of „inequality" by means of the formula

$$x \leqslant y \equiv y - x \in S.$$

This relation fulfils the following conditions:

(3) for every $x \in E$ we have $x \leqslant x$,

(4) if $x \leqslant y$, $y \leqslant z$, then $x \leqslant z$,

(5) if $x \leqslant y$ and z is arbitrary, then $x+z \leqslant y+z$,

(6) if $\lambda \geqslant 0$ and $x \leqslant y$, then $\lambda x \leqslant \lambda y$.

Conversely, if the relation „$\leqslant$" satisfies (3)-(6), then the set

$$S = \mathop{E}_{x} \{\Theta \leqslant x\},$$

if it is closed, is a cone.

In the following, in order to simplify our considerations we assume that in E the second axiom of Hausdorff is satisfied, *i. e.*, that for every $x \neq y$ there exist disjoint neighbourhoods of x and y. This enables us to introduce the definition (of Cauchy's type) of the limit $\lim_{t\to t_0} x(t)$ for functions of the real variable t with values in E.

We define $D^+x(t)$ by the formula

$$D^+x(t) = \lim_{h\to 0+} \frac{x(t+h)-x(t)}{h}.$$

The interior S^0 of the cone being non-void, one can introduce the relation

$$x < y \equiv y - x \in S^0.$$

The relation „$<$" possesses the following properties:

(7) if $x \leqslant y$ and $y < z$, then $x < z$,

(8) if $\lambda > 0$ and $x < y$, then $\lambda x < \lambda y$,

(9) if $\lim_{t\to t_0+} x(t) > x_0$, then for $t > t_0$ and t sufficiently near t_0 we have the relation $x(t) > x_0$.

Let us formulate some theorems implied by theorem 1 and theorem 2 (theorem 3 is true in locally convex topological spaces without the assumption $S^0 \neq 0$).

THEOREM 3. *Suppose that $x(t)$ is continuous in the interval Δ. Let S be a cone with a non-void interior. If the relation $\Theta \leqslant D^+x(t)$ holds in Δ except in an at most denumerable subset of Δ, then for $t_1 < t_2$ $(t_1, t_2 \in \Delta)$ the inequality $x(t_1) \leqslant x(t_2)$ holds.*

THEOREM 4. *Suppose that $x(t)$ is continuous in the interval Δ. Let S be a cone with a non-void interior. If the relation $\Theta < D^+x(t)$ holds in Δ except in an at most denumerable subset of Δ, then for $t_1 < t_2$ $(t_1, t_2 \in \Delta)$ we have the inequality $x(t_1) < x(t_2)$.*

Let us consider the function $f(t, y)$ defined on the Cartesian product $\Delta\times E$, Δ being an interval. Suppose that E fulfils the second axiom of separation of Hausdorff and S is a cone with a non-void interior. The function $f(t, y)$ is increasing in y if the inequality $y_1 \leqslant y_2$ implies $f(t, y_1) \leqslant f(t, y_2)$. Now we formulate the theorem about „strong" differential inequalities. We use the notation introduced previously.

THEOREM 5. *We assume that $f(t, y)$ defined in $\Delta\times E$, where $\Delta = \langle t_0, t_0+a)$, increases in y. Let $x(t)$ and $y(t)$ be continuous in Δ. Suppose that the following conditions are satisfied:*

$$x(t_0) \leqslant y(t_0), \tag{10}$$

$$D^+x(t) < f\big(t, x(t)\big) \quad \textit{for} \quad t \in \Delta, \tag{11}$$

$$f\big(t, y(t)\big) \leqslant D^+y(t) \quad \textit{for} \quad t\in\Delta. \tag{12}$$

Under our assumptions the inequality

$$x(t) < y(t) \tag{13}$$

holds for $t_0 < t < t_0+a$.

