STUDIA MATHEMATICA, T. XVII. (1958)

Local characteristics of generalized stochastic processes

by

K. URBANIK (Wrocław)

Let us consider generalized stochastic processes whose realizations are distributions. Using the notion of the value of distributions at a fixed point [3] it is easy to verify that almost all realizations of the derivative of a homogeneous Brownian motion process have no values at a fixed moment. Consequently, there is no probability distribution of this process at a fixed moment. But from an intuitive point of view the derivative of a Brownian motion process has at a fixed moment a "probability distribution" uniform on the line. In connection with this fact at the III-d All-Union Mathematical Congress in Moscow in June 1956 I. M. Gelfand raised the following problem:

To define the generalized probability distribution of generalized stochastic processes at a fixed moment such that the generalized probability distributions at all the fixed moments of $d^k f(\omega, t)/dt^k$ $(k \geqslant 1)$, *where* $f(\omega, t)$ *is a process with independent increments, give the probability characteristic of the increments of* $f(\omega, t)$.

The aim of the present paper is to give a solution of Gelfand's problem. In this paper we introduce the notion of local characteristics, which can be regarded as a generalization of the notion of probability distributions with finite moments.

We remark that the definition of local characteristics for deterministic generalized processes[1]) can be reduced to the definition of values at a fixed point of distributions. Therefore there exist generalized processes having no local characteristics.

The subject of Section I is the definition of the local characteristic at a fixed moment and the investigation of elementary properties of local characteristics. In Section II we shall analyse some local properties of homogeneous stochastic processes with independent increments. Section III contains some theorems concerning infinitely divisible distributions. Section IV contains a complete discussion of local character-

[1]) We say that a generalized process $\Phi(\omega, t)$ is *deterministic* if $\Phi(\omega, t) = \Phi(t)$, where $\Phi(t)$ is a distribution.

istics for derivatives of homogeneous processes with independent increments.

I wish to express my thanks to Professor I. M. Gelfand for his valuable remarks, which I have utilized in the present paper.

I. We shall use the following notations:

$$\Delta_h f(\omega, t) = f(\omega, t+h) - f(\omega, t),$$

(I.1)

$$D_{h_1, h_2, \ldots, h_k} f(\omega, t) = \frac{\Delta_{\lambda_1} \Delta_{\lambda_2} \ldots \Delta_{\lambda_k} f(\omega, t)}{\lambda_1 \lambda_2 \ldots \lambda_k},$$

where

(I.2) $$\lambda_k = h_k, \quad \lambda_j = h_j + \sum_{s=1}^{k-j} 2^{s-1} h_{j+s} \quad (j = 1, 2, \ldots, k-1).$$

The limit $\lim\limits_{h_1 \downarrow 0} \lim\limits_{h_2 \downarrow 0} \ldots \lim\limits_{h_k \downarrow 0}$ we shall briefly denote by $\operatorname*{Lim}\limits_{h_1, \ldots, h_k \downarrow 0}$.

Let $\mathfrak{M}$ be the class of measurable stochastic processes $f(\omega, t)$ with expectations $E|f(\omega, t)|^n$ $(n = 1, 2, \ldots)$ (Lebesgue-integrable over every finite interval) for which $\lim\limits_{h \to 0} E|\Delta_h f(\omega, t)|^n = 0$. Since almost all sample functions of processes belonging to $\mathfrak{M}$ are Lebesgue integrable over every finite interval (see Doob [1], p. 62), then $\mathfrak{M}$ is a subclass of the class of generalized stochastic processes (see Urbanik [6], § I.4).

In the sequel X_n $(n = 0, 1, \ldots)$ denotes the Banach space of functions φ continuous in $-\infty < x < \infty$, having the following limits:

(I.3) $$G_n^+(\varphi) = \lim_{x \to \infty} \frac{\varphi(x)}{x^n}, \qquad G_n^-(\varphi) = \lim_{x \to -\infty} \frac{\varphi(x)}{|x|^n}.$$

The norm in X_n will be defined by the formula

$$\|\varphi\| = \sup_{-\infty < x < \infty} \frac{|\varphi(x)|}{1+|x|^n}.$$

LEMMA I.1. *Let $f(\omega, t) \in \mathfrak{M}$ and*

(I.4) $$g(\omega, t) = \frac{1}{(s-1)!} \int_0^t (t-u)^{s-1} f(\omega, u)\, du.$$

If

(I.5) $$\operatorname*{Lim}_{h_1, \ldots, h_s \downarrow 0} D_{h_1, h_2, \ldots, h_s} g(\omega, t_0) = f(\omega, t_0)$$

in probability, then for each integer k and for each $\varphi \in \bigcup\limits_{n=0}^{\infty} X_n$ the equality

$$\operatorname*{Lim}_{h_{k+1}, \ldots, h_{k+s} \downarrow 0} \int_{-\infty}^{\infty} \varphi(x)\, dP\big(D_{h_1, h_2, \ldots, h_{k+s}} g(\omega, t_0) < x\big)$$

$$\int_{-\infty}^{\infty} \varphi(x)\, dP\big(D_{h_1, h_2, \ldots, h_k} f(\omega, t_0) < x\big)$$

holds.

Proof. Let

(I.6) $$H_n(h_1, \ldots, h_{k+s}; x) = \int_{-\infty}^{x} (1+|u|^n)\, dP\big(D_{h_1, h_2, \ldots, h_{k+s}} g(\omega, t_0) < u\big),$$

(I.7) $$H_n^*(h_1, \ldots, h_k; x) = \int_{-\infty}^{x} (1+|u|^n)\, dP\big(D_{h_1, h_2, \ldots, h_k} f(\omega, t_0) < u\big).$$

Equalities (I.1) and (I.4) imply

$$D_{h_1, h_2, \ldots, h_{k+s}} g(\omega, t_0)$$
$$= \frac{1}{\lambda_{k+1} \ldots \lambda_{k+s}} \int_{t_0}^{t_0+\lambda_{k+1}} \int_{x_1}^{x_1+\lambda_{k+2}} \int_{x_{s-1}}^{x_{s-1}+\lambda_{k+s}} \frac{\Delta_{\lambda_1} \ldots \Delta_{\lambda_k}}{\lambda_1 \ldots \lambda_k} f(\omega, u)\, du\, dx_1 \ldots dx_{s-1}.$$

Hence, in view of the convexity of the function $|x|^n$ $(n = 1, 2, \ldots)$,

$$|D_{h_1, h_2, \ldots, h_{k+s}} g(\omega, t_0)|^n$$
$$\leqslant \frac{1}{\lambda_{k+1} \ldots \lambda_{k+s}} \int_{t_0}^{t_0+\lambda_{k+1}} \int_{x_1}^{x_1+\lambda_{k+2}} \ldots \int_{x_{s-1}}^{x_{s-1}+\lambda_{k+s}} \left| \frac{\Delta_{\lambda_1} \ldots \Delta_{\lambda_k}}{\lambda_1 \ldots \lambda_k} f(\omega, u) \right|^n du\, dx_1 \ldots dx_{s-1}.$$

Thus the following inequality is satisfied:

$$H_n(h_1, \ldots, h_{k+s}; \infty) = 1 + E|D_{h_1, h_2, \ldots, h_{k+s}} g(\omega, t_0)|^n$$

$$\leqslant 1 + \frac{1}{\lambda_{k+1} \ldots \lambda_{k+1}} \times$$

$$\times \int_{t_0}^{t_0+\lambda_{k+1}} \int_{x_1}^{x_1+\lambda_{k+2}} \ldots \int_{x_{s-1}}^{x_{s-1}+\lambda_{k+s}} E \left| \frac{\Delta_{\lambda_1} \ldots \Delta_{\lambda_k}}{\lambda_1 \ldots \lambda_k} f(\omega, u) \right|^n du\, dx_1 \ldots dx_{s-1}.$$

Since $f(\omega, t) \in \mathfrak{M}$, the last inequality implies

$$\operatorname*{Lim\,sup}_{h_{k+1}, \ldots, h_{k+s} \downarrow 0} H_n(h_1, \ldots, h_{k+s}; \infty) \leqslant 1 + E|D_{h_1, h_2, \ldots, h_k} f(\omega, t_0)|^n.$$

Consequently, according to (I.7),

(I.8) $$\operatorname*{Lim\,sup}_{h_{k+1},\ldots,h_{k+s}\downarrow 0} H_n(h_1,\ldots,h_{k+s};\infty) \leqslant H_n^*(h_1,\ldots,h_k,\infty) < \infty .$$

From the assumption (I.5) we obtain

(I.9) $$\operatorname*{Lim}_{h_{k+1},\ldots,h_{k+s}\downarrow 0} \big(H_n(h_1,\ldots,h_{k+s};x_1)-H_n(h_1,\ldots,h_{k+s};x_1)\big)$$
$$= H_n^*(h_1,\ldots,h_k;x_1)-H_n^*(h_1,\ldots,h_k;x_2)$$

at all continuity points x_1, x_2 of the function $H_n^*(h_1,\ldots,h_k;x)$. Thus for each x_1, x_2 the inequality

$$\operatorname*{Lim\,inf}_{h_{k+1},\ldots,h_{k+s}\downarrow 0} H_n(h_1,\ldots,h_{k+s};\infty) \geqslant H_n^*(h_1,\ldots,h_k;x_1)-H_n^*(h_1,\ldots,h_k;x_2)$$

holds. Consequently,

$$\operatorname*{Lim\,inf}_{h_{k+1},\ldots,h_{k+s}\downarrow 0} H_n(h_1,\ldots,h_{k+s};\infty) \geqslant H_n^*(h_1,\ldots,h_k\,;\infty) .$$

Hence and from (I.8) it follows

(I.10) $$\operatorname*{Lim}_{h_{k+1},\ldots,h_{k+s}\downarrow 0} H_n(h_1,\ldots,h_{k+s};\infty) = H_n^*(h_1,\ldots,h_k;\infty) .$$

Since $H_n(h_1,\ldots,h_{k+s};-\infty) = 0 = H_n^*(h_1,\ldots,h_k;-\infty)$, the equalities (I.9) and (I.10) imply for each $\varphi \in X_0$

(I.11)
$$\operatorname*{Lim}_{h_{k+1},\ldots,h_{k+s}\downarrow 0} \int_{-\infty}^{\infty} \psi(x)\,dH_n(h_1,\ldots,h_{k+s};x) = \int_{-\infty}^{\infty} \psi(x)\,dH_n(h_1,\ldots,h_k;x) .$$

Putting $\psi(x) = \varphi(x)/(1+|x|^n)$ for $\varphi \in X_n$ $(n = 0,1,\ldots)$, from formulas (I.6), (I.7) and (I.11) we obtain the assertion of Lemma.

Let $\langle a_n, L_n\rangle$ $(n = 0,1,\ldots)$ denote a sequence where $a_n = a_n(\lambda)$ is a positive function defined for $\lambda > 0$ and L_n is a continuous linear functional in X_n satisfying the following conditions: $L_n(\varphi) \geqslant 0$ for $\varphi \geqslant 0$ $(\varphi \in X_n)$, $L_n(1+|x|^n) = 1$.

We say that the sequence $\langle a_n, L_n\rangle$ is *the local characteristic of the generalized stochastic process* $\Phi(\omega, t)$ [2]) *at the moment* t_0 if there exist an

[2]) In this paper we consider the generalized stochastic processes defined in the papers [6] and [7], i. e. the generalized derivatives of continuous stochastic processes.

integer k, a system of positive functions $A_{1n}(\lambda),\ldots,A_{kn}(\lambda)$ and a continuous stochastic process $f(\omega,t)\in\mathfrak{M}$ such that

$$\frac{d^k}{dt^k} f(\omega,t) = \Phi(\omega,t), \qquad a_n(\lambda) = \prod_{j=1}^{k} A_{jn}(\lambda) \qquad (n = 0,1,\ldots),$$

and for each $\varphi\in X_n$ $(n = 0,1,\ldots)$

$$\operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \prod_{j=1}^{k} A_{jn}(h_j) \int_{-\infty}^{\infty} \varphi(x)\,dP\big(D_{h_1,\ldots,h_k} f(\omega,t_0) < x\big) = L_n(\varphi) .$$

The author has been led to regard local characteristics (generalized distribution functions) as sequences of functions and sequences of functionals by the following intuitive reasoning. Distributions are realizations of generalized stochastic processes. From an intuitive point of view those realizations assume at a fixed point either numerical values or "infinite values". Infinite values may be of "different orders". E. g. the infinity of the distribution $\delta'(x)$ for $x = 0$ is stronger than the infinity of the distribution $\delta(x)$ for $x = 0$ because $x\delta(x) = 0$ and $x\delta'(x) = -\delta(x)$.

Besides the infinities $+\infty$, $-\infty$ we must also consider the infinity accompanied by two signs $\pm\infty$, which, for instance, is assumed by the distribution $\delta'(x)$ (dipole) for $x = 0$. This shows that the local characteristics would describe not only the distribution of numerical values but also the distribution of "infinite values" of various orders. Roughly speaking, the functional L_0 describes the distribution of numerical values and the functional L_n $(n = 1,2,\ldots)$ describes the distribution of "infinite values" of the same order as $\lim_{\lambda\downarrow 0} a_n^{-1}(\lambda)$.

We say that a local characteristic $\langle a_n, L_n\rangle$ is *equal* to a local characteristic $\langle \tilde{a}_n, \tilde{L}_n\rangle$ if $\tilde{L}_n = L_n$ and

$$\lim_{\lambda\downarrow 0} \frac{\tilde{a}_n(\lambda)}{a_n(\lambda)} = 1 \quad \text{for} \quad n = 0,1,\ldots$$

Now we prove the following

THEOREM I.1. *The local characteristic at a fixed moment of a generalized stochastic process is uniquely determined.*

Proof. Let $\Phi(\omega,t)$ be a generalized stochastic process. Let us suppose that there exist integers k_1, k_2, systems of functions $A_{1n}^{(1)}(\lambda),\ldots,A_{k_1 n}^{(1)}(\lambda)$, $A_{1n}^{(2)}(\lambda),\ldots,A_{k_2 n}^{(2)}(\lambda)$ and continuous stochastic processes $f_1(\omega,t)$, $f_2(\omega,t)$ belonging to $\mathfrak{M}$ and satisfying the following conditions:

(I.12) $$\frac{d^{k_1}}{dt^{k_1}} f_2(\omega,t) = \Phi(\omega,t) = \frac{d^{k_2}}{dt^{k_2}} f_2(\omega,t) ,$$

for each $\varphi \in X_n$ the limits

(I.13) $L_n^{(1)}(\varphi) = \operatorname*{Lim}_{h_1,\dots,h_{k_1}\downarrow 0} \prod_{j=1}^{k_1} A_{jn}^{(1)}(h_j) \int_{-\infty}^{\infty} \varphi(x)\, dP(D_{h_1,\dots,h_{k_1}} f_1(\omega, t_0) < x)$,

(I.14) $L_n^{(2)}(\varphi) = \operatorname*{Lim}_{h_1,\dots,h_{k_2}\downarrow 0} \prod_{j=1}^{k_2} A_{jn}^{(2)}(h_j) \int_{-\infty}^{\infty} \varphi(x)\, dP(D_{h_1,\dots,h_{k_2}} f_2(\omega, t_0) < x)$

exist, and

(I.15) $L_n^{(1)}(1+|x|^n) = 1 = L_n^{(2)}(1+|x|^n) \quad (n = 0, 1, \dots).$

Put

(I.16) $a_n^{(1)}(\lambda) = \prod_{j=1}^{k_1} A_{jn}^{(1)}(\lambda), \quad a_n^{(2)}(\lambda) = \prod_{j=1}^{k_2} A_{jn}^{(2)}(\lambda).$

To prove the theorem it is sufficient to prove that the equality

(I.17) $\langle a_n^{(1)}, L_n^{(1)}\rangle = \langle a_n^{(2)}, L_n^{(2)}\rangle$

holds.

First let us assume that the equality $k_1 = k_2 = k$ is true. Then from the formula (I.12) it follows that

$$f_1(\omega, t) = f_2(\omega, t) + \sum_{j=0}^{k-1} a_j(\omega) t^j$$

(see [6], § I.4). Hence, in view of (I.1), $D_{h_1,\dots,h_k} f_1(\omega, t) = D_{h_1,\dots,h_k} f_2(\omega, t)$ for every $h_1, \dots, h_k$. Consequently, according to (I.14),

(I.18) $L_n^{(2)}(\varphi) = \operatorname*{Lim}_{h_1,\dots,h_k\downarrow 0} \prod_{j=1}^{k} A_{jn}^{(2)}(h_j) \int_{-\infty}^{\infty} \varphi(x)\, dP(D_{h_1,\dots,h_k} f_1(\omega, t_0) < x)$

for $\varphi \in X_n$ $(n = 0, 1, \dots)$. The last equality and the formula (I.13) imply

$$\operatorname*{Lim}_{h_1,\dots,h_k\downarrow 0} \frac{\prod_{j=1}^{k} A_{jn}^{(1)}(h_j)}{\prod_{j=1}^{k} A_{jn}^{(2)}(h_j)} = \frac{L_n^{(1)}(\varphi)}{L_n^{(2)}(\varphi)}$$

for $L_n^{(2)}(\varphi) \neq 0$ $(\varphi \in X_n,\ n = 0, 1, \dots)$. Putting $\varphi(x) = 1+|x|^n$ we obtain, in view of (I.15) and (I.16),

$$\lim_{\lambda\downarrow 0} \frac{a_n^{(1)}(\lambda)}{a_n^{(2)}(\lambda)} = 1 \quad (n = 0, 1, \dots).$$

Hence, according to (I.13) and (I.18), we obtain the equality $L_n^{(1)} = L_n^{(2)}$ $(n = 0, 1, \dots)$.

The equality (I.17) is thus proved.

Now let us assume that $k_1 \neq k_2$. We may suppose that $k_1 = k_2 + s$ and $s \geqslant 1$. Let

$$g(\omega, t) = \frac{1}{(s-1)!} \int_0^t (t-u)^{s-1} f_2(\omega, u)\, du.$$

Then equality (I.12) implies

$$g(\omega, t) = f_1(\omega, t) + \sum_{j=0}^{k_1-1} a_j(\omega) t^j.$$

Hence for each $h_1, \dots, h_{k_1}$

(I.19) $D_{h_1,\dots,h_{k_1}} g(\omega, t) = D_{h_1,\dots,h_{k_1}} f_1(\omega, t).$

Since $f_2(\omega, t)$ is a continuous process, we have

$$\operatorname*{Lim}_{h_1,\dots,h_s\downarrow 0} D_{h_1,\dots,h_s} g(\omega, t) = f_2(\omega, t)$$

with probability 1. Consequently the assumption (I.5) of Lemma I.1 is satisfied. Therefore, in view of Lemma I.1 and the formula (I.19), the equality

$$\operatorname*{Lim}_{h_{k_2+1},\dots,h_{k_1}\downarrow 0} \int_{-\infty}^{\infty} \varphi(x)\, dP(D_{h_1,\dots,h_{k_1}} f_1(\omega, t_0) < x) = \int_{-\infty}^{\infty} \varphi(x)\, dP(D_{h_1,\dots,h_{k_2}} f_2(\omega, t_0) < x)$$

holds for each $\varphi \in \bigcup_{n=0}^{\infty} X_n$. Hence, according to (I.13), we obtain the convergence

(I.20) $A_n = \operatorname*{Lim}_{h_{k_2+1},\dots,h_{k_1}\downarrow 0} \prod_{j=k_2+1}^{k_1} A_{jn}^{(1)}(h_j) \quad (n = 0, 1, \dots),$

and for $\varphi \in X_n$ $(n = 0, 1, \dots)$

$$L_n^{(1)}(\varphi) = \operatorname*{Lim}_{h_1,\dots,h_{k_2}\downarrow 0} A_n \prod_{j=1}^{k_2} A_{jn}^{(1)}(h_j) \int_{-\infty}^{\infty} \varphi(x)\, dP(D_{h_1,\dots,h_{k_2}} f_2(\omega, t_0) < x).$$

Hence, in virtue of the first part of the proof and the formula (I.14), we obtain the equalities

$$L_n^{(1)} = L_n^{(2)}, \quad \lim_{\lambda\downarrow 0} \frac{A_n \prod_{j=1}^{k_2} A_{jn}^{(1)}(\lambda)}{a_n^{(2)}(\lambda)} = 1.$$

From this formula taking into account formula (I.20), we obtain

$$\lim_{\lambda\downarrow 0}\frac{a_n^{(1)}(\lambda)}{a_n^{(2)}(\lambda)}=1.$$

Consequently the equality (I.17) is true. The theorem is thus proved.

