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On a kind of character sums and their recurrence properties

by

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1. Introduction. For any integer $q \geq 3$, let χ denote any Dirichlet character modulo q. Then the *generalized Gauss sum* $G(m,k,\chi;q)$ is defined as

$$G(m,k,\chi;q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma^k}{q}\right), \quad \text{where} \quad e(y) = e^{2\pi i y}.$$

If $k = m = 1$, then $G(m,k,\chi;q) = \tau(\chi)$ reduces to the classical Gauss sum; see [1] for its definition and various elementary properties.

Many scholars have studied the arithmetical properties of $G(m,k,\chi;q)$ and obtained a series of interesting conclusions (see [2–9]). For example, according to A. Weil's classical results [5], one can obtain the upper bound

$$|G(m,k,\chi;p)| \leq (k+1)\sqrt{p}.$$

W. Zhang and H. Liu [9] obtained an exact computational formula for

$$\sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a^3}{p}\right) \right|^4, \quad \text{where} \quad p \equiv 1 \bmod 3.$$

In this paper, for any positive integers k and h, we consider the computation of the character sums, with an odd prime p,

$$A_k(h,\chi_1,\ldots,\chi_k;p) = \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{\substack{a_k=1 \\ a_1^h + a_2^h + \cdots + a_k^h \equiv 0 \bmod p}}^{p-1} \chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k), \tag{1}$$

where χ_i $(i = 1,\ldots,k)$ are all the Dirichlet characters modulo p.

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It seems that such character sums have not been studied yet. They are closely related to Jacobi sums. In fact, if $(h, p-1) = 1$, then there exists a unique integer $\overline{h}$ with $1 \le \overline{h} \le p-1$ such that $h\overline{h} \equiv 1 \bmod (p-1)$. Thus, from the properties of the reduced residue system modulo p we have

$$\begin{aligned} A_k(h, \chi_1, \dots, \chi_k; p) &= \sum_{\substack{a_1=1 \\ a_1^h + \cdots + a_k^h \equiv 0 \bmod p}}^{p-1} \cdots \sum_{a_k=1}^{p-1} \chi_1(a_1) \cdots \chi_k(a_k) \\ &= \sum_{\substack{a_1=1 \\ a_1 + \cdots + a_k \equiv 0 \bmod p}}^{p-1} \cdots \sum_{a_k=1}^{p-1} \chi_1^{\overline{h}}(a_1) \cdots \chi_k^{\overline{h}}(a_k). \end{aligned}$$

So, obviously (1) becomes the standard Jacobi sum.

We will use the properties of generalized Gauss sums to derive some results for (1). Under some conditions on p and the characters χ_i $(i = 1, \dots, k)$, we will give an exact computational formula for $A_k(3, \chi_2, \dots, \chi_2; p)$ where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre symbol modulo p. Throughout, we let $A_k(3, \chi_2, \dots, \chi_2; p) \equiv A_k(p)$ for convenience. Our main purpose is to prove that $A_k(p)$ satisfies an interesting third order linear recurrence formula.

We will use the analytic method and the properties of the classical Gauss sums to prove the following main results.

Theorem 1. *Let p be an odd prime with $3 \nmid (p-1)$. Then for any positive integer k, we have*

$$A_k(p) = \begin{cases} 0 & \text{if } k = 2h+1, \\ (-1)^{h(p-1)/2} \cdot (p-1) \cdot p^{h-1} & \text{if } k = 2h. \end{cases}$$

Theorem 2. *Let p be a prime with $p \equiv 1 \bmod 6$, and ψ be any third order character modulo p. Then*

$$A_2(p) = 3\tau^2(\chi_2) \cdot \frac{p-1}{p},$$

$$A_4(p) = 19p(p-1) + 4\tau(\chi_2) \cdot \frac{p-1}{p} \cdot \left(\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right).$$

For any integer $h \ge 3$, we have the recurrence formula

$$\begin{aligned} A_{2h}(p) &= 9\tau^2(\chi_2)A_{2h-2}(p) + \left[6\tau(\chi_2)\left(\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right) - 12p^2\right] \cdot A_{2h-4}(p) \\ &+ \left[\tau^6(\chi_2\psi) + \tau^6\left(\overline{\chi_2\psi}\right) - 4\tau^3(\chi_2)\left(\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right) + 6\tau^2(\chi_2)p^2\right] \cdot A_{2h-6}(p), \end{aligned}$$

where $A_0(p) = (p-1)/p$.

