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On bases and unconditional convergence of series in Banach spaces

by

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In the monograph of Banach [2] the following statement is given without proof:

(B) *Every infinitely dimensional Banach space contains an infinite dimensional subspace with a basis.*

The main result of § 1 of our work is Theorem 3, the corollaries to which are various generalizations and modifications of (B). § 1 contains also several applications of the "theory of bases" to the study of projections in Banach spaces.

In § 2, using the obtained results concerning bases, we prove that (Theorem 5) in every Banach space X weakly unconditional summability is equivalent to unconditional summability if and only if no subspace of the space X is isomorphic to the space c_0. From this fact we obtain simple proofs of some known theorems and other corollaries concerning unconditional summability.

The last part of this work contains generalizations of the results of previous paragraphs to the case of several classes of linear metric spaces.

Closely connected with this paper is the paper [6] concerning absolute bases in Banach spaces. This concerns especially the problems given at the end of [6].

1. Notation, definitions and basic properties [1]. X denotes a Banach space (except section 7). X^* denotes the space conjugate to X. Elements of X will be denoted by $u, w, x, y, z, \ldots$ Elements of X^* — by $f, g, h, \ldots$

The symbol $[x_n]$ will denote the smallest closed linear set spanned upon the elements (x_n).

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[1]) We intent to preserve the notation and terminology of S. Banach [2].

A sequence (x_n) is said to be a *basic sequence* (an *absolute basic sequence*) if (x_n) is a basis (an absolute basis) of the space $[x_n]$ [2]).

1.1. It is known that if (x_n) is a basic sequence, then every x in $[x_n]$ can be represented in the form

$$x = \sum_{i=1}^{\infty} f_i(x) x_i,$$

where $(f_n) \subset [x_n]^*$ is the sequence biorthogonal to (x_n).

In the sequel the symbol (f_n) will be reserved for denoting the biorthogonal sequence with respect to the basic sequence (x_n).

Suppose that (x_n) is a basic sequence. Let (p_n) be an increasing sequence of positive integers, (t_n) — a sequence of real numbers. A sequence (z_n) which is of the form

$$z_n = \sum_{i=p_n+1}^{p_{n+1}} t_i x_i, \quad z_n \neq 0 \quad (n = 1, 2, \ldots)$$

will be called a *block basis*.

According to criterion 1.3 given below, (z_n) is a basic sequence.

Definition 1. The basic sequences (x_n) and (y_n) are called *equivalent* if

$$\Big\{(t_n): \sum_{i=1}^{\infty} t_i x_i \text{ converges}\Big\} = \Big\{(t_n): \sum_{i=1}^{\infty} t_i y_i \text{ converges}\Big\}.$$

It is well known that [3])

1.2. *If basic sequences (x_n) and (y_n) are equivalent, then the spaces $[x_n]$ and $[y_n]$ are isomorphic.*

We know the following criteria for the sequence (x_n) to be a basic sequence or an absolute basic sequence:

1.3. *(x_n) is a basic sequence if and only if there is a constant $K \geqslant 1$ such that the inequality*

$$(1) \qquad \|t_1x_1 + t_2x_2 + \ldots + t_px_p\| \leqslant K \|t_1x_1 + t_2x_2 + \ldots + t_px_p + \ldots + t_qx_q\|$$

is satisfied for arbitrary positive integers p, q with $p \leqslant q$ and for arbitrary reals $t_1, t_2, \ldots, t_q$.

1.4. *(x_n) is an absolute basic sequence if and only if there is a constant $K_{ab} \geqslant 1$ such that the inequality*

$$(2) \qquad \sup_n |\vartheta_n| \|t_1x_1 + t_2x_2 + \ldots + t_px_p\| \leqslant K_{ab} \|\vartheta_1 t_1 x_1 + \vartheta_2 t_2 x_2 + \ldots + \vartheta_p t_p x_p\|$$

is satisfied for arbitrary positive integer p and for arbitrary reals $t_1, t_2, \ldots, t_p, \vartheta_1, \vartheta_2, \ldots, \vartheta_p$.

[2]) For the basic properties of absolute bases see [13].

[3]) This follows for instance from the consideration of [2], chap. VII, § 3.

