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SOME SPECIAL METRICS IN GENERAL TOPOLOGY

BY

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The notions to be discussed in this note lie mainly on the border-line between general topology and distance geometry. The problems raised do not seem to be too difficult. The author is indebted to E. Marczewski for several useful remarks.

1. Metrics characterizing dimension. It has been shown [3] that a separable metrizable space M has dimension $\leqslant n$ if and only if one can introduce a (topology-preserving) totally bounded metric ϱ in M with the following property: for every $n+3$ points $x, y_1, y_2, \ldots, y_{n+2}$ in M, there is a triplet of indices i, j, k such that

$$\varrho(y_i, y_j) \leqslant \varrho(x, y_k) \quad (i \neq j). \tag{1}$$

In particular, if M is a compactum, the condition of total boundedness can clearly be omitted in this criterion.

Observe that a real segment supplied with the ordinary topology satisfies our condition (for $n = 1$), but that the ordinary metric of a full square does not.

The necessity of the condition is proved in [3] by means of a rather complicated theorem of Nagata [9]. Hence the following problem arises:

P 255. Give a satisfactory proof of the necessity of the condition.

If $n = 0$, such a proof follows easily from the possibility of the embedding of a 0-dimensional M in the discontinuum of Cantor, in which a non-archimedean metric is introduced [2]. If $n > 0$, one has a fair chance to solve the question by suitable remetrisizing the universal compact n-dimensional space described by Menger [7] (see also Lefschetz [6]), which generalizes the discontinuum of Cantor.

Though sometimes useful — e. g. if $\dim M \leqslant n$, and $\overline{M}$, i. e. the completed M, is compact and $\dim \overline{M} \leqslant n$ — the metric defined above is very peculiar in many respects. Another way to characterize the dimension by means of a metric follows from results of E. Marczewski [10], see e. g. [5], p. 107:

A separable metric space has dimension $\leqslant n$ if and only if one can introduce a topology-preserving metric in which almost all of the spherical neighbourhoods of any point have boundaries of dimension $\leqslant n-1$.

2. Finitely additive topological properties. Let the topological space T be the union of a finite number of *closed* subspaces T_i. A topological property is called *finitely additive* if it holds for T whenever it holds for each of the T_i.

THEOREM. *Metrizability is a finitely additive property.*

The proof of this theorem easily follows from a metrical extension theorem of Hausdorff [4], apparently rediscovered by Bing [1]. A purely topological proof of this theorem and also a generalization of it has been given by Nagata [11]. The proof of the theorem follows clearly from the following two lemmas:

LEMMA I. *If A and B are metric spaces whose metrics ϱ_A and ϱ_B coincide on their intersection $D = A \cap B$, where we assume D to be closed both in A and B, then these metrics can be extended to a metric ϱ on $A \cup B$, i. e. on $A \cup B$ one can define a metric ϱ such that $\varrho = \varrho_A$ on A and $\varrho = \varrho_B$ on B.*

LEMMA II. *If a finite number of metrizable spaces intersect pairwise in sets closed in each, the union of those spaces can be defined in a natural way as a topological space* (1)*, and we contend that this space is metrizable.*

Observe that Lemma I fails if the intersection D is assumed to be open instead of closed.

Proof of Lemma I. We only have to define $\varrho(a, b)$ for points $a \in A$ and $b \in B$. Put, if $D \neq \emptyset$,

$$\varrho(a, b) = \inf_{d \in D} [\varrho_A(a, d) + \varrho_B(b, d)].$$

It can easily be shown that ϱ is a metric on $A \cup B$ satisfying the requirements.

If $D = \emptyset$, we take fixed points $p \in A$, $q \in B$ and define $\varrho(a, b)$ by

$$\varrho(a, b) = \varrho_A(a, p) + \varrho_B(b, q) + 1.$$

Observe that both A and B are closed subsets of the space $A \cup B$ with metric ϱ.

Proof of Lemma II. Here we use essentially a theorem of Hausdorff [4] and Bing [1], which states that a (topology-preserving) metric defined on a closed subset of a metrizable space can be extended to a topology-preserving metric defined on the whole space. Suppose A and B are two metrizable spaces intersecting in a set D, closed in each. Introduce a metric ϱ_A in A. This ϱ_A on D can be extended to a ϱ_B on B. Applying Lemma I we find a metric ϱ on $A \cup B$. If C is a third metrizable set intersecting both A and B in closed sets, the intersection $(A \cup B) \cap C$ is also closed, since A and B are closed in $A \cup B$.

