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On the error term concerning the number of subgroups of the groups $\mathbb{Z}_m \times \mathbb{Z}_n$ with $m, n \le x$

by

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1. Introduction. Let $\mathbb{Z}_m$ be the additive group of residue classes modulo m. For arbitrary positive integers m and n consider the group $G := \mathbb{Z}_m \times \mathbb{Z}_n$, which is isomorphic to $\mathbb{Z}_{\gcd(m,n)} \times \mathbb{Z}_{\operatorname{lcm}(m,n)}$. When $\gcd(m,n) = 1$, G is cyclic and isomorphic to $\mathbb{Z}_{mn}$. When $\gcd(m,n) > 1$, G has rank two. Let $s(m,n)$ and $c(m,n)$ denote the total number of subgroups and the number of cyclic subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$, respectively.

Here $s(m,n)$ and $c(m,n)$ are multiplicative functions of two variables, that is,

$$s(m,n) = \prod_p s(p^{\nu_p(m)}, p^{\nu_p(n)}), \tag{1.1}$$

$$c(m,n) = \prod_p c(p^{\nu_p(m)}, p^{\nu_p(n)}), \tag{1.2}$$

for all $m, n \in \mathbb{N}$. Furthermore, for the rank two p-group $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$ with $1 \le a \le b$, one has the following formulas:

$$s(p^a, p^b) = \frac{(b-a+1)p^{a+2} - (b-a-1)p^{a+1} - (a+b+3)p + (a+b+1)}{(p-1)^2} \tag{1.3}$$

and

$$c(p^a, p^b) = 2(1 + p + p^2 + \cdots + p^{a-1}) + (b-a+1)p^a. \tag{1.4}$$

Hence, one can compute $s(m,n)$ and $c(m,n)$ by using (1.1), (1.3) and (1.2), (1.4), respectively. However, the following more compact identities hold for all $m, n \in \mathbb{N}$:

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$$(1.5)\qquad s(m,n)=\sum_{d\mid m,e\mid n}\gcd(d,e)=\sum_{d\mid \gcd(m,n)}\phi(d)\tau(m/d)\tau(n/d),$$

$$(1.6)\qquad c(m,n)=\sum_{d\mid m,e\mid n}\phi(\gcd(d,e))=\sum_{d\mid \gcd(m,n)}(\mu*\phi)(d)\tau(m/d)\tau(n/d).$$

For general properties of the subgroup lattice of finite abelian groups, see R. Schmidt [10] and M. Suzuki [11]. We note that formula (1.3) was deduced by using Goursat's lemma for groups in [3, 9], and using the concept of the fundamental group lattice in [12, 13]. Formula (1.4) was given in [13]. Identities (1.5) and (1.6) were derived in [4] by a simple elementary method, and in [14] by using Goursat's lemma for groups.

W. G. Nowak and L. Tóth [7] studied the average orders of the functions $s(m,n)$ and $c(m,n)$. Suppose $x>0$ is a real number. Define

$$S^{(1)}(x):=\sum_{m,n\le x}s(m,n),\qquad S^{(2)}(x):=\sum_{\substack{m,n\le x\\ \gcd(m,n)>1}}s(m,n),$$

$$S^{(3)}(x):=\sum_{m,n\le x}c(m,n),\qquad S^{(4)}(x):=\sum_{\substack{m,n\le x\\ \gcd(m,n)>1}}c(m,n).$$

Here $S^{(2)}(x)$ and $S^{(4)}(x)$ denote the total number of subgroups and cyclic subgroups, respectively, of the groups $\mathbb{Z}_m\times\mathbb{Z}_n$ having rank two, with $m,n\le x$.

W. G. Nowak and L. Tóth [7] proved that for every j with $1\le j\le 4$,

$$(1.7)\qquad S^{(j)}(x)=x^2\sum_{r=0}^{3}A_{j,r}\log^r x+O(x^{1117/701+\varepsilon}),$$

where $A_{j,r}$ $(1\le j\le 4,\ 0\le r\le 3)$ are explicit constants, whose definitions are omitted here. Note that $1117/701=1.593437\ldots$. In fact, the error term in (1.7) is $O(x^{(3-\theta)/(2-\theta)+\varepsilon})$, where θ is the exponent in the Dirichlet divisor problem for $\tau(n)$. The exponent 1117/701 is obtained from $\theta=131/416$ of M. N. Huxley [5]. The asymptotic formula (1.7) holds for the slightly better exponent $4427/2779=1.593019\ldots$ by using the exponent $\theta=517/1648$ obtained in [2]. Note that the limit of this approach is $11/7=1.571428\ldots$ with $\theta=1/4$.