Proof([2]). According to (10), (11) and (12), since $f(t, y)$ increases in y we get $D^+x(t_0) < D^+y(t_0)$. Therefore for some $\delta > 0$ we have $x(t) < y(t)$ for $t_0 < t < t_0+\delta$. Suppose that the set

$$Z = \mathop{E}_{t}\{t\in(t_0, t_0+a),\ y(t)x-(t)\notin S^0\}$$

is non-void. Write $\tau = \inf Z$. We have $\tau \geqslant t_0+\delta$. Functions $x(t), y(t)$ are continuous and S is closed — therefore $x(\tau) \leqslant y(\tau)$. The inequality $x(\tau) \leqslant y(\tau)$ implies $D^+x(\tau) < D^+y(\tau)$. Hence for some $\eta > 0$ and for $t\in(\tau, \tau+\eta)$ we have the inequality $x(t) < y(t)$. We now see that $x(t) \leqslant y(t)$ for $t\in\langle t_0, \tau+\eta)$. The function $f(t, y)$ increases in y — according to the last inequality and to (11) and (12) we get

$$D^+x(t) < f\big(t, x(t)\big) \leqslant f\big(t, y(t)\big) \leqslant D^+y(t), \qquad t\in(t_0, \tau+\eta).$$

Applying theorem 4 to the function $z(t) = y(t)-x(t)$ we get $x(t) < y(t)$ for $t_0 < t < \tau+\eta$. One can infer therefore the inclusion $Z \subset \langle \tau+\eta, t_0+a)$ whence $\tau = \inf Z \geqslant \tau+\eta$, which is a contradiction.

II. Let us assume that E is a Banach space. We introduce the following assumption:

Assumption H. The function $f(t, x)$ defined for $t_0 \leqslant t \leqslant t_0+a$, $\|x-x_0\| \leqslant r$ is continuous and takes on values from a compact set $V\subset E$; moreover if $W = \operatorname{conv}\big(V\cup\{\Theta\}\big)$, then $a\delta(W) < r$ ([3]). We have the following existence theorem ([4])

([2]) In that proof we apply (7), (8) and (9).

THEOREM 6. *Suppose that the function $f(t, x)$ satisfies the assumption* H. *Then in the interval $t_0 \leqslant t \leqslant t_0+a$ there exists at least one solution of the differential equation $y' = f(t, y)$ satisfying the initial condition $y(t_0) = x_0$.*

Proof. The set W is compact (see [4]). Let C_E denote the space of continuous functions $\eta = x(t)$ defined for $t_0 \leqslant t \leqslant t_0+a$ with values from E, the norm being defined as usual: The subset R of C_E composed of those functions η for which $\|x(t)-x_0\| \leqslant r$ is convex and closed in C_E. Let us consider in R the following operation F:

$$\eta \to F(\eta) = x_0 + \int_{t_0}^{t} f\big(\tau, x(\tau)\big)\,d\tau.$$

Denoting by $y(t)$ the element $F(\eta)$ we have

$$y(t)\in x_0+(t-t_0)\mathop{\operatorname{conv}}_{t_0\leqslant\tau\leqslant t} f\big(\tau, x(\tau)\big) \subset x_0+(t-t_0)\operatorname{conv}W \subset x_0+a\operatorname{conv}W \subset x_0+aW$$

(the last inclusion follows by $\Theta\in W$). Hence $\|y(t)-x_0\| \leqslant a\delta(W) < r$. Thus the operation F maps the set R in a subset T of R; moreover $T \subset x_0+aW$, whence T is compact. Applying Schauder's fixed point theorem [6] we deduce that there exists an element $\eta\in R$ such that $\eta = F(\eta)$.

Now we assume that $S^0 \neq 0$, S is a cone, the assumption H being fulfilled. Suppose that function $x(t)$, continuous in $\langle t_0, t_0+\gamma)$ where $\gamma = r/\delta(W)$, satisfies the inequalities

$$x(t_0) \leqslant x_0, \quad D^+x(t) \leqslant f\big(t, x(t)\big), \quad t\in\langle t_0, t_0+\delta) \tag{α}$$

(we assume $\|x(t)-x_0\| \leqslant r$ for $t\in\langle t_0, t_0+a)$). Let us consider the equations

$$y' = f(t, y) + \frac{1}{n}y_0 \tag{β}$$

where $\Theta < y_0$ and $\|y_0\| = 1$. We form a sequence of solutions of (β) $x_n(t)$ such that $x_n(t_0) = x_0$; $x_n(t)$ is defined in $\langle t_0, t_0+a_n\rangle$ where

$$a_n = \frac{r}{1/n+\delta(W)}.$$

([3]) $\operatorname{conv} A$ denotes the smallest convex set containing the set A: $\delta(W)$ stands for the diameter of the set W.

