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C O M M U N I C A T I O N S

ON THE ISOMORPHISM OF LINDENBAUM ALGEBRAS WITH FIELDS OF SETS

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Let $\mathcal{S}$ be a system of the classical functional calculus of the first order, containing the following primitive symbols: an infinite set $\mathcal{V}$ of individual variables, a set $\mathcal{C}$ of individual constants, a set $\mathcal{F}$ of functors (*i. e.*, symbols for functions from individuals to individuals), a non-void set $\mathcal{R}$ of predicates (*i. e.*, symbols for relations), the signs ( ), $+$, $\cdot$, $\rightarrow$, $-$ and quantifiers $\prod_x$, $\sum_x$ where $x \in \mathcal{V}$. The set $\mathcal{C}$ or $\mathcal{F}$ may be empty.

Among the expressions constructed from these signs we distinguish, in the familiar way, terms and well formed formulae. Let $\mathcal{T}$ and $\mathcal{W}$ be, respectively, the set of all terms and the set of all well formed formulae of $\mathcal{S}$.

For every set $\mathcal{A} \subset \mathcal{W}$ let $\mathcal{S}(\mathcal{A})$ denote the elementary theory based on the system $\mathcal{S}$ of the functional calculus and on the set $\mathcal{A}$ of axioms. In particular, $\mathcal{S}(O)$ is identical with the system $\mathcal{S}$ of the functional calculus.

We shall always assume that the well formed formulae belonging to $\mathcal{A}$ are closed. Clearly, this hypothesis puts no restrictions on our investigations.

Let $\mathrm{Cn}(\mathcal{A})$ be the set of all theorems of the theory $\mathcal{S}(\mathcal{A})$. A well formed formula α is called *refutable* in $\mathcal{S}(\mathcal{A})$ when $-\alpha \in \mathrm{Cn}(\mathcal{A})$.

Given a consistent theory $\mathcal{S}(\mathcal{A})$, let $\mathfrak{L}(\mathcal{A})$ be the *Lindenbaum algebra* of this theory. More precisely, if we treat $\mathcal{W}$ as an abstract algebra with operations $+$, $\cdot$, $\rightarrow$, $-$, the Lindenbaum algebra $\mathfrak{L}(\mathcal{A})$ is the factor algebra $\mathcal{W}/\sim_{\mathcal{A}}$, where $\sim_{\mathcal{A}}$ is the congruence relation defined on $\mathcal{W}$ as follows[1]):

$\alpha \sim_{\mathcal{A}} \beta$ if and only if $\alpha \rightarrow \beta \in \mathrm{Cn}(\mathcal{A})$ and $\beta \rightarrow \alpha \in \mathrm{Cn}(\mathcal{A})$ for $\alpha, \beta \in \mathcal{W}$.

If $\alpha \in \mathcal{W}$, then $|\alpha|_{\mathcal{A}}$ denotes the element of $\mathfrak{L}(\mathcal{A})$ determined by α.

It is known that

[1]) The concept of constructing algebraic structures with formulae of deductive systems is due to Lindenbaum. It was never published by its author and came to be known thanks to Tarski. Lindenbaum's method was first published by McKinsey. For the description of Lindenbaum algebras see *e. g.* [11].

(1) $\mathfrak{L}(\mathcal{A})$ is a Boolean algebra with the unit element $e = |\alpha|_{\mathcal{A}}$ where $\alpha \in \mathrm{Cn}(\mathcal{A})$.

Hence for each $\beta \in \mathcal{W}$, β is refutable if and only if $|\beta|_{\mathcal{A}} = 0$.

(2) $|\alpha|_{\mathcal{A}} \subset |\beta|_{\mathcal{A}}$ if and only if $\alpha \to \beta \in \mathrm{Cn}(\mathcal{A})$ [2].

(3) $$\sum_{\tau \in \mathcal{T}} \left|\alpha\binom{\tau}{x}\right|_{\mathcal{A}} = \left|\sum_x \alpha\right|_{\mathcal{A}}, \qquad \prod_{\tau \in \mathcal{T}} \left|\alpha\binom{\tau}{x}\right|_{\mathcal{A}} = \left|\prod_x \alpha\right|_{\mathcal{A}} \text{ [3]},$$

where $x \in \mathcal{V}$, $\alpha\binom{\tau}{x}$ is a well formed formula which results from $\alpha \in \mathcal{W}$ by the substitution of the term τ for x (assuming that the necessary changes of the bound variables of α were performed before the operation of substitution) and $\sum_{\tau \in \mathcal{T}}$ as well $\prod_{\tau \in \mathcal{T}}$ on the left side of these equalities are the symbols of the infinite Boolean sums and products in $\mathfrak{L}(\mathcal{A})$.

The sums and products (3) are said bo the *sums* or *products corresponding to the logical quantifiers*, or simply the *l-sums* or *l-products*.

A Boolean isomorphism h of $\mathfrak{L}(\mathcal{A})$ onto a field of sets is said to be an *l-isomorphism* provided that it preserves all l-sums and l-products, *i. e.*, if

(4) $$h\left(\left|\sum_x \alpha\right|_{\mathcal{A}}\right) = \bigcup_{\tau \in \mathcal{T}} h\left(\left|\alpha\binom{\tau}{x}\right|_{\mathcal{A}}\right)$$

and

(5) $$h\left(\left|\prod_x \alpha\right|_{\mathcal{A}}\right) = \bigcap_{\tau \in \mathcal{T}} h\left(\left|\alpha\binom{\tau}{x}\right|_{\mathcal{A}}\right)$$

for $\alpha \in \mathcal{W}$, $x \in \mathcal{V}$, where $\bigcup_{\tau \in \mathcal{T}}$ and $\bigcap_{\tau \in \mathcal{T}}$ denotes set-theoretical unions and intersections.

A Lindenbaum algebra $\mathfrak{L}(\mathcal{A})$ is said to be *representable by a field of sets* if there exists an l-isomorphism of $\mathfrak{L}(\mathcal{A})$ onto a field of sets.

We shall discuss the question under what conditions $\mathfrak{L}(\mathcal{A})$ is representable by a field of sets.

Let $\langle \mathfrak{A}, +, \cdot, - \rangle$ be a Boolean algebra. A non-void subset $\mathfrak{f} \subset \mathfrak{A}$ is said to be a *filter* of this algebra when $\mathfrak{f} \neq \mathfrak{A}$ and following conditions are satisfied: 1° if $a, b \in \mathfrak{f}$, then $a \cdot b \in \mathfrak{f}$; 2° if $a \in \mathfrak{f}$ and $b \in \mathfrak{A}$, then $a+b \in \mathfrak{f}$.

A filter $\mathfrak{f}$ is called *prime* provided that if $a+b \in \mathfrak{f}$, then either $a \in \mathfrak{f}$ or $b \in \mathfrak{f}$.

[2]) The sign $\subset$ denotes here the Boolean inclusion between the elements of the Boolean algebra $\mathfrak{L}(\mathcal{A})$.

[3]) For the proof of (3) see *e. g.* [11], p. 70.