In this paper we shall prove the following theorem, which improves the above error terms.

THEOREM 1.1. *The asymptotic formulas*

$$S^{(j)}(x)=x^2\sum_{r=0}^{3}A_{j,r}\log^r x+O(x^{3/2}(\log x)^{6.5})$$

hold for every j *with* $1\le j\le 4$.

For the proof we use a multidimensional Perron formula and the complex integration method.

NOTATION. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers, ϕ is Euler's totient function, μ is the Möbius function, ζ denotes the Riemann zeta-function, and $\tau_k(n)$ denotes the number of ways n can be written as a product of k positive integers ($\tau(n) = \tau_2(n)$). Let $n = \prod_p p^{\nu_p(n)}$ denote the prime power factorization of $n \geq 2$, where the product is over the primes p and all but a finite number of the exponents $\nu_p(n)$ are zero.

2. Preliminary lemmas

LEMMA 2.1 ([7]). *Suppose* $\Re z, \Re w > 1$. *Then*

$$S(z,w) := \sum_{m\geq 1}\sum_{n\geq 1} \frac{s(m,n)}{m^z n^w} = \zeta^2(z)\zeta^2(w)\zeta(z+w-1)\zeta^{-1}(z+w),$$

$$C(z,w) := \sum_{m\geq 1}\sum_{n\geq 1} \frac{c(m,n)}{m^z n^w} = \zeta^2(z)\zeta^2(w)\zeta(z+w-1)\zeta^{-2}(z+w).$$

LEMMA 2.2. *Suppose that* $r \geq 2$ *is a fixed integer and* $f(n_1,\dots,n_r)$ *is an arithmetical function of* r *variables that is symmetric in* $n_1,\dots,n_r$ *and whose Dirichlet series*

$$F(z_1,\dots,z_r) := \sum_{n_1\geq 1} \cdots \sum_{n_r \geq 1} \frac{f(n_1,\dots,n_r)}{n_1^{z_1}\cdots n_r^{z_r}}$$

is absolutely convergent for $\Re z_j > \sigma_a$ $(1 \leq j \leq r)$ *with some* $\sigma_a > 0$. *Suppose* $x, T \geq 10$ *are two parameters, and define*

$$b = \sigma_a + \frac{1}{\log x}, \qquad T_j = 2^{j-1}T \quad (1 \leq j \leq r).$$

Then

$$\sum_{n_1\leq x} \cdots \sum_{n_r\leq x} f(n_1,\dots,n_r) h\left(\frac{x}{n_1}\right)\cdots h\left(\frac{x}{n_r}\right)$$
$$= \frac{1}{(2\pi i)^r} \int_{b-iT_1}^{b+iT_1} \cdots \int_{b-iT_r}^{b+iT_r} F(z_1,\dots,z_r) x^{z_1+\dots+z_r} \frac{dz_r\cdots dz_1}{z_r\cdots z_1} + O(x^{r\sigma_a}E_f(x,T)),$$

where

$$E_f(x,T) := \sum_{n_1\geq 1} \cdots \sum_{n_r\geq 1} \frac{|f(n_1,\dots,n_r)|(n_1\cdots n_r)^{-\sigma_a - 1/\log x}}{\min_{1\leq j\leq r} T|\log(x/n_j)| + 1}$$

and

$$h(y) := \begin{cases} 1 & \text{if } y > 1, \\ 1/2 & \text{if } y = 1, \\ 0 & \text{if } 0 < y < 1. \end{cases}$$

Proof. This is a multiple type Perron formula, which easily follows from [1, Propositions 5 and 6]. ■

LEMMA 2.3. *Suppose $\ell = 0$ or $\ell = 1$. For $\sigma > 1$ we have*

$$\zeta^{(\ell)}(\sigma + it) \ll \min\left(\frac{1}{(\sigma-1)^{1+\ell}}, \log^{1+\ell}(|t|+2)\right),$$
$$\zeta^{-1}(\sigma + it) \ll \min\left(\frac{1}{\sigma-1}, \log(|t|+2)\right).$$

Proof. The first estimate for $\ell = 0$ can be found in Pan and Pan [8, Chapter 7]. The first estimate for $\ell = 1$ follows from the result for $\ell = 0$ and Cauchy's theorem. The second estimate can also be found in Pan and Pan [8, Chapter 7]. ■