§ I

2. Theorem 1. *Let $(x_n) \subset X$ be a basic sequence. If the sequence $(y_n) \subset X$ fulfills the condition*

$$(3) \qquad \sum_{n=1}^{\infty} \|x_n - y_n\| \|f_n\| = \delta < 1,$$

then (y_n) is a basic sequence. (x_n) and (y_n) are equivalent [4]).

Proof. Since

$$|t_i| = |f_i(t_1x_1 + \ldots + t_px_p)| \leqslant \|t_1x_1 + \ldots + t_px_p\| \|f_i\| \qquad (i \leqslant p),$$

we obtain

$$\begin{aligned}\|t_1y_1 + \ldots + t_py_p\| &\leqslant \|t_1x_1 + \ldots + t_px_p\| + \sum_{i=1}^{\infty} |t_i| \|x_i - y_i\| \\ &\leqslant \|t_1x_1 + \ldots + t_px_p\| + \sum_{i=1}^{\infty} \|t_1x_1 + \ldots + t_px_p\| \|f_i\| \|x_i - y_i\| \\ &\leqslant (1+\delta) \|t_1x_1 + \ldots + t_px_p\|\end{aligned}$$

and

$$\begin{aligned}\|t_1y_1 + \ldots + t_qy_q\| &\leqslant \|t_1x_1 + \ldots + t_qx_q\| - \sum_{i=1}^{\infty} |t_i| \|x_i - y_i\| \\ &\leqslant (1-\delta) \|t_1x_1 + \ldots + t_qx_q\|\end{aligned}$$

for arbitrary positive integers p, q $(p \leqslant q)$ and for arbitrary reals $t_1, t_2, \ldots, t_q$.

Hence

$$\|t_1y_1 + t_2y_2 + \ldots + t_py_p\| \leqslant \|t_1y_1 + t_2y_2 + \ldots + t_qy_q\| \cdot K \frac{1+\delta}{1-\delta}.$$

Thus, according to 1.4, (y_n) is a basic sequence.

The equivalence of (x_n) and (y_n) is an immediate consequence of the condition (3).

Theorem 2. *Let (x_n) and (x_n') be basic sequences in a space X. If there exists a projection U of X onto $[x_n]$ and the condition*

$$(4) \qquad \|U\| \sum_{n=1}^{\infty} \|f_n\| \|x_n - x_n'\| < 1$$

is fulfilled, then $[x_n']$ is complemented in X [5]).

[4]) Results of a character similar to ours are given in [17]. Since that paper was not available we could not compare them.

[5]) A linear mapping U of the space X into itself is called a *projection* if $U^2 = U$. A subspace Y is said to be *complemented* in X if there is a projection of X onto Y.

Proof. The formula $A(x) = x - U(x) + \sum_{i=1}^{\infty} f_i(U(x))x'_i$ defines an isomorphic mapping of X onto itself.

Indeed, this follows from the fact that by (4)

$$\|I-A\| = \sup_{\|x\|\leqslant 1} \|x - A(x)\| = \sup_{\|x\|\leqslant 1} \Big\|x - \Big(x - U(x) + \sum_{i=1}^{\infty} f_i(U(x))x'_i\Big)\Big\|$$

$$= \sup_{\|x\|\leqslant 1} \Big\|\sum_{i=1}^{\infty} f_i(U(x))(x_i - x'_i)\Big\| \leqslant \|U\| \sum_{i=1}^{\infty} \|f_i\|\,\|x_i - x'_i\| < 1.$$

It is easily seen that $A([x_n]) = [x'_n]$; then the mapping $AUA^{-1}(x)$ is a projection of X onto $[x'_n]$.

THEOREM 3. *Let (x_n) be a basis of a space X. If a sequence $(y_n) \subset X$ satisfies the conditions*

$$\inf_n \|y_n\| = \varepsilon > 0, \tag{5}$$

$$f_i(y_n) \to 0 \quad (i = 1, 2, \ldots), \tag{6}$$

then there exists a subsequence (y_{p_n}) which is a basic sequence. This basic sequence is equivalent to a block basis (with respect to (x_n)).