Theorem 3. *Let p be a prime with $p \equiv 1 \bmod 6$. If 2 is a cubic residue modulo p, then*

$$A_2(p) = (-1)^{(p-1)/2} \cdot 3(p-1), \qquad A_4(p) = 27(p-1)(p-4b^2).$$

For any integer $h \geq 3$, we have the recurrence formula

$$A_{2h}(p) = (-1)^{(p-1)/2} 9p \cdot A_{2h-2}(p) - 162pb^2 \cdot A_{2h-4}(p) + (-1)^{(p-1)/2} 729pb^4 \cdot A_{2h-6}(p),$$

where b is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

From Theorem 3 we can also deduce the following three corollaries:

Corollary 1. *Let p be a prime with $p \equiv 1 \bmod 6$. If 2 is a cubic residue modulo p, then*

$$\sum_{\substack{a=1 \\ a^3+b^3+c^3 \equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)\left(\frac{c}{p}\right) = (-1)^{(p-1)/2} \cdot 27 \cdot (p - 4b^2).$$

Corollary 2. *Let p be a prime with $p \equiv 1 \bmod 6$. If 2 is a cubic residue modulo p, then*

$$\sum_{\substack{a=1 \\ a^3+b^3+c^3+d^3+e^3 \equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{e=1}^{p-1} \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)\left(\frac{c}{p}\right)\left(\frac{d}{p}\right)\left(\frac{e}{p}\right) = 243 \cdot (p^2 - 6pb^2 + 3b^4).$$

Corollary 3. *Let p be a prime with $p \equiv 1 \bmod 6$. If 2 is a cubic residue modulo p, then for any positive integer $h \geq 4$,*

$$A_h(p) \equiv 0 \bmod 3^h p.$$

Some notes. Firstly, in Theorem 2 we only consider the case of $A_{2h}(p)$, because for any integer $h \geq 0$, it is easy to prove that $A_{2h+1}(p) = 0$.

Secondly, if 2 is not a cubic residue modulo p, then the computations are more complicated. In this case we cannot give an exact computational formula for $\left(\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right)$. According to Theorem 2 it is easy to prove that $\tau(\chi_2)\left(\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right)$ is an integer divisible by p. To calculate its exact value is an interesting open problem.

As applications of our results, we can derive a formula for the number $N(h,k,p)$ of solutions of the congruence equation

$$x_1^h + \cdots + x_k^h \equiv 0 \bmod p,$$

where all x_i $(i = 1, \ldots, k)$ are quadratic residues (or quadratic non-residues) modulo p. For example, if p is a prime with $p \equiv 1 \bmod 12$ and $h = 3$, then

$$N(h,k,p) = N(3,2,p) = \frac{1}{4} \sum_{\substack{a=1 \\ a^3+b^3 \equiv 0 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \left(1 + \left(\frac{a}{p}\right)\right)\left(1 + \left(\frac{b}{p}\right)\right) = \frac{3}{2}(p-1).$$

If p is an odd prime with $3 \nmid (p-1)$ and $h = 1$, then

$$N(1,3,p) = \frac{1}{8} \sum_{\substack{a=1 \\ a+b+c\equiv 0 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \left(1 + \left(\frac{a}{p}\right)\right)\left(1 + \left(\frac{b}{p}\right)\right)\left(1 + \left(\frac{c}{p}\right)\right)$$
$$= \frac{1}{8}(p-1)\big(p - 2 - 3(-1)^{(p-1)/2}\big).$$

2. Some lemmas. In this section, we will deduce several simple lemmas, which are necessary in the proofs of our main results. We make use of elementary number theory and the properties of the classical Gauss sums and Dirichlet characters, which can be found in [1], [3] and [8].

LEMMA 1. *Let p be a prime with $p \equiv 1 \bmod 3$, and $C = \tau(\chi_2)$. For any integer m with $(m,p) = 1$, write $U(m,p) = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) e\left(\frac{ma^3}{p}\right)$. Then*

$$U^6(m,p) = 9C^2U^4(m,p) + \left[6C\big(\tau^3(\chi_2\psi) + \tau^3(\overline{\chi_2\psi})\big) - 12p^2\right] \cdot U^2(m,p)$$
$$+ 6C^2p^2 + \tau^6(\chi_2\psi) + \tau^6(\overline{\chi_2\psi}) - 4C^3\left[\tau^3(\chi_2\psi) + \tau^3(\overline{\chi_2\psi})\right],$$

where $\left(\frac{*}{p}\right) = \chi_2$ denotes the Legendre symbol modulo p.