LEMMA 2.4. *Suppose $\ell = 0$ or $\ell = 1$. Then for $1/2 \le \sigma \le 1$ we have*

$$\zeta^{(\ell)}(\sigma + it) \ll (|t|+2)^{(1-\sigma)/3} \log^{1+\ell}(|t|+2).$$

Proof. The estimate for $\ell = 0$ follows from the bounds

$$\zeta(1/2 + it) \ll (|t|+2)^{1/6},$$
$$\zeta(1 + it) \ll \log(|t|+2)$$

and the Phragmén–Lindelöf principle. The estimate for $\ell = 1$ follows from the result for $\ell = 0$ and Cauchy's theorem. ■

LEMMA 2.5. *Suppose $V > 10$ is a large parameter and $|u - 1/2| \le 1/\log V$. Then*

$$\int_{-V}^{V} |\zeta(u+iv)|^4 \, dv \ll V \log^4 V, \tag{2.1}$$

$$\int_{-V}^{V} |\zeta(u+iv)|^2 \, dv \ll V \log V, \tag{2.2}$$

$$\int_{-V}^{V} |\zeta'(1/2+iv)|^2 \, dv \ll V \log^3 V, \tag{2.3}$$

$$\int_{-V}^{V} |\zeta(u+iv)|^2 \, dv \ll V \qquad (0.6 < u < 2). \tag{2.4}$$

Proof. The estimates (2.1) and (2.2) can be found in Pan and Pan [8, Chapter 25]. The estimate (2.3) follows from (2.2) and Cauchy's theorem. Actually, (2.4) holds for $u > 1/2 + \varepsilon$: see for example Ivić [6, (8.112)]. ∎

3. Proof of Theorem 1.1. We only prove the theorem for the function $s(m,n)$, i.e., for the sums $S^{(1)}(x)$ and $S^{(2)}(x)$. The proof for $c(m,n)$ is similar.

By Lemmas 2.1 and 2.2 with $r = 2$ and $\sigma_a = 1$ we have

$$\sum_{m\le x}\sum_{n\le x} s(m,n) h\left(\frac{x}{m}\right) h\left(\frac{x}{n}\right) = I(x,T) + O(x^2 E(x,T)), \tag{3.1}$$

where

$$I(x,T) := \frac{1}{(2\pi i)^2} \int_{b-iT}^{b+iT} \int_{b-2iT}^{b+2iT} \frac{\zeta^2(z)\zeta^2(w)\zeta(z+w-1)x^{z+w}}{\zeta(z+w)wz}\, dw\, dz,$$

$$E(x,T) := \sum_{n\ge 1}\sum_{m\ge 1} \frac{s(m,n)(mn)^{-1-1/\log x}}{T|\log(x/n)|+1} + \sum_{n\ge 1}\sum_{m\ge 1} \frac{s(m,n)(mn)^{-1-1/\log x}}{T|\log(x/m)|+1},$$

and T is a parameter to be determined such that $10 \le T \le x/2$.

3.1. Estimate of $E(x,T)$. Since $s(m,n)$ is symmetric, that is, $s(m,n) = s(n,m)$, we have

$$E(x,T) = 2\sum_{n\ge 1}\sum_{m\ge 1} \frac{s(m,n)(mn)^{-1-1/\log x}}{T|\log(x/n)|+1}. \tag{3.2}$$

Write

$$\sum_{n\ge 1}\sum_{m\ge 1} \frac{s(m,n)(mn)^{-1-1/\log x}}{T|\log(x/n)|+1} = E_1 + E_2 + E_3,$$

where

$$E_1 := \sum_{n\le x/2}\sum_{m\ge 1} \frac{s(m,n)(mn)^{-1-1/\log x}}{T|\log(x/n)|+1},$$

$$E_2 := \sum_{x/2<n\le 2x}\sum_{m\ge 1} \frac{s(m,n)(mn)^{-1-1/\log x}}{T|\log(x/n)|+1},$$

$$E_3 := \sum_{n>2x}\sum_{m\ge 1} \frac{s(m,n)(mn)^{-1-1/\log x}}{T|\log(x/n)|+1}.$$

If $n \le x/2$ or $n > 2x$ then $|\log(x/n)| \gg 1$, so by Lemma 2.1 (with $z = w = 1 + 1/\log x$) and Lemma 2.3 we have

$$\begin{aligned}(3.3)\qquad E_1 + E_3 &\ll T^{-1} \sum_{n\ge 1}\sum_{m\ge 1} s(m,n)(mn)^{-1-1/\log x} \\ &= T^{-1}\zeta^4(1+1/\log x)\zeta(1+2/\log x)\zeta^{-1}(2+2/\log x) \\ &\ll T^{-1}\log^5 x.\end{aligned}$$