Proof. According to 1.3, we may assume that the inequality (1) is satisfied. Let us choose increasing sequences (p_n) and (q_n) of positive integers in such a way that

$$\frac{4K}{\varepsilon}\Big\|\sum_{i=q_n+1}^{\infty} f_i(y_{p_n})x_i\Big\| \leqslant \frac{1}{2^{n+2}}, \tag{7}$$

$$\frac{4K}{\varepsilon}\Big\|\sum_{i=1}^{q_n} f_i(y_{p_{n+1}})x_i\Big\| \leqslant \frac{1}{2^{n+2}} \quad (n = 1, 2, \ldots). \tag{8}$$

(This is possible by (6)).

Set

$$z_n = \sum_{i=q_n+1}^{q_{n+1}} f_i(y_{p_{n+1}})x_i.$$

By (5), (7) and (8),

$$\|z_n\| \geqslant \varepsilon/2 \quad (n = 1, 2, \ldots), \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{4K}{\varepsilon}\|z_n - y_{p_n}\| < \frac{1}{2}. \tag{10}$$

Let $(h_n) \subset [z_n]^*$ be a sequence biorthogonal to (z_n). Using the inequality (1) we find from (9) that $\|h_n\| \leqslant 2K/\tfrac{1}{2}\varepsilon = 4K/\varepsilon$ $(n = 1, 2, \ldots)$.

Therefore by (10) and Theorem 1, (y_{p_n}) is a basis sequence which is equivalent to the block basis (z_n).

3. THEOREM 4. *If the conjugate space X^* contains a subspace isomorphic to c_0, then there exists a projection of X onto a space Y which is isomorphic to l; therefore X^* contains a subspace which is isomorphic to the space m (i. e. all three conditions are equivalent).*

Proof. Suppose that X^* contains a subspace which is isomorphic to c_0. We shall prove that a subspace isomorphic to l is complemented in X. (This is the only non-trivial implication).

Let $(f''_n) \subset X^*$ be a basic sequence equivalent to the unit-vector-basis [6]) of c_0. We may suppose that

$$\|f''_1\| = \|f''_2\| = \ldots = 1, \quad \|t_1f''_1 + \ldots + t_nf''_n\| \leqslant C\sup_{i\leqslant n}|t_i| \quad (C \geqslant 1). \tag{11}$$

Let us note that

$$\sum_{n=1}^{\infty}|f''_n(x)| < +\infty, \quad f''_n(x) \to 0 \quad \text{for every} \quad x \in X, \tag{12}$$

because for every $F \in [f''_n]^*$, $\sum_{n=1}^{\infty}|F(f''_n)| < +\infty$ (this is an obvious property of the unit vectors in c_0).

According to (11) and (12) we can choose a subsequence (f'_n) of (f''_n) and a sequence $(y'_n) \subset X$ with $\|y'_n\| = 1$ $(n = 1, 2, \ldots)$ in such a way that for every n

$$|f'_n(y'_n) - 1| \leqslant 2^{-5}\cdot C^{-1}, \quad f'_{n+1}(y'_i) \leqslant 2^{-n-6}\cdot C^{-1} \quad (i \leqslant n). \tag{13}$$

The set of the numbers $f'_n(y'_i)$ is evidently bounded. Using the diagonal method we can choose a subsequence (r_n) of indices such that if $f_n = f'_{r_{2n}}$, $y_n = y'_{r_{2n}}$ $(n = 1, 2, \ldots)$ then for every i

$$|f_n(y_i) - f_n(y'_{r_{2i-1}})| \leqslant 2^{-n-5}\cdot C^{-1} \quad (n \leqslant i-1). \tag{14}$$

It follows from (13) that

$$|f_n(y_n) - 1| \leqslant 2^{-5}\cdot C^{-1}, \quad |f_{n+1}(y_i)| \leqslant 2^{-n-6}\cdot C^{-1} \quad (i \leqslant n). \tag{13'}$$

Let us set $z_n = y_n - y'_{r_{2n-1}}$ $(n = 1, 2, 3, \ldots)$. By (14), (13′) and (11),

$$|f_n(z_n) - 1| \leqslant |f_n(y_n) - 1| + |f_n(y'_{r_{2n-1}})| \leqslant 2^{-4}\cdot C^{-1}, \tag{15}$$

$$|f_n(z_i)| \leqslant C^{-1}\cdot\sum_{n=1}^{\infty}2^{-n-5} \leqslant 2^{-4}\cdot C^{-1}, \tag{16}$$

$$\|z_n\| \leqslant 2 \quad (n = 1, 2, \ldots). \tag{17}$$

[6]) The unit-vector-basis is composed of the elements $(1, 0, \ldots)$, $(0, 1, 0, \ldots)$, ...