Proof. From the properties of third order characters ψ modulo p and Gauss sums we have

$$(2) \quad U(m,p) = \sum_{a=1}^{p-1} \chi_2(a^3)e\Big(\frac{ma^3}{p}\Big) = \sum_{a=1}^{p-1} \chi_2(a)\big(1 + \psi(a) + \overline{\psi}(a)\big)e\left(\frac{ma}{p}\right)$$
$$= \chi_2(m)\tau(\chi_2) + \chi_2(m)\overline{\psi}(m)\tau(\chi_2\psi) + \chi_2(m)\psi(m)\tau(\overline{\chi_2\psi}).$$

Noting that $\psi^2 = \overline{\psi}$, $\chi_2^2 = \chi_0$, $C^2 = \tau^2(\chi_2) = \chi_2(-1)p$ and $\tau(\chi_2\psi)\tau(\overline{\chi_2\psi}) = \chi_2(-1)p = C^2$, from (2) we have

$$(3) \quad U^2(m,p) = \big(\chi_2(m)\tau(\chi_2) + \chi_2(m)\overline{\psi}(m)\tau(\chi_2\psi) + \chi_2(m)\psi(m)\tau(\overline{\chi_2\psi})\big)^2$$
$$= \tau^2(\chi_2) + 2\tau(\chi_2)\big(\overline{\psi}(m)\tau(\chi_2\psi) + \psi(m)\tau(\overline{\chi_2\psi})\big)$$
$$+ \psi(m)\tau^2(\chi_2\psi) + 2\chi_2(-1)p + \overline{\psi}(m)\tau^2(\overline{\chi_2\psi})$$
$$= 3C^2 + 2C\big(\overline{\psi}(m)\tau(\chi_2\psi) + \psi(m)\tau(\overline{\chi_2\psi})\big) + \psi(m)\tau^2(\chi_2\psi) + \overline{\psi}(m)\tau^2(\overline{\chi_2\psi})$$
$$= 3C^2 + \overline{\psi}(m)\big(2C\tau(\chi_2\psi) + \tau^2(\overline{\chi_2\psi})\big) + \psi(m)\big(2C\tau(\overline{\chi_2\psi}) + \tau^2(\chi_2\psi)\big).$$

By the identity $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, from (3) we have

$$U^6(m,p) - 9C^2U^4(m,p) + 27C^4U^2(m,p) - 27C^6 = (U^2(m,p) - 3C^2)^3$$
$$= \left[\overline{\psi}(m)\big(2C\tau(\chi_2\psi) + \tau^2(\overline{\chi_2\psi})\big) + \psi(m)\big(2C\tau(\overline{\chi_2\psi}) + \tau^2(\chi_2\psi)\big)\right]^3$$
$$= \big(2C\tau(\chi_2\psi) + \tau^2(\overline{\chi_2\psi})\big)^3 + \big(2C\tau(\overline{\chi_2\psi}) + \tau^2(\chi_2\psi)\big)^3$$
$$+ 3\big(2C\tau(\chi_2\psi) + \tau^2(\overline{\chi_2\psi})\big)\big(2C\tau(\overline{\chi_2\psi}) + \tau^2(\chi_2\psi)\big)(U^2(m,p) - 3C^2)$$

$$= 14C^3\left[\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right] + \tau^6(\chi_2\psi) + \tau^6\left(\overline{\chi_2\psi}\right) + 24C^2p^2 + 3\left[5p^2 + 2C\left(\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right)\right] \cdot (U^2(n,p) - 3C^2),$$

so

$$U^6(m,p) = 9C^2U^4(m,p) + \left[6C\left(\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right) - 12p^2\right] \cdot U^2(m,p) + 6C^2p^2 + \tau^6(\chi_2\psi) + \tau^6\left(\overline{\chi_2\psi}\right) - 4C^3\left[\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right].$$

This proves Lemma 1.