Examples. a) *The derivatives of a homogeneous Brownian motion process*. Let $f_1(\omega, t)$ be a homogeneous Brownian motion process with the variance $\sigma^2|t|$. Evidently, $f_1(\omega,t)\in\mathfrak{M}$. The derivative $\Phi_1(\omega,t)=d^kf_1(\omega,t)/dt^k$ $(k\geqslant 1)$ is a stationary generalized process with independent values (see [6], theorems 8 and 18). The local characteristic at an arbitrary moment of the process $\Phi_1(\omega,t)$ is given by the following formulas:

$$a_0(\lambda)=\tfrac{1}{2},\qquad L_0=\tfrac{1}{4}(G_0^+ + G_0^-),$$

$$a_n(\lambda)=\sqrt{\pi}\,\lambda^{n/2+k-1}\,2^{-k\frac{n}{2}}\,\sigma^{-n}\left\{\Gamma\left(\frac{n+1}{2}\right)\right\}^{-1}\qquad (n=1,2,\ldots),$$

$$L_n=\tfrac{1}{2}(G_n^+ + G_n^-),$$

where the functionals G_n^+, G_n^- are defined by the formula (I.3).

b) *The derivative of a homogeneous Poisson process.* Let $f_2(\omega,t)$ be a homogeneous Poisson process with expectation $a|t|$. Then $f(\omega,t)=\int_0^t f_2(\omega,u)du$ is a continuous process belonging to $\mathfrak{M}$. Therefore the derivative $\Phi_2(\omega,t)=df_2(\omega,t)/dt=d^2f(\omega,t)/dt^2$ has the following local characteristic at an arbitrary moment:

$$a_0(\lambda)=\tfrac{1}{2},\qquad L_0(\varphi)=\tfrac{1}{2}\varphi(0),$$

$$a_1(\lambda)=\frac{1}{1+a},\qquad L_1(\varphi)=\frac{1}{1+a}\varphi(0)+\frac{a}{1+a}G_1^+(\varphi),$$

$$a_n(\lambda)=\lambda^{n-1}/a,\qquad L_n=G_n^+\qquad (n=2,3,\ldots).$$

c) Let $f(\omega,t)$ be a continuous process belonging to $\mathfrak{M}$ and $p_{t_0}(x)=P\big(f(\omega,t_0)<x\big)$. The local characteristic at the moment t_0 of the process $f(\omega,t)$ is given by formulas

$$a_n(\lambda)=\Big\{1+\int_{-\infty}^{\infty}|x|^n dp_{t_0}(x)\Big\}^{-1},$$

$$(n=0,1,\ldots)$$

$$L_n(\varphi)=\Big\{1+\int_{-\infty}^{\infty}|x|^n dp_{t_0}(x)\Big\}^{-1}\int_{-\infty}^{\infty}\varphi(x)\,dp_{t_0}(x).$$

We see that all the functionals L_n are induced by the measure $p_{t_0}(x)$.

d) Now we shall give an example of a continuous stochastic process having a local characteristic whose functionals are not induced by the measure $p_{t_0}(x)$. Let $\nu(\omega)$ be a random variable with the Cauchy distribution

$$P\big(\nu(\omega)<x\big)=\tfrac{1}{2}+\frac{1}{\pi}\operatorname{arctg}x.$$

Now we define the continuous stochastic process

$$f(\omega,t)=\nu(\omega)\,|\log|\nu(\omega)||\sin\nu(\omega)t.$$

Put $g(\omega,t)=|\log|\nu(\omega)||\big(1-\cos\nu(\omega)t\big)$. It is easy to see that $g(\omega,t)\in\mathfrak{M}$,

$$\text{(I.21)}\qquad \frac{d}{dt}g(\omega,t)=f(\omega,t)$$

and

$$\text{(I.22)}\qquad E|g(\omega,t)|^n=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\log|x||^n(1-\cos xt)^n}{1+x^2}dx.$$

The formula (I.21) and the continuity of $f(\omega,t)$ imply the convergence

$$\text{(I.23)}\qquad \lim_{h\downarrow 0}\int_{-\infty}^{\infty}\varphi(x)\,dP\big(D_hg(\omega,0)<x\big)=\int_{-\infty}^{\infty}\varphi(x)\,dp(x)\qquad\text{for}\quad \varphi\in X_0,$$

where $p(x)=P\big(f(\omega,0)<x\big)$. Since $D_hg(\omega,0)=g(\omega,h)/h$, in view of (I.22) we obtain

$$\text{(I.24)}\qquad E|D_hg(\omega,0)|^{1-n}=\frac{1}{\pi}h^{1-n}|\log h|^n\Big\{\int_{-\infty}^{\infty}\frac{(1-\cos x)^n}{x^2}dx+o(1)\Big\}.$$

Consequently

$$\text{(I.25)}\qquad \lim_{h\downarrow 0}E|D_hg(\omega,0)|^n=\infty\qquad (n=1,2,\ldots).$$

Let $\varphi\in X_n$ $(n=1,2,\ldots)$. For every $\varepsilon>0$ there exists a number $x_0>0$ such that

$$\sup_{x>x_0}\left|\frac{\varphi(x)}{x^n}-G_n^+(\varphi)\right|<\frac{\varepsilon}{3}.$$

We can find, in view of (I.25), a number $h_0 > 0$ such that for $0 < h < h_0$

$$\Big|\int_0^{x_0} \varphi(x)\, dP\big(D_h g(\omega, 0) < x\big)\Big| < \frac{\varepsilon}{3} E\,|D_h g(\omega, 0)|^n$$

and

$$|G_n^+(\varphi)| \int_0^{x_0} x^n\, dP\big(D_h g(\omega, 0) < x\big) < \frac{\varepsilon}{3} E\,|D_h g(\omega, 0)|^n.$$

Then, taking into account the positivity of the random variable $D_h g(\omega, 0)$, we obtain for $0 < h < h_0$ the inequality

$$\Big|\int_{-\infty}^{\infty} \varphi(x)\, dP\big(D_h g(\omega, 0) < x\big) - G_n^+(\varphi) E\,|D_h g(\omega, 0)|^n\Big|$$

$$\leqslant \int_{x_0}^{\infty} \Big|\frac{\varphi(x)}{x^n} - G_n^+(\varphi)\Big|\, x^n\, dP\big(D_h g(\omega, 0) < x\big) +$$

$$+ \Big|\int_0^{x_0} \varphi(x)\, dP\big(D_h g(\omega, 0) < x\big)\Big| + |G_n^+(\varphi)| \int_0^{x_0} x^n\, dP\big(D_h g(\omega, 0) < x\big)$$

$$< \varepsilon E\,|D_h g(\omega, 0)|^n.$$

Consequently, for each $\varphi \epsilon X_n$ $(n = 1, 2, \ldots)$ the equality

$$\lim_{h \downarrow 0} \{E\,|D_h g(\omega, 0)|^n\}^{-1} \int_{-\infty}^{\infty} \varphi(x)\, dP\big(D_h g(\omega, 0) < x\big) = G_n^+(\varphi) \tag{I.26}$$

holds. Equalities (I.23), (I.24) and (I.26) imply that the local characteristic at the moment $t_0 = 0$ of the process $f(\omega, t)$ is given by the following formulas:

$$a_0(\lambda) = \tfrac{1}{2}, \quad L_0(\varphi) = \tfrac{1}{2} \int_{-\infty}^{\infty} \varphi(x)\, dp(x) = \tfrac{1}{2}\varphi(0),$$

$$a_n(\lambda) = \pi \lambda^{n-1} |\log \lambda|^{-n} \Big\{\int_{-\infty}^{\infty} \frac{(1-\cos x)^n}{x^2}\, dx\Big\}^{-1} \quad (n = 1, 2, \ldots),$$

$$L_n = G_n^+.$$

From the definition of local characteristics of generalized stochastic processes immediately follows

THEOREM I.2. *Let $\Phi(\omega, t)$ be a generalized stochastic process with the local characteristic $\langle a_n, L_n\rangle$. Then for every real number a the process $a\Phi(\omega, t)$ has the local characteristic $\langle \tilde{a}_n, \tilde{L}_n\rangle$, where, for $a = 0$,*

$$\tilde{a}_0(\lambda) = \tfrac{1}{2}, \quad \tilde{L}_0(\varphi) = \tfrac{1}{2}\varphi(0), \quad \tilde{a}_n(\lambda) = 1, \quad \tilde{L}_n(\varphi) = \varphi(0) \quad (n = 1, 2, \ldots);$$

for $a \neq 0$,

$$\left.\begin{aligned} \tilde{a}_n(\lambda) &= \frac{a_n(\lambda)}{|a|^n + (1-|a|^n) L_n(1)}, \\ \tilde{L}_n(\varphi) &= \frac{L_n(\varphi_n)}{|a|^n + (1-|a|^n) L_n(1)}, \end{aligned}\right\} \quad (n = 0, 1, \ldots),$$

and $\varphi_a(x) = \varphi(ax)$.

Now we prove the following

LEMMA I.2. *Let $F(\omega, t)$ be a continuous stochastic process belonging to $\mathfrak{M}$ and*

$$\frac{d^k}{dt^k} \tilde{F}(\omega, t) = F(\omega, t). \tag{I.27}$$

Then for each $\varphi \epsilon X_n$ $(n = 0, 1, \ldots)$ the equality

$$\operatorname*{Lim}_{h_1, \ldots, h_k \downarrow 0} \sup_{-\infty < y < \infty} \frac{1}{1+|y|^n} \Big|\int_{-\infty}^{\infty} \varphi(x+y)\, dP\big(D_{h_1, \ldots, h_k} \tilde{F}(\omega, t) < x\big) -$$

$$- \int_{-\infty}^{\infty} \varphi(x+y)\, dP\big(F(\omega, t) < x\big)\Big| = 0$$

holds.

Proof. Let

$$S^{(n)}_{h_1, \ldots, h_k}(x) = \int_{-\infty}^{x} (1+|u|^n)\, d\big(D_{h_1, \ldots, h_k} \tilde{F}(\omega, t) < u\big), \tag{I.28}$$

$$S^{(n)}(x) = \int_{-\infty}^{x} (1+|u|^n)\, dP\big(F(\omega, t) < u\big). \tag{I.29}$$

Then, in virtue of Lemma I.1, for each $\psi \epsilon X_0$ the equality

$$\operatorname*{Lim}_{h_1, \ldots, h_k \downarrow 0} \int_{-\infty}^{\infty} \psi(x)\, dS^{(n)}_{h_1, h_2, \ldots, h_k}(x) = \int_{-\infty}^{\infty} \psi(x)\, dS^{(n)}(x), \tag{I.30}$$

is true. Suppose that $\varphi \in X_n$ $(n = 0, 1, \ldots)$. Then the family of functions

$$\psi_y(x) = \frac{\varphi(x+y)}{1+|x|^n} \tag{I.31}$$

belonging to X_0 is compact for $|y| < y_0 < \infty$. Hence, in view of (I.30), we have

$$\operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \sup_{|y|<y_0} \Big| \int_{-\infty}^{\infty} \psi_y(x)\, dS^{(n)}_{h_1,\ldots,h_k}(x) - \int_{-\infty}^{\infty} \psi_y(x)\, dS^{(n)}(x) \Big| = 0 .$$

Consequently, taking into account the notations (I.28), (I.29) and (I.31), we have for each $\varphi \in X_n$ $(n = 0, 1, \ldots)$

$$\operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \sup_{|y|<y_0} \Big| \int_{-\infty}^{\infty} \varphi(x+y)\, dP\big(D_{h_1,\ldots,h_k} F(\omega, t) < x\big) - \int_{-\infty}^{\infty} \varphi(x+y)\, dP\big(F(\omega, t) < x\big) \Big| = 0 . \tag{I.32}$$

Let ε be an arbitrary positive number. Then, in view of (I.30), there exist numbers x_n such that for any sufficiently small $h_1, h_2, \ldots, h_k$

$$\int_{|x|>x_n} dS^{(n)}_{h_1,\ldots,h_k}(x) < \varepsilon \qquad (n = 0, 1, \ldots), \tag{I.33}$$

$$\int_{|x|>x_n} dS^{(n)}(x) < \varepsilon \qquad (n = 0, 1, \ldots). \tag{I.34}$$

It is easy to verify that for each $\varphi \in X_n$ $(n = 0, 1, \ldots)$ the inequality

$$|\varphi(x+y)| \leqslant \|\varphi\| (1 + |x+y|^n) \qquad (n = 0, 1, \ldots)$$

holds. Then, in view of (I.33) and (I.34), we obtain for sufficiently small $h_1, h_2, \ldots, h_k$

$$\Big| \int_{|x|>x_n} \psi_y(x)\, dS^{(n)}_{h_1,\ldots,h_k}(x) \Big| \leqslant \varepsilon \|\varphi\| (2^n + |y|^n), \tag{I.35}$$

$$\Big| \int_{|x|>x_n} \psi_y(x)\, dS^{(n)}(x) \Big| \leqslant \varepsilon \|\varphi\| (2^n + |y|^n). \tag{I.36}$$

Let us choose positive numbers y_n so large that for each $|x| < x$

$$\Big| \frac{\varphi(x+y)}{1+y^n} - G_n^+(\varphi) \Big| < \varepsilon \quad \text{if} \quad y > y_n ,$$

$$\Big| \frac{\varphi(x+y)}{1+|y|^n} - G_n^-(\varphi) \Big| < \varepsilon \quad \text{if} \quad y < -y_n .$$

Then

$$\sup_{|y|>y_n} \frac{1}{1+|y|^n} \Big| \int_{|x|<x_n} \psi_y(x)\, dS^{(n)}_{h_1,\ldots,h_k}(x) - \int_{|x|<x_n} \psi_y(x)\, dS^{(n)}(x) \Big|$$
$$\leqslant \varepsilon \int_{|x|<x_n} \frac{1}{1+|x|^n}\, dS^{(n)}_{h_1,\ldots,h_k}(x) + \varepsilon \int_{|x|<x_n} \frac{1}{1+|x|^n}\, dS^{(n)}(x) +$$
$$+ \big(G_n^+(\varphi) + G_n^-(\varphi)\big) \Big| \int_{|x|<x_n} \frac{1}{1+|x|^n}\, dS^{(n)}_{h_1,\ldots,h_k}(x) - \int_{|x|<x_n} \frac{1}{1+|x|^n}\, dS^{(n)}(x) \Big| .$$

The last inequality and formulas (I.30), (I.35), (I.36), considering that ε can be chosen arbitrarily small, imply the convergence

$$\operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \sup_{|y|>y_n} \frac{1}{1+|y|^n} \Big| \int_{-\infty}^{\infty} \psi_y(x)\, dS^{(n)}_{h_1,\ldots,h_k}(x) - \int_{-\infty}^{\infty} \psi_y(x)\, dS^{(n)}(x) \Big| = 0 .$$

Hence, taking into account the notations (I.28), (I.29), (I.31) and the equality (I.32), we obtain the assertion of Lemma.

THEOREM I.3. *Let $\Phi(\omega, t)$ be a generalized stochastic process with the local characteristic $\langle a_n, L_n \rangle$ at the moment t_0. Let $F(\omega, t)$ be a continuous process belonging to $\mathfrak{M}$ with a distribution function $p_t(x) = P\big(F(\omega, t) < x\big)$. Suppose that the processes $\Phi(\omega, t)$ and $F(\omega, t)$ are independent*[3]*). Then the process $\Phi(\omega, t) + F(\omega, t)$ has the local characteristic $\tilde{a}_n(\lambda) = a_n(\lambda)$, $\tilde{L}_n(\varphi) = L_n(\tilde{\varphi})$ $(n = 0, 1, \ldots)$, where*

$$\tilde{\varphi}(x) = \int_{-\infty}^{\infty} \varphi(x+y)\, dp_{t_0}(y) .$$

Proof. Let us suppose that there exist an integer k, a system of function $A_{1n}(\lambda), \ldots, A_{kn}(\lambda)$ and a continuous process $f(\omega, t)$ belonging to $\mathfrak{M}$ such that

$$\frac{d^k}{dt^k} f(\omega, t) = \Phi(\omega, t), \qquad a_n(\lambda) = \prod_{j=1}^{k} A_{jn}(\lambda) \qquad (n = 0, 1, \ldots),$$

and for each $\varphi \in X_n$ $(n = 0, 1, \ldots)$

$$\operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \prod_{j=1}^{k} A_{jn}(h_j) \int_{-\infty}^{\infty} \varphi(x)\, dP\big(D_{h_1,\ldots,h_k} f(\omega, t_0) < x\big) = L_n(\varphi) . \tag{I.37}$$

---

[3]) The independence of generalized stochastic processes is defined in [6], § II.1.

Put

$$\tilde{F}(\omega, t) = \frac{1}{(k-1)!} \int_0^t (t-u)^{k-1} F(\omega, u)\, du .$$

It is easy to prove that $\tilde{F}(\omega, t)$ is a continuous process belonging to $\mathfrak{M}$. Consequently, the continuous process

$$g(\omega, t) = f(\omega, t) + \tilde{F}(\omega, t) \tag{I.38}$$

belongs to $\mathfrak{M}$ and

$$\frac{d^k}{dt^k} g(\omega, t) = \Phi(\omega, t) + F(\omega, t).$$

The independence of the processes $\Phi(\omega, t)$ and $F(\omega, t)$ implies the independence of the random variables $D_{h_1,\ldots,h_k} f(\omega, t_0)$ and $D_{h_1,\ldots,h_k} \tilde{F}(\omega, t_0)$ (see [6], § II.1). Consequently, in view of (I.38)

$$\int_{-\infty}^{\infty} \varphi(x)\, dP\big(D_{h_1,\ldots,h_k} g(\omega, t_0) < x\big) \tag{I.39}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x+y)\, dP\big(D_{h_1,\ldots,h_k} \tilde{F}(\omega, t_0) < x\big)\, dP\big(D_{h_1,\ldots,h_k} f(\omega, t_0) < y\big).$$

Let

$$\tilde{\varphi}(y) = \int_{-\infty}^{\infty} \varphi(x+y)\, dP\big(\tilde{F}(\omega, t_0) < x\big).$$

If $\varphi \in X_n$, then also $\tilde{\varphi} \in X_n$ $(n = 0, 1, \ldots)$. Hence, in view of Lemma I.2 and the equalities (I.37), (I.39), we have for each $\varphi \in X_n$ $(n = 0, 1, \ldots)$

$$\operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \prod_{j=1}^{k} A_{jn}(h_j) \int_{-\infty}^{\infty} \varphi(x)\, dP\big(D_{h_1,\ldots,h_k} g(\omega, t_0) < x\big) = L_n(\tilde{\varphi}).$$

The theorem is thus proved.

II. In this Section we shall analyse some local properties of homogeneous stochastic processes with independent increments. In the sequel $\mathfrak{K}$ denotes the class of measurable homogeneous stochastic processes with independent increments, whose almost all sample functions are Lebesgue integrable over every finite interval. Since we shall consider derivatives of processes belonging to $\mathfrak{K}$, we can assume without loss of generality that

$$f(\omega, 0) = 0 \quad \text{for} \quad f(\omega, t) \in \mathfrak{K}. \tag{II.1}$$

By $\mathfrak{K}_s$ $(s = 1, 2, \ldots)$ we shall denote the class of derivatives of order s of processes belonging to $\mathfrak{K}$. It is well known that processes belonging to $\bigcup_{s=1}^{\infty} \mathfrak{K}_s$ are stationary generalized processes with independent values (see e. g. [6], theorem 8 and 18). It is also well known that the distributions of processes belonging to $\mathfrak{K}$ are infinitely divisible (see [1], p. 417). Consequently the logarithm of the characteristic function $R_t(z)$ of the random variable $f(\omega, t)$ $\big(f(\omega, t) \in \mathfrak{K},\ t \geqslant 0\big)$ is uniquely determined by the Lévy-Khintchine formula

$$\log R_t(z) = i\tilde{\gamma}_f tz + t \int_{-\infty}^{\infty} \left(e^{iuz} - 1 - \frac{iuz}{1+u^2}\right) \frac{1+u^2}{u^2}\, d\tilde{G}_f(u), \tag{II.2}$$

where $\tilde{\gamma}_f$ is a real constant and $\tilde{G}_f$ is a monotone non-decreasing bounded function, continuous on the right and normalized by supposing $\tilde{G}_f(-\infty) = 0$. The characteristic function $R_t(z)$ determines $\tilde{G}_f$ and $\tilde{\gamma}_f$ uniquely. From the Lévy-Khintchine expression it follows that

$$R_t(z) = \{R_1(z)\}^t \quad (t \geqslant 0). \tag{II.3}$$

If the process $f(\omega, t)$ belonging to $\mathfrak{K}$ has a finite variance, then the logarithm of the characteristic function $R_t(z)$ may be represented in the Kolmogorov form,

$$\log R_t(z) = i\gamma_f tz + t \int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz) \frac{1}{u^2}\, dG_f(u), \tag{II.4}$$

where G_f is monotone, non-decreasing, continuous on the right and bounded, with $G_f(-\infty) = 0$, and γ_f is a constant. It is easy to see that in this case

$$\gamma_f = \tilde{\gamma}_f + \int_{-\infty}^{\infty} u\, d\tilde{G}_f(u), \tag{II.5}$$

$$G_f(x) = \int_{-\infty}^{\infty} (1+u^2)\, d\tilde{G}_f(u). \tag{II.6}$$

Now we shall prove the following very simple

THEOREM II.1. *Let $f(\omega, t) \in \mathfrak{K}$. Then $f(\omega, t) \in \mathfrak{M}$ if and only if for every integer n the inequality*

$$\int_{-\infty}^{\infty} |u|^n d\tilde{G}_f(u) < \infty \tag{II.7}$$

holds.

In virtue of formula (II.6) condition (II.7) is equivalent to the following one:

$$\int_{-\infty}^{\infty} |u|^n dG_t(u) < \infty \qquad (n = 1, 2, \ldots).$$

Proof. From condition (II.1) it follows that the process $f(\omega, t)$ belonging to $\mathfrak{K}$ belongs also to $\mathfrak{M}$ if and only if the expectations $E|f(\omega, t)|^n$ $(n = 1, 2, \ldots)$ are continuous. In virtue of well known properties of characteristic functions the expectations $E|f(\omega, t)|^n$ $(n = 1, 2, \ldots)$ are continuous if and only if the characteristic function $R_t(z)$ is infinitely differentiable in z and its derivatives at $z = 0$ are continuous in t. Since, in view of equality (II.3), the derivatives at $z = 0$ of $R_t(z)$ are polynomials in t, we have $f(\omega, t) \epsilon \mathfrak{M}$ if and only if $R_t(z)$ is infinitely differentiable in z. The last condition, in view of the Lévy-Khintchine formula, is equivalent to condition (II.7), which was to be proved.

LEMMA II.1. *Let $f(\omega, t) \epsilon \mathfrak{K}$. Then the characteristic function $R_{h_1, \ldots, h_k}(z)$ of the difference $\Delta_{\lambda_1} \Delta_{\lambda_2} \ldots \Delta_{\lambda_k} f(\omega, t)$ where λ_j are determined by formula* (I.2) *is given by the formula*

$$R_{h_1, \ldots, h_k}(z) = |R_{h_k}(z)|^{2^{k-1}} \qquad (k = 2, 3, \ldots).$$

Proof. From the homogeneity of the process $f(\omega, t)$ if follows that the random variables

(II.8) $$\Delta_{\lambda_2} \ldots \Delta_{\lambda_k} f(\omega, t+\lambda_1), \; \Delta_{\lambda_2} \ldots \Delta_{\lambda_k} f(\omega, t) \qquad (k = 2, 3, \ldots)$$

are equidistributed. Since, in view of formula (I.2) $\lambda_1 = \lambda_2 + \ldots + \lambda_k$, the random variables (II.8) are mutually independent. Hence, taking into account the equality

$$\Delta_{\lambda_1} \Delta_{\lambda_2} \ldots \Delta_{\lambda_k} f(\omega, t) = \Delta_{\lambda_2} \ldots \Delta_{\lambda_k} f(\omega, t+\lambda_1) - \Delta_{\lambda_2} \ldots \Delta_{\lambda_k} f(\omega, t),$$

we have the following formula:

$$R_{h_1, \ldots, h_k}(z) = |R_{h_2, \ldots, h_k}(z)|^2 \qquad (k = 2, 3, \ldots).$$

From this formula we immediately obtain the assertion of the Lemma.

LEMMA II.2. *Let $f(\omega, t) \epsilon \mathfrak{K}$. If there exists a continuous process $g(\omega, t)$ belonging to $\mathfrak{M}$ and satisfying for some integers k, s the equality*

$$\frac{d^s}{dt^s} f(\omega, t) = \frac{d^{k+s}}{dt^{k+s}} g(\omega, t),$$

then $f(\omega, t) \epsilon \mathfrak{M}$.