So it suffices to bound E_2. We have

$$(3.4)\qquad E_2 \ll \sum_{x/2<n\le 2x} \frac{n^{-1}}{T|\log(x/n)|+1} \sum_{m\ge 1} s(m,n) m^{-1-1/\log x}.$$

Recall that $b = 1 + 1/\log x$. From (1.5) we have

$$\begin{aligned}\sum_{m\ge 1} s(m,n)m^{-b} &= \sum_{m\ge 1}\sum_{d|m,\, e|n} \gcd(d,e) m^{-b} = \sum_{m\ge 1} m^{-b} \sum_{dm_1=m,\, en_1=n} \gcd(d,e) \\ &\le \sum_{m\ge 1} m^{-b} \sum_{\varrho d_1 m_1 = m,\, \varrho e_1 n_1 = n} \varrho = \sum_{\varrho e_1 n_1 = n} \varrho \sum_{\varrho d_1 m_1 = m} (\varrho d_1 m_1)^{-b} \\ &= \sum_{\varrho e_1 n_1 = n} \varrho^{1-b} \sum_{d_1 m_1 = m} (d_1 m_1)^{-b} \ll \zeta^2(b)\tau_3(n) \ll \tau_3(n)\log^2 x.\end{aligned}$$

Inserting this estimate into (3.4) and noting that $\tau_3(n) \ll n^{\varepsilon}$ we have

$$(3.5)\qquad E_2 \ll \frac{\log^2 x}{x} \sum_{x/2<n\le 2x} \frac{\tau_3(n)}{T|\log(x/n)|+1} \ll \frac{x^{\varepsilon}}{x}(E_{21}+E_{22}+E_{23}),$$

say, where

$$\begin{aligned} E_{21} &:= \sum_{x/2<n\le xe^{-1/T}} \frac{1}{T|\log(x/n)|+1}, \\ E_{22} &:= \sum_{xe^{-1/T}<n\le xe^{1/T}} \frac{1}{T|\log(x/n)|+1}, \\ E_{23} &:= \sum_{xe^{1/T}<n\le 2x} \frac{1}{T|\log(x/n)|+1}. \end{aligned}$$

For E_{22} we have

$$E_{22} \ll \sum_{xe^{-1/T}<n\le xe^{1/T}} 1 \ll xe^{1/T} - xe^{-1/T} + 1 \ll x/T.$$

For E_{21} we have

$$\begin{aligned} E_{21} &\ll \frac{1}{T} \sum_{x/2<n\le xe^{-1/T}} \frac{1}{|\log(x/n)|} \\ &\ll \frac{1}{T} \sum_{[x]-xe^{-1/T}\le k\le x/2} \frac{1}{|\log(x/([x]-k))|} \qquad (n=[x]-k) \\ &\ll \frac{1}{T} \sum_{x/T\ll k\le x/2} \frac{1}{\log x - \log([x]-k)} \ll \frac{1}{T} \sum_{x/T\ll k\le x/2} \frac{x}{k} \ll \frac{x\log x}{T}. \end{aligned}$$

Similarly, we deduce

$$E_{23} \ll \frac{x\log x}{T}.$$

Inserting the above estimates into (3.5) we conclude that

$$E_2 \ll \frac{x^\varepsilon}{T}. \tag{3.6}$$

From (3.2), (3.3) and (3.6) we get the following proposition.

PROPOSITION 3.1. *If* $10 \le T \le x/2$, *then*

$$E(x,T) \ll x^\varepsilon T^{-1}.$$

3.2. Evaluation of the integral $I(x,T)$ for the variable w. Consider the rectangle domain formed by the four points $w = b\pm 2iT$, $w = 1/2\pm 2iT$. In this domain the integrand

$$g(z,w) := \frac{\zeta^2(z)\zeta^2(w)\zeta(z+w-1)x^{z+w}}{\zeta(z+w)wz} \tag{3.7}$$

has two poles, namely $w=1$, which is a pole of order 2, and $w=2-z$, a simple pole. By the residue theorem we get

$$I(x,T) = J_1(x,T) + J_2(x,T) + H_1(x,T) + H_2(x,T) - H_3(x,T), \tag{3.8}$$

where

$$J_1(x,T) := \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \operatorname{Res}_{w=1} g(z,w)\,dz,$$

$$J_2(x,T) := \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \operatorname{Res}_{w=2-z} g(z,w)\,dz,$$

$$H_1(x,T) := \frac{1}{(2\pi i)^2}\int_{b-iT}^{b+iT} dz \int_{1/2+2iT}^{b+2iT} g(z,w)\,dw,$$

$$H_2(x,T) := \frac{1}{(2\pi i)^2} \int_{b-iT}^{b+iT} dz \int_{1/2-2iT}^{1/2+2iT} g(z,w)\, dw,$$

$$H_3(x,T) := \frac{1}{(2\pi i)^2} \int_{b-iT}^{b+iT} dz \int_{1/2-2iT}^{b-2iT} g(z,w)\, dw.$$