Let $\varepsilon_n = \operatorname{sgn} f_n(z_n)$. By (15), (16), (17) and (11) we obtain

$$(18)\quad \|t_1z_1+\ldots+t_qz_q\| \geqslant (\varepsilon_1 f_1+\ldots+\varepsilon_q f_q)(t_1z_1+\ldots+t_qz_q)\|\varepsilon_1 f_1+\ldots+\varepsilon_q f_q\|^{-1}$$
$$\geqslant \frac{1-2^{-2}C^{-2}}{C}(|t_1|+\ldots+|t_q|) \geqslant \frac{1}{2C}(|t_1|+\ldots+|t_q|)$$

and

$$(19)\quad \|t_1z_1+\ldots+t_pz_p\| \leqslant 2(|t_1|+\ldots+|t_p|).$$

Thus 1° (z_n) is a basic sequence (according to 1.3), 2° (z_n) is equivalent to the unit-vector-basis of l.

According to 2° and (12) the formula

$$U(x) = \sum_{n=1}^{\infty} f_n(x) z_n$$

defines a linear mapping of X into $[z_n]$.

Let $x = \sum_{n=1}^{\infty} t_n z_n$ be an arbitrary element of $[z_n]$. According to (15)-(19),

$$\|x-U(x)\| = \Big\|\sum_{i=1}^{\infty} t_i z_i - \sum_{n=1}^{\infty} t_n f_n(z_n) z_n - \sum_{i=1}^{\infty} t_i \sum_{i\neq n=1}^{\infty} f_n(z_i) z_n\Big\|$$
$$\leqslant \sum_{n=1}^{\infty}|t_k|\cdot\sup_i\Big(|1-f_i(z_i)|+\sum_{i\neq n=1}^{\infty}|f_n(z_i)|\Big)\cdot\sup_n\|z_n\|$$
$$\leqslant 2C\|x\|\cdot(2^{-4}\cdot C^{-1}+2^{-4}\cdot C^{-1})\cdot 2 \leqslant \tfrac{1}{2}\|x\|.$$

To complete the proof it is enough to apply the following

LEMMA 1. *If U is a linear mapping of the space X into its subspace Y and*

$$(20)\quad \|U(y)-y\| \leqslant \delta < 1 \quad \text{for} \quad y \in Y, \quad \|y\| \leqslant 1,$$

then Y is complemented in X.

Proof. Let $A(y) = U(y)$ for $y \in Y$. It follows from (20) that A is an isomorphic mapping of Y onto itself. Thus the formula $V(x) = A^{-1}(U(x))$ defines a projection of X onto Y.

4. Corollaries. We have

C. 1. *If the sequence $(y_n) \subset X$ converges weakly to 0 and $\inf\|y_n\| > 0$ then a subsequence (y_{p_n}) is a basic sequence. Moreover if X can be imbedded in a Banach space with an absolute basis then (y_{p_n}) can be chosen in such a way that it is an absolute basic sequence.*

The first part of this corollary is an immediate consequence of Theorem 3 and the fact that the universal Banach space C has a basis (see [2], Chap. XI, Théorème 10, and [24]).

It is obvious that *a block basis with respect to an absolute basis is an absolute basic sequence*, which implies the second part of C. 1.

C. 2. *If the space Y of an infinite dimension is a subspace of a space X with a basis (x_n), then there is in Y a basic sequence (y_n) which is equivalent to a block basis with respect to (x_n).*

To prove this it is sufficient to note that in the infinitely dimensional space Y there exists a sequence (y'_n) such that $\|y'_n\| = 1$, $f_i(y'_n) = 0$ for $i \leqslant n$, $n = 1, 2, \ldots$ and to apply Theorem 3.

C. 3. *Each infinitely dimensional Banach space X contains an infinitely dimensional subspace with a basis.*

For the proof it is enough to imbed an arbitrary separable subspace of X in the space C and to apply C. 2.

