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Linear functionals on two-norm spaces

by

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This paper is the sequel to the paper [1] of the first author. That paper dealt mainly with the problem under what conditions the limit of a point-wise convergent sequence of γ-linear functionals is γ-linear.

In the present paper we investigate the behaviour of the γ-linear functionals. We prove a theorem, due to A. Wiweger, stating that there is a convex linear topology such that convergence γ is equivalent to convergence with respect to that topology, and such that γ-linear functionals are identical with the functionals linear with respect to that topology. The main result establishes the general form of γ-linear functionals as uniform limits in the sphere $S = \{x: \|x\| \leqslant 1\}$ of the functionals linear with respect to the weaker norm $\| \|^*$ (Theorems 4.2 and 4.3); thus the set Ξ_γ of γ-linear functionals is equal to the uniform closure of the set Ξ^* of the functionals continuous with respect to the weaker norm. This theorem enables us to deduce rapidly (in section 6) the general form of γ-linear functionals in several concrete two-norm spaces including some known ones.

A section is devoted to counter examples which seem to throw some light on the problems arising. In particular, it is shown that the γ-linear functionals do not have the extension property.

The first author is indebted to Mr W. Orlicz for having called his attention to the fact that proposition 3.1 of the paper [1] is false; we give the correct version and rectify a consequence deduced in [1] from 3.1.

1. Preliminaries. In this paper we deal with the following case of two-norm convergence ([1], p. 49). X is a real or complex linear space provided with a homogeneous norm $\| \|$; X considered as a linear metric space normed by $\|\cdot\|$ will be denoted by $\langle X, \| \|\rangle$ — this space is not supposed to be complete[1]). Let another (homogeneous) norm $\| \|^*$ be defined in X. A sequence $\{x_n\}$ of elements of X is called *γ-convergent*

[1]) unlike as in [1]; W. Orlicz has noticed that this hypothesis is superfluous (comp. [17]).

to x_0 (in symbols $x_n \xrightarrow{\gamma} x_0$ or $\gamma\text{-}\lim\limits_n x_n = x_0$) if $\sup\limits_{n=1,2,\ldots} \|x_n\| < \infty$ and $\lim\limits_{n\to\infty} \|x_n - x_0\|^* = 0$. The space X provided with the convergence γ will be denoted by $\langle X, \| \|, \| \|^*\rangle$ and will be called a *two-norm space*. Obviously, $\langle X, \| \|, \| \|^*\rangle$ is an $\mathcal{L}^*$-space of Fréchet (see, for instance, [9], p. 83); moreover, the addition of elements and multiplication by scalars are continuous.

In the sequel we make the following assumptions:

(n_0) The norm $\| \|$ is finer than $\| \|^*$ [2]),

(n) If $x_n \xrightarrow{\gamma} x_0$, then $\|x_0\| \leqslant \varliminf\limits_{n\to\infty} \|x_n\|$.

The condition (n) is satisfied when the following postulate is satisfied (see [16], p. 226):

(n_1) If $\|x_n\| \leqslant K$ and $\lim\limits_{p,q\to\infty} \|x_p - x_q\|^* = 0$, then there exists in X an element x_0 such that $\lim\limits_{p\to\infty} \|x_p - x_0\|^* = 0$ and $\|x_0\| \leqslant K$.

This postulate implies in turn the γ-completeness of $\langle X, \| \|, \| \|^*\rangle$; indeed, we have (see [2], p. 207)

1.1. Proposition. *Let* (n_1) *be satisfied; if* $\{x_n\}$ *is a sequence such that* $p_n \to \infty$, $q_n \to \infty$ *implies* $\gamma\text{-}\lim\limits_{n\to\infty}(x_{p_n} - x_{q_n}) = 0$, *then there exists in* X *an element* x_0 *such that* $x_n \xrightarrow{\gamma} x_0$.

In his theory of Saks spaces (closely related to the two-norm spaces) W. Orlicz neither assumes the space $\langle X, \| \|\rangle$ to be complete nor postulates (n_0); (n_1) is postulated only. The above hypothesis, however, does not increase generality. Indeed, the considerations of W. Orlicz ([17], p. 1) imply

1.2. Proposition. *Let only the condition* (n_1) *be satisfied in the space* $\langle X, \| \|, \| \|^*\rangle$ *and let the norm* $\| \|^*$ *be equivalent on* $S = \{x : \|x\| \leqslant 1\}$ *to a homogeneous norm* $\| \|_1^*$. *Then the norm* $\| \|_1$ *defined by* $\|x\|_1 = \|x\| + \|x\|_1^*$ *is such that*:

1° *the space* $\langle X, \| \|_1\rangle$ *is complete*,

2° *the conditions* (n_0) *and* (n_1) *are satisfied in* $\langle X, \| \|_1, \| \|_1^*\rangle$,

3° *the notion of convergence* γ *is the same in the space* $\langle X, \| \|, \| \|^*\rangle$ *as in* $\langle X, \| \|_1, \| \|_1^*\rangle$.

The situation described above may be generalized. Suppose that $\langle X, \| \|^*\rangle$ is a B_0^*-space ([12], p. 185); this means that in X there is defined a sequence $\{[\ ]_i\}$ of (homogeneous) pseudonorms such that $\sum\limits_{i=1}^{\infty} [x]_i = 0$ implies $x = 0$. We set

$$\|x\|^* = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{[x]_k}{1 + [x]_k},$$

then $\| \|^*$ is an unhomogeneous norm; in the space $\langle X, \| \|^*\rangle$ the sequence $\{x_n\}$ is called *convergent to* x_0 if $\|x_n - x_0\|^* \to 0$ as $n \to \infty$ or, what is equivalent, if $\lim\limits_{n\to\infty} [x_n - x_0]_i = 0$ for $i = 1, 2, \ldots$ It may always be assumed that $[x]_1 \leqslant [x]_2 \leqslant \ldots$; we shall do this in certain cases. Now we can introduce the γ-convergence and the conditions (n_0), (n) and (n_1) as above without any alteration.

It is supposed throughout this paper that $\| \|$ is a homogeneous norm, that $\langle X, \| \|^*\rangle$ is a B_0^*-space and that the postulates (n_0) and (n) are satisfied; (n_1) is not supposed.

The conjugate spaces to $\langle X, \| \|\rangle$ and $\langle X, \| \|^*\rangle$ will be denoted by Ξ and Ξ^*, respectively. We shall deal in this paper with the γ-linear functionals in $\langle X, \| \|, \| \|^*\rangle$, *i. e.* with the distributive functionals such that $x_n \xrightarrow{\gamma} x_0$ implies $\xi(x_n) \to \xi(x_0)$; the set of the γ-linear functionals will be denoted by Ξ_γ. We obviously have $\Xi^* \subset \Xi_\gamma \subset \Xi$; this proves the existence of non-trivial γ-linear functionals, for such functionals are in Ξ^*.

In the sequel $\|\xi\|$ will denote the norm of ξ in Ξ, that is $\|\xi\| = \sup\limits_{\|x\|\leqslant 1} |\xi(x)|$.