LEMMA 2. *Let p be a prime with $p \equiv 1 \bmod 3$. Then*

$$\sum_{m=1}^{p-1} U^2(m,p) = 3\left(\frac{-1}{p}\right) \cdot p(p-1),$$

$$\sum_{m=1}^{p-1} U^4(m,p) = 19p^2(p-1) + 4C(p-1)\left(\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right).$$

Proof. From the orthogonality of characters modulo p and (3) we have

$$\sum_{m=1}^{p-1} U^2(m,p) = \sum_{m=1}^{p-1} 3C^2 + \sum_{m=1}^{p-1} \overline{\psi}(m)\left(2C\tau(\chi_2\psi) + \tau^2\left(\overline{\chi_2\psi}\right)\right) + \sum_{m=1}^{p-1} \psi(m)\left(2C\tau\left(\overline{\chi_2\psi}\right) + \tau^2(\chi_2\psi)\right)$$

$$= 3C^2(p-1) = 3\left(\frac{-1}{p}\right)p(p-1).$$

Similarly, noting that $C^4 = p^2$, from (3) we also have

$$\sum_{m=1}^{p-1} U^4(m,p)$$

$$= \sum_{m=1}^{p-1} 9p^2 + \sum_{m=1}^{p-1} 2\left(2C\tau(\chi_2\psi) + \tau^2\left(\overline{\chi_2\psi}\right)\right) \cdot \left(2C\tau\left(\overline{\chi_2\psi}\right) + \tau^2(\chi_2\psi)\right)$$

$$= 9p^2(p-1) + 4C(p-1)\left(\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right) + 10p^2(p-1)$$

$$= 19p^2(p-1) + 4C(p-1)\left(\tau^3(\chi_2\psi) + \tau^3\left(\overline{\chi_2\psi}\right)\right).$$

This proves Lemma 2.

LEMMA 3. *Let p be a prime with $p \equiv 1 \bmod 3$, and ψ be any third order character modulo p. Then*

$$\tau^3(\psi) + \tau^3(\overline{\psi}) = dp,$$

where d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

Proof. See [3] or [7, Lemma 3].

LEMMA 4. *Let p be a prime with $p \equiv 1 \bmod 6$, and ψ be any third order character modulo p. If 2 is a cubic residue modulo p, then*

$$\tau^3(\chi_2\psi) + \tau^3\big(\overline{\chi_2\psi}\big) = \frac{\tau^3(\chi_2)}{p} \cdot (d^2 - 2p),$$

where d is as in Lemma 3.

Proof. The prime p can be uniquely expressed as $p = x^2 + 3y^2$ and $x \equiv -1 \bmod 3$ (see [2, Theorem 3.3]). If 2 is a cubic residue modulo p, then from the result of Gauss (see [3, p. 120]) we know that $3 \mid y$. Let

$$G(k) = \sum_{a=0}^{p-1} e\left(\frac{a^k}{p}\right).$$

Then from the important work of B. C. Berndt and R. J. Evans [3] we have

$$G(6) = G(3) + \frac{G(2)}{p} \cdot (G^2(3) - p). \tag{4}$$

On the other hand, from the properties of Gauss sums,

$$\begin{aligned} G(6) &= 1 + \sum_{a=1}^{p-1}\left(1 + \left(\frac{a}{p}\right)\right) e\left(\frac{a^3}{p}\right) = G(3) + \sum_{a=1}^{p-1}\left(\frac{a}{p}\right) e\left(\frac{a^3}{p}\right) \\ &= G(3) + \sum_{a=1}^{p-1} \chi_2(a)\big(1 + \psi(a) + \overline{\psi}(a)\big) e\left(\frac{a}{p}\right) \\ &= G(3) + \tau(\chi_2) + \tau(\chi_2\psi) + \tau\big(\overline{\chi_2\psi}\big), \end{aligned} \tag{5}$$

$$G(3) = 1 + \sum_{a=1}^{p-1}(1 + \psi(a) + \overline{\psi}(a)) e\left(\frac{a}{p}\right) = \tau(\psi) + \tau(\overline{\psi}). \tag{6}$$