C. 4. *Each infinitely dimensional Banach space which is a subspace of a Banach space with an absolute basis contains an infinitely dimensional subspace with an absolute basis.*

(This follows from the remark which has been made during the proof of C. 1.)

DEFINITION 2. The basis (x_n) is called a *perfectly homogeneous basis* if $\|x_n\| = 1$ $(n = 1, 2, \ldots)$ and each block basis (z_n) with $\|z_n\| = 1$ $(n = 1, 2, \ldots)$ is equivalent to the basis (x_n).

C. 5. *If the space X has a perfectly homogeneous basis then all the infinitely dimensional subspaces of X are of the same linear dimension.*

In particular, if $X = c_0$ or $X = l^p$ $(p \geqslant 1)$ and Y is an infinitely dimensional subspace of X, then the linear dimensions of the spaces X and Y are equal ([2], Chap. XII, Théorème 1).

C. 6. *Let X be a separable Banach space. If there exists a subspace Y which is isomorphic to c_0, then Y contains a subspace Y_1, isomorphic to c_0, which is complemented in X* [7]).

Proof. It may be supposed that X has a basis (x_n) (because X can be imbedded in the universal space C having a basis). Assume that the basis (y_n) of the space Y is equivalent to the unit-vector-basis of the space c_0 and $\|y_n\| = 1$ $(n = 1, 2, \ldots)$. Since the unit-vector-basis in c_0 fulfils the hypothesis of C. 1, there is a basic sequence (y_{p_n}) which is equivalent to a block basis (z_n):

$$z_n = \sum_{i=q_n+1}^{q_{n+1}} t_i x_i \quad (n = 1, 2, \ldots).$$

[7]) This is a particular case of Sobczyk's theorem ([25], Theorem 5).

Moreover, we may suppose (see the proof of Theorem 3) that

$$(21)\qquad 3K\sum_{n=1}^{\infty}\|h_n\|\,\|z_n-y_n\|<1, \qquad \|z_n\|=1 \qquad (n=1,2,\ldots),$$

where $(h_i)\subset[z_n]^*$ is the sequence biorthogonal to (z_n), K being a constant satisfying (1).

Let $Z_n=[x_{q_n+1}, x_{q_n+2}, \ldots, x_{q_n+1}]$. The linear functionals (h_n) can be extended to linear functionals (h'_n) defined in the whole space X in such a way that

$$\sup_n\|h'_n\|<3K,$$

$$h'_n(x)=0 \quad \text{for every} \quad x\,\epsilon\,Z_m, \quad m\neq n.$$

Thus $h'_n(x)\to 0$ for every x in X. Therefore the formula

$$U(x)=\sum_{i=1}^{\infty}h'_i(x)z_i$$

defines a projection operation of X onto $[z_n]$ such that $\|U\|\leqslant 3K$.

Thus, by (21) and Theorem 2, $[y_{p_n}]$ is a projection of X.

C. 7. *Suppose that X can be imbedded in a Banach space with an absolute basis. If a subspace Y of X is equivalent to the space l, then Y contains a subspace Y_1, isomorphic to l, which is complemented in X.*

To prove this corollary it is enough to reproduce a fragment of James's proof ([15], Theorem 2) and to apply our Theorem 2.

Remark. The inclusion $Y_1\subset Y$ cannot be replaced by the equality: *There is a space X with an absolute basis, a subspace Y of which is isomorphic to l, but there is no projection of X onto Y.*

§ II

5. Definition 3. The series $\sum_{n=1}^{\infty}x_n$ is *w. u. c. (weakly unconditionally convergent)* if for every permutation (k_n) of indices the series $\sum_{n=1}^{\infty}x_{k_n}$ converges weakly (may be that the limit-element does not exist).

The series $\sum_{n=1}^{\infty}x_n$ is *u. c. (unconditionally convergent)* if for every permutation (k_n) the series $\sum_{n=1}^{\infty}x_{k_n}$ converges.

We shall need several properties of w. u. c. series [8]).

[8]) w. u. c. and u. c. series are considered in [20], [8], and [11]. Lemma 2 and further properties (IV-VI) of w. u. c. series can easily be deduced from the results given in those papers.