A prime filter $\mathfrak{f}$ of $\mathfrak{L}(\mathcal{A})$ is said to be *l-filter* when the condition $\left|\sum_x \alpha\right|_{\mathcal{A}} \in \mathfrak{f}$ implies that there exists such $\tau \in \mathcal{T}$ that $\left|\alpha\binom{\tau}{x}\right| \in \mathfrak{f}$.

By an easy analysis of the proof of Stone's [18] representation theorem for Boolean algebra we obtain

(i) *$\mathfrak{L}(\mathcal{A})$ is representable by a field of sets if and only if for every well formed formula $\beta \in \mathcal{W}$ non refutable in $\mathcal{S}(\mathcal{A})$ there is an l-filter $\mathfrak{f}$ such that $|\beta|_{\mathcal{A}} \in \mathfrak{f}$.*

In fact, if h is an l-isomorphism of $\mathfrak{L}(\mathcal{A})$ onto a field $\mathfrak{F}$ of subsets of a set X and $|\beta|_{\mathcal{A}} \neq 0$, then $h(|\beta|_{\mathcal{A}})$ is a non-void subset of X. Let $x_0 \in h(|\beta|_{\mathcal{A}})$ and let $\mathfrak{f}_0$ be the class of all sets $A \in \mathfrak{F}$ such that $x_0 \in A$. Since h is an l-isomorphism of $\mathfrak{L}(\mathcal{A})$ onto $\mathfrak{F}$, we infer that the set $\mathfrak{f} = h^{-1}(\mathfrak{f}_0)$ is the required l-filter of $\mathfrak{L}(\mathcal{A})$ containing $|\beta|_{\mathcal{A}}$.

Conversely, suppose that the condition mentioned in (i) is satisfied.

For every $\alpha \in \mathcal{W}$, let $\|\alpha\|_{\mathcal{A}}$ be the set of all l-filters $\mathfrak{f}$ of $\mathfrak{L}(\mathcal{A})$ such that $|\alpha|_{\mathcal{A}} \in \mathfrak{f}$. The mapping h which associates with every $|\alpha|_{\mathcal{A}} \in \mathfrak{L}(\mathcal{A})$ the set $\|\alpha\|_{\mathcal{A}}$ is an l-isomorphism of $\mathfrak{L}(\mathcal{A})$ onto the field $\mathfrak{L}_1(\mathcal{A})$ of all sets $\|\alpha\|_{\mathcal{A}}$ $(\alpha \in \mathcal{W})$.

This isomorphism will be called the *canonical isomorphism* of $\mathfrak{L}(\mathcal{A})$, and the field $\mathfrak{L}_1(\mathcal{A})$ — the *canonical representation* of $\mathfrak{L}(\mathcal{A})$.

In the sequel we shall use the notion of generalized model as well the notion of generalized algebraic model[4]) of a theory $\mathcal{S}(\mathcal{A})$. These notions being known, we shall give only an outline of their description in the language of theory of Boolean algebras.

Let J be a non-void set and let $\mathfrak{B}$ be a complete Boolean algebra. Let $\mathfrak{M}$ be a mapping associating with every $a \in \mathcal{C}$ an element $a_{\mathfrak{M}} \in J$, with every n-argument $f \in \mathcal{F}$ an n-argument function $f_{\mathfrak{M}}$ defined on J with values belonging to J, and with every k-argument $r \in \mathcal{R}$ a k-argument function $r_{\mathfrak{M}}$ defined on J with values belonging to $\mathfrak{B}$. The mapping $\mathfrak{M}$ will be called a *realization* of primitive signs of $\mathcal{S}$ in J and $\mathfrak{B}$.

The realization $\mathfrak{M}$ permits us to interpret every formula $\alpha \in \mathcal{W}$ as a function $\alpha_{\mathfrak{M}}$ of several variables running over J with values in $\mathfrak{B}$, viz. it suffices to treat the individual variables as variables running over J and the logical operations $+, \cdot, \to, -, \sum_x, \prod_x$ as the corresponding operations in $\mathfrak{B}$. In particular, the quantifiers $\sum_x$ and $\prod_x$ are interpreted as sums and products in $\mathfrak{B}$ where x runs through J. Clearly, if no free variable appear in α, then $\alpha_{\mathfrak{M}}$ is an element of $\mathfrak{B}$.

[4]) For the notion of a generalized algebraic model cf. [8] and [7].

Every mapping φ of the set $\mathcal{V}$ into J is called a *valuation* of $\mathcal{V}$ in the realization $\mathfrak{M}$. A valuation φ is said to be *valable for a formula* $\alpha\in\mathcal{W}$ when, after substitution of elements $\varphi(x)$ for variables x in the $\alpha_{\mathfrak{M}}$ respectively, the value of $\alpha_{\mathfrak{M}}$ is equal to the unit element of $\mathfrak{B}$. In that case we shall write $\alpha_{\mathfrak{M}\varphi}=e$. A formula α is said to be *satisfiable in* $\mathfrak{M}$ if there exists a valuation φ of $\mathcal{V}$ in the realization $\mathfrak{M}$, which is valable for α. A formula α is called *valid in* $\mathfrak{M}$, if every valuation φ of $\mathcal{V}$ in the realization $\mathfrak{M}$ is valable for α, *i. e.*, if $\alpha_{\mathfrak{M}}$ is identically equal to the unit element of $\mathfrak{B}$.

A realization $\mathfrak{M}$ is said to be a *generalized algebraic model* of a theory $\mathcal{S}(\mathcal{A})$ provided that every $\alpha\in\mathcal{A}$ is valid in $\mathfrak{M}$. A generalized algebraic model $\mathfrak{M}$ of a theory $\mathcal{S}(\mathcal{A})$ is called *functionally free* if, for every $\alpha\in\mathcal{W}$, the condition $\alpha\in\mathrm{Cn}(\mathcal{A})$ is equivalent to the condition: α is valid in $\mathfrak{M}$. A generalized algebraic model $\mathfrak{M}$ of a theory $\mathcal{S}(\mathcal{A})$ is said to be *of a power* $\mathfrak{m}$, if the set J is of the power $\mathfrak{m}$.

In the case of a realization $\mathfrak{M}$ in the two-element Boolean algebra $\mathfrak{B}_0$, the generalized algebraic model of $\mathcal{S}(\mathcal{A})$ will be called a *generalized model* of this theory. Notice that in a generalized model the symbol of identity need not be realized by the characteristic function of the relation of equality. Of course, it is always interpreted as the characteristic function of some congruence relation. If it is interpreted as the characteristic function of equality, we omit the word „generalized". In this case the definitions of satisfiability, validity and model are equivalent to the usual ones, since every function of k arguments defined on J with values in the two-element Boolean algebra $\mathfrak{B}_0$ can also be treated as a k-argument relation on J.