(1) Indeed, the closed sets of the topological space will be e. g. finite unions of closed subsets of each of the metrizable spaces.

So we can repeat the process and the proof follows by induction.

Many other topological properties are finitely additive, e. g. the property of being a Hausdorff space, resp. a regular space or a normal space (for normality see Mrówka [8]). The property of complete regularity is *not* finitely additive, as has been shown by H. de Vries. Indeed, Tychonof constructed a space which may be used to find a counterexample. Hence the following problem arises:

P 256. Inquire systematically which important notions in general topology are finitely additive.

3. Convexity in metric spaces. Let M be a metric space with distance function ϱ. A *subset* $S \subset M$ is called *convex* (relative to M) if the following condition holds:

If $x, y \in S$ and if there exists a $z \in M$ with $\varrho(x, z) + \varrho(z, y) = \varrho(x, y)$, then $z \in S$ for all such z.

If M is the Euclidean space E_n supplied with the ordinary metric, or, for example also with the equivalent distance function defined by

$$\varrho(x, y) = \max_i |x_i - y_i| \qquad (x = (x_i),\ y = (y_i)),$$

then convex sets in the ordinary sense coincide with convex sets in the way we defined above, which justifies our definition.

E. Marczewski kindly drew my attention to the fact that for each metric ϱ in M, a topology-preserving metric ϱ^* can easily be defined under which every subset is convex. In fact, put $\varrho^* = \sqrt{\varrho}$. This shows that the usefulness of this notion of convexity in general topology will be limited. However, it may be of some use in distance geometry and in problems of a mixed character. For example, let us mention the following problem:

P 257. Given a metrizable M and a closed subset $S \subset M$. Can every (topology-preserving) metric on S be extended to a (topology-preserving) metric on M under which S is convex? (2)

(2) This problem has been solved in the affirmative by W. Nitka.

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ON CONSTRUCTIBLE FALSITY IN THE CONSTRUCTIVE LOGIC WITH STRONG NEGATION

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This article is a continuation of paper [9] of Rasiowa, in which an algebraic characterization of the system $\mathcal{S}$ of the constructive propositional calculus with strong negation was given. The terminology and the notation is here the same as in [9]. The knowledge of [9] is here assumed.

The idea of the above-mentioned constructive logic with strong negation is due to David Nelson, who introduced in paper [7] a new constructive interpretation for logical connectives of the number theory and characterized a formal system of the number theory satisfying this interpretation. An analogical system of the number theory was later investigated by Markov [6]. The system of the propositional calculus with strong negation was examined by Vorobiev [12] and [13].

Under Nelson's interpretation of logical connectives the strong negation of a conjunction $\sim(\alpha\cdot\beta)$ is valid in case when at least one of the formulas $\sim\alpha$, $\sim\beta$ is valid and a formula $\sim\prod_{x_k}\alpha(x_k)$ is valid if and only if there exists such an x_p that $\sim\alpha\binom{x_p}{x_k}$ is valid.

We deal in this paper with the above-mentioned system $\mathcal{S}$ and with the system $\mathcal{S}^*$ of the functional calculus based on $\mathcal{S}$. The algebraic characterization of $\mathcal{S}$ is here generalized on $\mathcal{S}^*$. Using algebraic and topological methods we prove that according to the idea of Nelson a formula $\sim(\alpha\cdot\beta)$ is provable in $\mathcal{S}$ or in $\mathcal{S}^*$ if and only if at least one of the formulas $\sim\alpha$, $\sim\beta$ is provable. Similarly, a formula $\sim\prod_{x_k}\alpha(x_k)$ is provable in $\mathcal{S}^*$ if and only if for some x_p the formula $\sim\left(\alpha\binom{x_p}{x_k}\right)$ is provable. The above mentioned theorems are equivalent to the theorems stating that a disjunction $\alpha+\beta$ is provable in $\mathcal{S}$ or in $\mathcal{S}^*$ if and only if at least one of the formulas α, β is provable and that a formula $\sum_{x_k}\alpha(x_k)$ is provable if and only if for some x_p the formula $\alpha\binom{x_p}{x_k}$ is provable. The decidability of formulas of $\mathcal{S}^*$ having the prenex normal form follows from the last theorem.