1.3. Lemma. *Let* X_0 *be a dense subspace of a normed space* $\langle X, \| \|\rangle$ *and let* $\| \|^*$ *be a pseudonorm in* X, *coarser* [3]) *than* $\| \|$, *satisfying the condition* (n) *in* X_0. *Then the condition* (n) *is satisfied in* X. *If* $\| \|^*$ *is a norm in* X_0, *it is a norm in* X *too.*

Proof. Let $x_n \in X$, $\|x_n\| \leqslant K$, $\lim\limits_{n\to\infty} \|x_n - x_0\|^* = 0$. Choose $\varepsilon > 0$ freely and, then, $y_0, y_n, z_n \in X_0$ so that

$$\|x_0 - y_0\| < \varepsilon, \quad \|x_n - y_n\| < 1/n, \quad \|x_0 - z_n\| < 1/n.$$

Then $\|y_n - z_n\|^* \leqslant \|y_n - x_n\|^* + \|x_n - x_0\|^* + \|x_0 - z_n\|^*$; $\|x_n - y_n\|^*$ and $\|x_0 - z_n\|^*$ tend to 0, whence $\|y_n - z_n\|^* \to 0$. Also $\|y_n + y_0 - z_n\| \leqslant \|x_n\| + \|x_n - y_n\| + \|y_0 - x_0\| + \|x_0 - z_n\| \leqslant K + 1/n + \varepsilon + 1/n$, which implies $y_n + y_0 - z_n \xrightarrow{\gamma} y_0$. The condition (n) being satisfied in X_0, we get

$$\|y_0\| \leqslant \varliminf_{n\to\infty} \|y_n + y_0 - z_n\| \leqslant K + \varepsilon, \quad \|x_0\| \leqslant K + 2\varepsilon,$$

which obviously implies the condition (n) in X.

[2]) This means that $\|x_n\| \to 0$ implies $\|x_n\|^* \to 0$.

[3]) *i. e.* the norm $\| \|$ is finer than $\| \|^*$.

Now, let $\| \|^*$ be a norm in X_0 and let $\|x\|^* = 0$ for an x in X. Let $x_n \epsilon X_0$ be such that $\|x_n - x\| \to 0$; then $\|x_n - x\|^* \to 0$, whence $\|x_n\|^* \to 0$. Given $\varepsilon > 0$, choose P such that $\|x_p - x_q\| < \varepsilon$ for $q \geqslant p \geqslant P$. Then for $P \leqslant p < q$

$$\|(x_p - x_q) - x_p\|^* = \|x_q\|^* = \|x_q\|^* - \|x\|^* \leqslant \|x_q - x\|^* \to 0$$

as $q \to \infty$; the sequence $\{x_p - x_q\}_{q=1,2,\ldots}$ being obviously bounded in $\langle X, \| \|\rangle$, $x_p - x_q \xrightarrow{\gamma} x_p$ as $q \to \infty$. By the condition (n)

$$\|x_p\| \leqslant \lim_{q\to\infty} \|x_p - x_q\| \leqslant \varepsilon \quad \text{for} \quad p \geqslant P,$$

which implies in turn $\|x_p\| \to 0$, $\|x\| = 0$ and $x = 0$. Thus $\| \|^*$ is a norm in X.

The space $\langle X, \| \|\rangle$ is not supposed to be complete; this, however, may be assumed im many important cases, for the process of completion leads again to a two-norm space satisfying the assumed postulates and such that the spaces Ξ^*, Ξ_γ, and Ξ remain, roughly speaking, unaltered. The completion, $\tilde{X}$, of X is the set of all equivalence classes of Cauchy sequences $\tilde{x} = \{x_n\}$ in $\langle X, \| \|\rangle$, under the equivalence relation $\{x_n\} \sim \{x_n'\}$ if and only if $\lim\limits_{n\to\infty} \|x_n - x_n'\| = 0$. For every $\tilde{x} = \{x_n\} \epsilon \tilde{X}$, $\lim\limits_{n\to\infty} \|x_n\|$ exists and is defined as the norm $\|\tilde{x}\|$ of $\tilde{x}$. It is well known that the space $\langle \tilde{X}, \| \|\rangle$ is a Banach space. Now the condition (n_0) implies that every Cauchy sequence in $\langle X, \| \|\rangle$ is also a Cauchy sequence in $\langle X, \| \|^*\rangle$, whence for every $\tilde{x} \epsilon \tilde{X}$ there exists $\|\tilde{x}\|^* = \lim\limits_{n\to\infty} \|x_n\|^*$ independent of the representation $\tilde{x} = \{x_n\}$ of the element $\tilde{x}$.

1.4. PROPOSITION. *$\langle \tilde{X}, \| \|, \| \|^*\rangle$ is a two-norm space satisfying the conditions (n_0) and (n). The space $\langle \tilde{X}, \| \|\rangle$ is complete. Every linear functional on $\langle X, \| \|\rangle$ may be uniquely extended to a linear functional on $\langle \tilde{X}, \| \|\rangle$, and the same properties are possessed by the γ-linear functionals on X and by the linear functionals on $\langle X, \| \|^*\rangle$.*

Proof. The first part of the theorem follows by Lemma 1.3. To prove the second part it is sufficient to notice that for every Cauchy sequence $\tilde{x} = \{x_n\}$ in $\langle X, \| \|\rangle$ and every linear functional ξ on $\langle X, \| \|\rangle$ there exists the limit $\lim\limits_{n\to\infty} \xi(x_n) = \tilde{\xi}(\tilde{x})$ which is linear on $\langle \tilde{X}, \| \|\rangle$. This extension is easily seen to preserve the γ-linearity and the linearity on $\langle X, \| \|^*\rangle$.

1.5. PROPOSITION. *The set $Y = \{\zeta : \zeta \epsilon \Xi^*, \|\zeta\| = 1\}$ is norming*[4]) *in the space Ξ; more precisely*

$$\|x\| = \sup\{\zeta(x) : \zeta \epsilon Y\} \quad \textit{for each} \quad x.$$

[4]) A symmetric convex set $\Gamma \subset \Xi$ is called *norming in* $\langle X, \| \|\rangle$ if the norm $\|x\|_1 = \sup\{\xi(x) : \xi \epsilon \Gamma, \|\xi\| \leqslant 1\}$ is equivalent to the norm $\| \|$.

Proof. Let $\|x_0\| = 1$ and $\varepsilon > 0$. The set $S = \{x : \|x\| \leqslant 1\}$ is convex, symmetric and closed in the space $\langle X, \| \|^*\rangle$ by the condition (n). The element $(1+\varepsilon)x_0$ is not in S, whence ([13], p. 156) there is a functional $\zeta \epsilon \Xi^*$ such that

$$\zeta(x) \begin{cases} < 1 & \text{for} \quad x \epsilon S, \\ = 1 & \text{for} \quad x = (1+\varepsilon)x_0; \end{cases}$$

thus $\|x\| \leqslant 1$ implies $|\zeta(x)| < 1$, whence $\|\zeta\| \leqslant 1$, *i. e.* $\zeta \epsilon Y$; on the other hand, $\zeta(x_0) = \|x_0\|/(1+\varepsilon)$. This leads to the conclusion of the proposition, $\varepsilon > 0$ being arbitrary.

In the paper [1] the following example of a two-norm space was considered: X was the space $\boldsymbol{L}^2 = \boldsymbol{L}^2(0, 1)$,

$$\|x\| = \Big(\int_0^1 |x(t)|^2 dt\Big)^{1/2}, \qquad \|x\|^* = \int_0^1 |x(t)|\, dt.$$

Theorem 3.1 of this paper is to be read as follows:

1.6. PROPOSITION. *The general form of γ-linear functionals in $\langle \boldsymbol{L}^2, \| \|, \| \|^*\rangle$ is*

$$\xi(x) = \int_0^1 x(t) g(t)\, dt,$$

where $g(\cdot) \epsilon \boldsymbol{L}^2$.