We estimate $H_1(x,T)$ first. In this case by Lemmas 2.3 and 2.4 we have (noting that $|t| \le T$), uniformly for $1/2 \le u \le b = 1 + 1/\log x$,

$$\begin{aligned} g(z,w) &= g(b+it, u+2iT) \\ &\ll \frac{|\zeta(b+it)|^2}{|t|+1} \frac{x^{b+u}}{T} |\zeta^2(u+2iT)|\, |\zeta(u+1/\log x + i(t+2T))| \\ &\ll \frac{\log^2 x}{|t|+1} \times \frac{x^{1+u}}{T} T^{\max(1-u,0)} \log^3 T \\ &\ll \frac{x \log^5 x}{T} \times \frac{1}{|t|+1} \times x^u T^{\max(1-u,0)}. \end{aligned}$$

So we get

$$H_1(x,T) \ll \frac{x\log^5 x}{T} \int_{-T}^{T} \frac{1}{|t|+1}\, dt \Big(\int_{1/2}^{1} x^u T^{1-u}\, du + \int_1^b x^u\, du \Big) \ll \frac{x^2 \log^6 x}{T}. \tag{3.9}$$

Similarly, we have

$$H_3(x,T) \ll \frac{x^2 \log^6 x}{T}. \tag{3.10}$$

Now we estimate $H_2(x,T)$. In this case by Lemmas 2.3 and 2.4 we have, with $|t| \le T$, $|v| \le 2T$,

$$\begin{aligned} g(z,w) &= g(b+it, 1/2+iv) \\ &\ll \frac{|\zeta(b+it)|^2}{|t|+1} \frac{x^{b+1/2}}{|v|+1} |\zeta^2(1/2+iv)|\, |\zeta(1/2+1/\log x + i(t+v))| \\ &\ll x^{3/2} \log^2 x \times \frac{|\zeta(1/2+iv)|^2 |\zeta(1/2+1/\log x + i(t+v))|}{(|t|+1)(|v|+1)}. \end{aligned}$$

Hence

$$\begin{aligned} H_2(x,T) & \\ &\ll x^{3/2}\log^2 x \int_{-T}^{T} dt \int_{-2T}^{2T} \frac{|\zeta(1/2+iv)|^2 |\zeta(1/2+1/\log x + i(t+v))|}{(|t|+1)(|v|+1)}\, dv \\ &\ll x^{3/2} \log^2 x \left(\textstyle\int_1 + \int_2 \right), \end{aligned} \tag{3.11}$$

where $\int_1 = \int_{|v|\le|t|}$ and $\int_2 = \int_{|t| \le |v|}$.

We first estimate $\int_1$. Let $L_1(v) := \int_0^v |\zeta(1/2+iy)|^2\,dy$. By (2.2) and partial summation we get

$$(3.12)\qquad \int_{-V}^{V} \frac{|\zeta(1/2+iv)|^2}{|v|+1}\,dv \ll 1 + \int_1^V \frac{|\zeta(1/2+iv)|^2}{v}\,dv = 1 + \int_1^V \frac{dL_1(v)}{v}$$
$$\ll 1 + \frac{L_1(V)}{V} + \int_1^V \frac{L_1(v)}{v^2}\,dv \ll \log V + \int_1^V \frac{\log v}{v}\,dv \ll \log^2 V.$$

From the estimate (2.2) and Cauchy's inequality we get

$$\int_0^T |\zeta(1/2+1/\log x+iy)|\,dy \ll T\log^{1/2} T,$$

which by partial summation yields

$$(3.13)\qquad \int_{-2T}^{2T} \frac{|\zeta(1/2+1/\log x+iy)|\,dy}{|y|+1} \ll \log^{1.5} T.$$

Note that in $\int_1$ we have

$$|t+v|+1 \le |t|+|v|+1 \le 2(|t|+1).$$

Thus from (3.12) and (3.13) we get

(3.14)

$$\int_1 = \int_{|v|\le|t|} \frac{|\zeta(1/2+iv)|^2|\zeta(1/2+1/\log x+i(t+v))|}{(|v|+1)(|t+v|+1)} \times \frac{|t+v|+1}{|t|+1}\,dv\,dt$$
$$\le 2\int_{-T}^{T} \frac{|\zeta(1/2+iv)|^2}{|v|+1}\,dv \int_{|v|\le|t|} \frac{|\zeta(1/2+1/\log x+i(t+v))|}{|t+v|+1}\,dt$$
$$\le 2\int_{-T}^{T} \frac{|\zeta(1/2+iv)|^2}{|v|+1}\,dv \int_{-2T}^{2T} \frac{|\zeta(1/2+1/\log x+iy)|}{|y|+1}dy$$
$$\ll (\log T)^{3.5} \ll (\log x)^{3.5}.$$