Noting that $G(2) = \tau(\chi_2)$ and $\tau(\psi)\tau(\overline{\psi}) = p$, from (4)–(6) we immediately deduce that

$$\begin{aligned} \tau(\chi_2\psi) + \tau\big(\overline{\chi_2\psi}\big) &= \frac{\tau(\chi_2)}{p} \cdot G^2(3) - 2\tau(\chi_2) \\ &= \frac{\tau(\chi_2)}{p} \cdot (\tau(\psi) + \tau(\overline{\psi}))^2 - 2\tau(\chi_2) = \frac{\tau(\chi_2)}{p} \cdot (\tau^2(\psi) + \tau^2(\overline{\psi})). \end{aligned} \tag{7}$$

Since $\tau(\chi_2\psi)\tau\big(\overline{\chi_2\psi}\big) = \tau^2(\chi_2)$, from (7) and Lemma 3 we get

$$\begin{aligned} &\tau^3(\chi_2\psi) + \tau^3\big(\overline{\chi_2\psi}\big) + 3\tau(\chi_2\psi)\tau\big(\overline{\chi_2\psi}\big) \cdot \big(\tau(\chi_2\psi) + \tau\big(\overline{\chi_2\psi}\big)\big) \\ &= \tau^3(\chi_2\psi) + \tau^3\big(\overline{\chi_2\psi}\big) + 3\tau^2(\chi_2) \cdot \big(\tau(\chi_2\psi) + \tau\big(\overline{\chi_2\psi}\big)\big) \\ &= \frac{\tau^3(\chi_2)}{p^3} \cdot \big[\tau^6(\psi) + 3\tau^2(\psi)\tau^2(\overline{\psi})(\tau^2(\psi) + \tau^2(\overline{\psi})) + \tau^6(\overline{\psi})\big] \end{aligned} \tag{8}$$

$$= \frac{\tau^3(\chi_2)}{p^3} \cdot \left[\tau^6(\psi) + 3p^2(\tau^2(\psi) + \tau^2(\overline{\psi})) + \tau^6(\overline{\psi})\right]$$
$$= \frac{\tau^3(\chi_2)}{p^3} \cdot \left[(\tau^3(\psi) + \tau^3(\overline{\psi}))^2 - 2p^3 + 3p^2(\tau^2(\psi) + \tau^2(\overline{\psi}))\right]$$
$$= \frac{\tau^3(\chi_2)}{p^3} \cdot \left[d^2p^2 - 2p^3 + 3p^2(\tau^2(\psi) + \tau^2(\overline{\psi}))\right].$$

Combining (7) and (8) we deduce that

$$\begin{aligned}\tau^3(\chi_2\psi) + \tau^3\big(\overline{\chi_2\psi}\big) &= \frac{\tau^3(\chi_2)}{p} \cdot (d^2 - 2p) + \frac{3\tau^3(\chi_2)}{p} \cdot (\tau^2(\psi) + \tau^2(\overline{\psi})) \\ &\quad - 3\tau^2(\chi_2) \cdot \big(\tau(\chi_2\psi) + \tau\big(\overline{\chi_2\psi}\big)\big) \\ &= \frac{\tau^3(\chi_2)}{p} \cdot (d^2 - 2p).\end{aligned}$$

This proves Lemma 4.

3. Proofs of the theorems. We start with the proof of Theorem 1. Providing that $3 \nmid (p-1)$, we have $(3, p-1) = 1$. As a runs through a reduced residue system modulo p, a^3 runs through a reduced residue system modulo p as well. So for any integer m with $(m,p) = 1$, from the properties of Gauss sums we have

$$(9) \qquad \begin{aligned} U(m,p) &= \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) e\left(\frac{ma^3}{p}\right) = \sum_{a=1}^{p-1} \left(\frac{a^3}{p}\right) e\left(\frac{ma^3}{p}\right) \\ &= \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) e\left(\frac{ma}{p}\right) = \left(\frac{m}{p}\right)\tau(\chi_2). \end{aligned}$$

Combining (9) and the properties of trigonometric sums we get

$$\begin{aligned} A_k(p) &= \sum_{\substack{a_1=1 \\ a_1^3+\cdots+a_k^3 \equiv 0 \bmod p}}^{p-1} \cdots \sum_{a_k=1}^{p-1} \left(\frac{a_1 \cdots a_k}{p}\right) = \frac{1}{p} \sum_{m=0}^{p-1} \left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) e\left(\frac{ma^3}{p}\right)\right)^k \\ &= \frac{1}{p} \sum_{m=1}^{p-1} U^k(m,p) = \frac{1}{p} \tau^k(\chi_2) \sum_{m=1}^{p-1} \left(\frac{m}{p}\right)^k \\ &= \begin{cases} 0 & \text{if } k = 2h+1, \\ (-1)^{h(p-1)/2} \cdot (p-1) \cdot p^{h-1} & \text{if } k = 2h. \end{cases} \end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. By the method of proving Theorem 1 we have

$$(10)\qquad A_k(p) = \frac{1}{p}\sum_{m=0}^{p-1}\left(\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^3}{p}\right)\right)^k = \frac{1}{p}\sum_{m=1}^{p-1}U^k(m,p).$$