Proof. The assumption of the Lemma implies the equality

$$\int_0^t \int_0^{x_{k-1}} \ldots \int_0^{x_1} f(\omega, u) du dx_1 \ldots dx_{k-1} = \sum_{j=0}^{k+s-1} a_j(\omega) t^j + g(\omega, t).$$

Suppose that the numbers $\lambda_1, \lambda_2, \ldots, \lambda_{k+s}$ are given by formula (I.2). The last equality implies

$$\Delta_{\lambda_1} \Delta_{\lambda_2} \ldots \Delta_{\lambda_{k+s}} g(\omega, t)$$
$$= \Delta_{\lambda_1} \ldots \Delta_{\lambda_{s-1}} \int_t^{t+\lambda_{s+1}} \int_{x_{k-1}}^{x_{k-1}+\lambda_{s+2}} \ldots \int_{x_1}^{x_1+\lambda_{k+s}} \{f(\omega, u+\lambda_s) - f(\omega, u)\} du dx_1 \ldots dx_{k-1}.$$

From this formula we obtain by simple reasoning the following equality:

(II.9) $$\frac{(-1)^{s-1} \Delta_{\lambda_1} \Delta_{\lambda_2} \ldots \Delta_{\lambda_{k+s}} g(\omega, 0)}{\lambda_{s+1} \ldots \lambda_{k+s}} = f(\omega, \lambda_s) - f(\omega, \lambda_{s+1} + \ldots + \lambda_{k+s}) + Z(\omega);$$

the random variable $Z(\omega)$ is defined by the formula

$$Z(\omega) = \frac{1}{\lambda_{s+1} \ldots \lambda_{k+s}} \int_0^{\lambda_{s+1}} \int_{x_{k-1}}^{x_{k-1}+\lambda_{s+2}} \ldots \int_{x_1}^{x_1+\lambda_{k+s}} \{f(\omega, u+\lambda_s) - f(\omega, \lambda_s) +$$
$$+ f(\omega, \lambda_{s+1} + \ldots + \lambda_{k+s}) - f(\omega, u)\} du dx_1 \ldots dx_{k-1} +$$
$$+ \sum_{\langle i_1, \ldots, i_r \rangle} A_{i_1, \ldots, i_r} \int_{\lambda_{i_1} + \ldots + \lambda_{i_r}}^{\lambda_{i_1} + \ldots \lambda_{i_r} + \lambda_{i_{s+1}}} \int_{x_{k-1}}^{x_{k-1}+\lambda_{s+2}} \ldots \int_{x_1}^{x_1+\lambda_{k+s}}$$
$$\{f(\omega, u+\lambda_s) - f(\omega, u)\} du dx_1 \ldots dx_{k-1},$$

where the indices $i_1, i_2, \ldots, i_r$ run over all systems of integers $1, 2, \ldots, s-1$ and $A_{i_1, \ldots, i_r}$ are constants. The random variable $Z(\omega)$ is the limit in probability of a sequence of the form

(II.10) $$Z_n(\omega) = \sum_{j=1}^{k_n} B_{jn} \{f(\omega, u_{jn}) - f(\omega, t_{jn})\},$$

where B_{jn} $(j = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ are constants and numbers u_{jn}, t_{jn} satisfy the inequality

(II.11) $$\lambda_s = \min\{\lambda_s, \min_{\langle i_1, \ldots, i_r \rangle} (\lambda_{i_1} + \ldots + \lambda_{i_r})\} \leqslant t_{jn} \leqslant u_{jn}$$

or the inequality

(II.12) $$t_{jn} \leqslant u_{jn} \leqslant \lambda_{s+1} + \ldots + \lambda_{k+s}.$$

Since the process $f(\omega, t)$ has independent increments, it follows from inequalities (II.11), (II.12) and from equality (II.9) that for fixed n the random variables $Z_n(\omega)$ and $f(\omega,\lambda_s)-f(\omega,\lambda_{s+1}+\ldots+\lambda_{k+s})$ are mutually independent. Consequently the random variables $Z(\omega)$ and $f(\omega,\lambda_s)-f(\omega,\lambda_{s+1}+\ldots+\lambda_{k+s})$ are mutually independent. Moreover, the random variables $Z_n(\omega)$ are infinitely divisible, which implies the infinite divisibility of $Z(\omega)$.

By $\tilde{G}_f$ we shall denote the Lévy-Khintchine function for the process $f(\omega,t)$ and by $\tilde{G}_Z$ the Lévy-Khintchine function for the random variable $Z(\omega)$. Suppose that $\lambda_s-\lambda_{s+1}-\ldots-\lambda_{k+s}=1$. Then, in view of (II.9), the Lévy-Khintchine function for the random variable

$$\frac{(-1)^{s-1}\Delta_{\lambda_1}\Delta_{\lambda_2}\ldots\Delta_{\lambda_{h+s}}g(\omega,0)}{\lambda_{s+1}\ldots\lambda_{k+s}}\tag{II.13}$$

is given by the formula

$$\tilde{G}(x)=\tilde{G}_f(x)+\tilde{G}_Z(x).\tag{II.14}$$

Since $g(\omega,t)\in\mathfrak{M}$, the characteristic function of the random variable (II.13) is infinitely differentiable. Hence for all integers n $\int_{-\infty}^{\infty}|u|^n d\tilde{G}(u)<\infty$. Consequently, in view of (II.14), $\int_{-\infty}^{\infty}|u|^n d\tilde{G}_f(u)<\infty$. From this fact and from Theorem II.1 our assertion follows immediately.

LEMMA II.3. *Let $f(\omega,t)\in\mathfrak{K}$. Suppose that the derivative $d^kf(\omega,t)/dt^k$ $(k\geqslant 1)$ has the local characteristic $\langle a_n, L_n\rangle$ at the moment t_0. Thus there exist a system of functions $B_{1n}(\lambda),\ldots,B_{kn}(\lambda)$ such that*

$$a_n(\lambda)=\prod_{j=1}^{k}B_{jn}(\lambda)\qquad(n=0,1,\ldots)$$

and for each $\varphi\in X_n$ $(n=0,1,\ldots)$

$$\operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0}\prod_{j=1}^{k}B_{jn}(h_j)\int_{-\infty}^{\infty}\varphi(x)\,dP\big(D_{h_1,\ldots,h_k}f(\omega,t_0)<x\big)=L_n(\varphi).$$

Proof. From the assumptions of the Lemma it follows that there exist an integer s, a continuous process $F(\omega,t)$ belonging to $\mathfrak{M}$, and a system of positive functions $A_{1n}(\lambda),\ldots,A_{k+s,n}(\lambda)$ satisfying the conditions

$$\frac{d_k}{dt^k}f(\omega,t)=\frac{d^{k+s}}{dt^{k+s}}F(\omega,t),\tag{II.15}$$

$$a_n(\lambda)=\prod_{j=1}^{k+s}A_{jn}(\lambda)\qquad(n=0,1,\ldots),\tag{II.16}$$

$$\operatorname*{Lim}_{h_1,\ldots,h_{k+s}\downarrow 0}\prod_{j=1}^{k+s}A_{jn}(h_j)\int_{-\infty}^{\infty}\varphi(x)\,dP\big(D_{h_1,\ldots,h_{k+s}}F(\omega,t_0)<x\big)=L_n(\varphi)\tag{II.17}$$

for each $\varphi\in X_n$ $(n=0,1,\ldots)$. Equality (II.15), in view of Lemma II.2, implies $f(\omega,t)\in\mathfrak{M}$. Put

$$g(\omega,t)=\frac{1}{(s-1)!}\int_0^t(t-u)^{s-1}f(\omega,u)\,du.$$

From (II.15) it follows that for each $h_1,\ldots,h_{k+s}$

$$D_{h_1,\ldots,h_{k+s}}g(\omega,t)=D_{h_1,\ldots,h_{k+s}}F(\omega,t).\tag{II.18}$$

It is easy to verify that the assumptions of Lemma I.1 are fulfilled by the processes $f(\omega,t)$ and $g(\omega,t)$. Setting

$$B_{jn}(\lambda)=A_{jn}(\lambda)\Big\{\prod_{r=k+1}^{k+s}A_{rn}(\lambda)\Big\}^{1/k}\qquad(j=1,2,\ldots,k)$$

in view of Lemma I.1 and equalities (II.16), (II.17), (II.18) we obtain the assertion of the Lemma.

LEMMA II.4. *Let $f(\omega,t)\in\mathfrak{K}\cap\mathfrak{M}$. If there exists a system of functions $A_{1n}(\lambda),\ldots,A_{kn}(\lambda)$ such that*

$$L_n(\varphi)=\operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0}\prod_{j=1}^{k}A_{jn}(h_j)\int_{-\infty}^{\infty}\varphi(x)\,dP\big(D_{h_1,\ldots,h_k}f(\omega,t_0)<x\big)$$

is a continuous non-negative linear functional on X_n, with $L_n(1+|x|^n)=1$ $(n=0,1,\ldots)$, then the derivative $d^kf(\omega,t)/dt^k$ has the local characteristic $\langle a_n, L_n\rangle$, where

$$a_n(\lambda)=\prod_{j=1}^{k}A_{jn}(\lambda)\qquad(n=0,1,\ldots).$$

Proof. Put $g(\omega,t)=\int_0^t f(\omega,u)\,du$. The process $g(\omega,t)$ is continuous and belongs ot $\mathfrak{M}$. In virtue of Lemma I.1 it is easy to verify that the integer $k+1$, the system of functions $A_{1n}(\lambda),\ldots,A_{kn}(\lambda),A_{k+1,n}(\lambda)\equiv 1$ and the process $g(\omega,t)$ satisfy the definition of the local characteristic at the moment t_0 for the derivative $d^kf(\omega,t)/dt^k$. Moreover, as the local characteristic we obtain the sequence $\langle a_n, L_n\rangle$. The lemma is thus proved.

An immediate consequence of Lemmas II.2, II.3 and II.4 is the following

THEOREM II.2. *Let $f(\omega,t)\in\mathfrak{K}$. Then the derivative $d^kf(\omega,t)/dt^k$ $(k\geqslant 1)$ has the local characteristic $\langle a_n, L_n\rangle$ at the moment t_0 if and only if*

$f(\omega, t)\in\mathfrak{M}$ *and if there exists a system of functions* $A_{1n}(\lambda), \ldots, A_{kn}(\lambda)$ *such that*

$$a_n(\lambda) = \prod_{j=1}^{k} A_{jn}(\lambda) \qquad (n = 0, 1, \ldots),$$

$$\operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \prod_{j=1}^{k} A_{jn}(h_j) \int_{-\infty}^{\infty} \varphi(x)\, dP\big(D_{h_1,\ldots,h_k} f(\omega, t_0) < x\big) = L_n(\varphi)$$

for each $\varphi\in X_n$ $(n = 0, 1, \ldots)$.

III. In this Section we shall give some elementary properties of characteristic functions of infinitely divisible distributions.

By L we shall denote the space of functions that are Lebesgue-integrable over $-\infty < z < \infty$. In the sequel $R(z)$ denotes the characteristic function of an infinitely divisible distribution with a finite variance. The logarithm of this characteristic function is given by Kolmogorov's formula

$$\text{(III.1)} \qquad \log R(z) = i\gamma z + \int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz)\frac{1}{u^2}\, dG(u)\,.$$

For brevity let us introduce the notation

$$\text{(III.2)} \qquad Q_h(z) = \left\{R\left(\frac{z}{h}\right)\right\}^h.$$

LEMMA III.1. *If*

$$\limsup_{z\downarrow 0} \frac{G(z)-G(-z)}{z} = \infty\,,$$

then there is a sequence $h_1, h_2, \ldots$ $(h_n \downarrow 0)$ *such that for every* $\psi\in L$ *the equality*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} Q_{h_n}(z)\psi(z)\, dz = 0$$

holds.

Proof. Let $\psi\in L$. Then for arbitrarily small positive ε we choose a positive number A such that

$$\text{(III.3)} \qquad \int_{|z|>A} |\psi(z)|\, dz < \varepsilon/2\,.$$

From the assumption of Lemma it follows that for a sequence $h_1, h_2, \ldots$ $(h_n \downarrow 0)$ the inequality

$$\text{(III.4)} \qquad \frac{G(A^{-1}h_n) - G(-A^{-1}h_n)}{h_n} > n \qquad (n = 0, 1, \ldots)$$

holds. Thus there exists a positive integer n_0 such that for $n \geqslant n_0$ the inequality

$$\text{(III.5)} \qquad \int_{-A}^{A} |\psi(z)| \exp\left(-\frac{z^2}{\pi} n\right) dz < \frac{\varepsilon}{2}$$

is true. Since

$$\frac{\cos uz - 1}{u^2} \leqslant -\frac{z^2}{\pi} \quad \text{for} \quad |u| < A^{-1},\ |z| < A\,,$$

in view of (III.1), (III.2) and (III.4), we obtain for $|z| < A$ and $n \geqslant n_0$ the following inequality:

$$\text{(III.6)} \qquad |Q_{h_n}(z)| = \exp\left\{\frac{1}{h_n}\int_{-\infty}^{\infty} (\cos zu - 1)\frac{1}{u^2}\, dG(h_n u)\right\}$$

$$\leqslant \exp\left\{\frac{1}{h_n}\int_{-A^{-1}}^{A^{-1}} (\cos zu - 1)\frac{1}{u^2}\, dG(h_n u)\right\} \leqslant \exp\left\{-\frac{z^2}{\pi}\frac{G(A^{-1}h_n) - G(-A^{-1}h_n)}{h_n}\right\}$$

$$\leqslant \exp\left\{-\frac{nz^2}{\pi}\right\}.$$

Consequently, in view of inequalities (III.3), (III.5) and (III.6), we have for $n \geqslant n_0$ the following inequality:

$$\left|\int_{-\infty}^{\infty} Q_{h_n}(z)\psi(z)\, dz\right| \leqslant \int_{|z|<A} |Q_{h_n}(z)||\psi(z)|\, dz + \int_{|z|>A} |\psi(z)|\, dz \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Lemma is thus proved.

LEMMA III.2. *If*

$$\limsup_{z\downarrow 0} \frac{G(z)-G(-z)}{z} < \infty\,,$$

then

$$\liminf_{h\downarrow 0} |Q_h(z)| = e^{-a|z|}\,,$$

where a *is a non-negative constant.*

Proof. The assumption of the Lemma implies the inequality

$$M = \sup_{z>0} \frac{G(z)-G(-z)}{z} < \infty\,.$$

It is easy to prove that the inequalities

$$\frac{1}{h}\int_{-1}^{1}\frac{1-\cos u}{u^2}\,dG(hu)\leqslant M\,,$$

$$\frac{1}{h}\int_{|u|>1}\frac{1-\cos u}{u^2}\,dG(hu)\leqslant\frac{2}{h}\int_{1}^{\infty}\frac{1}{u^2}\,d\{G(hu)-G(-hu)\}$$

$$=2\frac{G(-h)-G(h)}{h}+4\int_{1}^{\infty}\frac{G(hu)-G(-hu)}{hu}\,\frac{du}{u^2}\leqslant 4M$$

are true. Hence

$$-\log|Q_h(1)|=\frac{1}{h}\int_{-\infty}^{\infty}\frac{1-\cos u}{u^2}\,dG(hu)\leqslant 5M\,.$$

Consequently

(III.7) $$\liminf_{h\downarrow 0}|Q_h(1)|\geqslant e^{-5M}>0\,.$$

Introducing the notation

(III.8) $$e^{-a}=\liminf_{h\downarrow 0}|Q_h(1)|$$

we have, in view of (III.7) and the inequality $|Q_h(z)|\leqslant 1$, $0\leqslant a<\infty$. Formula (III.2) immediately implies for $z>0$:

$$|Q_h(z)|=|Q_{h/z}(1)|^z\,.$$

Then, according to (III.8),

$$\liminf_{h\downarrow 0}|Q_h(z)|=e^{-az}\quad\text{for}\quad z>0\,.$$

Hence, in view of the equality $|Q_h(z)|=|Q_h(-z)|$, we obtain the assertion of Lemma.

By X_0^* we shall denote the space of all functions φ continuous in $-\infty<x<\infty$, with $\lim\limits_{x\to\infty}\varphi(x)=\lim\limits_{x\to-\infty}\varphi(x)=0$.

LEMMA III.3. *Let q_n $(n=0,1,\ldots)$ be a sequence of monotone non-decreasing bounded functions and*

$$Q_n(z)=\int_{-\infty}^{\infty}e^{ixz}\,dq_n(x)\quad(n=0,1,\ldots)\,.$$

If

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\varphi(x)\,dq_n(x)=\int_{-\infty}^{\infty}\varphi(x)\,dq_0(x)$$

for each $\varphi\in X_0^*$, then

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}Q_n(z)\,\psi(z)\,dz=\int_{-\infty}^{\infty}Q_0(z)\,\psi(z)\,dz$$

for each $\psi\in L$.

Proof. Let $\psi\in L$. It is well known that

$$\tilde\psi(x)=\int_{-\infty}^{\infty}e^{izx}\psi(z)\,dz$$

is continuous in $-\infty<x<\infty$ and, in view of the Riemann-Lebesgue theorem,

$$\lim_{x\to\infty}\tilde\psi(x)=\lim_{x\to-\infty}\tilde\psi(x)=0\,.$$

Consequently $\tilde\psi\in X_0^*$. The assertion of the Lemma is a direct consequence of following equality:

$$\int_{-\infty}^{\infty}Q_n(z)\,\psi(z)\,dz=\int_{-\infty}^{\infty}\tilde\psi(x)\,dq_n(x)\quad(n=0,1,\ldots).$$

LEMMA III.4. *If*

$$\limsup_{z\downarrow 0}\frac{G(z)-G(-z)}{z}<\infty\,,$$

then every sequence $h_1^*,h_2^*,\ldots$ $(h_n^*\downarrow 0)$ contains a subsequence $h_1,h_2,\ldots$ such that the sequence $|Q_{h_1}(z)|,|Q_{h_2}(z)|,\ldots$ converges uniformly in every finite interval.

Proof. Let $q_h(x)$ be the distribution function determined by the characteristic function $|Q_h(z)|$. The sequence $h_1^*,h_2^*,\ldots$, according to Helly's theorem, contains a subsequence $h_1,h_2,\ldots$ such that

(III.9) $$\lim_{n\to\infty}q_{h_n}(x)=q(x)\quad(-\infty<x<\infty)$$

at all continuity points of the limit function $q(x)$. To prove the Lemma it suffices to show that $q(\infty)-q(-\infty)=1$. Contrary to this statement let us suppose that the inequality

(III.10) $$q(\infty)-q(-\infty)<1$$

is true. Let

$$Q(z)=\int_{-\infty}^{\infty}e^{ixz}\,dq(x)\,.$$

Further, from (III.9) we obtain the convergence

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\varphi(x)\,dq_{h_q}(x) = \int_{-\infty}^{\infty}\varphi(x)\,dq(x)$$

for each $\varphi\in X_0^*$. Then, in view of Lemma III.3, for each $\psi\in L$ the equality

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|Q_{h_n}(z)|\psi(z)\,dz = \int_{-\infty}^{\infty}|Q(z)|\psi(z)\,dz$$

holds. Hence, according to Lemma III.2 and Fatou's Lemma, for every non-negative function $\psi\in L$ we obtain the inequality

$$\int_{-\infty}^{\infty}|Q(z)|\psi(z)\,dz \geqslant \int_{-\infty}^{\infty}\liminf_{n\to\infty}|Q_{h_n}(z)|\psi(z)\,dz = \int_{-\infty}^{\infty}e^{-a|z|}\psi(z)\,dz,$$

where a is a non-negative constant. Let ψ be a non-negative function belonging to L. Then also functions $\psi_n(z)=\psi(nz)$ are non-negative and belong to L. Then the last inequality implies

$$\int_{-\infty}^{\infty}|Q(z)|\psi(nz)\,dz \geqslant \int_{-\infty}^{\infty}e^{-a|z|}\psi(nz)\,dz,$$

for every integer n. Hence

$$\int_{-\infty}^{\infty}\left|Q\left(\frac{z}{n}\right)\right|\psi(z)\,dz \geqslant \int_{-\infty}^{\infty}\exp\left\{-\frac{a|z|}{n}\right\}\psi(z)\,dz \qquad (n=1,2,\ldots).$$

Further, the last inequality implies for $n\to\infty$

$$Q(0)\int_{-\infty}^{\infty}\psi(z)\,dz \geqslant \int_{-\infty}^{\infty}\psi(z)\,dz$$

for every non-negative function $\psi\in L$. Consequently $q(\infty)-q(-\infty) = Q(0)\geqslant 1$, which contradicts inequality (III.10). The Lemma is thus proved.

LEMMA III.4. *If*

$$\limsup_{z\downarrow 0}\frac{G(z)-G(-z)}{z}<\infty,$$

then there are a sequence $h_1, h_2, \ldots$ and a sequence $A_1, A_2, \ldots$ such that the sequence $\mathrm{Im}\{\log Q_{h_n}(z)\}-A_n z$ $(n=1,2,\ldots)$ *converges uniformly in every finite interval.*

Proof. The assumption of the Lemma implies the inequality

$$M=\sup_{z>0}\frac{G(z)-G(-z)}{z}<\infty.$$

Hence for each $u>0$ and $h>0$ we obtain

$$0\leqslant\frac{G(hu)-G(-hu)}{h}\leqslant Mu. \tag{III.11}$$

Thus according to Helly's theorem, there are a sequence $h_1, h_2, \ldots$ $(h_n\downarrow 0)$ and a monotone non-decreasing function $H(u)$ such that

$$\lim_{n\to\infty}\frac{G(h_n u)-G(-h_n u)}{h_n}=H(u)\qquad(0\leqslant u<\infty) \tag{III.12}$$

at all continuity points of $H(u)$. Obviously, in view of (III.11),

$$H(u)\leqslant Mu\qquad(u\geqslant 0). \tag{III.13}$$

Let ε be an arbitrarily small positive number and $u_\varepsilon=9M/\varepsilon$. Then, in view of (III.11),

$$\frac{1}{h}\int_{u_\varepsilon}^{\infty}\frac{1}{u^2}\,d\{G(hu)-G(-hu)\} \tag{III.14}$$

$$=\frac{G(-hu_\varepsilon)-G(hu_\varepsilon)}{u_\varepsilon^2 h}+2\int_{u_\varepsilon}^{\infty}\frac{G(hu)-G(-hu)}{hu}\frac{du}{u^2}\leqslant\frac{M}{u_\varepsilon}+2M\int_{u_\varepsilon}^{\infty}\frac{du}{u^2}=\frac{\varepsilon}{3}.$$

In the same way we obtain the inequality

$$\int_{u_\varepsilon}^{\infty}\frac{dH(u)}{u^2}\leqslant\frac{\varepsilon}{3}. \tag{III.15}$$

Hence, in particular, it follows for every z that

$$\left|\int_{1}^{\infty}\frac{\sin zu}{u^2}\,dH(u)\right|<\infty.$$

Let z_0 be an arbitrary positive number. Then, in view of (III.12), there is an integer n_0 such that for $n\geqslant n_0$ and $|z|<z_0$ the inequality

$$\left|\frac{1}{h_n}\int_0^1\frac{\sin zu-zu}{u^2}\,d\{G(h_n u)-G(-h_n u)\}+\right. \tag{III.16}$$

$$+\frac{1}{h_n}\int_1^{u_\varepsilon}\frac{\sin zu}{u^2}\,d\{G(h_n u)-G(-h_n u)\}-$$

$$\left.-\int_0^1\frac{\sin zu-zu}{u^2}\,dH(u)-\int_1^{u_\varepsilon}\frac{\sin zu}{u^2}\,dH(u)\right|<\frac{\varepsilon}{3}$$

is true. Introducing the notation

$$A_n = \gamma - \frac{1}{h_n}\int_1^\infty \frac{1}{u}\, d\{G(h_n u) - G(-h_n u)\} \qquad (n = 1, 2, \ldots)$$

and taking into account formulas (III.1), (III.14), (III.15) and (III.16) we obtain for $|z| < z_0$, $n \geqslant n_0$ the following inequality:

$$\left|\operatorname{Im}\{\log Q_{h_n}(z)\} - A_n z - \int_0^1 \frac{\sin zu - zu}{u^2}\, dH(u) - \int_1^\infty \frac{\sin zu}{u^2}\, dH(u)\right|$$

$$\leqslant \left|\frac{1}{h_n}\int_0^1 \frac{\sin zu - zu}{u^2}\, d\{G(h_n u) - G(-h_n u)\} + \right.$$

$$+ \frac{1}{h_n}\int_1^{u_\varepsilon} \frac{\sin zu}{u^2} d\{G(h_n u) - G(-h_n u)\} - \int_0^1 \frac{\sin zu - zu}{u^2}\, dH(u) - \left.\int_1^{u_\varepsilon} \frac{\sin zu}{u^2} dH(u)\right| +$$

$$+ \frac{1}{h_n}\int_{u_\varepsilon}^\infty \frac{1}{u^2}\, d\{G(h_n u) - G(-h_n u)\} + \int_{u_\varepsilon}^\infty \frac{1}{u^2}\, dH(u) \leqslant \varepsilon .$$

Consequently, the sequence $\operatorname{Im}\{\log Q_{h_n}(z)\} - A_n z$ $(n = 1, 2, \ldots)$ converges uniformly in every finite interval, q. e. d.