Now we estimate $\int_2$. Similar to (3.12), by (2.1) and (2.2) we get

(3.15)

$$\int_{-V}^{V} \frac{|\zeta(1/2+iv)|^4}{|v|+1}\,dv \ll \log^5 V, \qquad \int_{-V}^{V} \frac{|\zeta(1/2+1/\log x+iv)|^2}{|v|+1}\,dv \ll \log^2 V.$$

Note that in $\int_2$,

$$|t+v|+1 \le |t|+|v|+1 \le 2(|v|+1), \quad \text{so} \quad \frac{1}{|v|+1} \le \frac{2}{|t+v|+1}.$$

Thus via (3.15) we obtain

$$\begin{aligned}
\int_2 &= \int\limits_{|t|\le|v|} \frac{|\zeta(1/2+iv)|^2|\zeta(1/2+1/\log x+i(t+v))|}{(|t|+1)(|v|+1)}\,dv\,dt \\
&\le \int\limits_{-T}^{T} \frac{dt}{|t|+1} \int\limits_{|t|\le|v|\le 2T} \frac{|\zeta(1/2+iv)|^2|\zeta(1/2+1/\log x+i(t+v))|}{|v|+1}\,dv \\
&\le 2\int\limits_{-T}^{T} \frac{dt}{|t|+1} \int\limits_{|t|\le|v|\le 2T} \frac{|\zeta(1/2+iv)|^2}{(|v|+1)^{1/2}} \times \frac{|\zeta(1/2+1/\log x+i(t+v))|}{(|t+v|+1)^{1/2}}\,dv \\
&\le 2\int\limits_{-T}^{T} \frac{dt}{|t|+1} \int\limits_{-2T}^{2T} \frac{|\zeta(1/2+iv)|^2}{(|v|+1)^{1/2}} \times \frac{|\zeta(1/2+1/\log x+i(t+v))|}{(|t+v|+1)^{1/2}}\,dv \\
&\ll \int\limits_{-T}^{T} \frac{dt}{|t|+1} \left(\int\limits_{-2T}^{2T} \frac{|\zeta(1/2+iv)|^4}{|v|+1}\,dv\right)^{1/2} \\
&\quad \times \left(\int\limits_{-2T}^{2T} \frac{|\zeta(1/2+1/\log x+i(t+v))|^2}{|t+v|+1}\,dv\right)^{1/2} \\
&\ll \int\limits_{-T}^{T} \frac{dt}{|t|+1} \left(\int\limits_{-2T}^{2T} \frac{|\zeta(1/2+iv)|^4}{|v|+1}\,dv\right)^{1/2} \left(\int\limits_{-3T}^{3T} \frac{|\zeta(1/2+1/\log x+iy)|^2}{|y|+1}\,dv\right)^{1/2} \\
&\ll (\log T)^{4.5} \ll (\log x)^{4.5},
\end{aligned}$$

which combined with (3.14) and (3.11) gives

$$H_2(x,T) \ll x^{3/2}(\log x)^{6.5}. \tag{3.16}$$

Now we evaluate $J_2(x,T)$. Since $w = 2-z$ is a simple pole of $g(z,w)$, we have

$$\mathrm{Res}_{w=2-z}\, g(z,w) = \frac{x^2}{\zeta(2)} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)}.$$

From (2.4) by partial summation we get

$$\int\limits_{T}^{\infty} \frac{|\zeta(u+iv)|^2\,dv}{v^2} \ll T^{-1}, \qquad 0.6 < u < 2. \tag{3.17}$$

So from (3.17) with $u = 1 - 1/\log x$ and Lemma 2.3 we have

$$\begin{aligned} J_2(x,T) &= \frac{x^2}{\zeta(2)} \times \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)}\,dz \\ &= \frac{x^2}{\zeta(2)} \times \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)}\,dz \\ &\quad + O\biggl(x^2 \int_T^{\infty} \biggl| \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)} \biggr|\,|dz|\biggr) \\ &= \frac{x^2}{\zeta(2)} \times \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)}\,dz \\ &\quad + O\biggl(x^2 \int_T^{\infty} \biggl| \frac{\zeta^2(1+1/\log x+it)\zeta^2(1-1/\log x-it)}{t^2} \biggr|\,dt\biggr) \\ &= \frac{x^2}{\zeta(2)} \times \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)}\,ds + O(x^2T^{-1}\log^2 x). \end{aligned}$$