From (10) and Lemma 2 we get

$$(11)\qquad A_2(p) = \frac{1}{p}\sum_{m=1}^{p-1}U^2(m,p) = 3\left(\frac{-1}{p}\right)\cdot(p-1),$$

$$(12)\qquad A_4(p) = \frac{1}{p}\sum_{m=1}^{p-1}U^4(m,p)$$
$$= 19p(p-1) + \frac{4(p-1)\tau(\chi_2)}{p}\cdot\left(\tau^3(\chi_2\psi)+\tau^3\left(\overline{\chi_2\psi}\right)\right).$$

If $h \ge 3$, then from (10) and Lemma 1 we obtain

$$(13)\qquad A_{2h}(p) = \frac{1}{p}\sum_{m=1}^{p-1}U^{2h}(m,p) = \frac{1}{p}\sum_{m=1}^{p-1}U^{2h-6}(m,p)\cdot U^6(m,p)$$
$$= \frac{9C^2}{p}\sum_{m=1}^{p-1}U^{2h-2}(m,p)$$
$$+ \left[6C\left(\tau^3(\chi_2\psi)+\tau^3\left(\overline{\chi_2\psi}\right)\right) - 12p^2\right]\frac{1}{p}\sum_{m=1}^{p-1}U^{2h-4}(m,p)$$
$$+ \left[6C^2p^2 + \tau^6(\chi_2\psi)+\tau^6\left(\overline{\chi_2\psi}\right) - 4C^3\left(\tau^3(\chi_2\psi)+\tau^3\left(\overline{\chi_2\psi}\right)\right)\right]$$
$$\times\frac{1}{p}\sum_{m=1}^{p-1}U^{2h-6}(m,p)$$
$$= 9\cdot\left(\frac{-1}{p}\right)p\cdot A_{2h-2}(p) + \left[6\tau(\chi_2)\left(\tau^3(\chi_2\psi)+\tau^3\left(\overline{\chi_2\psi}\right)\right) - 12p^2\right]\cdot A_{2h-4}(p)$$
$$+ \left[6C^2p^2 + \tau^6(\chi_2\psi)+\tau^6\left(\overline{\chi_2\psi}\right) - 4C^3\left(\tau^3(\chi_2\psi)+\tau^3\left(\overline{\chi_2\psi}\right)\right)\right]\cdot A_{2h-6}(p),$$

where $A_0(p) = (p-1)/p$.

Now Theorem 2 follows from (11)–(13).

Noting that $\tau^2(\chi_2) = \left(\frac{-1}{p}\right)p = (-1)^{(p-1)/2}p$, $\tau(\chi_2\psi)\tau\left(\overline{\chi_2\psi}\right) = \tau^2(\chi_2)$ and

$$\tau^6(\chi_2\psi)+\tau^6\left(\overline{\chi_2\psi}\right) = \left(\tau^3(\chi_2\psi)+\tau^3\left(\overline{\chi_2\psi}\right)\right)^2 - 2p^2\tau^2(\chi_2),$$

from Theorem 2 and Lemma 4 we immediately deduce the formulas

$$A_2(p) = (-1)^{(p-1)/2}\cdot 3(p-1),\qquad A_4(p) = 27(p-1)(p-4b^2),$$

and for any integer $h \ge 3$ we have the recurrence formula

$$A_{2h}(p) = (-1)^{(p-1)/2} 9p \cdot A_{2h-2}(p) - 162pb^2 \cdot A_{2h-4}(p) + (-1)^{(p-1)/2} 729pb^4 \cdot A_{2h-6}(p).$$

This proves Theorem 3.

Since $p \equiv 1 \bmod 6$ and $A_{2h+1} = 0$, Corollary 3 follows from Theorem 3 by induction.

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