It is known (see *e. g.* [3]) that for every theory $\mathcal{S}(\mathcal{A})$, if there exists a generalized model of $\mathcal{S}(\mathcal{A})$ of the power $\mathfrak{m}$, and $\mathfrak{n}>\mathfrak{m}$, then there exists a generalized model of this theory of the power $\mathfrak{n}$. Moreover, if there exists for a theory $\mathcal{S}(\mathcal{A})$ a generalized model of power $\mathfrak{m}$, then there exists a model of $\mathcal{S}(\mathcal{A})$ of the power $\leqslant\mathfrak{m}$. Consequently, the sentences "there is a generalized model of $\mathcal{S}(\mathcal{A})$ of a power $\leqslant\mathfrak{m}$" and "there is a model of $\mathcal{S}(\mathcal{A})$ of a power $\leqslant\mathfrak{m}$" are equivalent. For our purpose it will be convenient to use the notion of the generalized model since there is no reason to distinguish the relation of equality.

It is known that

(ii) *The existence of an l-filter of $\mathfrak{L}(\mathcal{A})$ containing a given element $|\beta|_{\mathcal{A}}\neq 0$ implies that there exists a generalized model $\mathfrak{M}$ of $\mathcal{S}(\mathcal{A})$ in a set J of power $\overline{\overline{\mathcal{T}}}$, such that β is satisfiable in $\mathfrak{M}$.*

This remark immediately follows from the known fact (see *e. g.* [3], [10] and [8]) that the following realization $\mathfrak{M}$ in the set $\mathcal{T}$ and in the two-element Boolean algebra $\mathfrak{B}_0=\mathfrak{L}(\mathcal{A})/\mathfrak{f}$ (where $\mathfrak{f}$ is an l-filter of $\mathfrak{L}(\mathcal{A})$ containing $|\beta|_{\mathcal{A}}$) is the required generalized model of $\mathcal{S}(\mathcal{A})$: $a_{\mathfrak{M}}=a$ for every $a\in\mathcal{C}$, $f_{\mathfrak{M}}=f$ for every $f\in\mathcal{F}$, and if r is a symbol of an n-argument relation which belongs to $\mathcal{R}$, then $r_{\mathfrak{M}}$ is the characteristic function of an n-argument relation defined as follows: $r_{\mathfrak{M}}(\tau_1,\ldots,\tau_n)=[|r(\tau_1,\ldots,\tau_n)|_{\mathcal{A}}]$ for $\tau_1,\ldots$ $\ldots,\tau_n\in\mathcal{T}$, where for each $\alpha\in\mathcal{W}$, $[|\alpha|_{\mathcal{A}}]$ is the element of $\mathfrak{B}_0$ determined by $|\alpha|_{\mathcal{A}}\in\mathfrak{L}(\mathcal{A})$. The valuation φ associating with every $x\in\mathcal{V}$ the same element $x\in\mathcal{T}$ is valable for β.

(iii) *If there exists a generalized model $\mathfrak{M}$ of $\mathcal{S}(\mathcal{A})$ in a set J of power $\leqslant\overline{\overline{\mathcal{V}}}$, such that a well formed formula $\beta\in\mathcal{W}$ is satisfiable in $\mathfrak{M}$, then there exists an l-filter of $\mathfrak{L}(\mathcal{A})$ containing $|\beta|_{\mathcal{A}}$.*

Let us suppose that $\mathfrak{M}$ is a generalized model of $\mathcal{S}(\mathcal{A})$ in the set J of power $\leqslant\overline{\overline{\mathcal{V}}}$, such that β is satisfiable in $\mathfrak{M}$. Clearly, $|\beta|_{\mathcal{A}}\neq 0$. Let φ be a valuation of $\mathcal{V}$ in $\mathfrak{M}$, which is valable for β. Since $\overline{\overline{J}}\leqslant\overline{\overline{\mathcal{V}}}$, we can assume that $\varphi(\mathcal{V})=J$. Obviously, if $|\gamma|_{\mathcal{A}}=|\beta|_{\mathcal{A}}$ for some $\gamma\in\mathcal{W}$, then φ is also valable for γ. Let $\mathfrak{f}$ be a set of all $|\gamma|_{\mathcal{A}}\in\mathfrak{L}(\mathcal{A})$ such that φ is valable for γ. Then $\mathfrak{f}$ is an l-filter od $\mathfrak{L}(\mathcal{A})$ containing $|\beta|_{\mathcal{A}}$. In fact, it is easy to see that $\mathfrak{f}$ is a prime filter containing $|\beta|_{\mathcal{A}}$. Let us suppose that

$$\sum_{\tau\in\mathcal{T}}\left|\alpha\binom{\tau}{x}\right|_{\mathcal{A}}$$

belongs to $\mathfrak{f}$. Hence, by (3), $|\sum_{x}\alpha(x)|_{\mathcal{A}}\in\mathfrak{f}$. In consequence, the valuation φ is valable for $\sum_{x}\alpha(x)$. Thus

$$\sum_{j\in J}\alpha_{\mathfrak{M}\varphi'}=e,$$

where $\varphi'(x)$ is an arbitrary element $j\in J$ and $\varphi'(y)=\varphi(y)$ for every $y\in\mathcal{V}$ and $y\neq x$. Consequently, there exists such a valuation φ_0 that $\varphi_0(x)=$ $=j_0\in J$, $\varphi_0(y)=\varphi(y)$ for every $y\neq x$, $y\in\mathcal{V}$ and $\alpha_{\mathfrak{M}\varphi_0}=e$. Since $\varphi(\mathcal{V})=J$, there is such a $y\in\mathcal{V}$ that $\varphi(y)=\varphi_0(x)$. In consequence,

$$\alpha\binom{y}{x}_{\mathfrak{M}\varphi}=e\quad\text{ and }\quad\left|\alpha\binom{y}{x}\right|_{\mathcal{A}}\in\mathfrak{f}.$$

The following simple example shows that the theorem converse to (ii) is not true, and that $\overline{\overline{\mathcal{V}}}$ cannot be replaced by $\overline{\overline{\mathcal{T}}}$ in (iii). Let us consider a theory $\mathcal{S}(\mathcal{A})$ containing an enumerable set $\mathcal{V}$ of individual variables and a set $\mathcal{C}$ of individual constants such that $\overline{\overline{\mathcal{C}}}=2^{\aleph_0}$. Among all relation signs we have the sign E of equality. The set $\mathcal{R}$ of predicates contains also a subset $\mathcal{R}_0$ of unary relation signs, such that $\overline{\overline{\mathcal{R}_0}}=2^{\aleph_0}$. The set $\mathcal{A}$

of axioms is formed from the axioms of equality and from all the well formed formulae as follows:

(6) $E(a, b)$ where a, b are different signs belonging to $\mathcal{C}$,

(7) $\sum\limits_{x} r(x)$ where $r \in \mathcal{R}_0$,

(8) $\prod\limits_{x}\prod\limits_{y} r_1(x)\cdot r_2(y) \to -E(x, y)$ where r_1, r_2 are different signs belonging to $\mathcal{R}$.

Clearly, there exists a model of this theory whose power is $\overline{\overline{\mathcal{C}}} = 2^{\aleph_0}$. On the other hand, there is no generalized model of $\mathcal{S}(\mathcal{A})$ of a power $< 2^{\aleph_0}$. Let us suppose that there exists an l-filter of $\mathfrak{L}(\mathcal{A})$. Then by a construction analogical to that described in the proof of (ii) we obtain a generalized model of this theory in the set $\mathcal{T}$ of all terms. Identifying any two terms τ_1, τ_2 if $E(\tau_1, \tau_2) \in \mathrm{Cn}(\mathcal{A})$ it is possible to construct a model of $\mathcal{S}(\mathcal{A})$ of a power at most enumerable, which contradicts the previous remark.