We shall show that the integral in the last line is a constant. By the residue theorem we have

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)}\,dz = \frac{1}{2\pi i} \int_{4/3-i\infty}^{4/3+i\infty} \frac{\zeta^2(z)\zeta^2(2-z)}{z(2-z)}\,dz.$$

By (3.17) with $u = 2/3$ we see that the integral on the right-hand side is absolutely convergent. Hence

$$J_2(x,T) = Cx^2 + O(x^2T^{-1}\log^2 x), \tag{3.18}$$

where C is an absolute constant.

Finally, we evaluate $J_1(x,T)$. We shall use the following easy fact: if $G(s)$ is analytic at $s = 1$, then

$$\operatorname{Res}_{s=1} \zeta^2(s)G(s) = G'(1) + 2\gamma G(1), \tag{3.19}$$

where γ is the Euler constant.

Define

$$G_z(w) := \frac{\zeta(z+w-1)x^w}{\zeta(z+w)w}.$$

It is easy to see that

$$\begin{aligned} G_z'(w) &= \frac{\zeta'(z+w-1)x^w}{\zeta(z+w)w} + \frac{\zeta(z+w-1)x^w \log x}{\zeta(z+w)w} \\ &\quad - \frac{\zeta(z+w-1)\zeta'(z+w)x^w}{\zeta^2(z+w)w} - \frac{\zeta(z+w-1)x^w}{\zeta(z+w)w^2}. \end{aligned} \tag{3.20}$$

From (3.19) and (3.20) we have

$$\operatorname{Res}_{w=1} g(z,w) = \frac{\zeta^2(z)x^z}{z}(G_z'(1)+2\gamma G_z(1)) = \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)} + \frac{\zeta^3(z)x^{z+1}\log x}{z\zeta(z+1)} + h(z)\frac{\zeta^3(z)x^{z+1}}{z}, \tag{3.21}$$

where

$$h(z) := \frac{2\gamma}{\zeta(z+1)} - \frac{1}{\zeta(z+1)} - \frac{\zeta'(z+1)}{\zeta^2(z+1)}.$$

From (3.21) we have

$$J_1(x,T) = J_{11}(x,T) + J_{12}(x,T) + J_{13}(x,T), \tag{3.22}$$

where

$$J_{11}(x,T) := \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)}\,dz,$$

$$J_{12}(x,T) := \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \frac{\zeta^3(z)x^{z+1}\log x}{z\zeta(z+1)}\,dz,$$

$$J_{13}(x,T) := \frac{1}{2\pi i}\int_{b-iT}^{b+iT} h(z)\frac{\zeta^3(z)x^{z+1}}{z}\,dz.$$

By the residue theorem, we have

$$J_{11}(x,T) = \operatorname{Res}_{z=1} \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)} + L_1(x,T) + L_2(x,T) - L_3(x,T), \tag{3.23}$$

where

$$L_1(x,T) := \frac{1}{2\pi i}\int_{1/2+iT}^{b+iT} \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)}\,dz,$$

$$L_2(x,T) := \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT} \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)}\,dz,$$

$$L_3(x,T) := \frac{1}{2\pi i}\int_{1/2-iT}^{b-iT} \frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)}\,dz.$$

Similar to the estimate for $H_1(x,T)$, we have

$$L_1(x,T) \ll \frac{x^2\log^4 x}{T}, \qquad L_3(x,T) \ll \frac{x^2\log^4 x}{T}. \tag{3.24}$$

For $L_2(x,T)$, by (2.1) and (2.3), Cauchy's inequality and partial summation we obtain

$$(3.25)\quad L_2(x,T) \ll x^{3/2}\int_{-T}^{T}\frac{|\zeta(1/2+it)|^2|\zeta'(1/2+it)|}{|t|+1}\,dt$$
$$\ll x^{3/2}\Big(\int_{-T}^{T}\frac{|\zeta(1/2+it)|^4}{|t|+1}\,dt\Big)^{1/2}\Big(\int_{-T}^{T}\frac{|\zeta'(1/2+it)|^2}{|t|+1}\,dt\Big)^{1/2}$$
$$\ll x^{3/2}\log^{4.5}T \ll x^{3/2}\log^{4.5}x.$$