THEOREM III.1. *If, for every $\psi \in L$, the limit*

$$\Lambda(\psi) = \lim_{h \downarrow 0} \int_{-\infty}^\infty Q_h(z)\psi(z)\, dz$$

exists and $\Lambda \not\equiv 0$, then there are real constants a and b $(a \geqslant 0)$ such that

$$\lim_{h \downarrow 0} Q_h(z) = e^{-a|z| + ibz}$$

uniformly in every finite interval.

Proof. From the assumptions of the theorem, in view of Lemma III.1, follows the inequality

$$\limsup_{z \downarrow 0} \frac{G(z) - G(-z)}{z} < \infty .$$

Hence, in view of Lemma III.4, there are a sequence $h_1^*, h_2^*, \ldots$ $(h_n^* \downarrow 0)$ and a sequence $A_{h_1}^*, A_{h_2}^*, \ldots$ such that the sequence $\operatorname{Im}\{\log Q_{h_n}^*(z)\} - A_{h_n}^* z$ $(n = 1, 2, \ldots)$ converges uniformly in every finite interval. Further, according to Lemma III.3, the sequence $h_1^*, h_2^*, \ldots$ contains a subsequence $h_1, h_2, \ldots$ such that the sequence $|Q_{h_1}(z)|$, $|Q_{h_2}(z)|, \ldots$ converges uniformly in every finite interval. Consequently the sequence

$$Q_{h_n}(z)\exp\{-iA_{h_n}z\} = |Q_{h_n}(z)|\exp\{i\operatorname{Im}(\log Q_{h_n}(z)) - iA_{h_n}z\} \qquad (n = 1, 2, \ldots)$$

converges to a function $r(z)$ uniformly in every finite interval. Moreover,

(III.17) $$|r(z)| = \lim_{n\to\infty} |Q_{h_n}(z)| \leqslant 1$$

and

(III.18) $$\lim_{n\to\infty} |Q_{h_n}(z) - r(z)\exp(izA_{h_n})| = 0$$

uniformly in every finite interval.

We shall prove that the sequence $A_{h_1}, A_{h_2}, \ldots$ is bounded. Contrary to this statement let us suppose that the sequence $h_1, h_2, \ldots$ contains a subsequence $h_{k_1}, h_{k_2}, \ldots$ such that

(III.19) $$\lim_{n\to\infty} A_{h_{k_n}} = \infty \quad \text{or} \quad -\infty .$$

The assumption of the theorem and equalities (III.17) and (III.18) imply for each $\psi \in L$

(III.20) $$\lim_{n\to\infty} \int_{-\infty}^\infty \exp(iA_{h_n}z) r(z)\psi(z)\, dz = \Lambda(\psi) .$$

Further, for every $\psi \in L$, in view of (III.17), the function $r(z)\psi(z)$ is integrable in $-\infty < z < \infty$. Then equalities (III.19) and (III.20) imply $\Lambda \equiv 0$, which would contradicts the assumption of the theorem. Thus the sequence $A_{h_1}, A_{h_2}, \ldots$ is bounded. Thus there is a subsequence $h_{k_1}, h_{k_2}, \ldots$ such that the limit $A = \lim_{n\to\infty} A_{h_{k_n}}$ exists. Consequently, in view of formula (III.18), the sequence of characteristic functions $Q_{h_{k_1}}(z), Q_{h_{k_2}}(z), \ldots$ converges to a characteristic function $Q(z)$ uniformly in every finite interval. Hence and from the assumption of the theorem we obtain the equality

$$\Lambda(\psi) = \int_{-\infty}^\infty Q(z)\psi(z)\, dz .$$

Since $Q(0) = 1$, the last formula and the assumption of the theorem imply $\lim_{h\downarrow 0} Q_h(z) = Q(z)$ uniformly in every finite interval. To prove this, the same reasoning process must be applied as has been used in the proof of Lemma III.3. Taking into account equality (III.2), we have for $z > 0$

$$Q(z) = \lim_{h\downarrow 0} Q_h(z) = \lim_{h\downarrow 0} \{Q_{h/z}(1)\}^z = \{Q(1)\}^z .$$

Introducing the notation $e^{-a+ib} = Q(1)$, where a and b are real constants ($a \geqslant 0$), we have, for $z > 0$, $Q(z) = e^{-az+ibz}$. Hence, in view of the equality $Q(z) = \overline{Q(-z)}$, we obtain the assertion of the theorem.

For brevity let us introduce the notation

$$Q_{h_1,\ldots,h_k}(z) = \left\{R\left(\frac{z}{\lambda_1 \ldots \lambda_k}\right)\right\}^{h_k}, \tag{III.21}$$

where λ_j ($j = 1, 2, \ldots, k$) are given by formula (I.2).

THEOREM III.2. *Let $k \geqslant 2$. If for every $\psi \in L$ the limit*

$$\Lambda(\psi) = \operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \int_{-\infty}^{\infty} Q_{h_1,\ldots,h_k}(z)\psi(z)\,dz$$

exists and $\Lambda \not\equiv 0$, then

$$\operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} Q_{h_1,\ldots,h_k}(z) = 1$$

uniformly in every finite interval.

Proof. Equalities (III.2) and (III.21) imply

$$Q_{h_1,\ldots,h_k}(z) = Q_{h_k}\left(\frac{z}{\lambda_1 \ldots \lambda_{k-1}}\right). \tag{III.22}$$

Hence, in view of the assumption of the theorem, for every $\psi \in L$ the limit

$$\Lambda_0(\psi) = \lim_{h \downarrow 0} \int_{-\infty}^{\infty} Q_h(z)\psi(z)\,dz$$

exists and $\Lambda_0 \neq 0$. Consequently, according to theorem III.1, there are constants a and b such that

$$\lim_{h \downarrow 0} Q_h(z) = e^{-a|z|+ibz}$$

uniformly in every finite interval. Then, in view of equality (III.22),

$$\Lambda(\psi) = \operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \int_{-\infty}^{\infty} Q_{h_1,\ldots,h_k}(z)\psi(z)\,dz$$

$$= \operatorname*{Lim}_{h_1,\ldots,h_{k-1} \downarrow 0} \int_{-\infty}^{\infty} \exp\left\{\frac{-a|z|+ibz}{\lambda_1 \ldots \lambda_{k-1}}\right\}\psi(z)\,dz.$$

The limit on the right side of the last equality exists if and only if the equality $a = b = 0$ holds. Consequently

$$\operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} Q_{h_1,\ldots,h_k}(z) = 1$$

uniformly in every finite interval, q. e. d.

LEMMA III.6. *Let X_h ($h > 0$) be a family of infinitely divisible random variables with characteristic functions $\{R(z/h)\}^h$. Suppose that $E|X_h|^n < \infty$ ($n = 1, 2, \ldots$). Then*

$$\lim_{h \downarrow 0} h^{n-1}E(X_h^n) = \begin{cases} \gamma & \text{for } n = 1, \\ \int_{-\infty}^{\infty} x^{n-2}\,dG(x) & \text{for } n \geqslant 2, \end{cases} \tag{III.23}$$

where the constant γ and the function G are determined by the Kolmogorov expression of $\log R(z)$ (see (III.1)).

Moreover, if $\int_{-\infty}^{\infty} x^{n-2}\,dG(x) = 0$ for some integer $n \geqslant 2$, then the limit

$$\lim_{h \downarrow 0} h^{n-2}E(X_h^n) \tag{III.24}$$

exists. If

$$P(X_h \neq 0) > 0, \tag{III.25}$$

then

$$\liminf_{h \downarrow 0} E|X_h| > 0. \tag{III.26}$$

Proof. By simple calculations, in view of equality (III.1), we obtain

$$\frac{d^n}{dz^n}\left\{R\left(\frac{z}{h}\right)\right\}^h\bigg|_{z=0} = \begin{cases} i\gamma & \text{for } n = 1, \\ h^{1-n}i^n\left\{\int_{-\infty}^{\infty} x^{n-2}\,dG(x) + a_n h + o(h)\right\} & \text{for } n \geqslant 2, \end{cases}$$

where a_n are constants. Thus assertions (III.23) and (III.24) are a direct consequence of the well known equality

$$E(X_h^n) = i^{-n}\frac{d^n}{dz^n}\left\{R\left(\frac{z}{h}\right)\right\}^h\bigg|_{z=0}.$$

Now we shall prove inequality (III.26). Contrary to this inequality let us suppose that there is a sequence $h_1, h_2, \ldots$ ($h_n \downarrow 0$) for which the equality

$$\lim_{n \to \infty} E|X_{h_n}| = 0 \tag{III.27}$$

is true. By $q_h(x)$ we shall denote the distribution function of the random variable hX_h. Then the characteristic function of $q_h(x)$ is equal to $R(z)^h$. Put

$$k_n = [1/h_n]\,. \tag{III.28}$$

Let us consider the independent random variables $\xi_{1n}, \xi_{2n}, \ldots, \xi_{k_n}, n$, with the common distribution function $q_{h_n}(x)$. From the elementary inequalities

$$k_n \int\limits_{|x|>\varepsilon} dq_{h_n}(x) \leqslant \frac{k_n h_n}{\varepsilon} E|X_{h_n}|\,,$$

$$k_n \Big| \int\limits_{|x|<\varepsilon} x\,dq_{h_n}(x) \Big| \leqslant k_n h_n E|X_{h_n}|\,,$$

$$k_x\Big\{ \int\limits_{|x|<\varepsilon} x^2\,dq_{h_n}(x) - \Big(\int\limits_{|x|<\varepsilon} x\,dq_{h_n}(x)\Big)^2\Big\} \leqslant k_n \varepsilon \int\limits_{|x|<\varepsilon} |x|\,dq_{h_n}(x) \leqslant k_n h_n \varepsilon E|X_{h_n}|$$

and from (III.27) and (III.28) it follows for every positive number ε that

$$\lim_{n\to\infty} k_n \int\limits_{|x|>\varepsilon} dq_{h_n}(x) = 0\,, \qquad \lim_{n\to\infty} k_n \int\limits_{|x|<\varepsilon} x\,dq_{h_n}(x) = 0\,,$$

$$\lim_{n\to\infty} k_n \Big\{ \int\limits_{|x|<\varepsilon} x^2\,dq_{h_n}(x) - \Big(\int\limits_{|x|<\varepsilon} x\,dq_{h_n}(x)\Big)^2\Big\} = 0\,.$$

Consequently, in view of a well known theorem (see [2], § 27), the sequence of the sums

$$\eta_n = \xi_{1n} + \xi_{2n} + \ldots + \xi_{k_n, n}$$

converges to 0 in probability as $n \to \infty$. Since the characteristic function of the sum η_n is equal to $\{R(z)\}^{h_n k_n}$, then

$$\lim_{n\to\infty} \{R(z)\}^{h_n k_n} = 1\,.$$

Taking into account equality (III.28) we obtain $R(z) \equiv 1$, which contradicts inequality (III.25). The Lemma is thus proved.

LEMMA III.7. *Let $f(a,t) \epsilon \mathfrak{R} \frown \mathfrak{M}$ and*

$$\int\limits_{-\infty}^{\infty} x^2\,dG_f(x) > 0\,. \tag{III.29}$$

Suppose that $d^k f(\omega, t)/dt^k$ $(k \geqslant 1)$ has the local characteristic $\langle a_n, L_n \rangle$ at the moment t_0. Let $A_{1n}(\lambda), \ldots, A_{kn}(\lambda)$ be a system of function satisfying the assertion of theorem II.2. *Then*

$$\lim_{\lambda\downarrow 0} a_0(\lambda) = \tfrac{1}{2}\,, \tag{III.30}$$

$$\lim_{\lambda\downarrow 0} \lambda^{1-kn} a_n(\lambda) = \frac{L_n(x^n)}{m_n} \tag{III.31}$$

for $n \geqslant 2$, $m_n \neq 0$, where

$$m_n = \begin{cases} \displaystyle\int\limits_{-\infty}^{\infty} x^{n-2}\,dG_f(x) & \text{for} \quad k = 1\,, \\ \displaystyle 2^{k-2} \int\limits_{-\infty}^{\infty} x^{n-2}\,d\{G_f(x) - G_f(-x)\} & \text{for} \quad k \geqslant 2\,, \end{cases} \tag{III.32}$$

and

(III.33)

$$\operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \prod_{j=1}^{k} A_{jn}(h_j) = 0 \quad \textit{for} \quad n = 2, 3, \ldots \ \textit{or} \ n = 1,\ k = 2, 3, \ldots$$

Proof. According to theorem II.2 the system $A_{1n}(\lambda), \ldots, A_{kn}(\lambda)$ satisfies the conditions

$$a_n(\lambda) = \prod_{j=1}^{k} A_{jn}(\lambda) \qquad (n = 0, 1, \ldots)\,, \tag{III.34}$$

and

$$L_n(\varphi) = \operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \prod_{j=1}^{k} A_{jn}(h_j) \int\limits_{-\infty}^{\infty} \varphi(x)\,dP\big(D_{h_1,\ldots,h_k} f(\omega, t_0) < x\big) \tag{III.35}$$

for each $\varphi \epsilon X_n$ $(n = 0, 1, \ldots)$. Formula (III.30) is an immediate consequence of the equalities

$$\operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \prod_{j=1}^{k} A_{jn}(h_j) = L_0(1)\,, \qquad L_0(1 + |x|^0) = L_0(2) = 1\,.$$

Further, in view of (III.35) and Lemma II.1, we have

$$\begin{aligned} L_n(x^n) &= \operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \prod_{j=1}^{k} A_{jn}(h_j) E\big(D_{h_1,\ldots,h_k} f(\omega, t_0)\big)^n \\ &= \operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \prod_{j=1}^{k-1} h_j^{-n} \prod_{j=1}^{k} A_{jn}(h_j) E(Y_{h_k}^n)\,, \end{aligned} \tag{III.36}$$

where Y_h is a random variable with the characteristic function $\{R_f(z/h)\}^h$ in the case $k = 1$ and $|R_f(z/h)|^{2^{k-1}h}$ in the case $k \geqslant 2$. Hence and from Lemma III.7 it follows that

$$\operatorname*{Lim}_{h\downarrow 0} h^{n-1} E(Y_h^n) = m_n \qquad (n \geqslant 2)\,, \tag{III.37}$$

where the numbers m_n are given by formula (III.32). Let $m_n \neq 0$. Then from equalities (III.36) and (III.37) we obtain the formula

$$\text{(III.38)} \qquad m_n \operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} h_k \prod_{j=1}^{k} \{h_j^{-n} A_{jn}(h_j)\} = L_n(x^n).$$

For $k=1$, according to (III.34), formula (III.31) is an immediate consequence of the last equality. If $k \geqslant 2$, then, according to (III.32), $m_n = 0$ for odd indices n. From assumption (III.29) it follows that $m_{2s} \neq 0$ for $s = 1, 2, \ldots$ Taking into account equality (III.38), we obtain

$$1 = L_{2s}(1+x^{2s}) = L_{2s}(1)+L_{2s}(x^{2s})$$
$$= \operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} m_{2s} h_k \prod_{j=1}^{k} \{h_j^{-2s} A_{j,2s}(h_j)\} + \operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \prod_{j=1}^{k} A_{j,2s}(h_j).$$

The last equality implies the convergence

$$\lim_{\lambda \downarrow 0} \lambda^{1-2s} A_{k,2s}(\lambda) = c_{k,s},$$
$$\lim_{\lambda \downarrow 0} \lambda^{-2s} A_{k,2s}(\lambda) = c_{j,s} \qquad (j = 1, 2, \ldots, k-1),$$

where

$$\prod_{j=1}^{k} c_{j,s} = \frac{1}{m_{2s}} = \frac{L_{2s}(x^{2s})}{m_{2s}}.$$

This, in view of equality (III.34), implies formula (III.31) for $k \geqslant 2$. Assertion (III.31) is thus proved. Moreover, we obtain equality (III.33) for even indices n ($n \geqslant 2$). Further, formulas (III.31), (III.35) and (III.37) imply

$$L_{2s+1}(x^{2s}) = \operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \prod_{j=1}^{k-1} h_j^{-2s} \prod_{j=1}^{k} A_{j,2s+1}(h_j) \cdot E(Y_{h_k}^{2s})$$
$$= m_{2s} \operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} h_k \prod_{j=1}^{k} \{h_j^{-2s} A_{j,2s+1}(h_j)\} \qquad (s = 1, 2, \ldots).$$

This formula implies equality (III.33) for odd indices n ($n \geqslant 3$). In an analogous way, using Lemma III.6, we obtain for $k \geqslant 2$

$$L_1(1+|x|) = \operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \prod_{j=1}^{k} A_{j,1}(h_j) \Big\{1 + E|Y_{h_k}| \prod_{j=1}^{k-1} h_j^{-1}\Big\}$$
$$\geqslant \operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \prod_{j=1}^{k} A_{j,1}(h_j) \Big\{1 + c \prod_{j=1}^{k-1} h_j^{-1}\Big\},$$

where c is a positive constant. This inequality implies formula (III.33) for $k \geqslant 2$, $n = 1$. The Lemma is thus proved.

IV. This Chapter contains a complete discussion of the local characteristics of processes belonging to $\bigcup_{s=1}^{\infty} \mathfrak{R}_s$. It is easy to see that if a stationary process has a local characteristic at a fixed moment t_0, then it also has local characteristics at other moments. Therefore, for the sake of brevity, the words "at the moment t_0" will be omitted.

Let us introduce the following definitions: A local characteristic $\langle a_n, L_n \rangle$ is called

1) *a singulary local characteristic* if

$$L_n(\varphi) = \frac{1}{1+|a|^n} \varphi(a) \qquad (n = 0, 1, \ldots),$$

where a is a constant;

2) *a Poissonian local characteristic* if

$$L_0(\varphi) = \tfrac{1}{2}\varphi(a), \qquad L_1(\varphi) = c\varphi(a) + a_1 G_1^+(\varphi) + b_1 G_1^-(\varphi),$$
$$L_n = a_n G_n^+ + b_n G_n^- \qquad (n = 2, 3, \ldots),$$

where a, c, a_n, b_n are constants and $c > 0$;

3) *a quasipoissonian local characteristic* if

$$L_0(\varphi) = \tfrac{1}{2}\varphi(a), \qquad L_n = a_n G_n^+ + b_n G_n^- \qquad (n = 1, 2, \ldots)$$

where a, a_n and b_n are constants;

4) *a Cauchy local characteristic* if

$$L_0(\varphi) = \frac{a}{2\pi} \int_{-\infty}^{\infty} \frac{\varphi(x)\,dx}{a^2+(x-b)^2}, \qquad L_n = a_n G_n^+ + b_n G_n^- \qquad (n = 1, 2, \ldots),$$

where a, b, a_n, b_n are constants and $a > 0$;

5) *a uniform local characteristic* if the functionals L_n ($n = 0, 1, \ldots$) are invariant under translations, i. e., for each a, $L_n(\varphi) = L_n(\varphi_a)$ ($n = 0, 1, \ldots$), where $\varphi_a(x) = \varphi(x+a)$.

It is easy to prove that the necessary and sufficient condition of the uniformity of the local characteristic $\langle a_n, L_n \rangle$ is the equality $L_n = a_n G_n^+ + b_n G_n^-$ ($n = 0, 1, \ldots$).

THEOREM IV.1. *The local characteristics of generalized stochastic processes belonging to $\mathfrak{R}_1$ are uniform, Poissonian, quasipoissonian, singular or of the Cauchy type.*

The local characteristics of generalizaed stochastic processes belonging to $\bigcup_{s=2}^{\infty} \mathfrak{R}_s$ are uniform, singular or quasipoissonian.

Proof. Suppose that the generalized process $\Phi(\omega, t)$ belonging to $\bigcup_{s=1}^{\infty}\mathfrak{R}_s$ has the local characteristic $\langle a_n, L_n\rangle$. Thus, in view of theorem II.2, there are a process $f(\omega, t)$ belonging to $\mathfrak{R}\cap\mathfrak{M}$, an integer k and a system of functions $A_{1n}(\lambda), \ldots, A_{kn}(\lambda)$ such that

$$\frac{d^k}{dt^k}f(\omega, t) = \Phi(\omega, t), \quad a_n(\lambda) = \prod_{j=1}^{k} A_{jn}(\lambda) \quad (n = 0, 1, \ldots)$$

and for each $\varphi \in X_n$ $(n = 0, 1, \ldots)$

$$\text{(IV.1)} \quad \operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \prod_{j=1}^{k} A_{jn}(h_j) \int_{-\infty}^{\infty} \varphi(x)\, dP\left(D_{h_1,\ldots,h_k} f(\omega, t) < x\right) = L_n(\varphi).$$

If $f(\omega, t)$ is a Brownian motion process, then $\Phi(\omega, t)$ has a uniform local characteristic (see example a, Chapter I). If $f(\omega, t)$ is a deterministic process, then, in view of example c (Chapter I), $\Phi(\omega, t)$ has a singular local characteristic. It is well known that $f(\omega, t)$ is a Brownian motion process or a deterministic process if and only if the equality

$$G_f(\infty) - G_f(+0) + G_f(-0) - G_f(-\infty) = 0$$

is true. Consequently in the sequel we may suppose that the inequality

$$\text{(IV.2)} \quad \int_{-\infty}^{\infty} |x|\, dG_f(x) > 0$$

holds.

It is easy to see that the space X_n $(n = 0, 1, \ldots)$ is isomorphic with the space C of all functions continuous in $0 \leqslant z \leqslant 1$. An isomorphism of X_n onto C is given by the formula

$$\varphi \leftrightarrow \tilde{\varphi} \quad (\varphi \in X_n,\ \tilde{\varphi} \in C),$$

where

$$\tilde{\varphi}(z) = \begin{cases} \lim\limits_{x\to-\infty} \dfrac{\varphi(x)}{1+|x|^n} & \text{for} \quad z = 0, \\ \dfrac{\varphi(\operatorname{tg}(\pi z - \pi/2))}{1+|\operatorname{tg}(\pi z-\pi/2)|^n} & \text{for} \quad 0 < z < 1, \\ \lim\limits_{x\to\infty} \dfrac{\varphi(x)}{1+x^n} & \text{for} \quad z = 1. \end{cases}$$

Hence the continuous linear functionals L_n $(n = 0, 1, \ldots)$ may be written in the form

$$\text{(IV.3)} \quad L_n(\varphi) = \int_{-\infty}^{\infty} \frac{\varphi(x)}{1+|x|^n}\, d\mu_n(x) + a_n G_n^+(\varphi) + b_n G_n^-(\varphi) \quad (n = 0, 1, \ldots),$$

where a_n, b_n are constants and μ_n are functions of bounded variation on $-\infty < x < \infty$, normalized by supposing $\mu_n(-\infty) = 0$. Moreover, the non-negativity of functionals L_n implies that a_n, b_n are non-negative constants and μ_n are monotone non-decreasing bounded functions.