It is easy to see that $\overline{\overline{\mathcal{T}}}$ cannot be replaced by $\overline{\overline{\mathcal{V}}}$ in theorem (ii). In fact, it suffices to consider a theory $\mathcal{S}(\mathcal{A})$ containing an enumerable set $\mathcal{V}$ of individual variables, a set $\mathcal{C}$ of the power $2^{\aleph_0}$ of individual constants and the sign E of equality. Let the set $\mathcal{A}$ of axioms be formed from the axioms of equality and from all well formed formulae of the following form:

(9) $-E(a, b)$ where a, b are different signs belonging to $\mathcal{C}$.

Let β be an axiom of $\mathcal{S}(\mathcal{A})$. Clearly, there is no generalized model of $\mathcal{S}(\mathcal{A})$ of the power $\overline{\overline{\mathcal{V}}}$. On the other hand, there is a model $\mathfrak{M}$ of $\mathcal{S}(\mathcal{A})$ in the set J of the power $\overline{\overline{\mathcal{C}}}$, viz. it suffices to put $a_{\mathfrak{M}} \neq b_{\mathfrak{M}}$ for $a \neq b$, $a, b \in J$, and to interpret E as the characteristic function of equality. We may assume that for every $j \in J$ there exists such an $a \in \mathcal{C}$ that $a_{\mathfrak{M}} = j$. By a similar argument to that used in the proof of (iii) we infer that there exists an l-filter of $\mathfrak{L}(\mathcal{A})$ containing $|\beta|_{\mathcal{A}}$.

For every set $\mathcal{A} \subset \mathcal{W}$, let $\mathcal{C}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{A}}$ be respectively the set of all individual constants and the set of all functors which occur in the well formed formulae belonging to $\mathcal{A}$. Theorem (iii) may be strengthend as follows:

(iv) *If there exists a generalized model $\mathfrak{M}$ of $\mathcal{S}(\mathcal{A})$ in a set J of a power $\leqslant \overline{\overline{\mathcal{V}}} + \overline{\overline{\mathcal{C}-\mathcal{C}_{\mathcal{A}}}} + \overline{\overline{\mathcal{F}-\mathcal{F}_{\mathcal{A}}}}$ in which a well formed formula β is satisfiable, then there exists an l-filter of $\mathfrak{L}(\mathcal{A})$ containing $|\beta|_{\mathcal{A}}$.*

The proof is similar to that of (iii). Suppose that $\mathfrak{M}$ is a generalized model of $\mathcal{S}(\mathcal{A})$ in a set J of a power $\leqslant \overline{\overline{\mathcal{V}}} + \overline{\overline{\mathcal{C}-\mathcal{C}_{\mathcal{A}}}} + \overline{\overline{\mathcal{F}-\mathcal{F}_{\mathcal{A}}}}$ and that β

is satisfiable in $\mathfrak{M}$. Let $x_1, \ldots, x_n \in \mathcal{V}$ be all the free variables in β. Let $a_1, \ldots, a_m$ be all the individual constants appearing in β which belong to $\mathcal{C}-\mathcal{C}_{\mathcal{A}}$ and let $g_1, \ldots, g_k$ be all the functors occurring in β which belong to $\mathcal{F}-\mathcal{F}_{\mathcal{A}}$. Interpreting every $g \in (\mathcal{F}-\mathcal{F}_{\mathcal{A}})-(g_1, \ldots, g_k)$ as a function assuming for arbitrary values of its arguments a constant value belonging to J, we can transform the set

$$\big(\mathcal{V}-(x_1, \ldots, x_n)\big) + \big((\mathcal{C}-\mathcal{C}_{\mathcal{A}})-(a_1, \ldots, a_m)\big) + \big((\mathcal{F}-\mathcal{F}_{\mathcal{A}})-(g_1, \ldots, g_k)\big)$$

onto the set J. Let ψ be such a mapping. It is easy to see that the realization $\mathfrak{M}'$ such that

$$\begin{aligned}
a_{\mathfrak{M}'} &= a_{\mathfrak{M}} \text{ for } a \in \mathcal{C}_{\mathcal{A}} \text{ and for } a = a_j \text{ where } j = 1, \ldots, m,\\
a_{\mathfrak{M}'} &= \psi(a) \text{ for } a \in (\mathcal{C}-\mathcal{C}_{\mathcal{A}})-(a_1, \ldots, a_m),\\
g_{\mathfrak{M}'} &= g_{\mathfrak{M}} \text{ for } g \in \mathcal{F}_{\mathcal{A}} \text{ and for } g = g_i \text{ where } i = 1, \ldots, k,\\
g_{\mathfrak{M}'} &= \psi(g) \text{ for } g \in (\mathcal{F}-\mathcal{F}_{\mathcal{A}})-(g_1, \ldots, g_k),\\
r_{\mathfrak{M}'} &= r_{\mathfrak{M}} \text{ for } r \in \mathcal{R}
\end{aligned}$$

is also generalized model of $\mathcal{S}(\mathcal{A})$. Moreover, if a valuation φ of $\mathcal{V}$ in $\mathfrak{M}$ is valable for β, then the valuation φ' in $\mathfrak{M}'$ such that $\varphi'(x_j) = \varphi(x_j)$ for $j = 1, \ldots, n$ and $\varphi'(x) = \psi(x)$ for $x \in \mathcal{V}-(x_1, \ldots, x_n)$ is also valable for β. Let us consider the set $\mathfrak{f}$ of all $|\gamma|_{\mathcal{A}}$, $\gamma \in \mathcal{W}$, such that the valuation φ' is valable for γ. It is easy to see that $\mathfrak{f}$ is the required l-filter of $\mathfrak{L}(\mathcal{A})$ containing $|\beta|_{\mathcal{A}}$.

From (i) and (ii) we find that

(v) *If $\mathfrak{L}(\mathcal{A})$ is representable as a field of sets, then for every formula β non refutable in $\mathcal{S}(\mathcal{A})$ there exists a generalized model $\mathfrak{M}$ of $\mathcal{S}(\mathcal{A})$ in a set J of a power $\leqslant \overline{\overline{\mathcal{T}}}$ such that β is satisfiable in $\mathfrak{M}$.*

As a corollary of (i) and (iv) we find that

(vi) *If for every formula $\beta \in \mathcal{W}$ non refutable in $\mathcal{S}(\mathcal{A})$ there exists a generalized model $\mathfrak{M}$ of $\mathcal{S}(\mathcal{A})$ in a set J of a power $\leqslant \overline{\overline{\mathcal{V}}} + \overline{\overline{\mathcal{C}-\mathcal{C}_{\mathcal{A}}}} + \overline{\overline{\mathcal{F}-\mathcal{F}_{\mathcal{A}}}}$ (in particular of a power $\leqslant \overline{\overline{\mathcal{V}}}$) such that β is satisfiable in $\mathfrak{M}$, then $\mathfrak{L}(\mathcal{A})$ is representable as a field of sets.*