Since $z=1$ is the pole of $\zeta^2(z)\zeta'(z)$ of degree 4, we have

$$\mathrm{Res}_{z=1}\frac{\zeta^2(z)\zeta'(z)x^{z+1}}{z\zeta(z+1)} = x^2\sum_{j=0}^{3}a_j\log^j x,$$

where a_j are computable constants. So from (3.23)–(3.25) we get

$$(3.26)\quad J_{11}(x,T) = x^2\sum_{j=0}^{3}a_j\log^j x + O\Big(\frac{x^2\log^4 x}{T}+x^{3/2}\log^{4.5}x\Big).$$

Similarly,

$$(3.27)\quad J_{12}(x,T) = x^2\sum_{j=0}^{3}b_j\log^j x + O\Big(\frac{x^2\log^4 x}{T}+x^{3/2}\log^{4.5}x\Big),$$

$$(3.28)\quad J_{13}(x,T) = x^2\sum_{j=0}^{3}c_j\log^j x + O\Big(\frac{x^2\log^4 x}{T}+x^{3/2}\log^{4.5}x\Big),$$

where b_j and c_j are constants such that $b_0=c_3=0$.

From (3.1), (3.8)–(3.10), (3.12), (3.16), (3.18), (3.22), (3.26)–(3.28) and Proposition 3.1 we get

$$(3.29)\quad \sum_{m,n\le x}s(m,n)h\Big(\frac{x}{m}\Big)h\Big(\frac{x}{n}\Big)$$
$$= x^2\sum_{r=0}^{3}A_{1,r}\log^r x + O\Big(\frac{x^{2+\varepsilon}}{T}+x^{3/2}\log^{6.5}x\Big)$$
$$= x^2\sum_{r=0}^{3}A_{1,r}\log^r x + O(x^{3/2}\log^{6.5}x)$$

by choosing $T=x/4$, where

$$A_{1,r}=a_r+b_r+c_r\quad(r=1,2,3),\qquad A_{1,0}=a_0+b_0+c_0+C.$$

3.3. Completion of proof. Suppose $N \geq 10$ is an integer. We shall give an upper bound of the sum

$$\sum_{m\leq N} s(m,N).$$

By (1.5) we have

$$\begin{aligned}\sum_{m\leq N} s(m,N) &= \sum_{m\leq N}\sum_{d|m,\, e|N} \gcd(d,e) = \sum_{m\leq N}\sum_{\varrho d_1 m_1=m,\, \varrho e_1 n_1=N} \varrho \\ &= \sum_{\varrho e_1 n_1=N} \varrho \sum_{\varrho d_1 m_1\leq N} 1 = \sum_{\varrho e_1 n_1=N} \varrho \sum_{e_1 m_1\leq N/\varrho} 1 \\ &\ll \sum_{\varrho e_1 n_1=N} \varrho \times \frac{N}{\varrho}\log\frac{N}{\varrho} \ll N\tau_3(N)\log N \ll N^{1+\varepsilon}.\end{aligned}$$

If $x > 1$ is not an integer, then

$$\sum_{m,n\leq x} s(m,n) = \sum_{m,n\leq x} s(m,n) h\Big(\frac{x}{m}\Big) h\Big(\frac{x}{n}\Big).$$

If $x > 1$ is an integer, then

$$\begin{aligned}\sum_{m,n\leq x} s(m,n) = &\sum_{m,n\leq x} s(m,n) h\Big(\frac{x}{m}\Big) h\Big(\frac{x}{n}\Big) \\ &+ \frac{1}{2}\sum_{n\leq x} s(x,n) + \frac{1}{2}\sum_{m\leq x} s(m,x) - s(x,x)/4.\end{aligned}$$

From the above three formulas we see that for any $x > 1$,

$$\sum_{m,n\leq x} s(m,n) = \sum_{m,n\leq x} s(m,n) h\Big(\frac{x}{m}\Big) h\Big(\frac{x}{n}\Big) + O(x^{1+\varepsilon}),$$

which combined with (3.29) completes the proof of the asymptotic formula for $S^{(1)}(x) = \sum_{m,n\leq x} s(m,n)$.

Furthermore, from (1.5) we have

$$S^{(2)}(x) = S^{(1)}(x) - U(x), \quad \text{where} \quad U(x) := \sum_{\substack{m,n\leq x \\ \gcd(m,n)=1}} \tau(m)\tau(n). \tag{3.30}$$

From [7, Lemma 3.3] we have

$$U(x) = x^2(b_2\log^2 x + b_1\log x + b_0) + O(x^{4/3+\varepsilon}), \tag{3.31}$$

where b_j $(j = 0,1,2)$ are explicit constants.

Now the required asymptotic formula for $S^{(2)}(x)$ follows from (3.30), (3.31) and our result for $S^{(1)}(x)$.

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