Lemma 2. *The following conditions are equivalent:*

(I) $\sum_{n=1}^{\infty}x_n$ *is w. u. c.,*

(II) *there is a constant C such that for every bounded real sequence (t_n) the inequality*

$$\sup_n\Big\|\sum_{i=1}^{n}t_ix_i\Big\|\leqslant C\sup_i|t_i|$$

holds,

(III) *for every real sequence (t_n) tending to 0 the series $\sum_{n=1}^{\infty}t_nx_n$ converges.*

Proof. 1° (I) implies (II). Let

$$Z_k=\Big\{f\,\epsilon\,X^*: \sum_{i=1}^{\infty}|f(x_i)|\leqslant k\Big\} \qquad (k=1,2,\ldots).$$

The sets Z_k are closed and $Z_1+Z_2+\ldots=X^*$ (because $\sum_{n=1}^{\infty}x_n$ is w. u. c.). Using Baire's theorem we find that $\sum_{n=1}^{\infty}|f(x_n)|\leqslant C$ for some $C>0$ and for every f with $\|f\|\leqslant 1$.

Thus

$$\Big\|\sum_{i=1}^{n}t_ix_i\Big\|=\sup_{|f|\leqslant 1}\Big|f\Big(\sum_{i=1}^{n}t_ix_i\Big)\Big|\leqslant\sup_{\|f\|\leqslant 1}\sum_{i=1}^{n}|f(x_i)|\cdot\sup_m|t_m|\leqslant C\sup_m|t_m| \qquad (n=1,2,\ldots).$$

2° (II) implies (III). According to (II), if $t_n\to 0$, then

$$\Big\|\sum_{n=p}^{q}t_nx_n\Big\|\leqslant C\cdot\sup_{p\leqslant n\leqslant q}|t_n|\to 0 \quad \text{for} \quad p,q\to+\infty \quad (p<q),$$

i. e. the series $\sum_{n=1}^{\infty}i_nx_n$ fulfils the Cauchy condition.

3° (III) implies (I). According to (II) and the well-known property of numerical series, we have

$$\sum_{n=1}^{\infty}|f(x_n)|<+\infty,$$

q. e. d.

Every w. u. c. series $\sum_{n=1}^{\infty}x_n$ has also the following properties (which can easily be reduced to the properties of numerical series):

(IV) $\sum_{n=1}^{\infty}x_{k_n}$ is w. u. c. for an arbitrary permutation (k_n) of indices,

(V) if (q_n) is an arbitrary increasing sequence of indices and $y_n = \sum_{i=q_n+1}^{q_{n+1}} x_i$, then the series $\sum_{n=1}^{\infty} y_n$ is w. u. c.

(VI) $\sum_{n=1}^{\infty} x_{r_n}$ is w. u. c. for every subsequence (r_n) of indices.

THEOREM 5. *The following conditions are equivalent:*

(α) *there exists in the space X a w. u. c. series which is not u. c.;*

(β) *there exists in the space X a w. u. c. series $\sum_{n=1}^{\infty} x_n$ such that $\inf_n \|x_n\| > 0$;*

(γ) *X contains a subspace which is isomorphic to c_0.*

For the proof we shall need the following

LEMMA 3[9]. *If a basic sequence (x_n) fulfils the conditions*

$$\inf_n \|x_n\| > 0, \tag{22}$$

$$\sum_{n=1}^{\infty} x_n \text{ is w. u. c.}, \tag{23}$$

then the basic sequence (x_n) is equivalent to the unit-vector-basis of the space c_0; therefore the space $[x_n]$ is isomorphic to c_0.

Proof of the lemma. It follows from (12) that if the series $\sum_{n=1}^{\infty} t_n x_n$ is convergent, then $t_n \to 0$. From (III) follows the converse implication. Thus the proposition 1.2 implies the statement of the lemma.

Proof of Theorem 5. 1° (α) implies (β). Indeed, if the w. u. c. series $\sum_{n=1}^{\infty} x_n$ is not u. c., then for some permutation (k_n) of indices the series $\sum_{n=1}^{\infty} x_{k_n}$ is not convergent. It follows that there is an increasing sequence of indices (q_n) such that

$$\inf_n \Big\| \sum_{i=q_n+1}^{q_{n+1}} x_{k_i} \Big\| > 0.$$

Hence according to (IV) and (V) the series $\sum_{n=1}^{\infty} y_n$, where $y_n = \sum_{i=q_n+1}^{q_{n+1}} x_{k_i}$, satisfies the condition (β).