Let

$$\operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \prod_{j=1}^{k} A_{jn}(h_j) = 0.$$

Then from the equality

$$L_n(1) = \operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \prod_{j=k}^{k} A_{jn}(h_j)$$

and from (IV.3) it follows that

$$\int_{-\infty}^{\infty} \frac{d\mu_n(x)}{1+|x|^n} = 0.$$

Consequently

$$\text{(IV.4)} \quad \mu_n = 0 \quad \text{if} \quad \operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \prod_{j=1}^{k} A_{jn}(h_j) = 0.$$

Hence, in view of Lemma III.7, and the inequality (IV.2),

$$\text{(IV.5)} \quad L_n = a_n G_n^+ + b_n G_n^- \text{ for } n = 2, 3, \ldots \text{ and for } n = 1,\ k = 2, 3, \ldots$$

Now we shall examine the functional L_0. From Lemma III.7 and from formulas (IV.1), (IV.3) it follows that for each $\varphi \in X_0^*$ the equality

$$\text{(IV.6)} \quad \operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} \int_{-\infty}^{\infty} \varphi(x)\, dP\left(D_{h_1,\ldots,h_k} f(\omega, t) < x\right) = 2\int_{-\infty}^{\infty} \varphi(x)\, d\mu_0(x)$$

holds. Let $R_{h_1,\ldots,h_k}(z)$ denote the characteristic function of the distribution function $P\left(D_{h_1,\ldots,h_k} f(\omega, t) < x\right)$. Let

$$\text{(IV.7)} \quad Q(z) = 2\int_{-\infty}^{\infty} e^{ixz}\, d\mu_0(x).$$

Then, in view of Lemma III.3 for each $\psi \in L$

$$\text{(IV.8)} \quad \operatorname*{Lim}_{h_1,\ldots,h_k\uparrow 0} \int_{-\infty}^{\infty} R_{h_1,\ldots,h_k}(z)\, \psi(z)\, dz = \int_{-\infty}^{\infty} Q(z)\, \psi(z)\, dz.$$

Equalities (I.1), (II.3) and Lemma II.1 imply the formula

$$\text{(IV.9)}\qquad R_{h_1,\dots,h_k}(z)=\begin{cases}\{R_f(z/h_1)\}^{h_1} & \text{if}\quad k=1,\\ \left\{R_f\left(\dfrac{z}{\lambda_1\dots\lambda_k}\right)\right\}^{2^{k-1}h_k} & \text{if}\quad k\geqslant 2,\end{cases}$$

where λ_j $(j=1,2,\dots,k)$ are given by formula (I.2).

Let $k\geqslant 2$, i. e. $\Phi(\omega,t)\in\bigcup_{s=2}^{\infty}\mathfrak{K}_s$. If $\mu_0\equiv 0$, then, in view of formulas (IV.3) and (IV.5), $\Phi(\omega,t)$ has a uniform local characteristic. If $\mu_0\not\equiv 0$, then $Q(z)\not\equiv 0$. Hence and from equalities (IV.8), (IV.9), using theorem III.2, we obtain $Q(z)\equiv 1$. Consequently, according to (IV.7),

$$\mu_0(x)=\begin{cases}\frac{1}{2} & \text{for}\quad x>0,\\ 0 & \text{for}\quad x\leqslant 0.\end{cases}$$

Hence, in view of (IV.3), $1=L_0(1+|x|^0)=L_0(2)=1+2a_0+2b_0$. Since $a_0\geqslant 0$ and $b_0\geqslant 0$, then the last equality implies $a_0=b_0=0$. Thus $L_0(\varphi)=\frac{1}{2}\varphi(0)$. Hence and from equality (IV.5) we infer that $\Phi(\omega,t)$ has a quasipoissonian local characteristic.

Let $\Phi(\omega,t)\in\mathfrak{K}_1$. If $\mu_0\equiv 0$, then, in view of (IV.3)

$$L_0=a_0G_0^{+}+b_0G_0^{-}.$$

Suppose that $\mu_0\not\equiv 0$. Consequently $Q(z)\not\equiv 0$. Hence, in view of equalities (IV.8), (IV.9) and theorem III.1, we obtain the formula $Q(z)=e^{-a|z|+ibz}$, where a,b are constants and $a\geqslant 0$. Consequently, according to (IV.7)

$$\mu_0(x)=\begin{Bmatrix}\frac{1}{2} & \text{for} & x>b\\ 0 & \text{for} & x\leqslant b\end{Bmatrix}\quad\text{if}\quad a=0,$$

$$\mu_0(x)=\frac{a}{2\pi}\int_{-\infty}^{\infty}\frac{du}{a^2+(u-b)^2}\quad\text{if}\quad a>0.$$

Hence, in view of (IV.3), $1=L_0(1+|x|^0)=L_0(2)=1+2a_0+2b_0$. Since $a_0\geqslant 0$ and $b_0\geqslant 0$, the last equality implies $a_0=b_0=0$. Consequently we have the following formulas:

$$\text{(IV.10)}\qquad L_0(\varphi)=\tfrac{1}{2}\varphi(b),$$

$$\text{(IV.11)}\qquad L_0(\varphi)=\frac{a}{2\pi}\int_{-\infty}^{\infty}\frac{\varphi(x)\,dx}{a^2+(x-b)^2}.$$

Now we shall examine the functionals L_1 for $\Phi(\omega,t)\in\mathfrak{K}_1$. If $\lim_{\lambda\downarrow 0}a_1(\lambda)=0$, then, in view of (IV.4), $L_1=a_1G_1^{+}+b_1G_1^{-}$. Consequently, according to (IV.5), (IV.9), (IV.10) and (IV.11), the local characteristic of $\Phi(\omega,t)$ is uniform, quasipoissonian or of the Cauchy type. Further we may suppose that $\lim_{\lambda\downarrow 0}a_1(\lambda)>0$. Hence, taking into account formula (IV.1) for $n=0$ and $n=1$, we obtain $L_1(\varphi)=cL_0(\varphi)$ for $\varphi\in X_0$, where c is a constant. This equality implies, in view of (IV.3),

$$\int_{-\infty}^{\infty}\frac{\varphi(x)}{1+|x|}\,d\mu_1(x)=c\int_{-\infty}^{\infty}\varphi(x)\,d\mu_0(x)\qquad\text{for}\qquad\varphi\in X_0^*.$$

Consequently

$$\text{(IV.12)}\qquad \mu_1(x)=c\int_{-\infty}^{x}(1+|u|)\,d\mu_0(u).$$

Hence in the case (IV.9) we obtain $\mu_1\equiv 0$. Consequently, in virtue of (IV.3), (IV.5) and (IV.9), the local characteristic of $\Phi(\omega,t)$ is uniform.

In the case (IV.10) it follows from (IV.12) that

$$\mu_1(x)=\begin{cases}c(1+|b|) & \text{for}\quad x>b,\\ 0 & \text{for}\quad x\leqslant b.\end{cases}$$

Then, in view of formulas (IV.3), (IV.5), in this case the local characteristic of $\Phi(\omega,t)$ is Poissonian (if $c\neq 0$) or quasipoissonian (if $c=0$).

In the case (IV.11) we have the equality

$$\int_{-\infty}^{\infty}(1+|u|)\,d\mu_0(u)=\frac{a}{2\pi}\int_{-\infty}^{\infty}\frac{(1+|u|)\,du}{a^2+(u-b)^2}=\infty.$$

Since $\mu_1(x)$ is bounded, it follows from (IV.12) that $c=0$. Hence $\mu_1\equiv 0$. Consequently, according to (IV.3), (IV.5), in this case $\Phi(\omega,t)$ has a Cauchy local characteristic. The Theorem is thus proved.

Now we shall prove three Lemmas and then give examples of all the local characteristics described by the preceding theorem.

LEMMA IV. 1. *Let Y_h $(h>0)$ be a family of infinitely divisible random variables with characteristic functions $\{R(z)\}^h$. Suppose that $E|Y_h|^n<\infty$ $(n=1,2,\dots)$ and for some x_0*

$$\text{(IV.13)}\qquad \int_0^{x_0}\frac{1}{u}\,dG(u)=\infty,$$

where $G(u)$ is a function determined by the Kolmogorov expression of $\log R(z)$ (see (III.1)). Let $q_h(x)$ denote the distribution function of the random variable Y_h. Then

$$\lim_{h\downarrow 0}\frac{1}{h^n}\left|\int_0^{x_0} x^n dq_h(x)\right| = \infty \qquad (n = 1, 2, \ldots).$$

Proof. Integrating by parts we obtain the following inequality:

$$\int_\varepsilon^{x_0}\int_x^{x_0}\frac{1}{u^2}dG(u)\,dx = -\varepsilon\int_\varepsilon^{x_0}\frac{1}{u^2}\,dG(u)+\int_\varepsilon^{x_0}\frac{1}{u}\,dG(u)$$

$$=\int_\varepsilon^{x_0}\left(1-\frac{\varepsilon}{u}\right)\frac{1}{u}\,dG(u) \geqslant \int_{2\varepsilon}^{x_0}\left(1-\frac{\varepsilon}{u}\right)\frac{1}{u}\,dG(u) \geqslant \frac{1}{2}\int_{2\varepsilon}^{x_0}\frac{1}{u}dG(u),$$

where $0 < |\varepsilon| < |x_0|$ and $\operatorname{sgn}\varepsilon\operatorname{sgn}x_0 = 1$. Hence, if $\varepsilon\to 0$, it follows that

$$\int_0^{x_0}\int_x^{x_0}\frac{1}{u^2}\,dG(u)\,dx \geqslant \tfrac{1}{2}\int_0^{x_0}\frac{1}{u}\,dG(u).$$

Consequently, in view of the assumption of the Lemma,

$$\text{(IV.14)}\qquad \int_0^{x_0}\int_x^{x_0}\frac{1}{u^2}\,dG(u)\,dx = \infty.$$

Put

$$\text{(IV.15)}\qquad k_h = [1/h].$$

Let $\xi_{1,h}, \xi_{2,h}, \ldots, \xi_{k_h,h}$ be independent random variables with a common distribution function $q_h(x)$. Since the characteristic function of the sum

$$\zeta_h = \xi_{1,h}+\xi_{2,h}+\ldots+\xi_{k_h,h}$$

is equal to $\{R(z)\}^{hk_h}$, then the distribution functions of the sums ζ_h converges to $q_1(x)$ if $h\downarrow 0$ at all continuity points x $(-\infty\leqslant x\leqslant\infty)$ of $q_1(x)$. Let

$$\text{(IV.16)}\qquad M(x) = \begin{cases} \displaystyle\int_{-\infty}^{x}\frac{1}{u^2}\,dG(u) & \text{for} \quad x<0,\\ \displaystyle -\int_x^{\infty}\frac{1}{u^2}\,dG(u) & \text{for} \quad x>0.\end{cases}$$

Then, according to a theorem of Gnedenko ([2], § 25, theorem 4),

$$\lim_{h\downarrow 0}\sum_{j=1}^{k_h}P(\xi_{j,h}<x) = M(x) \quad \text{for} \quad x<0,$$

$$\lim_{h\downarrow 0}\sum_{j=1}^{k_h}P(\xi_{j,h}\geqslant x) = -M(x) \quad \text{for} \quad x>0$$

at all continuity points of $M(x)$. Consequently, in view of (IV.15),

$$\lim_{h\downarrow 0}\frac{1}{h}\int_{\operatorname{sgn}x\infty}^{x} dq_h(u) = M(x) \quad \text{for} \quad x\neq 0,$$

at all continuity points of $M(x)$.

Without loss of generality we can assume that x_0 is a continuity point of $M(x)$. Then the last equality implies

$$\lim_{h\downarrow 0}\frac{1}{h}\int_x^{x_0} dq_h(u) = M(x_0)-M(x)$$

for all continuity points of $M(x)$ satisfying the conditions $0<|x|<|x_0|$, $\operatorname{sgn}x\operatorname{sgn}x_0 = 1$. Hence, in view of definition (IV.16), we obtain

$$\lim_{h\downarrow 0}\frac{1}{h}\int_x^{x_0} dq_h(u) = \int_x^{x_0}\frac{1}{u^2}\,dG(u).$$

Consequently, according to Fatou's Lemma,

$$\liminf_{h\downarrow 0}\frac{1}{h}\int_0^{x_0}\int_x^{x_0} dq_h(u)\,dx \geqslant \int_0^{x_0}\int_x^{x_0}\frac{1}{u^2}\,dG(u)\,dx \quad \text{if} \quad x_0>0,$$

$$\liminf_{h\downarrow 0}\frac{1}{h}\int_{x_0}^{0}\int_{x_0}^{x} dq_h(u)\,dx \geqslant \int_{x_0}^{0}\int_{x_0}^{x}\frac{1}{u^2}\,dG(u)\,dx \quad \text{if} \quad x_0<0.$$

Hence, in view of equality (IV.14), we obtain

$$\lim_{h\downarrow 0}\frac{1}{h}\int_0^{x_0}\int_x^{x_0} dq_h(u)\,dx = \infty.$$

Integrating by parts we obtain from the last equality

$$\text{(IV.17)}\qquad \lim_{h\downarrow 0}\frac{1}{h}\int_0^{x_0} x\,dq_h(x) = \infty.$$

According to the convexity of the function $|x|^n$ $(n = 1, 2, \ldots)$, we have the inequality

$$\Big|\int_0^{x_0} x\,dq_h(x)\Big|^n \leqslant \Big|\int_0^{x_0} x^n\,dq_h(x)\Big| \qquad (n = 1, 2, \ldots).$$

Hence and from (IV.17) it follows that

$$\lim_{h\downarrow 0}\frac{1}{h^n}\Big|\int_0^{x_0} x^n\,dq_h(x)\Big| = \infty.$$

The Lemma is thus proved.

LEMMA IV.2. *Let $f(\omega, t)\in\mathfrak{K}\cap\mathfrak{M}$ and let k be an integer. Suppose that for some x_0*

$$\text{(IV.18)}\qquad \begin{aligned} &\int_0^{x_0}\frac{1}{u}\,dG_f(u) = \infty && \textit{if}\quad k = 1,\\ &\int_0^{x_0}\frac{1}{u}\,d\{G_f(u) - G_f(-u)\} = \infty && \textit{if}\quad k \geqslant 2, \end{aligned}$$

and the limits

$$\text{(IV.19)}\qquad c_n = \lim_{h_1,\ldots,h_k\downarrow 0}\frac{\int_0^{\infty} x^n\,dP\big(D_{h_1,\ldots,h_k}f(\omega, t) < x\big)}{\int_{-\infty}^{\infty}|x|^n\,dP\big(D_{h_1,\ldots,h_k}f(\omega, t) < x\big)} \qquad (n = 1, 2, \ldots),$$

$$\text{(IV.20)}\qquad \Lambda(\varphi) = \operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0}\int_{-\infty}^{\infty}\varphi(x)\,dP\big(D_{h_1,\ldots,h_k}f(\omega, t) < x\big)$$

exist for each $\varphi\in X_0$. Then the derivative $d^kf(\omega, t)/dt^k$ has the local characteristic $\langle a_n, L_n\rangle$, where

$$L_0 = \tfrac{1}{2}\Lambda, \qquad L_n = c_nG_n^+ + (1 - c_n)G_n^- \qquad (n = 1, 2, \ldots).$$

Proof. Let $R(z)$ be the characteristic function of the random variable $f(\omega, 1)$. Then, in view of Lemma II.1, the characteristic function of the difference $\Delta_{\lambda_1}\Delta_{\lambda_2}\ldots\Delta_{\lambda_k}f(\omega, t)$ is equal to $R(z)^{h_k}$ if $k = 1$ and is equal to $|R(z)|^{2^{k-1}h_k}$ if $k\geqslant 2$. Let Y_h $(h > 0)$ be a family of random variables with the characteristic function $R(z)^h$ if $k = 1$ and $|R(z)|^{2^{k-1}h}$ if $k\geqslant 2$.

From (IV.18) it follows that the assumption (IV.13) of Lemma IV.1 is fulfilled. Put

$$\text{(IV.21)}\qquad q_h(x) = P(Y_h < x) = P\big(\Delta_{\lambda_1}\ldots\Delta_{\lambda_{k-1}}\Delta_h f(\omega, t) < x\big).$$

Then, in view of Lemma IV.1, for some x_0

$$\text{(IV.22)}\qquad \lim_{h\downarrow 0}\frac{1}{h^n}\Big|\int_0^{x_0} x^n\,dq_h(x)\Big| = \infty.$$

We introduce the following notation:

$$\text{(IV.23)}\qquad \beta_n(h) = \frac{h^n}{\int_{-\infty}^{\infty}|x|^n\,dq_h(x)} \qquad (n = 1, 2, \ldots).$$

From (IV.22) it follows that

$$\text{(IV.24)}\qquad \lim_{h\downarrow 0}\beta_n(h) = 0 \qquad (n = 1, 2, \ldots).$$

Since $f(\omega, t)$ belongs to $\mathfrak{M}$, we have

$$\text{(IV.25)}\qquad \gamma_h = \int_{-\infty}^{\infty}|x|\,dq_h(x)\to 0 \quad\text{for}\quad h\downarrow 0.$$

The inequality

$$\gamma_h^n = \Big\{\int_{-\infty}^{\infty}|x|\,dq_h(x)\Big\}^n \leqslant \int_{-\infty}^{\infty}|x|^n\,dq_h(x) \qquad (n = 1, 2, \ldots)$$

implies

$$\frac{\int_{-\gamma_h}^{\gamma_h}|x|^n\,dq_h(x)}{\int_{-\infty}^{\infty}|x|^n\,dq_h(x)} \leqslant \gamma_h^{-n}\int_{-\gamma_h}^{\gamma_h}|x|^n\,dq_h(x) = \int_{-1}^{1}|z|^n\,dq_h(\gamma_h z) \qquad (n = 1, 2, \ldots).$$

Consequently, in view of (IV.25),

$$\text{(IV.26)}\qquad \lim_{h\downarrow 0}\frac{\int_{-\gamma_h}^{\gamma_h}|x|^n\,dq_h(x)}{\int_{-\infty}^{\infty}|x|^n\,dq_h(x)} = 0 \qquad (n = 1, 2, \ldots).$$

Let $\varphi\in X_n$ $(n = 1, 2, \ldots)$. Then we have the inequalities $|\varphi(x)| \leqslant \|\varphi\|(1 + |x|^n)$, $|G_n^+(\varphi)| \leqslant \|\varphi\|$, $|G_n^-(\varphi)| \leqslant \|\varphi\|$, where $\|\ \|$ denotes the norm in X_n. Hence, using the formula (IV.23), we obtain the following inequality:

$$\beta_n(h)\left|\int_{-\infty}^{\infty}\varphi\left(\frac{x}{h}\right)dq_h(x)-G_n^+(\varphi)\int_0^{\infty}\left(\frac{x}{h}\right)^n dq_h(x)-G_n^-(\varphi)\int_{-\infty}^{\infty}\left(\frac{x}{h}\right)^n dq_h(x)\right|$$

$$\leqslant \beta_n(h)\int_{\gamma_h}^{\infty}\left|\frac{\varphi(x/h)}{(x/h)^n}-G_n^+(\varphi)\right|\left(\frac{x}{h}\right)^n dq_h(x)+$$

$$+\beta_h(h)\int_{-\infty}^{\gamma_h}\left|\frac{\varphi(x/h)}{|x/h|^n}-G_n^-(\varphi)\right|\left|\frac{x}{h}\right|^n dq_h(x)+$$

$$+\beta_n(h)\int_{-\gamma_h}^{\gamma_h}\left|\varphi\left(\frac{x}{h}\right)\right|dq_h(x)+\beta_n(h)|G_n^+(\varphi)|\int_0^{\gamma_h}\left(\frac{x}{h}\right)^n dq_h(x)+$$

$$+\beta_n(h)|G_n^-(\varphi)|\int_{-\gamma_h}^{0}\left|\frac{x}{h}\right|dq_h(x)\leqslant\sup_{x\geqslant\gamma_h/h}\left|\frac{\varphi(x)}{x^n}-G_n^+(\varphi)\right|+$$

$$+\sup_{x<-\gamma_h/h}\left|\frac{\varphi(x)}{|x|^n}-G_n^-(\varphi)\right|+\beta_n(h)\|\varphi\|+3\,\frac{\int_{-\gamma_h}^{\gamma_h}|x|^n dq_h(x)}{\int_{-\infty}^{\infty}|x|^n dq_h(x)}.$$

Since $\gamma_h/h = 1/\beta_1(h)\to\infty$ when $h\downarrow 0$, in view of (IV.24) and (IV.26), the last inequality implies

$$\text{(IV.27)}\qquad \lim_{h\downarrow 0}\beta_n(h)\left|\int_{-\infty}^{\infty}\varphi\left(\frac{x}{h}\right)dq_h(x)-G_n^+(\varphi)\int_0^{\infty}\left(\frac{x}{h}\right)^n dq_h(x)\right.$$

$$\left.-D_n^-(\varphi)\int_{-\infty}^{0}\left|\frac{x}{h}\right|^n dq_h(x)\right| = 0\qquad (n = 1, 2, \ldots)$$

for each $\varphi\in X_n$. From definitions (IV.21) and (IV.23) it follows that

$$\beta_n(h)\int_0^{\infty}\left(\frac{x}{h}\right)^n dq_h(x) = \frac{\int_0^{\infty}x^n dP\left(D_{h_1,\ldots,h_{k-1},h}f(\omega,t)<x\right)}{\int_{-\infty}^{\infty}|x|^n dP\left(D_{h_1,\ldots,h_{k-1},h}f(\omega,t)<x\right)},$$

$$\beta_n(h)\int_{-\infty}^{0}\left|\frac{x}{h}\right|^n dq_h(x) = \frac{\int_{-\infty}^{0}|x|^n dP\left(D_{h_1,\ldots,h_{k-1},h}f(\omega,t)<x\right)}{\int_{-\infty}^{\infty}|x|^n dP\left(D_{h_1,\ldots,h_{k-1},h}f(\omega,t)<x\right)}.$$

Consequently, the assumption (IV,19) and the equality (IV.27) imply for each $\varphi\in X_n$ $(n = 1, 2, \ldots)$

$$\text{(IV.28)}\qquad \lim_{h\downarrow 0}\beta_n(h)\int_{-\infty}^{\infty}\varphi\left(\frac{x}{h}\right)dq_h(x) = c_nG_n^+(\varphi)+(1-c_n)G_n^-(\varphi).$$

Putting $A_{jn}(\lambda)\equiv 1$ $(j = 1, 2, \ldots, k-1)$, $A_{kn}(\lambda) = \beta_n(\lambda)$, we have, according to (IV.21),

$$\prod_{j=1}^{k}A_{jn}(h_j)\int_{-\infty}^{\infty}\varphi(x)dP\left(D_{h_1,\ldots,h_k}f(\omega,t)<x\right) = \beta_n(h_k)\int_{-\infty}^{\infty}\varphi\left(\frac{x}{\lambda_1\ldots\lambda_{k-1}h_k}\right)dq_{h_k}(x).$$

Hence, in virtue of (IV.28),

$$\operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0}\prod_{j=1}^{k}A_{jn}(h_j)\int_{-\infty}^{\infty}\varphi(x)dP\left(D_{h_1,\ldots,h_k}f(\omega,t)<x\right) = c_nG_n^+(\varphi)+(1-c_n)G_n^-(\varphi)$$

for each $\varphi\in X_n$ $(n = 1, 2, \ldots)$. Taking into account assumption (IV.20), we find in virtue of theorem II.2 that the derivative $d^kf(\omega,t)/dt^k$ has the local characteristic $\langle a_n, L_n\rangle$, where $L_0 = \frac{1}{2}\Lambda$, $L_n = c_nG_n^+ + (1-c_n)G_n^-$ $(n = 1, 2, \ldots)$.