Proof. Every functional of this form being in Ξ, only the sufficiency is to be proved. Let $x_n \xrightarrow{\gamma} 0$; thus

$$\int_0^1 |x_n(t)|^2 dt \leqslant K, \qquad \int_0^1 |x_n(t)|\, dt \to 0.$$

Choose $\varepsilon > 0$; then there is a $\delta > 0$ such that $|H| < \delta$ implies $\int_H |g(t)|^2 dt < \varepsilon^2$. If $E_n = \{t : |x_n(t)| < \varepsilon\}$, then $|E_n'| = |[0, 1] \setminus E_n| \to 0$, whence $|E_n'| < \delta$ for $n > N$. Thus

$$\begin{aligned} |\xi(x_n)| &= \Big|\int_0^1 x_n(t) g(t)\, dt\Big| \leqslant \Big|\int_{E_n} x_n(t) g(t)\, dt\Big| + \Big|\int_{E_n'} x_n(t) g(t)\, dt\Big| \\ &\leqslant \varepsilon \int_{E_n} |g(t)|\, dt + \Big(\int_{E_n'} |x_n(t)|^2 dt\Big)^{1/2} \Big(\int_{E_n'} |g(t)|^2 dt\Big)^{1/2} \\ &\leqslant \varepsilon \Big(\int_0^1 |g(t)|\, dt + K^{1/2}\Big). \end{aligned}$$

The above example shows that it may happen that $\Xi_\gamma = \Xi$. In [1] it was deduced from the false theorem 3.1 that it is possible to have $\Xi_\gamma = \Xi^*$. We shall show in section 2 that this can only happen in the trivial case, viz. when the norms $\| \|$ and $\| \|^*$ are equivalent.

2. The theorem of Wiweger. In this section we shall topologize the convergence γ; such topologization (even in a more general case) is due to A. Wiweger [20]. We shall construct another linear convex Hausdorff topology generating the γ-convergence. The idea of our procedure is derived from a paper of J. Mařík [11] and is based upon the following lemma contained implicitly in Mařík's paper:

2.1. Lemma. *Let $\mathcal{F}$ be a family of functions defined on an arbitrary set Q and such that $\sup\{|f(q)|: f\in\mathcal{F}\} < \infty$ for every $q\in Q$, and let $\{q_n\}$ be a sequence of elements of Q. The following propositions are equivalent:*

(a) *for each sequence of positive numbers $\{a_n\}$ tending to ∞ and for each sequence $\{f_n\}$ of elements of $\mathcal{F}$ there is an M such that*

$$|f_n(q_m)| < a_n \quad \text{for} \quad n = 1, 2, \ldots, \; m > M;$$

(b) $\sup\limits_{n=1,2,\ldots} \sup\{|f(q_n)|: f\in\mathcal{F}\} < \infty$ *and for each $f\in\mathcal{F}$ we have* $\lim\limits_{n\to\infty} f(q_n) = 0$.

Proof. (a)$\Rightarrow$(b). Choose $\varepsilon > 0$ and $f\in\mathcal{F}$ arbitrarily and write $f_n = f$, $a_n = n\varepsilon$ for $n = 1, 2, \ldots$; by (a) we deduce $|f_n(q_m)| < n\varepsilon$ for $m \geqslant M$ and $n = 1, 2, \ldots$, which gives, for $n = 1$, $|f(q_m)| < \varepsilon$ for $m \geqslant M$. Thus $\lim\limits_{n\to\infty} f(q_n) = 0$.

Suppose now that

$$\sup_{n=1,2,\ldots} \sup\{|f(q_n)|: f\in\mathcal{F}\} = \infty;$$

then for each k there must exist an n_k and an $f_k\in\mathcal{F}$ such that $|f_k(q_{n_k})| > k$. Let us write $a_k = k$; then $0 < a_k \to \infty$ and $|f_k(q_{n_k})| > a_k$. The sequence $\{n_k\}$ cannot contain any integer, say l, infinitely many times, for $\sup\limits_{k=1,2,\ldots} |f_k(q_l)| < \infty$. Thus n_k must tend to ∞, whence the inequality $|f_k(q_m)| < a_k$ is not satisfied by an infinite multitude of indices m.

(b)$\Rightarrow$(a). Let $0 < a_n \to \infty$ and let $f_n\in\mathcal{F}$. By (b) there is an N such that $|f_n(q_m)| < N$ for $n, m = 1, 2, \ldots$ Since $N < a_n$ for $n > A$, $|f_n(q_m)| < a_n$ for $n > A$. For $n = 1, 2, \ldots, A$, $\lim\limits_{m\to\infty} f_n(q_m) = 0$, whence there is an M such that $|f_n(q_m)| < a_n$ for $m > M$ and $n = 1, 2, \ldots, A$; consequently $|f_n(q_m)| < a_n$ for $m > M$, $n = 1, 2, \ldots$

Remark. The lemma remains true if we suppose the sequence $\{a_n\}$ to be increasing.

Suppose that $\langle X, \tau\rangle$ is a topological space, τ being its topology. The sequence $\{x_n\}$ of elements of X is called *τ-convergent to x_0* (in symbols $x_n \xrightarrow{\tau} x_0$) if every neighbourhood of x_0 contains almost all the elements of the sequence.

Let $\xi_n\in\Xi_\gamma$, $\|\xi_n\| \leqslant 1$, $0 < a_n \to \infty$; let us write

$$V(m, \varepsilon, \{\xi_n\}, \{a_n\}) = \{x: [x]_1 + \ldots + [x]_m < \varepsilon\} \cap \bigcap_{n=1}^{\infty} \{x: |\xi_n(x)| < a_n\}.$$

It is easily proved that if $\{\xi_n\}$ and $\{a_n\}$ run through all the possible sequences (satisfying the above conditions) and if m are arbitrary positive integers and $\varepsilon > 0$, then the sets $V(m, \varepsilon, \{\xi_n\}, \{a_n\})$ compose the basis of neighbourhoods of zero of a convex linear Hausdorff topology on X, which will be denoted by μ.

2.2. Theorem. *The topology μ is such that $x_n \xrightarrow{\gamma} x_0$ is equivalent to $x_n \xrightarrow{\mu} x_0$.*

Proof. Let $x_n \xrightarrow{\gamma} 0$; then $\|x_n\|^* \to 0$ implies $[x_n]_1 + \ldots + [x_n]_m \to 0$ for each m, whence $x_n \in \{x: [x]_1 + \ldots + [x_m] < \varepsilon\}$ for $n > N_1$. If $\xi\in\Xi_\gamma$, then $\lim\limits_{n\to\infty} \xi(x_n) = 0$ and for each $\xi\in\Omega = \{\xi: \xi\in\Xi_\gamma, \|\xi\| = 1\}$ we have

$$\sup_{n=1,2,\ldots} \sup\{|\xi(x_n)|: \xi\in\Omega\} \leqslant \sup_{n=1,2,\ldots} \|x_n\| < \infty.$$

Now $\xi_n\in\Omega$ and $0 < a_n \to \infty$ implies $|\xi_n(x_p)| < a_n$ for $p > N_2$ by Lemma 2.1, and it follows that each neighbourhood of zero for the topology μ contains almost all the elements of $\{x_n\}$, whence $x_n \xrightarrow{\mu} 0$. Suppose now, conversely, that $x_n \xrightarrow{\mu} 0$. For each m and $\varepsilon > 0$ we have $[x_p]_1 + \ldots + [x_p]_m < \varepsilon$ for $p > P$, which gives $\|x_p\|^* \to 0$. Arguing as in the first part of this proof we infer that

$$\sup_{n=1,2,\ldots} \sup\{|\xi(x_n)|: \xi\in\Omega\} < \infty.$$

Therefore $\sup\limits_{n=1,2,\ldots} \|x_n\| < \infty$ in virtue of the Proposition 1.5.

Now let ξ be a γ-linear functional. Set $\xi_n = \xi(1+\|\xi\|)^{-1}$ and $a_n = n(1+\|\xi\|)^{-1}$; then

$$V = V(1, 1, \{\xi_n\}, \{a_n\}) \subset \{x: |\xi(x)| < 1\},$$

and thus $x\in V$ implies $|\xi(x)| < 1$. Hence every γ-linear functional is continuous in the topology μ. The converse being obvious, we may state

2.3. Theorem. *The γ-linear functionals are identical with the functionals linear with respect to the topology μ.*

We shall see in section 5 that the requirements that the topology be such that the conclusions of the theorems 2.2 and 2.3 be satisfied do not determine uniquely the topology.

Now we are able to answer in which case we have $\Xi^* = \Xi_\gamma$.