It immediately follows from (v) and (vi) that

(vii) *If $\mathcal{S}$ is a system of the functional calculus such that $\overline{\overline{\mathcal{V}}} = \overline{\overline{\mathcal{T}}} = \mathfrak{m}$, then the Lindenbaum algebra $\mathfrak{L}(\mathcal{A})$ of a theory $\mathcal{S}(\mathcal{A})$ is representable as a field of sets if and only if for every $\beta \in \mathcal{W}$ non refutable in $\mathcal{S}(\mathcal{A})$ there exists a generalized model $\mathfrak{M}$ of $\mathcal{S}(\mathcal{A})$ in a set J of a power $\leqslant \mathfrak{m}$, such that β is satisfiable in $\mathfrak{M}$.*

Observe that the conditions:

1° There exists a generalized model of the power $\mathfrak{m}$ for a theory $\mathcal{S}(\mathcal{A})$, 2° $\beta \in \mathcal{W}$ is non refutable in $\mathcal{A}$,

do not imply in general that there exists a generalized model of this theory of the same power $\mathfrak{m}$, in which β is satisfiable.

The following example is due to Mostowski. Suppose that $\overline{\overline{\mathcal{C}}} > \mathfrak{m}$ and that the set $\mathcal{R}$ is formed from two signs: the sign E of equality and the sign I of two-argument relation. The set $\mathcal{F}$ is empty and $\overline{\overline{\mathcal{V}}} = \aleph_0$. The set $\mathcal{A}$ is formed from the axioms of equality and from all well formed formulae of the following form: $I(a, b)$, where a, b are different signs of individual constants belonging to $\mathcal{C}$.

Clearly, the theory $\mathcal{S}(\mathcal{A})$ has a model of the power $\mathfrak{m}$ (more generally: of every power $\geqslant 1$). Let β be the formula

$$I(x, y) \to -E(x, y).$$

Obviously β is not satisfiable in any generalized model of $\mathcal{S}(\mathcal{A})$ of a power $< \overline{\overline{\mathcal{C}}}$, in particular of the power $\mathfrak{m}$.

The Gödel-Skolem-Löwenheim-Malcev [5]) theorem can be formulated as follows:

(viii) *If $\beta \in \mathcal{W}$ is not refutable in a theory $\mathcal{S}(\mathcal{A})$, then there exists a generalized model $\mathfrak{M}$ of $\mathcal{S}(\mathcal{A})$ in a set of a power $\mathfrak{m} \leqslant \max(\aleph_0, \overline{\overline{\mathcal{A}}})$ such that β is satisfiable in $\mathfrak{M}$.*

It follows from (vi) and (viii) that

(ix) *If $\overline{\overline{\mathcal{A}}} \leqslant \aleph_0$, then $\mathfrak{L}(\mathcal{A})$ is always representable as a field of sets.*

From (vi) and (viii) we obtain the following corollary:

(x) *If $\overline{\overline{\mathcal{V}}} \geqslant \max(\aleph_0, \overline{\overline{\mathcal{A}}})$, then $\mathfrak{L}(\mathcal{A})$ is always representable as a field of sets.*

It also follows from (vi) and (viii) that

(xi) *If $\overline{\overline{\mathcal{V}}} + \overline{\overline{\mathcal{C} - \mathcal{C}_{\mathcal{A}}}} + \overline{\overline{\mathcal{F} - \mathcal{F}_{\mathcal{A}}}} \geqslant \max(\aleph_0, \overline{\overline{\mathcal{A}}})$, then $\mathfrak{L}(\mathcal{A})$ is always representable as a field of sets.*

The following theorem results from theorem (x) and from theorem 3.6. of Rasiowa [8]:

(xii) *For every consistent theory $\mathcal{S}(\mathcal{A})$ there exists a functionally free generalized algebraic model of this theory in a set $J \neq 0$ and in a field $\mathfrak{F}$ of all subsets of a set Y.*

[5]) By the Gödel-Skolem-Löwenheim theorem we mean the result stating that for every denumerable elementary theory there exists a model of a power $\leqslant \aleph_0$. Malcev [4] has considered formal systems containing a non-denumerable number of symbols in the case of formulae of the sentential calculus and he has proved a theorem analogous to Gödel's theorem on the existence of models. The first statement and proof of the theorem equivalent to (viii) was given by Henkin [3] and later independently by Robinson [15].

Indeed, let $\mathcal{S}(\mathcal{A})$ be a consistent theory. Clearly, we can assume without any restriction of the generality of this theorem that $\overline{\overline{\mathcal{V}}} \geqslant \max(\aleph_0, \overline{\overline{\mathcal{A}}})$ (if not, we can extend the functional calculus on the basis of which the considered theory is formalized, increasing the set of individual variables). By (x), $\mathfrak{L}(\mathcal{A})$ is representable as a field of subsets of a set Y. Let h be an l-isomorphism of $\mathfrak{L}(\mathcal{A})$ into the field $\mathfrak{F}$ of all subsets of the set Y. Then the realization $\mathfrak{M}$ in the set $\mathcal{T}$ and in the field $\mathfrak{F}$, defined as follows:

$$a_{\mathfrak{M}} = a \quad \text{for} \quad a \in \mathcal{C}, \qquad f_{\mathfrak{M}} = f \quad \text{for} \quad f \in \mathcal{F},$$
$$r_{\mathfrak{M}}(\tau_1, \ldots, \tau_n) = h(|r(\tau_1, \ldots, \tau_n)|_{\mathcal{A}})$$

for every n-argument $r \in \mathcal{R}$ and $\tau_1, \ldots, \tau_n \in \mathcal{T}$, is the required functionally free generalized algebraic model of $\mathcal{S}(\mathcal{A})$.

It is easy to show that

(xiii) *There are non-enumerable theories (i. e., with a non-enumerable set $\mathcal{A}$ of axioms) such that $\mathfrak{L}(\mathcal{A})$ is not representable as a field of sets.*

In fact, let us consider the theory $\mathcal{S}(\mathcal{A})$ containing a denumerable set $\mathcal{V}$ of individual variables, the equality sign E, and a non-denumerable set of unary relation signs. The set $\mathcal{A}$ of axioms is formed from the axioms of equality and from all formulae of the form (7) and (8). It follows from (i) and (ii) that $\mathfrak{L}(\mathcal{A})$ is not representable as a field of sets.

We see from the above theorems and examples that $\mathfrak{L}(\mathcal{A})$ is representable as a field of sets whenever either the set of all variables or the set of individual constants not appearing in the axioms or the set of all functors not appearing in the axioms is rich enough.

If the functional calculus $\mathcal{S}$ is enumerable, then $\mathfrak{L}(\mathcal{A})$ is also at most enumerable and the class of all l-sums and l-products is enumerable. Therefore it is representable as a field of sets on account of the general theorem on Boolean algebras:

(xiv) *Each Boolean algebra can be isomorphically mapped on a field of sets in such a way that a given enumerable set of infinite sums and products is preserved* (see [11], 9.1).