2° (β) implies (γ). Let $\sum_{n=1}^{\infty} x_n$ be a w. u. c. series such that $\inf_n \|x_n\| = \delta > 0$.

[9]) Results similar to this lemma are given in [1].

From the Definition 3 it follows that (x_n) weakly converges to 0. Hence, by C. 1, the sequence (x_{p_n}) fulfils all conditions of Lemma 1; therefore the space $[x_{p_n}]$ is isomorphic to c_0.

3° (γ) implies (α). In the space c_0 the series composed of unit vectors is w. u. c. without being u. c. Since the weak unconditional convergence of the series $\sum_{n=1}^{\infty} x_n$ is invariant under isomorphic mappings, the last implication is proved, q. e. d.

6. Consequences of Theorem 5. We have

C. 8. THEOREM OF ORLICZ ([20], Satz 2). *If the space X is weakly complete, then every w. u. c. series in X is u. c.*

This immediately follows from the fact that every subspace of a weakly complete space is weakly complete but the space c_0 is not weakly complete.

C. 9. *If for every subsequence of indices (p_n) the series $\sum_{n=1}^{\infty} x_{p_n}$ converges weakly to an element, then $\sum_{n=1}^{\infty} x_n$ is u. c.*[10]).

According to Theorem 5, it is enough to establish this fact for the space c_0. We omit the easy proof.

C. 10. *In separable conjugate spaces every w. u. c. series is u. c.*

This is an immediate consequence of Theorems 4 and 5.

Since $X \subset X^{**}$, C. 10 implies

C. 11. *If the second conjugate space X^{**} is separable, then in the space X every w. u. c. series is u. c.*

Let us note that if the space X has the "extension property"[11]), then no projection of X is isomorphic to the space l, because in the opposite case the space l would be a projection of the space C, which is not true by [3]. Thus, by Theorems 4 and 5,

C. 12. *If X has the "extension property", then in the conjugate space X^* every w. u. c. series is u. c.*

Since there are Banach spaces which are not weakly complete but have separable second conjugate spaces, C. 11 implies

C. 13. *The converse of Orlicz's theorem* C. 8 *is not valid* (cf. [6], Theorem 2).

[10]) This result of Orlicz can be deduced, for instance, from the proof of "Satz 2" in [20].

[11]) For the properties of the spaces having "extension property" see [12]. According to Theorem 3.2 of this paper, if X is complemented in a space with "extension property", then it is complemented in every space in which it can be imbedded.

Remark. It is proved in [21] that in every space which has the property (β) the functional $\|x\|$ cannot be uniformly approximated by polynomials in any sphere. In view of Theorem 5 this problem reduces to the case of the space c_0 (see [7]).

Suppose that

$$U(X) = \{(x_n) \subset X \colon \sum_{n=1}^{\infty} x_n \text{ is u. c.}\},$$

$$B(X) = \{(x_n) \subset X \colon \sum_{n=1}^{\infty} x_n \text{ is w. u. c.}\},$$

$$B_w(X) = \{(x_n) \in B(X) \colon \sum_{n=1}^{\infty} x_n \text{ weakly converges to an element}\},$$

$$B_s(X) = \{(x_n) \in B(X) \colon \sum_{n=1}^{\infty} x_n \text{ is convergent}\},$$

$$IS(X) = \{(x_n) \in B_s(X) \colon \text{if for some permutation } (k_n) \text{ of indices } \sum_{n=1}^{\infty} x_{k_n} \text{ converges, then } \sum_{n=1}^{\infty} x_{k_n} = \sum_{n=1}^{\infty} x_n\}.$$

In [19] the following inclusions are proved:

$$U(X) \subset B_s(X) = IS(X) \frown B(X) \subset B_w(X) \subset B(X). \tag{24}$$

C. 14. *All the inclusion (24) are proper if the space X contains a subspace which is isomorphic to c_0. Otherwise all these inclusions may be replaced by equalities.*

This follows from Theorem 5 and the corresponding properties of the space c_0.

7. Generalizations. Let us denote:

by $\mathfrak{R}$ the class of all spaces of type F in which there exists a bounded neighbourhood (see [23]),

by $\mathfrak{M}$ the class of all spaces of type B_0 which possess a continuous homogeneous norm (see [4] and [5]).