2.4. Theorem. *If $\Xi^* = \Xi_\gamma$, then the norms $\| \;\|$ and $\| \;\|^*$ are equivalent.*

Proof. By Proposition 1.4 the space $\langle X, \| \;\|\rangle$ may be supposed to be complete. Let us suppose first that the norm $\| \;\|^*$ is homogeneous. We may suppose freely that $\|x\|^* \leqslant \|x\|$. It is well known that between all the convex linear topologies on X for which the conjugate space is

$\Xi^* = \Xi_\gamma$ there is a coarsest one, $\sigma(X, \Xi^*)$ ([5], p. 109), and a finest, $\tau(X, \Xi^*)$ (called the *Mackey-Arens topology*, see [10], p. 523, [3], p. 790, [6], p. 323). The basis of closed neighbourhoods of zero for $\tau(X, \Xi^*)$ consists of all the sets of form

$$\bigcap_{\xi \in \Phi} \{x : |\zeta(x)| \leqslant 1\},$$

where Φ runs through all subsets of Ξ^* compact in the topology $\sigma(\Xi^*, X)$. In the case considered now the set Φ must be bounded for the norm

$$\|\xi\|^* = \sup\{\xi(x) : \|x\|^* \leqslant 1\}.$$

Indeed, the topology $\sigma(\Xi^*, X)$ is identical with the induced topology by $\sigma(\Xi, X)$ into Ξ^*; the compactness of Φ for the topology $\sigma(\Xi^*, X)$ implies the same for the topology $\sigma(\Xi, X)$, which, in turn, implies the boundedness of Φ in $\sigma(\Xi, X)$, and finally the boundedness of Φ in $\langle \Xi^*, \|\;\|\rangle$. The set $\Xi^* = \Xi_\gamma$ is closed in $\langle \Xi, \|\;\|\rangle$ ([19], p. 57), whence Ξ^* is complete with respect to both norms, $\|\;\|$ and $\|\;\|^*$. Obviously $\|\xi\| \leqslant \|\xi\|^*$, whence by Banach's theorem ([4], p. 41) the spaces $\langle \Xi^*, \|\;\|\rangle$ and $\langle \Xi^*, \|\;\|^*\rangle$ are isomorphical. This implies the boundedness of Φ in $\langle \Xi^*, \|\;\|^*\rangle$, *i. e.* Φ is contained in a sphere $\Sigma_r = \{\xi : \|\xi\|^* \leqslant r\}$, whence

$$\bigcap_{\zeta \in \Phi} \{x : |\zeta(x)| \leqslant 1\} \supset \bigcap_{\zeta \in \Sigma_r} \{x : |\zeta(x)| \leqslant 1\} = \{x : \|x\|^* \leqslant r^{-1}\},$$

the last identity being obvious. Thus the topology $\tau(X, \Xi^*)$ is coarser than the topology of the norm $\|\;\|^*$. The converse follows by the Mackey-Arens theorem.

Suppose now that $\langle X, \|\;\|^*\rangle$ is a B_0^*-space with the sequence $\{[\;]_n\}$ of pseudonorms defining its topology. Since every pseudonorm $[\;]_n$ is coarser than $\|\;\|$, there exist constants $K_n > 1$ such that $[x]_n \leqslant K_n\|x\|$ for $x \in X$, $n = 1, 2, \ldots$ ([12], p. 194). Let us write

$$\|x\|_0^* = \sum_{n=1}^{\infty} \frac{1}{2^n K_n} [x]_n.$$

The norm $\|\;\|_0^*$ is finer than $\|\;\|^*$ and $\|x\|_0^* \leqslant \|x\|$, moreover both norms are equivalent on bounded subsets of $\langle X, \|\;\|\rangle$; thus the γ-convergences in $\langle X, \|\;\|, \|\;\|^*\rangle$ and $\langle X, \|\;\|, \|\;\|_0^*\rangle$ are identical. Let Ξ_0^* stand for the conjugate space of $\langle X, \|\;\|_0^*\rangle$; then $\Xi^* \subset \Xi_0^* \subset \Xi_\gamma$.

Now $\Xi^* = \Xi_\gamma$ implies $\Xi_0^* = \Xi_\gamma$, which gives, by the first part of this proof, the equivalence of the norms $\|\;\|$ and $\|\;\|_0^*$, whence the γ-convergence is metrical, which leads to the conclusion of the proposition by 1.2 and 1.3 of [1].

2.5. Remark. Theorem 2.4 in no longer true if the space $\langle X, \|\;\|^*\rangle$ is supposed to be only an F-space (in the sense of Banach). Indeed, let X be the space $\boldsymbol{L}$ of integrable functions in $[0,1]$ and let

$$\|x\| = \int_0^1 |x(t)|\,dt, \qquad \|x\|^* = \int_0^1 \frac{|x(t)|}{1+|x(t)|}\,dt.$$

Then Ξ^* and Ξ_γ consist only of the trivial functional 0 ([1], p. 54), the norms $\|\;\|$ and $\|\;\|^*$, however, are non-equivalent.

3. Null-sets of linear functionals. Let α be the notion of convergence in a linear space X such that $\langle X, \alpha\rangle$ is an $\mathcal{L}$-space of Fréchet and such that addition and multiplication by scalars are continuous. If the sequence $\{x_n\}$ converges to x_0 in the space $\langle X, \alpha\rangle$, we shall write, according to the practice adopted hitherto, $x_n \xrightarrow{\alpha} x_0$ or $\alpha\text{-}\lim_n x_n = x_0$. The α-closure of the set Y is the set of all α-limits of elements of Y; every set containing its α-closure will be called α-closed. The functional ξ on X will be called *α-linear* if it is distributive and if $x_n \xrightarrow{\alpha} x_0$ implies $\xi(x_n) \to \xi(x_0)$.

3.1. Lemma. *Let X_1 be a linear subset of X and suppose that x_0 is not in the α-closure of X_1. Let X_2 be the set of all the elements of form $x = z + \lambda x_0$ where $z \in X_1$ and $\lambda = \lambda(x)$ is a scalar. Then the functional λ is α-linear on $\langle X_2, \alpha\rangle$.*

Proof. It is well known that the set X_2 is linear and that the functional λ is uniquely determined, whence it is distributive. It suffices to prove that $x_n \xrightarrow{\alpha} 0$ implies $\lambda(x_n) \to 0$. Suppose that it is not so; then for a subsequence, say $\{x_n'\}$, there is a $\delta > 0$ such that $|\lambda(x_n')| > \delta$. There is a subsequence $\{x_n''\}$ of $\{x_n'\}$ such that $[\lambda(x_n'')]^{-1}$ converges to a limit μ. Let us write $x_n'' = z_n'' + \lambda(x_n'')x_0$, where $z_n'' \in X_1$; then $z_n'' + \lambda(x_n'')x_0 \xrightarrow{\alpha} 0$ and

$$\frac{z_n''}{\lambda(x_n'')} + x_0 \xrightarrow{\alpha} \mu \cdot 0 = 0,$$

whence x_0 is the α-limit of the elements $-z_n''[\lambda(x_n'')]^{-1}$ belonging to X_1, contrary to the hypothesis.

3.2. Theorem. *The set $H \subset X$ is the null-set of a non-trivial α-linear functional if and only if it is α-closed, linear and of deficiency* 1.

Proof. The necessity is obvious. The sufficiency results from Lemma 3.1.

Theorem 3.2 holds for the γ-convergence in any two-norm space. Given any set $A \subset X$, write $A^{\mathrm{I}} = \bigcup_{n=1}^{\infty} \overline{A \cap S_n}^*$ where $S_n = \{x : \|x\| \leqslant n\}$

and $\overline{B}^*$ is the closure of the set B in the space $\langle X, \| \|^* \rangle$. The following proposition is true for the γ-closed sets:

3.3. PROPOSITION. *The set $H \subset X$ is γ-closed if and only if $H = H^{\mathrm{I}}$.*

Proof. Necessity. By (n) the set $\overline{H \cap S_n}^*$ is contained in H, whence $H \supset H^{\mathrm{I}}$. On the other hand $H \cap S_n \subset \overline{H \cap S_n}^*$ implies $H = \bigcup_{n=1}^{\infty} H \cap S_n \subset H^{\mathrm{I}}$.