By (vii) this implies the Gödel-Skolem-Löwenheim theorem. This method of the proof of Gödel's completeness theorem and of the Gödel-Skolem-Löwenheim theorem was proposed by the authors [9, 10] and by L. Rieger [12, 13, 14]. The method described above can also be applied in the case where $\mathcal{S}$ is either enumerable or not, but $\mathcal{A}$ is at most enumerable. In fact, to prove the existence of a model for a given consistent theory $\mathcal{S}(\mathcal{A})$, it suffices to consider the theory with the same set $\mathcal{A}$ of axioms, but formalized on the basis of the enumerable functional calculus $\mathcal{S}'$ containing the enumerable set of individual variables and only these primitive constants which appear in the well formed formulae belonging to $\mathcal{A}$.

It follows from (xiv) that this method of proof of the Gödel-Skolem-Löwenheim-Malcev theorem cannot be applied in the general case of non-enumerable theories with a non-enumerable set of axioms, if the set $\mathcal{W}+(\mathcal{C}-\mathcal{C}_{\mathcal{A}})+(\mathcal{F}-\mathcal{F}_{\mathcal{A}})$ is too small. This explains the fact that other known methods (see [3, 15, 2, 6, 7]) of the proof of that theorem require the addition of a set of constants of a sufficiently great power.

Now we shall examine the case where the set $\mathcal{A}$ is empty. In this case we shall omit the letter $\mathcal{A}$ everywhere in the notation assumed above, *e. g.*, we shall write $|\alpha|$ instead of $|\alpha|_{\mathcal{A}}$, $\|\alpha\|$ instead of $\|\alpha\|_{\mathcal{A}}$, $\mathfrak{L}$ instead of $\mathfrak{L}(\mathcal{A})$, $\mathfrak{L}_1$ instead of $\mathfrak{L}_1(\mathcal{A})$, etc.

It follows from the above remarks that the following theorem can easily be proved by the category method of the authors (cf. [11]):

(xv) *The Lindenbaum algebra* $\mathfrak{L}$ *is representable as a field of sets.*

No hypothesis regarding the power of $\mathcal{W}$ is here assumed.

Following Marczewski [5], we say that two fields of sets $\mathfrak{F}_1$ and $\mathfrak{F}_2$ (of subsets of spaces X_1 and X_2, respectively) are *equivalent* if there is a one-to-one mapping ψ of X_1 onto X_2 such that

$$\psi(A)\in\mathfrak{F}_2 \quad \text{if and only if} \quad A\in\mathfrak{F}_1. \tag{10}$$

Clearly the representation

$$h(A) = \psi(A) \quad \text{for} \quad A\in\mathfrak{F}_1 \tag{11}$$

is then a Boolean isomorphism of $\mathfrak{F}_1$ onto $\mathfrak{F}_2$ preserving all infinite unions and intersections. Consequently the equivalence of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ implies the isomorphism of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ (the converse statement is not true, in general).

In particular, if a field of sets $\mathfrak{L}_2(\mathcal{A})$ is equivalent to the canonical representation $\mathfrak{L}_1(\mathcal{A})$ of $\mathfrak{L}(\mathcal{A})$, then $\mathfrak{L}_2(\mathcal{A})$ is also a representation of $\mathfrak{L}(\mathcal{A})$ in the sense defined on p. 144.

Rieger [12, 14] has found a very interesting field of sets $\mathfrak{L}_2$ equivalent to the canonical representation $\mathfrak{L}_1$ of the Lindenbaum algebra $\mathfrak{L}$. Since he has formulated his result only in the case of $\overline{\overline{\mathcal{W}}} = \aleph_0$, we recall his representation in the general case. Rieger's proof [14] (based essentially on (xiii)) is not valid if $\overline{\overline{\mathcal{W}}} > \aleph_0$. The proof given below is based on another idea.

Let $\mathcal{E}$ be the set of all elementary well formed formulae α of $\mathcal{S}$. Notice that if $\alpha_1, \alpha_2 \in \mathcal{E}$ and $\alpha_1 \neq \alpha_2$, then also $|\alpha_1| \neq |\alpha_2|$ and consequently $\|\alpha_1\| \neq \|\alpha_2\|$.

Let D be the class of all functions u defined on $\mathcal{E}$, assuming as values exclusively the integers 0 and 1. In other words, D is the Cartesian product

$$D = \mathbf{P}_{\alpha\in\mathcal{E}} U_\alpha \tag{12}$$

where U_α is the set formed only of integers 0 and 1. Clearly U_α can be considered as a Hausdorff space. Consequently D is also a bicompact totally disconnected Hausdorff space, the so-called *Cantor discontinuum*. For every $\alpha\in\mathcal{E}$, let

$$D_\alpha = \mathop{\mathrm{E}}_{u}(u(\alpha) = 1).$$

The canonical representation $\mathfrak{L}_1$ of $\mathfrak{L}$ is the class of all subsets $\|\beta\|$ of the set L of all l-filters of $\mathfrak{L}$ (see p. 145). $\mathfrak{L}$ is a kind of free Boolean algebra with generators $|\alpha|$, where $\alpha\in\mathcal{E}$. More exactly, for every mapping f of the elements $|\alpha|\in\mathfrak{L}$, where $\alpha\in\mathcal{E}$, onto a class of subsets of a space X, there exists a Boolean homomorphism h of $\mathfrak{L}$ into the field $\mathfrak{B}$ of all subsets of X such that

(a) h is an extension of f,

(b) the equalities (4) and (5) hold.

This follows from the fact that if α is refutable in $\mathcal{S}$, then $\alpha_{\mathfrak{M}}$ is identically equal to $0\in\mathfrak{B}$ for every realization $\mathfrak{M}$ in $\mathfrak{B}$ and in arbitrary $J\neq 0$.

By isomorphism, the representation $\mathfrak{L}_1$ of $\mathfrak{L}$ has the same property. Consequently, there exists a homomorphism h of $\mathfrak{L}_1$ onto a field $\mathfrak{L}_2$ of subsets of D such that:

(a′) $h(\|\alpha\|) = D_\alpha$ for every $\alpha\in\mathcal{E}$,

(b′) the equalities

$$h\Big(\Big\|\sum_x \alpha\Big\|\Big) = \bigcup_{\tau\in\mathcal{T}}\Big(h\Big\|\alpha\binom{\tau}{x}\Big\|\Big), \qquad h\Big(\Big\|\prod_x \alpha\Big\|\Big) = \bigcap_{\tau\in\mathcal{T}}\Big(h\Big\|\alpha\binom{\tau}{x}\Big\|\Big)$$

hold for every $\alpha\in\mathcal{W}$.

Moreover, this homomorphism h is induced (cf. [11]) by a point mapping, *i. e.*, there exists a mapping ξ of D into the set L of all l-filters of $\mathfrak{L}$, such that

$$h(A) = \xi^{-1}(A) \quad \text{for every} \quad A\in\mathfrak{L}_1. \tag{13}$$

The definition of ξ is as follows. Let $u_0\in D$. The class $\mathfrak{f}_0$ of all elements $|\alpha|\in\mathfrak{L}$ such that $u_0\in h(\|\alpha\|)$ is an l-filter of $\mathfrak{L}$, *i. e.*, a point of L. We define the value of ξ at the point $u_0\in D$ by the formula $\xi(u_0) = \mathfrak{f}_0$.