7.1. The Theorems 1, 2 and 3 may be generalized as follows:

THEOREM 1′. *Let (x_n) be a basic sequence in a space X which belongs to the class $\mathfrak{M}$ or $\mathfrak{R}$. Then there exists a sequence (ε_n) of positive numbers such that every sequence $(y_n) \subset Y$ which fulfils the inequalities $\varrho(x_n, y_n) \leqslant \varepsilon_n$ $(n = 1, 2, \ldots)$ is a basic sequence equivalent to (x_n) (ϱ denotes the metric of X).*

THEOREM 2′. *Let (x_n) be a basic sequence in a space X which belongs to the class $\mathfrak{M}$ or $\mathfrak{R}$. Suppose that $[x_n]$ is complemented in X. Then there exists a sequence (ε_n) of positive numbers such that the condition $\varrho(x_n, x_n') \leqslant \varepsilon_n$ $(n = 1, 2, \ldots)$ implies that $[x_n']$ is complemented in X.*

Remark. It is enough to assume that X is a B_0-space and $[x_n] \in \mathfrak{M}$.

THEOREM 3′. *Let (x_n) be a basis of the space X which either belongs to the class $\mathfrak{R}$ or is a B_0-space. If a sequence $(y_n) \subset X$ satisfies the condition*

$$\inf_n \varrho(y_n, 0) > 0, \quad f_i(y_n) \to 0 \quad (i = 1, 2, \ldots),$$

then there exists a subsequence (y_{p_n}) which is a basic sequence. (y_{p_n}) is equivalent to a block basis.

7.2. The corollaries C. 1.-C. 6 may be extended to the case of arbitrary B_0-spaces, C. 2, C. 4, C. 5 — to the case of the class $\mathfrak{R}$.

Remark. It follows from the generalization of C. 5 that if $X = l^p$ $(0 < p < \infty)$ (for the definition see [18], 1.621) and Y is an infinitely dimensional subspace of X, then the linear dimensions of the spaces X and Y are equal.

7.3. Theorem 5 and Lemma 3 hold true for B_0-spaces. We must only replace condition (II) by the condition

$$(\mathrm{II})' \quad |t_1x_1 + t_2x_2 + \ldots + t_nx_n|_k < C \sup_{i \leqslant n} |t_i|,$$

where $|\cdot|_k$ denotes the k-th pseudonorm of the space X.

Added in proof. During the print of this paper we have obtained some new corollaries to theorems 4 and 5. They have been published in the paper *Some remarks on conjugate spaces containing subspaces isomorphic to the space c_0*, Bull. Acad. Pol. Sci. 6 (1958), p. 249-250.

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A generalization of results of R. C. James concerning absolute bases in Banach spaces

by

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The main results obtained by R. C. James in his paper [14][1]) can be presented as follows:

Let X be a Banach space with an absolute basis.

(a) X is weakly complete if and only if no subspace of X is isomorphic to c_0.

(b) Every bounded set in X is weakly conditionally compact[2]) if and only if no subspace of X is isomorphic to l.

The purpose of this paper is to show that the propositions (a) and (b) hold true also in the case where the space X can be imbedded in a space with an absolute basis.

The last part of this paper contains several questions which are connected with the present paper and with the one entitled *On bases and unconditional convergence of series in Banach spaces* (this fasc., p. 151-164) which is subsequently denoted by [0].

Terminology and notation used in this paper are the same as in [0].

1. In the following we shall need several lemmas.

1.1. *In order that an absolute basic sequence (x_n) with $\sup_n \|x_n\| < +\infty$ be equivalent to the unit-vector-basis of the space l it is necessary and sufficient that there exist a linear functional f such that $\inf_n |f(x_n)| > 0$.*

Sufficiency. 1° If $\sum_{n=1}^{\infty}|t_n| < +\infty$, then $\sum_{n=1}^{\infty} t_n x_n$ is convergent.

[1]) The numbers in brackets [ ] refer to the "References" of the paper *On bases and unconditional convergence of series in Banach spaces*; in this fasc., p. 163.

[2]) A set $Z \subset Y$ is said to be *weakly conditionally compact* if every bounded sequence composed of elements of Z contains a weakly convergent subsequence.