Sufficiency. $H = H^{\mathrm{I}}$ implies $H \supset \overline{H \cap S_n}^*$ for any n. Let $x_p \in H$, $x_p \xrightarrow{\gamma} x_0$; then $x_p \in S_{n_0}$ for some n_0, whence $\|x_p - x_0\|^* \to 0$ implies $x_0 \in H$.

4. Representation of γ-linear functionals. In this section we shall represent the γ-linear functionals as limits of convergent sequences of functionals of Ξ^*. We need the following

4.1. LEMMA. *Let H be a linear closed subset of the space $\langle X, \| \| \rangle$ and let $x_0 \notin H$. There exists a constant A such that $h \in H$, $\|\lambda x_0 + h\| \leqslant 1$ imply $\|h\| \leqslant A$.*

Proof. Let X_1 be the linear set spanned upon the set $H \cup \{x_0\}$, i. e. the set of the elements of the form $h + \lambda x_0$, $h \in H$. Every element of X_1 may be uniquely represented as $x = h(x) + \lambda(x) x_0$, where $h(x) \in H$, and the functional λ is linear on $\langle X_1, \| \| \rangle$, for its null-set is closed in $\langle X_1, \| \| \rangle$. Hence

$$\|h\| = \|h(x)\| = \|x - \lambda(x) x_0\| \leqslant \|x\| + \|\lambda\| \, \|x\| \, \|x_0\| \leqslant 1 + \|\lambda\| \, \|x_0\| = A.$$

4.2. THEOREM. *The general form of γ-linear functionals in $\langle X, \| \|, \| \|^* \rangle$ is*

$$\xi(x) = \lim_{n \to \infty} \zeta_n(x),$$

where $\zeta_n \in \Xi^*$ and $\|\xi - \zeta_n\| \to 0$.

Proof. It suffices to consider only non-trivial functionals ξ. Let H be the null-set of ξ and let $\xi(x_0) = 1$. By Theorem 3.2 and Proposition 3.3 the set $Z_n = H \cap S_n$ is closed in the space $\langle X, \| \|^* \rangle$; this set is convex, symmetric and $x_0 \notin Z_n$, whence there exists a functional $\zeta_n \in \Xi^*$ such that

$$\zeta_n(x) \begin{cases} < 1 & \text{for} \quad x \in Z_n, \\ = 1 & \text{for} \quad x = x_0. \end{cases}$$

This implies $|\zeta_n(x)| < 1/n$ for $x \in H \cap S_1 = Z_1$.

Now set $\xi_n(x) = \xi(x) - \zeta_n(x)$; then

$$|\xi_n(x)| \begin{cases} \leqslant 1/n & \text{for} \quad x \in Z_1, \\ = 0 & \text{for} \quad x = x_0. \end{cases}$$

Let x be an arbitrary element such that $\|x\| \leqslant 1$. Then $x = h + \lambda x_0$ with $h \in H$, and, by Lemma 4.1, $\|h\| \leqslant A$, whence $h/A \in Z_1$. It follows that

$$|\xi_n(x)| = |\xi_n(h)| = A \, |\xi_n(h/A)| \leqslant A/n,$$

i. e. $\|\xi_n\| \leqslant A/n$. This evidently yields $\xi(x) = \lim_{n\to\infty} \zeta_n(x)$ for every x.

It remains to prove the sufficiency of the representation. Let $x_p \xrightarrow{\gamma} 0$: then $\sup_{p=1,2,\ldots} \|x_p\| = K < \infty$ and $\|x_p\|^* \to 0$, whence

$$|\xi(x_p)| \leqslant |\zeta_n(x_p)| + |(\zeta_n - \xi)(x_p)| \leqslant |\zeta_n(x_p)| + K \|\zeta_n - \xi\|,$$

$$\overline{\lim_{p\to\infty}} |\xi(x_p)| \leqslant K \|\zeta_n - \xi\|,$$

the functionals ζ_n belonging to Ξ^*. Now let n tend to ∞; then

$$\lim_{p\to\infty} |\xi(x_p)| = 0.$$

Theorem 4.2 shows that the space Ξ_γ is identical with the closure of the subspace Ξ^* in the space $\langle \Xi, \| \| \rangle$. Let us notice that this is not the case when the norm is supposed to be an F-norm. Indeed, let X be the space $\boldsymbol{L}^\infty$ of essentially bounded functions in $[0, 1]$, and let

$$\|x\| = \operatorname*{ess\,sup}_{0 \leqslant t \leqslant 1} |x(t)|, \qquad \|x\|^* = \int_0^1 \frac{|x(t)|}{1 + |x(t)|} \, dt.$$

Then the set Ξ^* consists only of the trivial functional 0; there are, however, non-trivial functionals in Ξ_γ, for the norm $\|x\|_1^* = \int_0^1 |x(t)| \, dt$, as a starred norm $\| \|^*$, leads to the same γ-convergence.

Theorem 4.2 shows that *the functional ξ is γ-linear if and only if for each $\varepsilon > 0$ it may be represented in the form*

$$\xi(x) = \zeta(x) + \eta(x),$$

where $\zeta \in \Xi^*$, $\eta \in \Xi$, and $\|\eta\| < \varepsilon$. The sufficiency part of this theorem has been proved by Orlicz ([18], p. 274).

If Y_0 is a linear subset of a linear normed space $\langle Y, \| \| \rangle$ and if y_0 is the limit of a sequence of elements of Y_0 then, as can be immediately shown, there exist elements $y_n \in Y_0$ such that

$$y_0 = \sum_{n=1}^{\infty} y_n \quad \text{and} \quad \sum_{n=1}^{\infty} \|y_n\| < \infty;$$

moreover, for any $\varepsilon > 0$, this representation may be chosen so that $\sum_{n=1}^{\infty} \|y_n\| \leqslant \|y_0\| + \varepsilon$. This proposition yields the following alternative formulation of Theorem 4.2:

4.3. THEOREM. *The general form of γ-linear functionals in $\langle X, \| \|, \| \|^* \rangle$ is*

$$\xi(x) = \sum_{n=1}^{\infty} \zeta_n(x),$$

where $\zeta_n \in \Xi^*$ and $\sum_{n=1}^{\infty} \|\zeta_n\| < \infty$. For any $\varepsilon > 0$ this representation may be chosen so that $\sum_{n=1}^{\infty} \|\zeta_n\| \leqslant \|\xi\| + \varepsilon$.

Let us suppose that the pseudonorms $[\ ]_i$ form a non-increasing sequence; then the space Ξ^* is the union $\Xi^* = \bigcup_{n=1}^{\infty} \Xi^{(n)}$ where $\Xi^{(n)}$ is the set of all distributive functionals on X satisfying the condition $|\zeta(x)| \leqslant k[x]_n$ with k independent of x ([13], p. 139). Then we have

4.4. PROPOSITION. *The functionals ζ_n in 4.3 may be chosen so that $\zeta_n \in \Xi^{(n)}$.*

Proof. $\Xi^{(1)} \subset \Xi^{(2)} \subset \ldots$ implies that there are k_n such that $\zeta_n \in \Xi^{(k_n)}$ and $k_1 < k_2 < \ldots$ We set

$$\zeta_n' = \begin{cases} \zeta_i & \text{for} \quad n = k_i,\ i = 1, 2, \ldots, \\ 0 & \text{elsewhere;} \end{cases}$$

$\xi = \sum_{n=1}^{\infty} \zeta_n'$ is the desired representation.