On the other hand, the field $\mathfrak{F}$ of all subsets of D is a free completely additive field of sets with D_α as generators, *i. e.*, for every mapping f

of the class of all sets D_a ($a \in \mathcal{C}$) onto a class of subsets of a space X, there exists a homomorphism g of $\mathfrak{F}$ into the field of all subsets of X such that

(c) g is an extension of f,

(d) g preserves all infinite unions and intersections, *i. e.*,

$$g\Big(\bigcup_t A_t\Big) = \bigcup_t g(A_t), \tag{14}$$

$$g\Big(\bigcap_t A_t\Big) = \bigcap_t g(A_t) \tag{15}$$

for every class of sets $A_t \subset D$.

Moreover, g is induced by a point mapping, *i. e.*, there exists a mapping η of X into D such that

$$g(A) = \eta^{-1}(A) \quad \text{for every} \quad A \subset D. \tag{16}$$

The proof of this fact is similar to that of an analogous theorem of Sikorski [17]. Let η_a be the characteristic function of the set $f(D_a) \subset X$. We define η by the formula

$$\eta(x) = \{\eta_a(x)\} \in D$$

and the homomorphism g by (16).

In particular, there is a mapping η of L into D such that (16) defines a homomorphism g of the field $\mathfrak{F}$ of all subsets of D into the field of all subsets of L, such that

(c′) $g(D_a) = \|a\|$ for every $a \in \mathcal{C}$,

(d′) g preserves all infinite unions and intersections, *i. e.*, (14) and (15) hold.

By (a′) and (c′) we have

$$gh(\|a\|) = \|a\| \quad \text{for every} \quad a \in \mathcal{C}. \tag{16}$$

Since the class of all $\|a\|$ where $a \in \mathcal{C}$ generates $\mathfrak{L}_1$, we infer that

$$gh(\|a\|) = \|a\| \quad \text{for every} \quad a \in \mathcal{W}. \tag{17}$$

This implies that h is a one-to-one mapping, *i. e.*, an isomorphism of $\mathfrak{L}_1$ onto $\mathfrak{L}_2$ and g restricted to $\mathfrak{L}_2$ is identical with h^{-1}, *i. e.*, $h^{-1} = g|\mathfrak{L}_2$.

If $u_1, u_2 \in D$ and $u_1 \neq u_2$, then there is an $a \in \mathcal{C}$ such that only one of these points belongs to D_a, for instance

$$u_1 \in D_a = h(\|a\|) = \xi^{-1}(\|a\|), \qquad u_2 \in D - D_a = h(\|-a\|) = \xi^{-1}(L - \|a\|).$$

Consequently, $\xi(u_1) \in \|a\|$ and $\xi(u_2) \in L - \|a\|$. This proves that ξ is one-to-one.

By a similar argument we infer that η is one-to-one.

It follows from (a′), (c′), (13) and (16) that

$$\xi^{-1}\big(\eta^{-1}(D_a)\big) = D_a \quad \text{for every} \quad a \in \mathcal{C}. \tag{18}$$

Consequently

$$\xi^{-1}\big(\eta^{-1}(A)\big) = A \quad \text{for every} \quad A \subset D. \tag{19}$$

Substituting any one-point set for A in this equality, we infer that

$$\eta = \xi^{-1}, \tag{20}$$

i. e., η is a one-to-one mapping of L onto D and

$$h(A) = \eta(A) \quad \text{for every} \quad A \in \mathfrak{L}_2. \tag{21}$$

This proves the Rieger's theorem:

(xvi) $\mathfrak{L}_1$ *is equivalent to* $\mathfrak{L}_2$.

This theorem explains the structure of $\mathfrak{L}_1$, since the structure of $\mathfrak{L}_2$ is rather obvious: $\mathfrak{L}_2$ is the least field (of subsets of D) containing all sets D_a and closed with respect to all infinite unions and intersections corresponding to the logical quantifiers.

Let $\mathfrak{L}_0$ be the subalgebra of all $|a| \in \mathfrak{L}$, where a is an open well formed formula of $\mathcal{S}$.

For every $a \in \mathcal{W}$ let $|||a|||$ be the set $h(\|a\|) \subset D$, where h is the isomorphism defined in the proof of (xvi). By definition $|||a||| = D_a$ for $a \in \mathcal{C}$. If a is an open formula, then $|||a|||$ is both open and closed in the bicompact, totally disconnected space D, and conversely, each subset simultaneously open and closed in D is of this form, since it is formed from the sets D_a by means of finite unions, intersections and complementations. This implies that D is homeomorphic with Stone's representation space constructed for the Boolean algebra $\mathfrak{L}_0$. More exactly, for every prime filter $\mathfrak{f}_0$ in $\mathfrak{L}_0$ there exists exactly one point u_0 such that

(22) $|a| \in \mathfrak{f}_0$ if and only if $u_0 \in |||a|||$, where a is an open well formed formula.

On the other hand, each point $u_0 \in D$ determines uniquely an l-filter $\mathfrak{f}$ of $\mathfrak{L}$, viz. the filter $\mathfrak{f}$ determined as follows:

$$|a| \in \mathfrak{f} \text{ if and only if } u_0 \in |||a||| \quad (a \in \mathcal{W}).$$

Consequently:

(xvii) *Every prime filter of* $\mathfrak{L}_0$ *can be uniquely extended to an* l*-filter of* $\mathfrak{L}$.

Let $\mathcal{A}$ be any set of closed formulae belonging to $\mathcal{W}$. By the same method as in the case of theorems (ii) and (iii) we prove that

(xviii) *If there is a point* $u_0 \in D$ *such that* $u_0 \in |||a|||$ *for every* $a \in \mathcal{A}$, *then there exists a model of* $\mathcal{S}(\mathcal{A})$ *of a power* $\leqslant \overline{\overline{\mathcal{T}}}$.

(xix) *If there is a model of $\mathcal{S}(\mathcal{A})$ of a power $\leqslant \overline{\overline{\mathcal{V}}}$, then there is a point $u_0 \in D$ such that $u_0 \in |||\alpha|||$ for every $\alpha \in \mathcal{A}$.*

Obviously, D may be replaced by L in (xviii) and (xix) (of course, $|||\alpha|||$ should then be replaced by $||\alpha||$).

