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On the degree of regularity of a particular linear equation

by

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To Robert Tijdeman on his 75th birthday

1. Introduction. For given $a_1, \dots, a_k$ and b in the set $\mathbb{Z}$ of integers, we consider the linear Diophantine equation L:

$$\sum_{i=1}^{k} a_i x_i = b.$$

Following [6], given $n \in \mathbb{N}_+$, the set of positive integers, equation L is said to be *n-regular* if, for every n-coloring of $\mathbb{N}_+$, there exists a *monochromatic* solution $x = (x_1, \dots, x_k) \in \mathbb{N}_+^k$ to L.

The *degree of regularity* of L is the largest integer $n \geq 0$, if any, such that L is n-regular. This (possibly infinite) number is denoted by $\mathrm{dor}(L)$. If $\mathrm{dor}(L) = \infty$, then L is said to be *regular*.

A well-known and challenging conjecture (known as *Rado's Boundedness Conjecture*) due to Rado [6] states that there is a function $r\colon \mathbb{N}_+ \to \mathbb{N}_+$ such that, given any $n \in \mathbb{N}_+$ and any equation $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ with integer coefficients, if this equation is not regular over $\mathbb{N}_+$, then it fails to be $r(n)$-regular. Even though there is a more general version, we state it here for a single homogeneous equation, as it has been proved by Rado [6] that if the conjecture is true for a single equation, then it is true for a system of finitely many linear equations, and as Fox and Kleitman [5] have shown, if the conjecture is true for a linear homogeneous equation, then it is true for any linear equation.

The first nontrivial case of the conjecture has been proved by Fox and Kleitman [5] by establishing the bound $r(3) \leq 24$. In the same paper, the

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authors made the following conjecture for a very specific linear Diophantine equation.

Conjecture 1.1. *Let $k \geq 1$. There exists an integer $b_k \geq 1$ such that the degree of regularity of the $2k$-variable equation $L_k(b_k)$ given by*

$$x_1 + \cdots + x_k - y_1 - \cdots - y_k = b_k$$

is exactly $2k - 1$.

Fox and Kleitman [5] had proved the following.

Proposition 1.2. *For any $b \in \mathbb{N}_+$, the equation $L_k(b)$ is not $2k$-regular.*

After some initial results [2], [1] for small values of k, the full conjecture of Fox and Kleitman has very recently been established by Schoen and Taczała [7] by generalizing a theorem of Eberhard et al. [4].

In [3], Bialostocki et al. considered equation L, that is, $\sum_{i=1}^{k} a_i x_i = b$, where $\sum_{i=1}^{k} a_i = 0$ and $b \neq 0$. Among other things, they computed $\operatorname{dor}(x_1 + x_2 - 2y_1 = b)$ under the condition $x_1 < y_1 < x_2$. Here in Section 2, following some arguments in [2], we furnish a somewhat different proof for the result on $\operatorname{dor}(x_1 + x_2 - 2y_1 = b)$; because of Proposition 1.2, the result here is unconditional.

2. The equation $x_1 + x_2 - 2y_1 = b$. As mentioned in the introduction, Bialostocki et al. [3] computed $\operatorname{dor}(x_1 + x_2 - 2y_1 = b)$ under the condition $x_1 < y_1 < x_2$. Here, following the line of arguments in [2], we give a proof of the following.

Theorem 2.1. *Consider the equation $L'(b)$:*

$$x_1 + x_2 - 2y_1 = b.$$

For all positive integers b, we have

$$\operatorname{dor}(L'(b)) = \begin{cases} 1 & \text{if } b \equiv 1 \pmod{2}, \\ 2 & \text{if } b \equiv 2, 4 \pmod{6}, \\ 3 & \text{if } b \equiv 0 \pmod{6}. \end{cases}$$

Proof. Because of Proposition 1.2, $\operatorname{dor}(L'(b)) \leq \operatorname{dor}(L_2(b)) \leq 3$. Again, since $L'(b)$ is solvable in $\mathbb{N}_+$, we have $1 \leq \operatorname{dor}(L'(b))$. Thus,

$$1 \leq \operatorname{dor}(L'(b)) \leq 3.$$

The proof will be complete with the following observations.

Observation 1. Consider the 2-coloring of $\mathbb{N}_+$ given by coloring each integer according to its residue class modulo 2. Let $(\lambda_1, \lambda_2, \lambda_3)$ be a monochromatic solution to $L'(b)$ under this coloring. This will imply

$$\lambda_1 + \lambda_2 - 2\lambda_3 \equiv 0 \pmod{2}.$$

Therefore, if b is odd, there cannot be a monochromatic solution in $\mathbb{N}_+^3$, and hence

$$\mathrm{dor}(L'(b)) = 1$$

in this case.

OBSERVATION 2. Let b be even and write $h = b/2$ with $h \in \mathbb{N}_+$. The following three vectors in $\mathbb{N}_+^3$ are solutions to $L'(b)$:

$$(b+1, 1, 1),$$
$$(h+1, h+1, 1),$$
$$(b+1, b+1, h+1).$$

Since, for any 2-coloring of $\mathbb{N}_+$, at least two elements in the set $\{b+1, h+1, 1\}$ must be of the same color, at least one of the above three solutions must be monochromatic, and hence $\mathrm{dor}(L'(b)) \geq 2$ when b is even.