Theorems 2.4 and 4.2 imply the following proposition, not involving the notion of the two-norm convergence:

4.5. PROPOSITION. *Let the topology of the B^*-space $\langle X, \| \| \rangle$ be finer than that of the B_0^*-space $\langle X, \| \|^* \rangle$, and let Ξ and Ξ^* denote the conjugate spaces of these spaces. If Ξ^* is closed in Ξ, then the norms $\| \|$ and $\| \|^*$ are equivalent. The condition* (n) *is, of course, supposed.*

Proof. By Theorem 4.2 $\Xi^* = \Xi_\gamma$, which implies by 2.4 the equivalence of the norms.

5. Counter examples. We shall prove first that the extension theorem is in general not true for γ-linear functionals.

Let X be the space l^1 of the sequences $x = \{x_n\}$ such that $\|x\| = \sum_{n=1}^{\infty} |x_n| < \infty$, and let us set

$$\|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n|.$$

The convergence γ in $\langle l^1, \| \|, \| \|^* \rangle$ may be characterized as follows: let $x_n = \{x_{n\nu}\}_{\nu=1,2,\ldots}$, $x_0 = \{x_{0\nu}\}_{\nu=1,2,\ldots}$; then $x_n \xrightarrow{\gamma} x_0$ if and only if $\sup_{n=1,2,\ldots} \|x_n\| < \infty$ and $\lim_{n\to\infty} x_{n\nu} = x_{0\nu}$ for $\nu = 1, 2, \ldots$

The space $\langle l^1, \| \| \rangle$ is conjugate to the space c_0 of null-convergent sequences. We immediately observe that the γ-convergence is identical with the *-weak convergence in l^1 (called also the *weak convergence* as functionals), this convergence is in turn equivalent to the convergence with respect to the topology $\sigma(l^1, c_0)$.

Mazurkiewicz [14] has constructed an important example of a linear set in l^1. Let e_n denote the n-th unit vector in l^1, let us arrange all the pairs (i, k) of positive integers in a single sequence, and let $N(i, k)$ be the place occupied there by the pair (i, k). Set

$$x_{ik} = \frac{e_1}{2^1} + \ldots + \frac{e_{2i-1}}{2^i} + i e_{2N(i,k)}.$$

The set $\mathfrak{M}$ of Mazurkiewicz is the linear span of the set of all the elements x_{ik} $(i, k = 1, 2, \ldots)$. Let us write

$$x_0 = \sum_{j=1}^{\infty} \frac{e_{2j-1}}{2^j}, \qquad x_n = \sum_{j=1}^{n} \frac{e_{2j-1}}{2^j}.$$

Mazurkiewicz has shown that $x_n \xrightarrow{\gamma} x_0$ and that x_n but not x_0 are in the γ-closure of the set $\mathfrak{M}$.

Let H be the set of all elements of the form $x = y + \lambda x_0$ with $y \in \mathfrak{M}$; this set is linear. The space $\langle H, \| \|, \| \|^* \rangle$ satisfies the condition (n). The element x_0 is not in the γ-closure of the set $\mathfrak{M}$, whence by Lemma 3.1 there exists a γ-linear functional $\lambda(x)$ on H such that $\lambda(x) = 0$ for $x \in \mathfrak{M}$ and $\lambda(x_0) = 1$. There exists, however, no extension ξ of λ onto l^1, which is γ-linear. For if such a functional existed, we should have $\xi(x_n) = 0$, whence from $x_n \xrightarrow{\gamma} x_0$ it would follow that $\xi(x_0) = 0 \neq \lambda(x_0)$. Thus we have proved

5.1. PROPOSITION. *In the space $\langle l^1, \| \|, \| \|^* \rangle$ there exist a linear subset H and a γ-linear functional on $\langle H, \| \|, \| \|^* \rangle$ which cannot be extended to the whole of l^1 with the preservation of γ-linearity.*

Let us modify the topology μ by assuming that the functionals ξ_n belong to Ξ^*. The (coarser) topology obtained in this fashion will be denoted by μ^*. It is easily verified that μ^* is such that $x_n \xrightarrow{\gamma} x_0$ is equivalent to $x_n \xrightarrow{\mu^*} x_0$. The topology μ^* depends only on the set Ξ^*, whence for every linear subset Z the topology μ^* constructed for the space $\langle Z, \| \|, \| \|^* \rangle$ is identical with the induced μ^*-topology for the whole of $\langle X, \| \|, \| \|^* \rangle$. We deduce

5.2. Proposition. *The topology μ^* in $\langle \boldsymbol{l}^1, \| \|, \| \|^* \rangle$ is such that $x_n \xrightarrow{\gamma} x_0$ is equivalent to $x_n \xrightarrow{\mu^*} x_0$. The γ-linear functionals, however, are not identical with the μ^*-linear ones.*

Proof. If every γ-linear functional were μ^*-linear, then the γ-linear functionals might be extended from arbitrary linear subsets of $\boldsymbol{l}^1$ (by the above properties of the topology μ^*).

The topology τ on the space $\langle X, \| \|, \| \|^* \rangle$ will be called *appropriate* if it is convex, linear, Hausdorff, if for sequences γ-convergence is equivalent to τ-convergence, and if the class Ξ_γ is identical with the class of functionals linear in the topology τ.

5.3. Proposition. *There may exist different appropriate topologies for the space $\langle X, \| \|, \| \|^* \rangle$.*

Proof. In the space $\langle \boldsymbol{l}^1, \| \|, \| \|^* \rangle$ let us consider the topologies μ and $\sigma = \sigma(\boldsymbol{l}^1, \boldsymbol{c}_0)$; both are appropriate for this space; however, μ is strictly finer than σ. Evidently σ is coarser than μ; on the other hand there are neighbourhoods of zero in μ containing no neighbourhood of zero in σ. Indeed, for $x = \{x_n\}$, let $\xi_n(x) = x_n$, and write $a_n = n$; then

$$V = V(1, 1, \{\xi_n\}, \{a_n\}) = \{x : \|x\|^* < 1\} \cap \bigcap_{n=1}^{\infty} \{x : |x_n| < n\}$$

is a neighbourhood of zero in μ. Suppose that a neighbourhood W of zero in σ is contained in V. W is of form

$$W = \bigcap_{k=1}^{m} \{x : |\eta_i(x)| < 1\},$$

where $\eta_i(x) = \sum_{\nu=1}^{\infty} a_{\nu i} x_\nu$ and $\lim_{\nu\to\infty} a_{\nu i} = 0$ for $i = 1, 2, \ldots, m$. Every finite set of linear functionals on $\boldsymbol{l}^1$ is not total, whence there is a constant k such that $\eta_i(e_k) = 0$ for $i = 1, 2, \ldots, m$; then $k^2 e_k \in W$ but $k^2 e_k \notin V$, because the k-th coordinate of this element is greater than k.

Let us consider on $\boldsymbol{l}^1$ another starred norm $\|x\|_1^* = \sup_{n=1,2,\ldots} |x_n|$, and let us denote the γ-convergence in $\langle X, \| \|, \| \|_1^* \rangle$ by γ_1. The γ_1-convergence implies the γ-convergence but not conversely. Indeed, the sequence $\{e_n\}$ is γ-convergent to zero but is not γ_1-convergent.

It is easily seen that the general form of γ-linear functionals in $\langle \boldsymbol{l}^1, \| \|, \| \|^* \rangle$ and in $\langle \boldsymbol{l}^1, \| \|, \| \|_1^* \rangle$ is the same:

$$\xi(x) = \sum_{n=1}^{\infty} a_n x_n, \quad \text{where} \quad \lim_{n\to\infty} a_n = 0.$$

Thus γ-convergences in the spaces $\langle X, \| \|, \| \|^* \rangle$ and $\langle X, \| \|, \| \|_1^* \rangle$ may be different and may produce the same set of γ-linear functionals.