Let us consider D as the topological space with another topology. Previously the neighbourhoods were the sets corresponding to open well formed formulae. Now we define the neighbourhoods as the subsets of D corresponding to closed well formed formulae. This topological space will be denoted by D^*. The space D^* is not a T_0-space, since there are points $u_1, u_2 \in D^*$ which are not separated by any subset of D^* of the form $|||\alpha|||$, where α is a closed well formed formula of $\mathcal{S}$. In fact, let $r \in \mathcal{R}$ be a sign of a k-argument relation. Let $x_1, x_2, \ldots, x_{k+1}$ be different individual variables of $\mathcal{S}$ and let β_1 be the formula

$$r(x_1, x_3, \ldots, x_{k+1}) \cdot -r(x_2, x_3, \ldots, x_{k+1}).$$

Since β_1 is not refutable in $\mathcal{S}$ we infer from (i) that there exists an l-filter $\mathfrak{f}_1$ of $\mathfrak{L}$ such that $|\beta_1| \in \mathfrak{f}_1$. Let β_2 be the formula

$$-r(x_1, x_3, \ldots, x_{k+1}) \cdot r(x_2, x_3, \ldots, x_{k+1}).$$

Let $\mathfrak{f}_2$ be the set of all $|\alpha| \in \mathfrak{L}$ for which the following condition is satisfied: α results by the change of variables x_1 and x_2 from some $\gamma \in \mathcal{W}$, such that $|\gamma| \in \mathfrak{f}_1$. Then $\mathfrak{f}_2$ is an l-filter of $\mathfrak{L}$ containing $|\beta_2|$. Since $|\beta_1| \neq |\beta_2|$ and $|\beta_1| \cdot |\beta_2| = 0$ we infer that $\mathfrak{f}_1 \neq \mathfrak{f}_2$. On the other hand, for every closed well formed formula α the following equivalence holds: $|\alpha| \in \mathfrak{f}_1$ if and only if $|\alpha| \in \mathfrak{f}_2$. Hence $\mathfrak{f}_1 \in ||\alpha||$ if and only if $\mathfrak{f}_2 \in ||\alpha||$. Thus we have demonstrated that there are points $\mathfrak{f}_1, \mathfrak{f}_2 \in L$ which are not separated by any subset $||\alpha||$ of L where α is a closed well formed formula. By (xvi) an analogical statement holds for the space D^*, which proves that D^* is not a T_0-space.

It is easy to see that D^* is a Hausdorff space after identification of points $u_1, u_2 \in D^*$ which are not separated by a set $|||\alpha|||$, where α is a closed well formed formula.

(xx) *If $\overline{\overline{\mathcal{W}}} = \overline{\overline{\mathcal{V}}}$, then D^* is a bicompact space.*

It suffices to prove that if $\mathcal{B}$ is a collection of closed well formed formulae such that

(23) $D = D^* \neq |||\alpha_1||| + |||\alpha_2||| + \ldots + |||\alpha_m|||$ for any finite sequence $\alpha_1, \ldots, \alpha_m \in \mathcal{B}$, then

$$D \neq \bigcup_{\alpha \in \mathcal{B}} |||\alpha|||.$$

Condition (23) means that the set $\mathcal{A}$ of all formulae $-\alpha$ where $\alpha \in \mathcal{B}$ is a consistent set of formulae. By (viii) there exists a model of $\mathcal{S}(\mathcal{A})$ of the power $\leqslant \overline{\overline{\mathcal{V}}}$, *i. e.* (see (iii)), there is an l-filter $\mathfrak{f}_0$ of $\mathfrak{L}$ containing all elements $|-\alpha|$ where $\alpha \in \mathcal{B}$. Let u_0 be the point of D such that $|\alpha| \in \mathfrak{f}_0$ if and only if $u_0 \in |||\alpha|||$. We have $u_0 \in |||-\alpha||| = D - |||\alpha|||$ for every $\alpha \in \mathcal{B}$. Consequently,

$$u_0 \notin \bigcup_{\alpha \in \mathcal{B}} |||\alpha|||,$$

which completes the proof of (xx).

(xxi) *If D^* is compact, then for every consistent set $\mathcal{A}$ of closed well formed formulae of $\mathcal{S}$ the theory $\mathcal{S}(\mathcal{A})$ posesses a model of a power $\leqslant \overline{\overline{\mathcal{T}}}$.*

Since the theory $\mathcal{S}(\mathcal{A})$ is consistent, we have

$$0 \neq |||\alpha_1 \cdot \ldots \cdot \alpha_m||| = |||\alpha_1||| \cdot \ldots \cdot |||\alpha_m|||$$

for every finite sequence $\alpha_1, \ldots, \alpha_m \in \mathcal{A}$. Since D^* is bicompact, there is a point $u_0 \in D^* = D$ such that $u_0 \in \bigcap_{\alpha \in \mathcal{A}} |||\alpha|||$, *i. e.*, $u_0 \in |||\alpha|||$ for every $\alpha \in \mathcal{A}$.

The l-filter $\mathfrak{f}_0$ of $\mathfrak{L}$ determined by u_0 (*i. e.*, the class of all $|\alpha| \in \mathfrak{L}$ such that $u_0 \in |||\alpha|||$) contains all elements $|\alpha|$, where $\alpha \in \mathcal{A}$. Consequently, by (xviii), there is a model of $\mathcal{S}(\mathcal{A})$ of a power $\leqslant \overline{\overline{\mathcal{T}}}$.

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THE COMPLETENESS OF THE HOMEOMORPHISMS GROUP OF A COMPLETE SPACE

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Let X be any completely regular space. It admits a uniform structure $\{V_a\}$ defined by all neighbourhoods V_a of the set $\Delta = \{(x, x): x \in X\}$ in the product topology in $X \times X$ [2].

A filter $\{U_\tau\}$ of a space which admits a uniform structure $\{V_a\}$ is called a *Cauchy filter* if, for each a and some τ, $U_\tau \times U_\tau \subset V_a$. If every Cauchy filter of the space converges, then this space will be called *complete* (in the uniform structure $\{V_a\}$).

Given any topological group H, one can define a uniform structure $\{{}^*\mathfrak{V}_a^*\}$ by saying that $(x, y) \in {}^*\mathfrak{V}_a^*$ if $x \in \mathfrak{V}_a y \cap y\mathfrak{V}_a$ and $\mathfrak{V}_a$ are neighbourhoods of the unity in H. A topological group is called *complete in the sense of Raïkov* if it is complete in the structure $\{{}^*\mathfrak{V}^*\}$ [3].

Let H be the homeomorphisms group of X. It is known that the family of sets $\mathfrak{V}_a = \{h: (x, h(x)) \in V_a \text{ for all } x \in V\}$ makes H a topological group, where $\mathfrak{V}_a$ is the system of neighbourhoods of the identity transformation of H [4].

The aim of this note is the proof of the following

THEOREM[1]). *If the space X is complete (in the maximal uniform structure of all neighbourhoods V_a in $X \times X$) and the homeomorphisms group H of X is topologized by the system of neighbourhoods of the unity $\mathfrak{V}_a$, then H is complete in the sense of Raïkov.*

The proof is based on the following two lemmas:

LEMMA 1. *Let $\{U_\tau\}$ be a Cauchy filter in H; then, for each a there is a τ such that $h, f \in U_\tau$ implies $(h(x), f(x)) \in V_a$ for all $x \in X$.*

Proof. Since $\{U_\tau\}$ is a Cauchy filter, $h, f \in U_\tau$ implies, for each a and some τ, $f \in \mathfrak{V}_a h$ or $(x, f(h^{-1}(x))) \in V_a$ for all $x \in X$, thus $(h(x), f(x)) \in V_a$.

[1]) The completeness of the homeomorphisms group in the g-topology of a locally compact space was proved by R. Arens [1]. His proof is based on the local compactness of the space and cannot be transferred to our case.