OBSERVATION 3. If $b \not\equiv 0 \pmod 3$, then coloring each integer according to its residue class modulo 3 gives a coloring of $\mathbb{N}_+$ for which there cannot be any monochromatic solution to $L'(b)$, and hence $\mathrm{dor}(L'(b)) \leq 2$ in this case.

OBSERVATION 4. Here we consider the case $b \equiv 0 \pmod 6$. Since the sum of the coefficients in $L'(b)$ is zero, it is easy to see that if $L'(6)$ is proved to be 3-regular, then so is $L'(b)$.

Let $c\colon \mathbb{N}_+ \to \{0,1,2\}$ be an arbitrary 3-coloring of $\mathbb{N}_+$. Consider the following families of special solutions to $L'(6)$ parametrized by $a \in \mathbb{N}_+$:

$$\begin{aligned} &(a+6, a, a), && (a+3, a+3, a),\\ &(a+5, a+1, a), && (a+8, a, a+1),\\ &(a+4, a+2, a), && (a+1, a+9, a+2). \end{aligned}$$

The underlying sets for each of these solutions can be assumed to be multichromatic, and thus all sets from

$$\begin{aligned} \mathcal{E} = \big\{ &\{a, a+3\}, \{a, a+6\}, \{a, a+2, a+4\}, \{a, a+1, a+5\},\\ &\qquad \{a, a+1, a+8\}, \{a+1, a+9, a+2\} \big\}, \end{aligned}$$

where a ranges through $\mathbb{N}_+$, are multichromatic sets under c.

As just observed, the integer a must be colored distinctly from both $a+3$ and $a+6$. Moreover, if $c(a+6) = c(a+3)$, then we would obtain the monochromatic solution $(a+6, a+6, a+3)$. It follows that

$$\{c(a), c(a+3), c(a+6)\} = \{0,1,2\} = \{c(a+3), c(a+6), c(a+9)\},$$

with the second equality justified by the same argument used for the first, only replacing a by $a+3$. Hence

$$c(a) = c(a+9).$$

Thus the color of an integer only depends on its residue class modulo 9. So, denoting the elements of $\mathbb{Z}/9\mathbb{Z}$ by $0, 1, \ldots, 8$ and their respective colors under c by $c_0, c_1, \ldots, c_8$ (with indices modulo 9), we may depict the distribution of colors by the following table:

Table 1. The color table C

c_0	c_1	c_2
c_3	c_4	c_5
c_6	c_7	c_8

Since the sets $\{a, a+2, a+4\}$, $\{a, a+1, a+5\}$ and $\{a+1, a+2, a+9\}$ belong to $\mathcal{E}$ for all $a \in \mathbb{N}_+$, and are assumed to be multichromatic under c, for all $i \in \mathbb{Z}/9\mathbb{Z}$ we have

$$|\{c_i, c_{i+2}, c_{i+4}\}| \geq 2, \tag{1}$$

$$|\{c_i, c_{i+1}, c_{i+5}\}| \geq 2, \tag{2}$$

$$|\{c_i, c_{i+1}, c_{i+2}\}| \geq 2. \tag{3}$$

We may assume that the first column (c_0, c_3, c_6) of C is equal to $(0, 1, 2)$ and the table is as follows:

Table 2

0	c_1	c_2
1	c_4	c_5
2	c_7	c_8

The second and third columns of C being permutations of its first column, there are nine possible pairs holding the remaining two 0's in C:

$$\begin{array}{c} (c_1, c_2),\ (c_1, c_5),\ (c_1, c_8); \\ (c_4, c_2),\ (c_4, c_5),\ (c_4, c_8); \\ (c_7, c_2),\ (c_7, c_5),\ (c_7, c_8). \end{array} \tag{4}$$

However, recalling that $c_0 = 0$, we have

$$\begin{array}{lll} |\{c_0, c_1, c_2\}| \geq 2 \text{ by (3)}, & |\{c_0, c_1, c_5\}| \geq 2 \text{ by (2)}, & |\{c_8, c_0, c_1\}| \geq 2 \text{ by (3)}; \\ |\{c_0, c_2, c_4\}| \geq 2 \text{ by (1)}, & |\{c_4, c_5, c_0\}| \geq 2 \text{ by (2)}, & |\{c_8, c_0, c_4\}| \geq 2 \text{ by (2)}; \\ |\{c_7, c_0, c_2\}| \geq 2 \text{ by (1)}, & |\{c_5, c_7, c_0\}| \geq 2 \text{ by (1)}, & |\{c_7, c_8, c_0\}| \geq 2 \text{ by (3)}. \end{array}$$

Hence none of the pairs from (4) can equal $(0, 0)$, contradicting the fact that the two remaining 0's in C must lie in one of the pairs from (4). ■

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