In $\langle X, \| \|, \| \|^* \rangle$ pointwise limits of γ-linear functionals may not be γ-linear. We know certain sufficient conditions for the limit of each pointwise convergent sequence of γ-linear functionals on $\langle X, \| \|, \| \|^* \rangle$ to be again γ-linear ([1], p. 55, [16], p. 267). In general, pointwise limits of γ-linear functionals form a larger set than Ξ_γ; for example, in $\langle \boldsymbol{l}^1, \| \|, \| \|^* \rangle$ every element of Ξ is the pointwise limit of functionals of Ξ_γ, but $\Xi \neq \Xi_\gamma$.

6. General form of γ-linear functionals. Now we shall show that Theorem 4.3 easily yields the general form of γ-linear functionals in several concrete spaces.

A. Let X be the space $C^*_{(-\infty,\infty)}$ of continuous and bounded functions $x = x(t)$ in $(-\infty, \infty)$ and set

$$\|x\| = \sup_{-\infty<t<\infty} |x(t)|, \qquad [x]_n = \max_{-n\leqslant t\leqslant n} |x(t)|.$$

The set $\Xi^{(n)}$ (see section 4) consists of the functionals of the form

$$\zeta(x) = \int_{-\infty}^{\infty} x(t)\, dg(t),$$

where g is a function of finite variation, continuous on the right, vanishing for $t = 0$, and constant in each of the intervals $(-\infty, -n]$ and $[n, \infty)$. The norm of ζ is

$$\|\zeta\| = \operatorname*{var}_{-\infty<t<\infty} g(t).$$

By Theorem 4.3 each functional of Ξ_γ is of the form

$$\xi(x) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} x(t)\, dg_n(t) \quad \text{where} \quad \sum_{n=1}^{\infty} \operatorname*{var}_{-\infty<t<\infty} g_n(t) < \infty.$$

This implies the uniform convergence of the series $\sum_{n=1}^{\infty} g_n(t) = g(t)$; moreover

$$\operatorname*{var}_{-\infty<t<\infty} g(t) \leqslant \sum_{n=1}^{\infty} \operatorname*{var}_{-\infty<t<\infty} g_n(t) < \infty.$$

Conversely, let g be of finite variation in $(-\infty, \infty)$. Let us write

$$\xi(x) = \int_{-\infty}^{\infty} x(t)\, dg(t), \qquad \zeta_n(x) = \int_{-n}^{n} x(t)\, dg(t);$$

then $\xi \in \Xi$, $\zeta_n \in \Xi^*$ and

$$\|\zeta_n - \xi\| \leqslant [\operatorname*{var}_{-\infty<t\leqslant n} g(t) + \operatorname*{var}_{n\leqslant t<\infty} g(t)] \to 0.$$

Thus we have proved the following theorem of J. Musielak and W. Orlicz [15]:

The general form of γ-linear functionals in $\langle C^*_{(-\infty,\infty)}, \|\,\|, \|\,\|^*\rangle$ *is*

$$\xi(x) = \int_{-\infty}^{\infty} x(t)\,dg(t),$$

where g is a function of finite variation in $(-\infty, \infty)$.

B. Let Ω be a completely regular σ-compact Hausdorff space, and let $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ be its representation as a union of compact sets such that $\Omega_1 \subset \Omega_2 \subset \ldots$ Let us denote by $C^*(\Omega)$ the space of all bounded real functions $x = x(t)$ continuous on Ω. Let us set

$$\|x\| = \sup_{t\in\Omega} |x(t)|, \qquad [x]_n = \max_{t\in\Omega_n} |x(t)|.$$

Then the sequence $\{x_n\}$ is γ-convergent to x_0 if and only if it is uniformly bounded on Ω and converges uniformly to $x_0(t)$ on every set Ω_n, and $x_0(t)$ is continuous. The condition (n) is satisfied, the condition (n_1), however, is not necessarily fulfilled (it is satisfied when Ω is locally compact).

The set $\Xi^{(n)}$ consists of the functionals of the form

$$\zeta(x) = \int_{\Omega} x(t)\,d\mu,$$

where μ is a signed measure defined on all Borel subsets of Ω, vanishing for subsets of $\Omega \setminus \Omega_n$, and $\|\zeta\| = \operatorname*{var}_{\Omega} \mu$ ([8], p. 1008-1012). An argument quite similar to that used in the proof of A immediately gives the following theorem of J. Mařík [5]):

The general form of γ-linear functionals in $\langle C^*(\Omega), \|\,\|, \|\,\|^*\rangle$ *is*

$$\xi(x) = \int_{\Omega} x(t)\,d\mu,$$

where μ is a signed measure defined on Borel subsets of Ω and such that $\operatorname{var}\mu < \infty$.

C. Let $X = L^{\infty}(-\infty, \infty)$ be the space of the functions $x = x(t)$ bounded and measurable on $(-\infty, \infty)$; set

$$\|x\| = \operatorname*{ess\,sup}_{-\infty<t<\infty} |x(t)|, \qquad [x]_n = \operatorname*{ess\,sup}_{-n<t<n} |x(t)|.$$

[5]) [11], p. 90; it was conjectured by C. Ryll-Nardzewski, May 11th 1955, at the meeting of the Toruń Section of the Polish Mathematical Society.

The set $\Xi^{(n)}$ consists of the functionals of the form

$$\zeta(x) = \int_{-\infty}^{\infty} x(t)\,d\mu,$$

where μ is a finitely additive set function of finite variation defined on all Lebesgue measurable sets, vanishing for sets lying outside the interval $[-n, n]$ and for any set of measure zero. The norm is

$$\|\zeta\| = \operatorname*{var}_{(-\infty,\infty)} \mu.$$

Arguing similarly as in A we can state that

The general form of γ-linear functionals in $\langle L^{\infty}_{(-\infty,\infty)}, \|\,\|, \|\,\|^*\rangle$ *is*

$$\xi(x) = \int_{-\infty}^{\infty} x(t)\,d\mu,$$

where μ is a finitely additive set function of finite variation, defined for all Lebesgue measurable sets, vanishing for sets of measure zero and such that

$$\lim_{n\to\infty} \big(\operatorname*{var}_{(-\infty,-n)} \mu + \operatorname*{var}_{(n,\infty)} \mu\big) = 0.$$

(The last condition follows by the passage to the limit from the fact that it is fulfilled for every set function μ_n; the condition is essential, for μ is finitely additive).

D. Let $1 \leqslant \beta < \alpha < \infty$, $X = L^{\alpha}$,

$$\|x\| = \|x\|_{\alpha}, \qquad \|x\|^* = \|x\|_{\beta}$$

where, generally, $\|x\|_{\delta} = \big(\int_0^1 |x(t)|^{\delta}dt\big)^{1/\delta}$. Marking the conjugate exponent by an apostrophe we have

$$\Xi^* = L^{\beta'}, \qquad \Xi = L^{\alpha'}.$$

Since $L^{\beta'} \subset L^{\alpha'}$ and $L^{\beta'}$ is dense in $\langle L^{\alpha'}, \|\,\|_{\alpha'}\rangle$, we obtain

$$\Xi_{\gamma} = L^{\alpha'} = \Xi.$$

In the limit case $\alpha = \infty$, $\beta = 1$ we have $\Xi_{\gamma} = L^1$; this has been proved by Fichtenholz ([7], p. 199).

E. Analogous arguments for the space l^{α} and $1 < \alpha < \beta < \infty$ give also $\Xi_{\gamma} = l^{\alpha'} = \Xi$.

F. Let $1 < \alpha \leqslant \infty$, $X = L^{\alpha}$, $\beta_n \nearrow \alpha$ $(\beta_n > 1)$,

$$\|x\| = \|x\|_{\alpha}, \qquad [x]_n = \|x\|_{\beta_n}.$$

Then $\Xi^{(n)} = L^{\beta_n'}$, $\Xi = L^{\alpha'}$; the set $\Xi^* = \bigcup_{n=1}^{\infty} L^{\beta_n'}$ is evidently dense in $\langle L^{\alpha'}, \|\ \|_{\alpha'}\rangle$. Thus again $\Xi_\gamma = L^{\alpha'} = \Xi$.