The Lemma is thus proved.

COROLLARY. *Let $f(\omega,t)\in\mathfrak{K}\cap\mathfrak{M}$ and let k be an integer. Suppose that in the case $k = 1$ for some x_0 the equality*

$$\int_0^{x_0}\frac{1}{u}\,dG_f(u) = \infty$$

is true and the increments $\Delta_hf(\omega,t)$ are symmetrically distributed, i. e. the characteristic function of $\Delta_hf(\omega,t)$ is real. Further, suppose that in the case $k\geqslant 2$ the equality

$$\int_0^{x_0}\frac{1}{u}\,d\{G_f(u)-G_f(-u)\} = \infty$$

holds. If for each $\varphi\in X_0$ the limit

$$\Lambda(\varphi) = \operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0}\int_{-\infty}^{\infty}\varphi(x)dP\left(D_{h_1,\ldots,h_k}f(\omega,t)<x\right)$$

exists, then the derivative $d^kf(\omega,t)/dt^k$ has the local characteristic $\langle a_n, L_n\rangle$, where $L_0 = \frac{1}{2}\Lambda$, $L_n = \frac{1}{2}G_n^+ + \frac{1}{2}G_n^-$ $(n = 1, 2, \ldots)$.

In fact, from Lemma II.2 it follows that $D_{h_1,\ldots,h_k} f(\omega, t)$ for $k \geqslant 2$ are symmetrically distributed. Hence, for every integer k,

$$c_n = \frac{\int\limits_0^\infty x^n dP(D_{h_1,\ldots,h_k} f(\omega, t) < x)}{\int\limits_{-\infty}^\infty |x|^n dP(D_{h_1,\ldots,h_k} f(\omega, t) < x)} = \tfrac{1}{2},$$

and our assertion is a direct consequence of Lemma IV.2.

We shall use the following theorem of Gnedenko ([2], § 19, theorem 1):

Let $q_0(x), q_1(x), \ldots$ be a sequence of infinitely divisible distribution functions. Then

$$\lim_{n\to\infty} q_n(x) = q_0(x)$$

at all continuity points x $(-\infty \leqslant x \leqslant \infty)$ of $q_0(x)$ if and only if

$$\lim_{n\to\infty} \tilde{G}_n(x) = \tilde{G}_0(x),$$

at all continuity points x $(-\infty \leqslant x \leqslant \infty)$ of $G_0(x)$ and

$$\lim_{n\to\infty} \tilde{\gamma}_n = \tilde{\gamma}_0,$$

where the functions $\tilde{G}_n(x)$ and the constants $\tilde{\gamma}_n$ $(n = 0, 1, \ldots)$ are determined by the Lévy-Khintchine expression of the logarithm of the characteristic functions of distributions functions $q_n(x)$ $(n = 0, 1, \ldots)$.

From this theorem, using theorem III.1, Lemmas II.1 and III.5 and equalities (II.3), (II.5) and (II.6), we obtain the following

LEMMA IV.3. *Let $f(\omega, t) \in \mathfrak{R} \frown \mathfrak{M}$. Then the limit*

$$\Lambda(\varphi) = \operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \int\limits_{-\infty}^\infty \varphi(x) dP(D_{h_1,\ldots,h_k} f(\omega, t) < x)$$

exists for each $\varphi \in X_0$ and $\Lambda(\varphi) \not\equiv 0$ for $\varphi \in X_0^*$ if and only if

(a) *in the case $k = 1$ the limits*

$$\lim_{h\downarrow 0} (h^2-1) \int\limits_{-\infty}^\infty \frac{u\, dG_f(u)}{(h^2+u^2)(1+u^2)}, \quad \lim_{h\downarrow 0} \frac{1}{h} \int\limits_{-\infty}^x \frac{dG_f(uh)}{1+u^2}$$

exist at all continuity points x $(-\infty \leqslant x \leqslant \infty)$ of the limit function;

(b) *in the case $k \geqslant 2$ the limit*

$$\lim_{h\downarrow 0} \frac{1}{h} \int\limits_{-\infty}^x \frac{d\{G_f(uh) - G_f(-uh)\}}{1+u^2}$$

exists at all continuity points x $(-\infty \leqslant x \leqslant \infty)$ of the limit function.

Let $G(u)$ be an arbitrary monotone non-decreasing bounded function continuous on the right, normalized by supposing $G(-\infty) = 0$ and having the finite moments

$$\text{(IV.29)} \qquad \int\limits_{-\infty}^\infty |x|^n dG(x) < \infty \qquad (n = 1, 2, \ldots).$$

Put

$$\text{(IV.30)} \qquad R(z) = \exp\Big\{\int\limits_{-\infty}^\infty (e^{izu} - 1 - izu)\frac{1}{u^2} dG(u)\Big\}.$$

From the well known theorem of Kolmogorov concerning the extension of measures in product spaces it follows that there is a stochastic process $f^*(\omega, t)$ with independent increments, defined on the ω-space of all real-valued functions, such that $\{R(z)\}^h$ is the characteristic function of the increment $f^*(\omega, t+h) - f^*(\omega, t)$. We may suppose that $f^*(\omega, 0) = 0$. Since $\lim\limits_{h\downarrow 0}\{R(z)\}^h = 1$, then the increments $f^*(\omega, t+h) - f^*(\omega, t)$ converge to 0 in probability, when $h \downarrow 0$. Consequently, in view of a theorem of Doob ([1], II, theorem 2.6) there is a process $f(\omega, t)$, defined on the same ω-space, which is measurable and $P(f^*(\omega, t) = f(\omega, t)) = 1$ for each t. From formulas (IV.29) and (IV.30) it follows that $E(f(\omega, t))^2$ is integrable over every finite interval. Then, in view of the inequality

$$E\int\limits_a^b |f(\omega, t)|\, dt = \int\limits_a^b E|f(\omega, t)|\, dt \leqslant \int\limits_a^b \big(1 + E(f(\omega, t))^2\big)\, dt,$$

almost all sample functions of the process $f(\omega, t)$ are integrable over every finite interval. Consequently, $f(\omega, t)$ belongs to $\mathfrak{R}$ and, in view of (IV.29) and theorem II.1, belongs also to $\mathfrak{M}$. Thus we have the following assertion:

For every monotone non-decreasing bounded function $G(x)$, continuous on the right, with finite moments and normalized by supposing $G(-\infty) = 0$, there exists a stochastic process $f(\omega, t)$ belonging to $\mathfrak{R} \frown \mathfrak{M}$ such that $G_f(x) = G(x)$.

In virtue of this assertion, in the examples of processes belonging to $\mathfrak{R} \frown \mathfrak{M}$ we shall define the Kolmogorov function $G(x)$ only.

Examples. (a) We shall give an example of a process $f(\omega, t)$ belonging to $\mathfrak{R} \frown \mathfrak{M}$ such that the local characteristics of all derivatives $d^k f(\omega, t)/dt^k$ $(k = 1, 2, \ldots)$ are quasipoissonian.

Let

$$G_f(x) = \int\limits_{-\infty}^\infty \frac{e^{-|u|}\, du}{1 + |\log|u||}.$$

Since

$$\int_{-\infty}^{\infty} |x|^n dG_f(x) < \infty \qquad (n = 1, 2, \ldots),$$

$f(\omega, t) \in \mathfrak{K} \cap \mathfrak{M}$. In view of Lemma II.1, the characteristic functions $R_{h_1,\ldots,h_k}(z)$ of increments $\Delta_{\lambda_1} \ldots \Delta_{\lambda_k} f(\omega, t)$ are given by the formula

$$R_{h_1,\ldots,h_k}(z) = \exp\left\{2^{k-1} h_k \int_{-\infty}^{\infty} (\cos zu - 1) \frac{e^{-|u|}\,du}{u^2(1+|\log|u||)}\right\}.$$

Then the characteristic functions $Q_{h_1,\ldots,h_k}(z)$ of $D_{h_1,\ldots,h_k} f(\omega, t)$ are given by the formula

$$Q_{h_1,\ldots,h_k}(z) = \exp\left\{2^k \int_0^{\infty} \left(\cos \frac{zu}{\lambda_1 \ldots \lambda_{k-1}} - 1\right) \frac{e^{-h_k u}\,du}{u^2(1+|\log h_k u|)}\right\}.$$

Hence

$$\operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} Q_{h_1,\ldots,h_k}(z) = 1,$$

uniformly in every finite interval. Consequently for each $\varphi \in X_0$

$$\operatorname*{Lim}_{h_1,\ldots,h_k \downarrow 0} \int_{-\infty}^{\infty} \varphi(x)\, dP\big(D_{h_1,\ldots,h_k} f(\omega, t) < x\big) = \varphi(0).$$

Since $R_{h_1,\ldots,h_k}(z)$ is real and

$$\frac{1}{2}\int_0^1 \frac{1}{u}\, d\{G_f(u) - G_f(-u)\} = \int_0^1 \frac{1}{u}\, dG_f(u) = \int_0^1 \frac{e^{-u}\,du}{u(1+|\log u|)} = \infty,$$

then in view of the Corollary to Lemma IV.2 the derivatives $d^k f(\omega, t)/dt^k$ $(k = 1, 2, \ldots)$ have the quasipoissonian local characteristic $\langle a_n^{(k)}, L_n^{(k)}\rangle$, where $L_0^{(k)}(\varphi) = \frac{1}{2}\varphi(0)$, $L_n^{(k)} = \frac{1}{2}G_n^+ + \frac{1}{2}G_n^-$ $(n = 1, 2, \ldots)$.

(b) Now we shall give some example of a process $f(\omega, t)$ belonging to $\mathfrak{K} \cap \mathfrak{M}$ such that the derivative $df(\omega, t)/dt$ has a Cauchy local characteristic.

Put

$$G_f(x) = \tfrac{1}{2} \int_{-\infty}^{x} e^{-|u|}\,du.$$

Since

$$\int_{-\infty}^{\infty} |x|^n dG_f(x) < \infty \qquad (n = 1, 2, \ldots),$$

we have $f(\omega, t) \in \mathfrak{K} \cap \mathfrak{M}$. Further, the characteristic function $R_h(z)$ of the random variable $D_h f(\omega, t)$ is given by the formula

$$R_h(z) = \exp\left\{h \int_0^{\infty} \left(\cos \frac{zu}{h} - 1\right) \frac{e^{-u}\,du}{u^2}\right\} = \exp\left\{-\int_0^{|z|} \exp\left(-\frac{h^2}{2u^2}\right) du\right\}.$$

Hence we obtain $\lim\limits_{h \downarrow 0} R_h(z) = e^{-|z|}$ uniformly in every finite interval. Consequently, for each $\varphi \in X_0$,

$$\lim_{h \downarrow 0} \int_{-\infty}^{\infty} \varphi(x)\, dP(D_h f(\omega, t) < x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(x)\,dx}{1+x^2}.$$

Since $R_h(z)$ is real and

$$\int_0^1 \frac{1}{u}\, dG_f(u) = \frac{1}{2}\int_0^1 \frac{e^{-u}\,du}{u} = \infty,$$

then in view of the Corollary to Lemma IV.2 we infer that $df(\omega, t)/dt$ has the Cauchy local characteristic $\langle a_n, L_n\rangle$, where

$$L_0(\varphi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varphi(x)\,dx}{1+x^2}, \qquad L_n = \frac{1}{2} G_n^+ + \frac{1}{2} G_n^- \qquad (n = 1, 2, \ldots).$$

(c) In Chapter I we have seen that all the derivatives of a Brownian motion process have the uniform local characteristics. It is easy to prove that *if a process $g(\omega, t)$ belonging to $\mathfrak{K}$ contains a Gaussian component, i. e. $\sigma^2 = G_f(+0) - G_f(-0) > 0$, and the derivative $d^k g(\omega, t)/dt^k$ has a local characteristic, then this local characteristic is uniform.* In fact, if the local characteristic is not uniform, then $L_0(\tilde{\varphi}) \not\equiv 0$ for $\varphi \in L$, where

$$\tilde{\varphi}(x) = \int_{-\infty}^{\infty} e^{ixz} \varphi(z)\, dz.$$

Hence, in view of theorems III.1 and III.2, the characteristic functions $R_{h_1,\ldots,h_k}(z)$ of $D_{h_1,\ldots,h_k} g(\omega, t)$ converge to $e^{-a|z|+ibz}$ $(a \geqslant 0)$. But

$$R_{h_1,\ldots,h_k}(z) = \exp\left\{-\frac{\sigma^2 z^2}{2h_k}\right\} Q_{h_1,\ldots,h_k}(z),$$

where $Q_{h_1,\ldots,h_k}(z)$ is a characteristic function. Hence

$$\operatorname*{Lim}_{h_1,\ldots,h_k\downarrow 0} R_{h_1,\ldots,h_k}(z) = 0 \quad \text{for} \quad z \neq 0,$$

which is impossible. Consequently, the local characteristic is uniform.

Now we shall give an example of a process $f(\omega, t)$, belonging to $\mathfrak{K}\cap\mathfrak{M}$ and having no Gaussian component, such that the derivative $df(\omega,t)/dt$ has a uniform local characteristic.

Put

$$G_f(x) = \int_{-\infty}^{x} \frac{e^{-|u|}du}{|u|^{1/2}}.$$

Obviously, $\sigma^2 = G_f(+0)-G_f(-0) = 0$. Since

$$\int_{-\infty}^{\infty} |x|^n dG_f(x) < \infty \qquad (n = 1, 2, \ldots),$$

we have $f(\omega, t)\in\mathfrak{K}\cap\mathfrak{M}$. Let $Q_h(z)$ be the characteristic function of $\Delta_h f(\omega,t)/h^{2/3}$. Then

$$Q_h(z) = \exp\left\{h\int_{-\infty}^{\infty}\left(e^{izuh^{-2/3}}-1-izuh^{-2/3}\right)\frac{e^{-|u|}du}{|u|^{5/2}}\right\}$$

$$= \exp\left\{-2|z|^{3/2}\int_0^{\infty}\frac{1-\cos u}{u^{5/2}}\exp\left(-\frac{uh^{2/3}}{|z|}\right)du\right\}.$$

Consequently

$$\lim_{h\downarrow 0} Q_h(z) = \exp\left\{-2|z|^{3/2}\int_0^{\infty}\frac{1-\cos u}{u^{5/2}}du\right\}$$

uniformly in every finite interval. Since the limit characteristic function is integrable over $-\infty < x < \infty$, there is a density function $r(x)$ such that for every x

$$\text{(IV.31)} \qquad \lim_{h\downarrow 0} P\left(\frac{\Delta_h f(\omega,t)}{h^{2/3}} < x\right) = \int_{-\infty}^{x} r(u)\,du.$$

Moreover, since the characteristic function of $r(x)$ is real,

$$\text{(IV.32)} \qquad \int_{-\infty}^{0} r(u)\,du = \int_0^{\infty} r(u)\,du = \tfrac{1}{2}.$$

We have the following inequality for $\varphi\in X_0$:

$$\left|\int_{-\infty}^{\infty}\varphi(x)\,dP\big(D_h f(\omega,t) < x\big)-\frac{1}{2}G_0^+(\varphi)-\frac{1}{2}G_0^-(\varphi)\right|$$

$$\leqslant \int_{\varepsilon}^{\infty}\left|\varphi\left(\frac{x}{h^{1/3}}\right)-G_0^+(\varphi)\right|dP\left(\frac{\Delta_h f(\omega,t)}{h^{2/3}} < x\right)+$$

$$+|G_0^+(\varphi)|\left|\frac{1}{2}-P\left(\frac{\Delta_h f(\omega,t)}{h^{2/3}}\geqslant\varepsilon\right)\right|+\int_{-\infty}^{-\varepsilon}\left|\varphi\left(\frac{x}{h^{1/3}}\right)-G_0^-(\varphi)\right|dP\left(\frac{\Delta_h f(\omega,t)}{h^{2/3}} < x\right)+$$

$$+|G_0^-(\varphi)|\left|\frac{1}{2}-P\left(\frac{\Delta_h f(\omega,t)}{h^{2/3}} < -\varepsilon\right)\right|+\|\varphi\|P\left(\left|\frac{\Delta_h f(\omega,t)}{h^{2/3}}\right|\leqslant\varepsilon\right).$$

Hence, in view of (IV.31) and (IV.32), we obtain for each $\varphi\in X_0$ and each positive ε

$$\limsup_{h\downarrow 0}\left|\int_{-\infty}^{\infty}\varphi(x)\,dP\big(D_h f(\omega,t) < x\big)-\frac{1}{2}G_0^+(\varphi)-\frac{1}{2}G_0^-(\varphi)\right|$$

$$\leqslant |G_0^+(\varphi)|\int_0^{\varepsilon} r(u)\,du+|G_0^-(\varphi)|\int_{-\varepsilon}^{0} r(u)\,du+\|\varphi\|\int_{-\varepsilon}^{\varepsilon} r(u)\,du.$$

Consequently, when $\varepsilon\to 0$, we obtain for each $\varphi\in X_0$

$$\text{(IV.33)} \qquad \lim_{h\downarrow 0}\int_{-\infty}^{\infty}\varphi(x)\,dP\big(D_h f(\omega,t) < x\big) = \frac{1}{2}G_0^+(\varphi)+\frac{1}{2}G_0^-(\varphi).$$

Since the increments $D_h f(\omega,t)$ are symmetrically distributed and

$$\int_0^1 \frac{1}{u}\,dG_f(u) = \int_0^1 \frac{e^{-u}du}{u^{3/2}} = \infty,$$

in view of (IV.33) and the Corollary to Lemma IV.2, we infer that $df(\omega,t)/dt$ has a uniform local characteristic.

(d) As an example of a process belonging to $\mathfrak{K}\cap\mathfrak{M}$ for which no derivative has a local characteristic let us consider the following. Let

$$\text{(IV.34)} \qquad G_f(x) = \int_{-\infty}^{\infty}\big(2-\sin(\log|u|)\big)e^{-|u|}du.$$

Since $\int_{-\infty}^{\infty}|x|^n dG_f(x) < \infty$ $(n = 1, 2, \ldots)$, then $f(\omega, t)\epsilon \mathfrak{K}\cap\mathfrak{M}$. The characteristic function $R_{h_1,\ldots,h_k}(z)$ of $D_{h_1,\ldots,h_k}f(\omega, t)$, in view of Lemma II.1, is given by the formula

$$(\text{IV.35})\quad R_{h_1,\ldots,h_k}(z) = \exp\left\{2^k h_k\int_0^{\infty}\left(\cos\frac{zu}{\lambda_1\ldots\lambda_k}-1\right)\frac{1}{u^2}e^{-u}\big(2-\sin(\log|u|)\big)du\,.\right.$$

Let us suppose that for some k the derivative $d^k f(\omega, t)/dt^k$ has a local characteristic. Then, in view of theorem II.2 and the equality $\lim\limits_{\lambda\downarrow 0}\alpha_0(\lambda) = \frac{1}{2}$, the limit

$$(\text{IV.36})\qquad \Lambda_{h_1,\ldots,h_{k-1}}(\varphi) = \lim_{h_k\downarrow 0}\int_{-\infty}^{\infty}\varphi(x)\,dP\big(D_{h_1,\ldots,h_k}f(\omega, t) < x\big)$$

exists for each $\varphi\epsilon X_0$. Let $\psi\epsilon L$ and

$$\tilde{\psi}(x) = \int_{-\infty}^{\infty}e^{ixz}\psi(z)\,dz\,.$$

Then $\tilde{\psi}\epsilon X_0^*$ and

$$(\text{IV.37})\qquad \int_{-\infty}^{\infty}\tilde{\psi}(x)\,dP\big(D_{h_1,\ldots,h_k}f(\omega, t) < x\big) = \int_{-\infty}^{\infty}\psi(z)R_{h_1,\ldots,h_k}(z)\,dz\,.$$

It is easy to verify, in view of (IV.35), that

$$(\text{IV.38})\qquad \log R_{h_1,\ldots,h_k}(z) = 2^{k+1}\int_0^{\infty}\left(\cos\frac{zu}{\lambda_1\ldots\lambda_{k-1}}-1\right)\frac{1}{u^2}e^{-h_k u}du -$$
$$-2^k\sin(\log h_k)\int_0^{\infty}\left(\cos\frac{zu}{\lambda_1\ldots\lambda_{k-1}}-1\right)\frac{\cos(\log u)e^{-h_k u}}{u^2}du -$$
$$-2^k\cos(\log h_k)\int_0^{\infty}\cos\left(\frac{zu}{\lambda_1\ldots\lambda_{k-1}}-1\right)\frac{\sin(\log u)e^{-h_k u}du}{u^2}.$$

Putting

$$(\text{IV.39})\qquad h_k^{(n)} = \exp\left(\frac{-4n+1}{2}\pi\right)\qquad (n = 1, 2, \ldots)$$

we have, in view of (IV.36), (IV.37) and (IV.38),

$$\Lambda_{h_1,\ldots,h_{k-1}}(\tilde{\psi}) = \lim_{n\to\infty}\int_{-\infty}^{\infty}\psi(z)R_{h_1,\ldots,h_{k-1},h_k^{(n)}}(z)\,dz$$
$$= \int_{-\infty}^{\infty}\psi(z)\exp\left\{2^{k+1}\int_0^{\infty}\left(\cos\frac{zu}{\lambda_1\ldots\lambda_{k-1}}-1\right)\frac{du}{u^2}+\right.$$
$$\left.+2^k\int_0^{\infty}\left(\cos\frac{zu}{\lambda_1\ldots\lambda_{k-1}}-1\right)\frac{\cos(\log u)\,du}{u^2}\right\}.$$

Consequently $\Lambda_{h_1,\ldots,h_k}(\varphi)\not\equiv 0$ for $\varphi\epsilon X_0^*$. Hence, according to Lemma IV.3, the limit

$$H(x) = \begin{cases}\lim\limits_{h\downarrow 0}\dfrac{1}{h}\displaystyle\int_{-\infty}^{\infty}\frac{dG_f(uh)}{1+u^2} & \text{if}\quad k = 1,\\[2ex] \dfrac{1}{2}\lim\limits_{h\downarrow 0}\dfrac{1}{h}\displaystyle\int_{-\infty}^{x}\frac{d\{G_f(uh)-G_f(-uh)\}}{1+u^2} & \text{if}\quad k\geqslant 2,\end{cases}$$

exists at all continuity points of $H(x)$. From (IV.34) it follows that

$$(\text{IV.40})\qquad H(x) = 2\int_{-\infty}^{x}\frac{du}{1+u^2}-\lim_{h\downarrow 0}\left\{\sin(\log h)\int_{-\infty}^{x}\frac{\cos(\log|u|)e^{-h|u|}}{1+u^2}du+\right.$$
$$\left.+\cos(\log h)\int_{-\infty}^{x}\frac{\sin(\log|u|)e^{-h|u|}}{1+u^2}du\right\}.$$

Since

$$\int_{-\infty}^{0}\frac{\cos(\log|u|)}{1+u^2}du = 2\int_0^{\infty}\frac{\cos t}{e^t+e^{-t}}dt = \frac{\pi}{e^{\pi/2}+e^{\pi/2}},$$

we get for arbitrarily small $\varepsilon > 0$

$$(\text{IV.41})\qquad \int_{-\infty}^{-\varepsilon}\frac{\cos(\log|u|)}{1+u^2}du > 0\,.$$

Assume that $-\varepsilon$ is a continuity point of $H(x)$. From equalities (IV.39) and (IV.40) it follows that for the sequence $h_k^{(n)}$ $(n = 1, 2, \ldots)$

$$(\text{IV.42})\qquad H(-\varepsilon) = 2\int_{-\infty}^{-\varepsilon}\frac{du}{1+u^2}+\int_{-\infty}^{-\varepsilon}\frac{\cos(\log|u|)}{1+u^2}du\,.$$

Further, putting

$$h^{(n)} = \exp\left(-\frac{4n+3}{2}\pi\right) \qquad (n = 1, 2, \ldots)$$

we have

$$H(-\varepsilon) = 2\int_{-\infty}^{-\varepsilon} \frac{du}{1+u^2} - \int_{-\infty}^{-\varepsilon} \frac{\cos(\log|u|)}{1+u^2}\, du.$$

Hence and from equality (IV.42) it follows that

$$\int_{-\infty}^{-\varepsilon} \frac{\cos(\log|u|)}{1+u^2}\, du = 0,$$

which contradicts inequality (IV.41). Consequently none of the derivatives $d^k f(\omega, t)/dt^k$ have local characteristics.