([4]) This theorem has been communicated to me by A. Alexiewicz to replace a less general theorem of my own. I am indebted to Prof. Alexiewicz for his permission to publish this theorem here.

We have

$$(\gamma) \qquad f\big(t, x_n(t)\big) < x_n'(t).$$

Suppose that $f(t, x)$ increases in x. By theorem 5 and (γ), according to (α), we obtain

$$(\delta) \qquad x(t) < x_n(t), \qquad t_0 < t < t_0 + a_n.$$

On the other hand, we have the equality

$$(\varepsilon) \qquad x_n(t) = x_0 + \int_{t_0}^{t} f\big(\tau, x_n(\tau)\big)\,d\tau + \frac{1}{n}\,y_0(t-t_0).$$

But $x(t)$ are equicontinuous and their diagrams are in a compact set. From Arzela's theorem we conclude that there exists a partial sequence $x_{a_n}(t)$ converging in $\langle t_0, t_0+\gamma)$ almost uniformly to a certain function $y(t)$. From (ε) we get $y'(t) = f\big(t, y(t)\big)$ in $\langle t_0, t_0+\gamma)$. Because of (δ) we have $x(t) \leqslant y(t)$.

We introduce the following definition:

DEFINITION. The solution $y(t)$ of the equation $y' = f(t, y)$ such that $y(t_0) = x_0$ valid in $\langle t_0, t_0+a)$ is called the *right maximal integral* of this equation for the interval $\langle t_0, t_0+a)$ if for every solution $x(t)$ of the equation $y' = f(t, y)$ passing through (t_0, x_0), valid in $\langle t_0, t_0+a)$, we have the inequality $x(t) \leqslant y(t)$.

Let us assume that the following condition holds:

(14) If $x \leqslant y$ and $y \leqslant x$, then $x = y$.

If (14) is satisfied, then S is called a *proper cone*.

According to our previous discussion we formulate the following theorems:

THEOREM 7. *Suppose that $f(t,y)$ increases in y and satisfies the assumption* H. *Let S be a proper cone with a non-void interior. Then there exists a unique right maximal integral of the equation $y' = f(t, y)$ for the interval $\langle t_0, t_0+\gamma)$ $(\gamma = r/\delta(W))$ passing through the point (t_0, x_0).*

THEOREM 8. *Suppose that the assumptions of theorem 7 are fulfilled. Let the function $x(t)$ be continuous in $\langle t_0, t_0+\gamma)$ and satisfy the following conditions:*

$$\|x(t) - x_0\| \leqslant r, \qquad x(t_0) \leqslant x_0, \qquad D^+x(t) \leqslant f\big(t, x(t)\big).$$

Then for $t \in \langle t_0, t_0+\gamma)$ we have the inequality $x(t) \leqslant y(t)$ where $y(t)$ is the right maximal integral of the equation $y' = f(t, y)$ for the interval $\langle t_0, t_0+\gamma)$, passing through the point (t_0, x_0).

Remark 1. In a similar way one can introduce the notion of the right minimal integral and formulate theorems analogous to theorems 7 and 8.

Remark 2. If $f(t, y)$ increasing in y satisfies the Lipschitz condition, a theorem analogous to theorem 8 may be proved by the method of succesive aproximations. In that case the H assumption and the assumption $S^0 \neq 0$ are superfluous.

Remark 3. If $y' = f(t, y)$ posseses a unique solution passing through the point (t_0, x_0), the H assumption is satisfied and $f(t, y)$ increases in y, then a theorem analogous to theorem 8 holds, if we assume in addition that the following condition is satisfied: if $x_n \to y_0$, $z_n \to y_0$ and $x_n \leqslant y_n \leqslant z_n$, then $y_n \to y_0$. In that case the assumption $S^0 \neq 0$ is superfluous.

Remark 4. The theorems presented in this paper may be generalized to the case of derivatives of the form

$$D_\varphi^+ x(t) = \lim_{h\to 0+} \frac{x(t+h) - x(t)}{\varphi(t+h) - \varphi(t)}$$

$\varphi(t)$ being a real valued, stricly increasing function. Left-sided derivatives may be considered.

References

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