G. The same result holds for the space l^α with $1 \leqslant \alpha < \infty$, $\beta_n \searrow \alpha$.

H. Let X be the space CV_0 of continuous functions of finite variation in $[0,1]$, vanishing for $t = 0$. Let us set

$$\|x\| = \operatorname*{var}_{0\leqslant t\leqslant 1} x(t), \qquad \|x\|^* = \max_{0\leqslant t\leqslant 1} |x(t)|.$$

Any functional $\zeta \in \Xi^*$ may be represented in the form

$$\zeta(x) = \int_0^1 x(t)\,dh(t),$$

where the function h is of finite variation, continuous on the right in $(0,1)$ and $h(1) = 0$. Write

$$g(t) = \begin{cases} h(t) & \text{for} \quad 0 < t \leqslant 1, \\ \lim\limits_{h\to 0+} h(u) & \text{for} \quad t = 0; \end{cases}$$

then

$$\int_0^1 x(t)\,dg(t) = \int_0^1 x(t)\,dh(t) \qquad \text{for} \qquad x \in X.$$

To find the norm $\|\zeta\|$ let us notice first that

$$\zeta(x) = g(1)x(1) - g(0)x(0) - \int_0^1 g(t)\,dx(t) = -\int_0^1 g(t)\,dx(t) \leqslant \sup_{0\leqslant t\leqslant 1}|g(t)|\,\|x\|,$$

whence $\|\zeta\| \leqslant \sup\limits_{0\leqslant t\leqslant 1}|g(t)| = G$. On the other hand, for $\varepsilon > 0$, there is a t_0 such that $0 < t_0 < 1$ and $|g(t_0)| \geqslant G - \varepsilon$. Set

$$x_n(t) = \begin{cases} 1 & \text{for} \quad t \in [t_0 + 1/n, 1], \\ 0 & \text{for} \quad t \in [0, t_0), \\ \text{linear} & \text{for} \quad t \in [t_0, t_0 + 1/n] \end{cases}$$

(x_n are defined for sufficiently large n); then

$$\zeta(x_n) = -n \int_{t_0}^{t_0+1/n} g(t)\,dt \to -g(t_0),$$

which finally gives $\|\zeta\| = \sup\limits_{0\leqslant t\leqslant 1} |g(t)|$.

Let $\xi \in \Xi_\gamma$; then, by Theorem 4.3, there exist a g_n such that

$$\xi(x) = \sum_{n=1}^{\infty} \int_0^1 g_n(t)\,dx(t) \qquad \text{and} \qquad \sum_{n=1}^{\infty} \sup_{0\leqslant t\leqslant 1} |g_n(t)| < \infty.$$

Write $g(t) = \sum_{n=1}^{\infty} g_n(t)$; then g is obviously 1° bounded, 2° continuous on the right for $0 \leqslant t < 1$, 3° vanishing at $t = 1$, 4° having only discontinuities of the first kind, and 5° for every $\varepsilon > 0$ the set

$$\{t\colon 0 < t < 1, |g(t) - g(t-0)| > \varepsilon\}$$

is finite.

Conversely, it is easily seen that every function satisfying 1°-5° is the sum of a uniformly convergent series of step functions vanishing at $t = 1$. Hence we can state that

The general form of γ-linear functionals in $\langle CV_0, \|\ \|, \|\ \|^*\rangle$ is

$$\xi(x) = \int_0^1 g(t)\,dx(t)$$

where the function g satisfies the conditions 1°-5°.

J. Let $\langle A, \mathfrak{C}, \mu\rangle$ be a measure space with a σ-finite measure μ. Suppose that $A = \bigcup_{n=1}^{\infty} A_n$ with measurable A_n such that $A_1 \subset A_2 \subset \ldots$; suppose, moreover, that $\mu(A_1) > 0$. $L^1(A, \mathfrak{C}, \mu)$ will denote the space of functions $x = x(t)$, μ-integrable on A. Let us write

$$\|x\| = \int_A |x(t)|\,d\mu, \qquad [x]_n = \int_{A_n} |x(t)|\,d\mu.$$

Every linear functional of Ξ is of the form

$$\xi(x) = \int_A x(t)\varphi(t)\,d\mu,$$

where φ is an essentially bounded function, μ-measurable on A and $\|\zeta\| = \operatorname*{ess\,sup}\limits_{t\in A} |\varphi(t)|$. The functional ξ belongs to $\Xi^{(n)}$ if and only if $\varphi(t) = 0$ for $t \in A \setminus A_n$. Hence if $\xi \in \Xi_\gamma$, then

$$\xi(x) = \sum_{n=1}^{\infty} x(t)\,d\varphi_n(t), \qquad \sum_{n=1}^{\infty} \operatorname*{ess\,sup}_{t\in A} |\varphi_n(t)| < \infty.$$

This implies the essentially uniform convergence of the series $\sum \varphi_n$; it follows also that $\lim\limits_{n\to\infty} \operatorname*{ess\,sup}\limits_{t\in A\setminus A_n} |\varphi(t)| = 0$. Thus

The general form of γ-linear functionals on $\langle L^1(A, \mathfrak{C}, \mu), \|\ \|, \|\ \|^*\rangle$ is

$$\xi(x) = \int_A x(t)\varphi(t)\,d\mu,$$

where φ is a μ-measurable function essentially bounded on A such that $\lim\limits_{n\to\infty} \operatorname*{ess\,sup}\limits_{t\in A\setminus A_n} |\varphi(t)| = 0$.

The sufficiency of such representation is obvious.

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Zu linearen Limitierungsverfahren

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In der vorliegenden Mitteilung werden die notwendigen und hinreichenden Bedingungen dafür angegeben, daß aus der A-Summierbarkeit der Reihe $\sum_{\nu=0}^{\infty} a_\nu$ die A-Summierbarkeit der Reihe $\sum_{\nu=1}^{\infty} a_\nu$ folgt und umgekehrt. Es wird der Zusammenhang zwischen den verallgemeinerten Summen $A-\sum_{\nu=0}^{\infty} a_\nu$ und $A-\sum_{\nu=1}^{\infty} a_\nu$ festgestellt. Dabei bedeutet $A = \|a_{\mu\nu}\|$ eine normale Matrix, für die der Grenzwert

$$\lim_{\mu\to\infty} \sum_{\nu=0}^{\mu} a_\nu \tag{1}$$

existiert.

Bei den Betrachtungen wird wesentlich ein bemerkenswerter Satz von Mazur verwendet, der in seiner Arbeit *Über lineare Limitierungsverfahren* (Mathematische Zeitschrift 28 (1928), S. 599-611) bewiesen worden ist.

Für jede normale Matrix $A = \|a_{\mu\nu}\|$ besitzt das unendliche Gleichungssystem

$$y_\mu = \sum_{\nu=0}^{\mu} a_{\mu\nu} x_\nu \qquad (\mu = 0, 1, \ldots)$$

eine einzige Lösung. Es sei

$$\xi_\nu = x_\nu \quad \text{für} \quad y_\mu = 1 \qquad (\mu, \nu = 0, 1, \ldots),$$

$$\xi_{\nu\mu} = x_\nu \quad (\mu, \nu = 0, 1, \ldots) \quad \text{für} \quad y_\nu = \begin{cases} 1 & \text{bei} \quad \nu = \mu, \\ 0 & \text{bei} \quad \nu \neq \mu. \end{cases}$$

Wenn $B = \|b_{\mu\nu}\|$ eine beliebige Matrix ist, und falls $\bar{A}$ und $\bar{B}$ die Konvergenzfelder von A und B bedeuten, so gilt folgender

SATZ 1 (von Mazur). *Die notwendingen und hinreichenden Bedingungen dafür, daß $\bar{A} \subset \bar{B}$, sind:*