(e) Now we shall give an example of a process $f(\omega, t)$ belonging to $\mathfrak{K}\cap\mathfrak{M}$ such that the derivative $df(\omega, t)/dt$ does not have a local characteristic but the derivatives $d^k f(\omega, t)/dt^k$ $(k \geqslant 2)$ have local characteristics.

Let

$$\text{(IV.43)} \qquad G_f(x) = \int_{-\infty}^{\infty} \big(2 - \operatorname{sgn} u \cdot \sin(\log|u|)\big) e^{-|u|}\, du.$$

Since

$$\int_{-\infty}^{\infty} |x|^n\, dG_f(x) < \infty \qquad (n = 1, 2, \ldots),$$

then $f(\omega, t) \in \mathfrak{K}\cap\mathfrak{M}$. Assume that $df(\omega, t)/dt$ has a local characteristic. Then, in view of theorem II.2 and the equality $\lim\limits_{\lambda\downarrow 0} \alpha_0(\lambda) = \frac{1}{2}$, the limit

$$\text{(IV.44)} \qquad \Lambda(\varphi) = \lim_{h\downarrow 0} \int_{-\infty}^{\infty} \varphi(x)\, dP\big(D_h f(\omega, t) < x\big)$$

exists for each $\varphi \in X_0$. Let $\psi \in L$ and

$$\tilde{\psi}(x) = \int_{-\infty}^{\infty} e^{ixz} \psi(z)\, dz.$$

Then $\tilde{\psi} \in X_0^*$ and

$$\text{(IV.45)} \qquad \int_{-\infty}^{\infty} \tilde{\psi}(x)\, dP\big(D_h f(\omega, t) < x\big) = \int_{-\infty}^{\infty} \psi(z) \left\{R\left(\frac{z}{h}\right)\right\}^h dz,$$

where

$$R(z) = \exp \int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz) \frac{\big(2 - \operatorname{sgn} u \cdot \sin(\log|u|)\big) e^{-|u|}}{u^2}\, du.$$

It is easy to verify that

$$\text{(IV.46)} \qquad \log\left\{R\left(\frac{z}{h}\right)\right\}^h = 4\int_0^{\infty} (\cos uz - 1) \frac{e^{-hu}}{u^2}\, du - 2i\sin(\log h) \int_0^{\infty} \frac{\sin uz - uz}{u^2} e^{-hu} \cos(\log u)\, du - 2i\cos(\log h)\int_0^{\infty} \frac{\sin uz - uz}{u^2} e^{-hu} \sin(\log u)\, du.$$

Let $h^{(n)}$ be defined by formula (IV.39). Then, in view of (IV.44), (IV.45) and (IV.46), for each

$$\Lambda(\tilde{\psi}) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \psi(z) \left\{R\left(\frac{z}{h^{(n)}}\right)\right\}^{h^{(n)}} dz = \int_{-\infty}^{\infty} \psi(z) \exp = \left\{-4|z| + 2i\int_0^{\infty} \frac{\sin uz - uz}{u^2} \cos(\log u)\, du\right\} dz.$$

Consequently $\Lambda(\varphi) \not\equiv 0$ for $\varphi \in X_0^*$. Hence, according to Lemma IV.3, the limit

$$H(x) = \lim_{h\downarrow 0} \frac{1}{h} \int_{-\infty}^{x} \frac{dG_f(hu)}{1+u^2}$$

exists at all continuity points of $H(x)$. From definition (IV.43) we obtain

$$\frac{1}{h}\int_{-\infty}^{x} \frac{dG_f(hu)}{1+u^2} = 2\int_{-\infty}^{x} \frac{e^{-h|u|}}{1+u^2}\, du - \sin(\log h) \int_{-\infty}^{x} \frac{\operatorname{sgn} u \cdot \cos(\log|u|) e^{-h|u|}}{1+u^2}\, du - \cos(\log h) \int_{-\infty}^{x} \frac{\operatorname{sgn} u \cdot \sin(\log|u|) e^{-h|u|}}{1+u^2}\, du.$$

In the same way as in the preceding example we obtain for arbitrarily small $\varepsilon > 0$ the equality

$$\int_{-\infty}^{-\varepsilon} \frac{\cos(\log|u|)}{1+u^2}\, du = 0,$$

which contradicts inequality (IV.41). Consequently, the derivative $df(\omega,t)/dt$ does not have a local characteristic.

Now we shall prove that the derivatives $d^k f(\omega,t)/dt^k$ $(k \geqslant 2)$ have local characteristics. According to (IV.43), the equality

$$\frac{1}{h}\int_{-\infty}^{x} \frac{d\{G_f(hu)-G_f(-hu)\}}{1+u^2} = 2\int_{-\infty}^{x} \frac{e^{-h|u|}}{1+u^2}\, du$$

holds. Hence, for each x,

$$\lim_{h\downarrow 0} \frac{1}{h}\int_{-\infty}^{x} \frac{d\{G_f(hu)-G_f(-hu)\}}{1+u^2} = 2\int_{-\infty}^{x} \frac{du}{1+u^2}.$$

Consequently, in view of Lemma IV.3, for each $\varphi \in X_0$ and $k \geqslant 2$ the limit

$$\operatorname*{Lim}_{h_1,\dots,h_k\downarrow 0} \int_{-\infty}^{\infty} \varphi(x)\, dP(D_{h_1,\dots,h_k} f(\omega,t) < x)$$

exists. Since

$$\int_0^1 \frac{1}{u}\, d\{G_f(u)-G_f(-u)\} = 2\int_0^1 \frac{e^{-u}}{u}\, du = \infty,$$

then, in view of the Corollary to Lemma IV.2, the local characteristics of $d^k f(\omega,t)/dt^k$ exist for $k = 2, 3, \dots$, q. e. d.

A measurable homogeneous stochastic process $f(\omega,t)$ with independent increments is called *a composed Poisson process* if the characteristic function $R_h(z)$ of increments $f(\omega,t+h)-f(\omega,t)$ has the form

(IV.47) $$R_h(z) = \exp\Big\{ih\gamma_f^* z + h\int_{u\neq 0} (e^{iuz}-1)\, d\mu_f(u)\Big\},$$

where γ_f^* is a real constant and μ_f is a σ-finite measure such that

(IV.48) $$0 < \int_{-\infty}^{\infty} \frac{|z|}{1+|z|}\, d\mu_f(z) < \infty.$$

It is easy to verify that the composed Poisson process $f(\omega,t)$ belongs to $\mathfrak{M}$ if and only if

(IV.49) $$\int_{-\infty}^{\infty} |z|^n\, d\mu_f(z) < \infty \quad (n = 1, 2, \dots).$$

Moreover, the equalities

(IV.50) $$\gamma_f = \gamma_f^* + \int_{-\infty}^{\infty} u\, d\mu_f(u), \quad G_f(x) = \int_{-\infty}^{x} u^2\, d\mu_f(u)$$

hold. Consequently, according to (IV.48) and (IV.50), a process $f(\omega,t)$ belonging to $\mathfrak{K}\cap\mathfrak{M}$ is a composed Poisson process if and only if

$$0 < \int_{-\infty}^{\infty} \frac{1}{|u|}\, dG_f(u) < \infty.$$

Now we shall prove the following

LEMMA IV.4. *Let $g(\omega,t)$ be a composed Poisson process belonging to $\mathfrak{M}$ and $\gamma_g^* = 0$. Then for each $\varepsilon > 0$*

(IV.51) $$\lim_{h\downarrow 0} P(|D_h g(\omega,t)| > \varepsilon) = 0,$$

and for $n = 1, 2, \dots$

(IV.52) $$\lim_{h\downarrow 0} h^{n-1}\int_0^{\infty} x^n\, dP(D_h g(\omega,t) < x) = \int_0^{\infty} u^n\, d\mu_g(u),$$

(IV.53) $$\lim_{h\downarrow 0} h^{n-1}\int_{-\infty}^{0} |x|^n\, dP(D_h g(\omega,t) < x) = \int_{-\infty}^{0} |u|^n\, d\mu_g(u).$$

Proof. We shall use the following notations:

(IV.54) $$Q_h(z) = \exp\Big\{h\int_{-\infty}^{\infty} (e^{iuz/h}-1)\, d\mu_g(u)\Big\},$$

(IV.55) $$Q_h^{(1)}(z) = \exp\Big\{h\int_0^{\infty} (e^{iuz/h}-1)\, d\mu_g(u)\Big\},$$

(IV.56) $$Q_h^{(2)}(z) = \exp\Big\{h\int_{-\infty}^{0} (e^{iuz/h}-1)\, d\mu_g(u)\Big\}.$$

Obviously, $Q_h(z)$ is the characteristic function of the random variable $D_h g(\omega, t)$. Let $q_h^{(j)}(x)$ denote the distribution function generated by the characteristic function $Q_h^{(j)}(z)$ $(j = 1, 2)$. It is easy to prove that

$$q_h^{(1)}(x) = 0 \text{ for } x < 0, \qquad q_h^{(2)}(x) = 0 \text{ for } x > 0. \tag{IV.57}$$

From inequality (IV.49) it follows that

$$\lim_{h\downarrow 0} Q_h(z) = \lim_{h\downarrow 0} Q_h^{(1)}(z) = \lim_{h\downarrow 0} Q_h^{(2)}(z) = 1,$$

uniformly in every finite interval, which implies equality (IV.51) and

$$\lim_{h\downarrow 0} \int_{-\infty}^{\infty} \varphi(x)\, dq_h^{(j)}(x) = \varphi(0) \qquad (j = 1, 2) \tag{IV.58}$$

for each $\varphi \in X_0$.

Since the proof of equality (IV.53) is analogous to the proof of equality (IV.52), we shall prove equality (IV.52) only. Putting for brevity

$$M_h^{(n)} = \int_0^{\infty} x^n\, dP\big(D_h g(\omega, t) < x\big) \qquad (n = 1, 2, \ldots)$$

we obtain, in view of (IV.54), (IV.55), (IV.56) and (IV.57),

$$M_h^{(n)} = \int_0^{\infty} z^n\, d\int_0^{\infty} q_h^{(2)}(z - x)\, dq_h^{(1)}(x) = \int_0^{\infty}\int_{-x}^{0} (x + y)^n\, dq_h^{(2)}(y)\, dq_h^{(1)}(x) \tag{IV.59}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \int_0^{\infty} x^k \int_{-x}^{0} y^{n-k}\, dq_h^{(2)}(y)\, dq_h^{(1)}(x).$$

Further, equalities (IV.55) and (IV.57) imply

$$\int_0^{\infty} x^k\, dq_h^{(1)}(x) = \int_{-\infty}^{\infty} x^k\, dq_h^{(1)}(x) = i^{-k} \frac{d^k}{dz^k} Q_h^{(1)}(z)\Big|_{z=0} \tag{IV.60}$$

$$= h^{1-k} \int_0^{\infty} u^k\, d\mu_g(u) \qquad (k = 1, 2, \ldots).$$

Let ε be an arbitrary positive number. Then for each $x > 0$ the inequality

$$\Big| \int_{-\infty}^{0} y^{n-k}\, dq_h^{(2)}(y) \Big| \leqslant \int_{-\varepsilon}^{0} |y|^{n-k}\, dq^{(2)}(y) + \int_{-x}^{-\varepsilon} |y|^{n-k}\, dq_h^{(2)}(y) \leqslant \varepsilon^{n-k} + x^{n-k} \int_{-\infty}^{-\varepsilon} dq_h^{(2)}(y)$$

holds. Hence, in view of (IV.60), we obtain for $k < n$

$$\limsup_{h\downarrow 0} h^{n-1} \Big| \int_0^{\infty} x^k \int_{-x}^{0} y^{n-k}\, dq_h^{(2)}(y)\, dq_h^{(1)}(x) \Big|$$

$$\leqslant \varepsilon^{n-k} \limsup_{h\downarrow 0} h^{n-1} \int_0^{\infty} x^k\, dq_h^{(1)}(x) + \int_{-\infty}^{-\varepsilon} dq_h^{(2)}(y) \limsup_{h\downarrow 0} h^{n-1} \int_0^{\infty} x^n\, dq_h^{(1)}(x)$$

$$\leqslant \int_{-\infty}^{-\varepsilon} dq^{(2)}(y) \int_0^{\infty} u^n\, d\mu_g(u).$$

Consequently, according to (IV.58),

(IV.61)

$$\lim_{h\downarrow 0} h^{n-1} \int_0^{\infty} x^k \int_{-x}^{0} y^{n-k}\, dq_h^{(2)}(y)\, dq_h^{(1)}(x) = 0 \quad \text{ for } \quad k = 0, 1, \ldots, n-1.$$

From equality (IV.57) it follows that

$$\int_0^{\infty} x^n \int_{-x}^{0} dq_h^{(2)}(y)\, dq_h^{(1)}(x) = \int_0^{\infty} x^n\, dq_h^{(1)}(x) - \int_0^{\infty} x^n \int_{-\infty}^{-x} dq_h^{(2)}(y)\, dq_h^{(1)}(x).$$

Hence

$$h^{n-1} \Big| \int_0^{\infty} x^n \int_{-x}^{0} dq_h^{(2)}(y)\, dq_h^{(1)}(x) - \int_0^{\infty} x^n\, dq_h^{(1)}(x) \Big|$$

$$= h^{n-1} \int_0^{\infty} x^n \int_{-\infty}^{-x} dq_h^{(2)}(y)\, dq_h^{(1)}(x) \leqslant h^{n-1} \int_0^{1} x^n\, dq_h^{(1)}(x) + h^{n-1} \int_{-\infty}^{-1} dq_h^{(2)}(y) \int_0^{\infty} x^n\, dq_h^{(1)}(x).$$

Thus, according to (IV.58) and (IV.60),

$$\lim_{h\downarrow 0} h^{n-1} \int_0^{\infty} x^n \int_{-x}^{0} dq_h^{(2)}(y)\, dq_h^{(1)}(x) = \int_0^{\infty} u^n\, d\mu_g(u) \qquad (n = 1, 2, \ldots).$$

Hence and from (IV.59) and (IV.61) we obtain

$$\lim_{h\downarrow 0} h^{n-1} M_h^{(n)} = \int_0^{\infty} u^n\, d\mu_g(u) \qquad (n = 1, 2, \ldots).$$

The Lemma is thus proved.

THEOREM IV.2. *A generalized stochastic process $\Phi(\omega, t)$ belonging to $\bigcup_{s=1}^{\infty}\mathfrak{R}_s$ has a Poissonian local characteristic if and only if*

$$\Phi(\omega, t)=\frac{d}{dt}f(\omega, t),$$

where $f(\omega, t)$ is a composed Poisson process belonging to $\mathfrak{M}$.

Moreover, the local characteristic $\langle a_n, L_n\rangle$ of $\Phi(\omega, t)$ is given by the following formulas:

$$a_0(\lambda)=\tfrac{1}{2},\qquad a_1(\lambda)=\frac{1}{1+\int_{-\infty}^{\infty}|u|\,d\mu_f(u)},\qquad a_n(\lambda)=\frac{\lambda^{n-1}}{\int_{-\infty}^{\infty}|u|^n\,d\mu_f(u)},$$

$$L_0(\varphi)=\tfrac{1}{2}\varphi(\gamma_f^*),$$

$$L_1(\varphi)=\frac{\varphi(\gamma_f^*)}{1+\int_{-\infty}^{\infty}|u|\,d\mu_f(u)}+\frac{\int_{0}^{\infty}u\,d\mu_f(u)}{1+\int_{-\infty}^{\infty}|u|\,d\mu_f(u)}G_1^+(\varphi)+\frac{\int_{-\infty}^{0}|u|\,d\mu_f(u)}{1+\int_{-\infty}^{\infty}|u|\,d\mu_f(u)}G_1^-(\varphi),$$

$$L_n=\frac{\int_{0}^{\infty}u^n\,d\mu_f(u)}{\int_{-\infty}^{\infty}|u|^n\,d\mu_f(u)}G_n^+ +\frac{\int_{-\infty}^{0}|u|^n\,d\mu_f(u)}{\int_{-\infty}^{\infty}|u|^n\,d\mu_f(u)}G_n^-\qquad (n=2,3,\ldots).$$

Proof. First we shall prove the sufficiency of the condition. Let $f(\omega, t)$ be a composed Poisson process belonging to $\mathfrak{M}$. Put

$$\Phi(\omega, t)=\frac{d}{dt}f(\omega, t).\qquad g(\omega, t)=f(\omega, t)-\gamma_f^* t,\qquad \Psi(\omega, t)=\frac{d}{dt}g(\omega, t).$$

Evidently, $g(\omega, t)$ is a composed Poisson process belonging to $\mathfrak{M}$ with

(IV.62) $$\gamma_g^*=0,\qquad \mu_g=\mu_f.$$

Moreover, the equality

(IV.63) $$\Phi(\omega, t)=\Psi(\omega, t)-\gamma_f^*$$

is true.

Let us first prove that the process $\Psi(\omega, t)$ has a Poissonian local characteristic. Suppose that $\varphi\in X_n$ $(n=1,2,\ldots)$. Then for every positive number A we have the following inequality:

$$\Big|h^{n-1}\int_{-\infty}^{\infty}\varphi(x)\,dP\big(D_hg(\omega, t)<x\big)-h^{n-1}\varphi(0)-G_n^+(\varphi)\int_{0}^{\infty}u^n\,d\mu_g(u)-$$
$$-G_n^-(\varphi)\int_{-\infty}^{0}|u|^n\,d\mu_g(u)\Big|\leqslant h^{n-1}\Big|\int_{-A}^{A}\varphi(x)\,dP\big(D_hg(\omega, t)<x\big)-\varphi(0)\Big|+$$
$$+h^{n-1}\int_{A}^{\infty}\Big|\frac{\varphi(x)}{x^n}-G_n^+(\varphi)\Big|x^n\,dP\big(D_hg(\omega, t)<x\big)+$$
$$+h^{n-1}\int_{-\infty}^{-A}\Big|\frac{\varphi(x)}{|x|^n}-G_n^-(\varphi)\Big||x|^n\,dP\big(D_hg(\omega, t)<x\big)+$$
$$+|G_n^+(\varphi)|\Big|h^{n-1}\int_{A}^{\infty}x^n\,dP\big(D_hg(\omega, t)<x\big)-\int_{0}^{\infty}u^n\,d\mu_g(u)\Big|+$$
$$+|G_n^-(\varphi)|\Big|h^{n-1}\int_{-\infty}^{-A}|x|^n\,dP\big(D_hg(\omega, t)<x\big)-\int_{-\infty}^{0}|u|^n\,d\mu_g(u)\Big|$$
$$\leqslant h^{n-1}\Big|\int_{-A}^{A}\varphi(x)\,dP\big(D_hg(\omega, t)<x\big)-\varphi(0)\Big|+$$
$$+h^{n-1}\sup_{x>A}\Big|\frac{\varphi(x)}{x^n}-G_n^+(\varphi)\Big|\int_{A}^{\infty}x^n\,dP\big(D_hg(\omega, t)<x\big)+$$
$$+h^{n-1}\sup_{x<-A}\Big|\frac{\varphi(x)}{|x|^n}-G_n^-(\varphi)\Big|\int_{-\infty}^{-A}|x|^n\,dP\big(D_hg(\omega, t)<x\big)+$$
$$+|G_n^+(\varphi)|\Big|h^{n-1}\int_{0}^{\infty}x^n\,dP\big(D_hg(\omega, t)<x\big)-\int_{0}^{\infty}u^n\,d\mu_g(u)\Big|+$$
$$+|G_n^-(\varphi)|\Big|h^{n-1}\int_{-\infty}^{0}|x|^n\,dP\big(D_hg(\omega, t)<x\big)-\int_{-\infty}^{0}|u|^n\,d\mu_g(u)\Big|+$$
$$+|G_n^+(\varphi)|\int_{0}^{A}x^n\,dP\big(D_hg(\omega, t)<x\big)+|G_n^-(\varphi)|\int_{-A}^{0}|x|^n\,dP\big(D_hg(\omega, t)<x\big).$$

Hence, in view of Lemma IV.4, equality (IV.62) and the arbitrariness of the number A, we obtain for each $\varphi \in X_n$ $(n = 1, 2, \ldots)$

$$\lim_{h\downarrow 0} h^{n-1}\Big\{\int_{-\infty}^{\infty}\varphi(x)dP\big(D_h g(\omega,t)<x\big)-\varphi(0)\Big\} = G_n^+(\varphi)\int_0^{\infty}u^n d\mu_f(u)+G_n^-(\varphi)\int_{-\infty}^{0}|u|^n d\mu_f(u).$$

From Lemma IV.4 it follows also that

$$\lim_{h\downarrow 0}\int_{-\infty}^{\infty}\varphi(x)dP\big(D_h g(\omega,t)<x\big) = \varphi(0)$$

for each $\varphi \in X_0$. Consequently, according to theorem II.2, the process $\Psi(\omega,t)$ has the local characteristic $\langle\tilde a_n, \tilde L_n\rangle$, where

$$\tilde a_0(\lambda) = \tfrac{1}{2}, \qquad \tilde L_0(\varphi) = \tfrac{1}{2}\varphi(0),$$

$$\tilde a_1(\lambda) = \frac{1}{1+\int_{-\infty}^{\infty}|u|\,d\mu_f(u)},$$

$$\tilde L_1(\varphi) = \frac{\varphi(0)}{1+\int_{-\infty}^{\infty}|u|\,d\mu_f(u)} + \frac{\int_0^{\infty}u\,d\mu_f(u)}{1+\int_{-\infty}^{\infty}|u|\,d\mu_f(u)}G_1^+(\varphi)+\frac{\int_{-\infty}^{0}|u|\,d\mu_f(u)}{1+\int_{-\infty}^{\infty}|u|\,d\mu_f(u)}G_1^-(\varphi),$$

$$\tilde a_n(\lambda) = \frac{\lambda^{n-1}}{\int_{-\infty}^{\infty}|u|^n d\mu_f(u)},$$

$$\tilde L_n(\varphi) = \frac{\int_0^{\infty}u^n d\mu_f(u)}{\int_{-\infty}^{\infty}|u|^n d\mu_f(u)}G_n^+(\varphi)+\frac{\int_{-\infty}^{0}|u|^n d\mu_f(u)}{\int_{-\infty}^{\infty}|u|^n d\mu_f(u)}G_n^-(\varphi) \qquad (n = 2, 3, \ldots).$$

Hence and from equality (IV.63) and theorem I.3 it follows that the process $\Phi(\omega,t)$ has the Poissonian local characteristic $\langle a_n, L_n\rangle$, which is given by the following formulas:

$$a_n(\lambda) = \tilde a_n(\lambda) \quad (n = 0, 1, \ldots), \qquad L_n(\varphi) = \tilde L_n(\varphi^*),$$

where $\varphi^*(x) = \varphi(x+\gamma_f^*)$. The sufficiency of the condition is thus proved.

Now we shall prove the necessity of the condition. Suppose that a process $\Phi(\omega,t)$ belonging to $\bigcup_{s=1}^{\infty}\mathfrak{K}_s$ has the Poissonian local characteristic

$\langle a_n, L_n\rangle$. In view of Theorem IV.1 there exists a process $f(\omega,t)$ belonging to $\mathfrak{K}\cap\mathfrak{M}$ such that

$$\Phi(\omega,t) = \frac{d}{dt}f(\omega,t).$$

Further, in view of theorem II.2, the limits

$$\lim_{h\downarrow 0}\int_{-\infty}^{\infty}\varphi(x)dP\big(D_h f(\omega,t)<x\big) = 2L_0(\varphi)$$

for $\varphi \in X_0$ and

$$c_n = \lim_{h\downarrow 0}\frac{\int_0^{\infty}x^n dP\big(D_h f(\omega,t)<x\big)}{\int_{-\infty}^{\infty}|x|^n dP\big(D_h f(\omega,t)<x\big)} \qquad (n = 1, 2, \ldots)$$

exist. Since

$$L_1(\varphi) = c\varphi(a)+a_1G_1^+(\varphi)+b_1G_1^-(\varphi),$$

where $c > 0$, it follows from Lemma IV.2 that

$$0 < \int_{-\infty}^{\infty}\frac{1}{|u|}\,dG_f(u) < \infty.$$

Consequently, $f(\omega,t)$ is a composed Poisson process. The Theorem is thus proved.

Corollary. *The derivative of a homogeneous process $f(\omega,t)$ with independent increments almost all sample functions of which are of bounded variation has a Poissonian local characteristic if and only if $f(\omega,t)\in\mathfrak{M}$.*

By a theorem of Ryll-Nardzewski [3] a process with independent increments almost all sample functions of which are of bounded variation is a composed Poisson process. Consequently our statement is a direct consequence of theorem IV.2.

Now we shall investigate the relation between the local characteristic of $d^k f(\omega,t)/dt^k$ and the moments of the increments of $f(\omega,t)$. First we shall prove the following

Lemma IV.5. *Let $f(\omega,t)\in\mathfrak{K}\cap\mathfrak{M}$ and*

$$\text{(IV.64)} \qquad \int_{-\infty}^{\infty}|x|\,dG_f(x) > 0.$$

Suppose that $df(\omega, t)/dt$ has the local characteristic $\langle a_n, L_n\rangle$. Then for $n \geqslant 2$ the equality $L_n(x^n) = 0$ is equivalent to the equality

$$m_n = \int_{-\infty}^{\infty} u^{n-2}\, dG_f(u) = 0 .$$

Proof. Since for even indices n, in virtue of (IV.64), $m_n > 0$ and, in virtue of theorem IV.1, $L_n(x^n) = L_n(1+|x|^n) = 1$, then in the sequel we shall consider only odd indices n.

From theorem II.2 it follows that

$$\lim_{h\downarrow 0} a_n(h) \int_{-\infty}^{\infty} |x|^n dP\big(D_h f(\omega, t) < x\big) = \operatorname*{Lim}_{h\downarrow 0} a_n(h)^{-n} E|\Delta_h f(\omega, t)|^n = L_n(|x|^n).$$

Hence, in view of theorem IV.1,

$$\lim_{h\downarrow 0} a_n(h) h^{-n} E|\Delta_h f(\omega, t)|^n = L_n(1+|x|^n) = 1 \tag{IV.65}$$

for $n \geqslant 2$.

It is easy to prove the inequality

$$E|\Delta_h f(\omega, t)|^{2s+1} \leqslant E|\Delta_h f(\omega, t)|^{2s} + E|\Delta_h f(\omega, t)|^{2s+2}.$$

Consequently, in view of Lemma III.6,

$$E|\Delta_h f(\omega, t)|^{2s+1} \leqslant h\Big\{\int_{-\infty}^{\infty} u^{2s-2} dG_f(u) + \int_{-\infty}^{\infty} u^{2s} dG_f(u) + o(1)\Big\} \qquad s = 1, 2, \ldots)$$

Hence, according to (IV.65),

$$\liminf_{h\downarrow 0} a_{2s+1}(h) h^{-2s} > 0 \qquad (s = 1, 2, \ldots). \tag{IV.66}$$

Further, we obtain in the same way

$$\lim_{h\downarrow 0} a_{2s+1}(h) h^{-2s}\big(m_{2s+1} + o(1)\big) = L_{2s+1}(x^{2s+1}) \qquad (s = 1, 2, \ldots).$$

Consequently, in view of (IV.66), $L_{2s+1}(x^{2s+1}) = 0$ implies $m_{2s+1} = 0$ $(s = 1, 2, \ldots)$.

From theorem II.2 it follows that

$$\lim_{h\downarrow 0} a_{2s+1}(h) E\{D_h f(\omega, t)\}^{2s+1} = L_{2s+1}(x^{2s+1}) \tag{IV.67}$$

and, according to theorem IV.1,

$$\lim_{h\downarrow 0} a_{2s+1}(h) E\{D_h f(\omega, t)\}^{2s} = L_{2s+1}(x^{2s}) = 0 . \tag{IV.68}$$

Contrary to the statement of the Lemma let us suppose that $m_{2s+1} = 0$ and

$$L_{2s+1}(x^{2s+1}) \neq 0 \tag{IV.69}$$

for some $s \geqslant 1$. Then, in view of Lemma III.6,

$$\lim_{h\downarrow 0} h^{2s-1} E\{D_h f(\omega, t)\}^{2s} = \int_{-\infty}^{\infty} u^{2s-2} dG_f(u) \tag{IV.70}$$

and the limit

$$\lim_{h\downarrow 0} h^{2s-1} E\{D_h f(\omega, t)\}^{2s+1} \tag{IV.71}$$

exists. From (IV.67), (IV.69) and (IV.71) it follows that

$$\liminf_{h\downarrow 0} h^{1-2s} a_{2s+1}(h) > 0 .$$

Hence, in view of (IV.68),

$$\lim_{h\downarrow 0} \frac{a_{2s+1}(h) E\{D_h f(\omega, t)\}^{2s}}{a_{2s-1}(h) h^{1-2s}} = \lim_{h\downarrow 0} h^{2s-1} E\{D_h f(\omega, t)\}^{2s} = 0 .$$

Then, according to (IV.70),

$$\int_{-\infty}^{\infty} u^{2s-2} dG_f(u) = 0 ,$$

which contradicts inequality (IV.64). Consequently the equality $m_{2s+1} = 0$ implies $L_{2s+1}(x^{2s+1}) = 0$ $(s = 1, 2, \ldots)$. The Lemma is thus proved.

THEOREM IV.3. *Let $\Phi(\omega, t)$ be a generalized stochastic process belonging to $\bigcup_{s=1}^{\infty} \mathfrak{K}_s$ with the local characteristic $\langle a_n, L_n\rangle$. Then the limit*

$$s_0 = \lim_{\lambda\downarrow 0} \frac{\log a_2(\lambda)}{\log \lambda}$$

exists.

Put

$$k_0 = \begin{cases} s_0 & \text{if} \quad \liminf\limits_{\lambda\downarrow 0} \dfrac{\lambda^2 a_2(\lambda)}{a_4(\lambda)} = 0 , \\[2ex] \left[\dfrac{s_0+1}{2}\right] & \text{if} \quad \liminf\limits_{\lambda\downarrow 0} \dfrac{\lambda^2 a_2(\lambda)}{a_4(\lambda)} > 0 . \end{cases}$$

If $k_0 = 0$, then $\Phi(\omega, t)$ is a constant process. If $k_0 \geqslant 1$, then there is a process $f(\omega, t)$ belonging to $\mathfrak{K} \cap \mathfrak{M}$ such that

$$\frac{d^{k_0}}{dt^{k_0}} f(\omega, t) = \Phi(\omega, t),$$

and the local characteristic $\langle a_n, L_n \rangle$ determines all the moments of the increments $\Delta_{h_1}\Delta_{h_2} f(\omega, t)$ $(h_1, h_2 > 0)$. Moreover, if $k_0 = 1$, then the local characteristic $\langle a_n, L_n \rangle$ determines all the moments of $\Delta_h f(\omega, t) - E\{\Delta_h f(\omega, t)\}$.

Proof. In virtue of theorem II.2 there exist an integer k and a process $f(\omega, t)$ belonging to $\mathfrak{K} \cap \mathfrak{M}$ such that

$$\frac{d^k}{dt^k} f(\omega, t) = \Phi(\omega, t). \tag{IV.72}$$

Suppose that $f(\omega, t)$ is a deterministic process, i.e. $f(\omega, t) = ct$, where c is a constant. Consequently $\Phi(\omega, t)$ is a constant process and $\lim\limits_{\lambda \downarrow 0} a_n(\lambda) > 0$ $(n = 0, 1, \ldots)$ (see Section I, example (c)). Thus

$$s_0 = 0, \quad \lim_{\lambda \downarrow 0} \frac{\lambda^2 a_2(\lambda)}{a_4(\lambda)} = 0 \quad \text{and} \quad k_0 = 0.$$

Now we shall prove that in other cases

$$k_0 = k, \tag{IV.73}$$

which implies $k_0 \geqslant 1$.

First we assume that $f(\omega, t)$ is a Brownian motion process. In Section I (example (a)) we have seen that in this case

$$\lim_{\lambda \downarrow 0} \frac{\sqrt{\pi}\, \lambda^{k-1+n/2}}{a_n(\lambda) 2^{kn/2} \sigma^n \Gamma\left(\frac{n+1}{2}\right)} = 1 \quad n = 1, 2, \ldots). \tag{IV.74}$$

Hence we obtain

$$s_0 = k, \quad \lim_{\lambda \downarrow 0} \frac{\lambda^2 a_2(\lambda)}{a_4(\lambda)} = 0,$$

and consequently $k_0 = k$. Now we may suppose that

$$\int_{-\infty}^{\infty} |x|\, dG_f(x) > 0. \tag{IV.75}$$

From theorem IV.1 it follows that $L_{2s}(x^{2s}) = L_{2s}(1 + x^{2s}) = 1$ $(s = 1, 2, \ldots)$. Then, in view of Lemma III.7,

$$\lim_{\lambda \downarrow 0} \lambda^{1-2sk} a_{2s}(\lambda) = \frac{1}{2^{k-1} \int\limits_{-\infty}^{\infty} x^{2s-2}\, dG_f(x)} \quad (s = 1, 2, \ldots), \tag{IV.76}$$

which implies

$$s_0 = 2k - 1, \quad \liminf_{\lambda \downarrow 0} \frac{\lambda^2 a_2(\lambda)}{a_4(\lambda)} > 0,$$

and consequently $k_0 = k$. Equality (IV.73) is thus proved.

Further, the characteristic function of the increments $\Delta_h \Delta_{h_2} f(\omega, t)$ $(h_1, h_2 > 0)$ is given, in view of Lemma II.1, by the formula

$$R_{h_1, h_2}(z) = \exp\left\{2 \min(h_1, h_2) \int_{-\infty}^{\infty} (\cos zu - 1)\, dG_f(u)\right\}.$$

Consequently $E\{\Delta_{h_1}\Delta_{h_2} f(\omega, t)\}^n = 0$ for $n = 1, 3, \ldots$ and

$$E\{\Delta_{h_1}\Delta_{h_2} f(\omega, t)\}^n \quad (n = 0, 2, \ldots)$$

are determined by the moments

$$\int_{-\infty}^{\infty} u^{2s}\, dG_f(u) \quad (s = 0, 1, \ldots).$$

Thus to prove that $\langle a_n, L_n \rangle$ determines the moments of $\Delta_{h_1}\Delta_{h_2} f(\omega, t)$ it suffices to show that the moments

$$\int_{-\infty}^{\infty} u^{2s}\, dG_f(u) \quad (s = 0, 1, \ldots)$$

are determined by the local characteristic $\langle a_n, L_n \rangle$.

From equalities (IV.74) and (IV.76) we obtain for $s = 0, 1, \ldots$

$$\int_{-\infty}^{\infty} u^{2s}\, dG_f(u) = \begin{cases} \lim\limits_{\lambda \downarrow 0} \dfrac{\lambda^2 (s+1) k_0 - 1}{2^{k_0 - 1} a_{2(s+1)}(\lambda)} & \text{if} \quad \liminf\limits_{\lambda \downarrow 0} \dfrac{\lambda^2 a_2(\lambda)}{a_4(\lambda)} > 0, \\ \lim\limits_{\lambda \downarrow 0} \dfrac{\lambda}{a_2(\lambda)} & \text{if} \quad s = 0 \text{ and } \liminf \dfrac{\lambda^2 a_2(\lambda)}{a_4(\lambda)} = 0, \\ 0 & \text{in other cases}. \end{cases}$$

Thus the local characteristic $\langle a_n, L_n \rangle$ determines the moments

$$\int_{-\infty}^{\infty} u^{2s}\, dG_f(u) \quad (s = 0, 1, \ldots)$$

and consequently determines the moments of $\Delta_{h_1}\Delta_{h_2} f(\omega, t)$ $(h_1, h_2 > 0)$.

Analogously, to prove that $\langle a_n, L_n\rangle$ determines the moments of $\Delta_h f(\omega,t)-E\{\Delta_h f(\omega,t)\}$ in the case $k_0=1$, it suffices to show that the moments

$$\int_{-\infty}^{\infty} u^n dG_f(u) \quad (n=0,1,\ldots)$$

are determined by $\langle a_n, L_n\rangle$.

Hitherto we have proved that the condition

$$\liminf_{\lambda\downarrow 0}\frac{\lambda^2 a_2(\lambda)}{a_4(\lambda)}>0$$

is equivalent to condition (IV.75). Consequently, in view of Lemmas III.7 and IV.5, in the case of

$$\liminf_{\lambda\downarrow 0}\frac{\lambda^2 a_2(\lambda)}{a_4(\lambda)}>0$$

we obtain

$$\int_{-\infty}^{\infty} u^n dG_f(u)=\begin{cases} L_{n+2}(x^{n+2})\lim\limits_{\lambda\downarrow 0}\dfrac{\lambda^{n+1}}{a_{n+2}(\lambda)} & \text{if} \quad L_{n+2}(x^{n+2})>0,\\ 0 & \text{if} \quad L_{n+2}(x^{n+2})=0,\end{cases}$$

and in the case of

$$\liminf_{\lambda\downarrow 0}\frac{\lambda^2 a_2(\lambda)}{a_4(\lambda)}=0$$

according to (IV.74),

$$\int_{-\infty}^{\infty} u^n dG_f(u)=\begin{cases}\lim\limits_{\lambda\downarrow 0}\dfrac{\lambda a_2(\lambda)}{3a_4(\lambda)} & \text{for} \quad n=0,\\ 0 & \text{for} \quad n=1,2,\ldots\end{cases}$$

The Theorem is thus proved.

Finally we give a simple example of processes $f_1(\omega,t)$, $f_2(\omega,t)$ belonging to $\mathfrak{R}\cap\mathfrak{M}$ whose derivatives $df_1(\omega,t)/dt$, $df_2(\omega,t)/dt$ have the same local characteristic, but the distribution function of $\Delta_h f_1(\omega,t)-E\{\Delta_h f_1(\omega,t)\}$ is not equal to the distribution function of $\Delta_h f_2(\omega,t)-E\{\Delta_h f_2(\omega,t)\}$.

Example. Let us consider the functions

$$s_1(x)=x^{-\log x} \quad (x\geqslant 0),$$

$$s_2(x)=x^{-\log x}(1+\sin(2\pi\log x))$$

for which the moment problem is indeterminate, i. e.

$$\int_0^{\infty} x^n s_1(x)\,dx=\int_0^{\infty} x^n s_2(x)\,dx \quad (n=0,1,\ldots). \tag{IV.77}$$

(This example is due to Stieltjes [5]). Let us consider the composed Poisson processes $f_1(\omega,t)$, $f_2(\omega,t)$ belonging to $\mathfrak{M}$ for which

$$\gamma_{f_1}^*=0,\qquad \mu_{f_1}(u)=\begin{cases}\int_0^u s_1(x)\,dx & \text{for} \quad u\geqslant 0,\\ 0 & \text{for} \quad u<0,\end{cases}$$

$$\gamma_{f_2}^*=0,\qquad \mu_{f_2}(u)=\begin{cases}\int_0^u s_2(x)\,dx & \text{for} \quad u\geqslant 0,\\ 0 & \text{for} \quad u<0.\end{cases}$$

Evidently the distribution function of $\Delta_h f_1(\omega,t)-E\{\Delta_h f_1(\omega,t)\}$ is not equal to the distribution function of $\Delta_h f_2(\omega,t)-E\{\Delta_h f_2(\omega,t)\}$. Further, from theorem IV.2 it follows that the derivatives $df_1(\omega,t)/dt$ and $df_2(\omega,t)/dt$ have the local characteristics $\langle a_n^{(1)}, L_n^{(1)}\rangle$ and $\langle a_n^{(2)}, L_n^{(2)}\rangle$, where

$$a_0^{(1)}(\lambda)=a_0^{(2)}(\lambda)=\tfrac{1}{2},\qquad L_0^{(1)}(\varphi)=L_0^{(2)}(\varphi)=\tfrac{1}{2}\varphi(0),$$

$$a_1^{(1)}(\lambda)=\frac{1}{1+\int_{-\infty}^{\infty} us_1(u)\,du},\qquad a_1^{(2)}(\lambda)=\frac{1}{1+\int_0^{\infty} us_2(u)\,du},$$

$$L_1^{(1)}(\varphi)=\frac{\varphi(0)}{1+\int_0^{\infty} us_1(u)\,du}+\frac{\int_0^{\infty} us_1(u)\,du}{1+\int_0^{\infty} us_1(u)\,du}G_1^+(\varphi),$$

$$L_1^{(2)}(\varphi)=\frac{\varphi(0)}{1+\int_0^{\infty} us_2(u)\,du}+\frac{\int_0^{\infty} us_2(u)\,du}{1+\int_0^{\infty} us_2(u)\,du}G_1^+(\varphi),$$

$$a_n^{(1)}(\lambda)=\frac{\lambda^{n-1}}{\int_0^{\infty} u^n s_1(u)\,du},\qquad a_n^{(2)}(\lambda)=\frac{\lambda^{n-1}}{\int_0^{\infty} u^n s_2(u)\,du},$$

$$L_n^{(1)}=L_n^{(2)}=G_n^+ \quad (n=2,3,\ldots).$$

Consequently, in view of (IV.77), $\langle a_n^{(1)}, L_n^{(1)}\rangle=\langle a_n^{(2)}, L_n^{(2)}\rangle$, q. e. d.

References

[1] J. L. Doob, *Stochastic processes*, New York-London 1953.

[2] Б. В. Гнеденко и К. Н. Колмогоров, *Предельные распределения для сумм независимых случайных величин*, Москва-Ленинград 1949.

[3] S. Łojasiewicz, *Sur la valeur et la limite d'une distribution en un point*, Studia Math. 16 (1957), p. 1-36.

[4] C. Ryll-Nardzewski, *On the non-homogeneous Poisson process*, Coll. Math. 3 (1955), p. 192-195.

[5] T. J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. de Toulouse 8 (1894), p. 1-122.

[6] K. Urbanik, *Generalized stochastic processes*, Studia Math. 16 (1957), p. 268-334.

[7] — *Случайные процессы, реализации которых являются обобщенными функциями*, Теория вероятностей и её применения I (1) (1956), p. 146-149.

Reçu par la Rédaction le 4. 9. 1957

STUDIA MATHEMATICA, T. XVII. (1958)

The conditional expectations and the ergodic theorem for strictly stationary generalized stochastic processes

by

K. URBANIK (Wrocław)

I. Introduction. In the present note we shall consider generalized stochastic processes defined in [2]. We say that a generalized stochastic process $\Phi(\omega, t)$ is *strictly stationary* if there exists a sequence $\{f_n(\omega, t)\}$ of strictly stationary continuous stochastic processes such that $\Phi(\omega, t) = [f_n(\omega, t)]$. Let $F(\omega, t)$ be a continuous stochastic process and set $\Delta_h F(\omega, t) = F(\omega, t+h) - F(\omega, t)$. Then it is easy to prove the following assertion:

The generalized process $d^k F(\omega, t)/dt^k$ $(k \geqslant 1)$ is strictly stationary if and only if for each $h_1, h_2, \ldots, h_k$ the process $\Delta_{h_1} \Delta_{h_2} \ldots \Delta_{h_k} F(\omega, t)$ is strictly stationary (in the usual sense).

By $\Xi(t_1, t_2, \ldots, t_k)$ we shall denote the space of all generalized stochastic processes depending on variables $t_1, t_2, \ldots, t_k$. Suppose that λ_{ij} $(i, j = 1, 2, \ldots, k)$ are real constants and $\det|\lambda_{ij}| \neq 0$. Let $\Phi(\omega, t_1, t_2, \ldots, t_k) = [f_n(\omega, t_1, t_2, \ldots, t_k)]$. Then the generalized stochastic process $\Phi(\omega, \sum_{j=1}^{k} \lambda_{1j} t_j, \ldots, \sum_{j=1}^{k} \lambda_{kj} t_j)$ is defined by the formula

$$\Phi\Big(\omega, \sum_{j=1}^{k} \lambda_{1j} t_j, \ldots, \sum_{j=1}^{k} \lambda_{kj} t_j\Big) = \Big[f_n\Big(\omega, \sum_{j=1}^{k} \lambda_{1j} t_j, \ldots, \sum_{j=1}^{k} \lambda_{kj} t_j\Big)\Big].$$

It is easy to verify that the convergence $\Phi_T(\omega, t_1, \ldots, t_k) \to \Phi(\omega, t_1, \ldots, t_k)$ when $T \to \infty$ implies the convergence

$$\Phi_T\Big(\omega, \sum_{j=1}^{k} \lambda_{1j} t_j, \ldots, \sum_{j=1}^{k} \lambda_{kj} t_j\Big) \to \Phi\Big(\omega, \sum_{j=1}^{k} \lambda_{1j} t_j, \ldots, \sum_{j=1}^{k} \lambda_{kj} t_j\Big).$$

(The convergence of generalized stochastic processes is defined in [2]). Hence in particular we obtain the following

LEMMA 1. *Let $\Phi_T(\omega, t) \in \Xi(t)$. Then $\Phi_T(\omega, t_1 + \ldots + t_k) \in \Xi(t_1, \ldots, t_k)$ and the convergence of $\Phi_T(\omega, t_1 + \ldots + t_k)$ when $T \to \infty$ implies the convergence of $\Phi_T(\omega, t_1)$ (in $(\Xi t_1, \ldots, t_k)$).*