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Distances from points to planes

by

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1. Introduction. The *Erdős–Falconer distance problem in* $\mathbb{F}_q^d$ is to determine how large $E \subset \mathbb{F}_q^d$ needs to be to ensure that the set

$$\Delta(E) = \{\|x-y\| : x, y \in E\},$$

with $\|x\| = x_1^2 + x_2^2 + \cdots + x_d^2$, is the whole field $\mathbb{F}_q$, or at least a positive proportion thereof. Here and throughout, $\mathbb{F}_q$ denotes the field with q elements and $\mathbb{F}_q^d$ is the d-dimensional vector space over this field.

The distance problem in vector spaces over finite fields was introduced by Bourgain, Katz and Tao [1]. In the form described above, it was introduced by the second listed author of the present paper and Misha Rudnev [4], who proved that $\Delta(E) = \mathbb{F}_q$ if $|E| > 2q^{(d+1)/2}$. It was shown in [3] that this exponent is essentially sharp for general fields when d is odd. When $d = 2$, it was proved in [2] that if $E \subset \mathbb{F}_q^2$ with $|E| \geq cq^{4/3}$, then $|\Delta(E)| \geq C(c)q$. We do not know if the exponent $(d+1)/2$ is best possible when $d \geq 4$ is even.

More generally, let $\mathrm{Graff}(k, d)$ denote the set of k-dimensional affine planes in $\mathbb{F}_q^d$. In this paper we shall focus on distances from points in subsets of $\mathrm{Graff}(0, d) = \mathbb{F}_q^d$ to $(d-1)$-dimensional planes in subsets of $\mathrm{Graff}(d-1, d)$. The set of distances from points to points (see e.g. [4]) can be defined as the set of equivalence classes of two-point configurations where two pairs (x, y) and (x', y') are equivalent if there exists a translation $\tau \in \mathbb{F}_q^d$ and a rotation $\theta \in O_d(\mathbb{F}_q^d)$ that takes one pair to the other. In the case of points and $(d-1)$-dimensional planes in $\mathbb{F}_q^d$, we may similarly define (x, h) and (x', h') to be equivalent, where x's are points and h's are planes, if after translating x to x', there exists a rotation $\theta \in O_d(\mathbb{F}_q)$ that takes h to h'. Denote the resulting set of equivalence classes by $\Delta(E, F)$.

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Before stating our main results, we need to say a few words about the parameterization of $(d-1)$-dimensional planes in $\mathbb{F}_q^d$. Every $(d-1)$-dimensional plane in $\mathbb{F}_q^d$ can be expressed in the form

$$H_{v,t} = \{y \in \mathbb{F}_q^d : y \cdot v = t\},$$

where we should think of v as a normal vector to the plane, and t as the distance to the origin. Note that the notion of distance from a point to a plane described above only makes sense if $\|v\| \neq 0$. We shall henceforth refer to planes with this property as *non-degenerate* planes. See Lemma 2.1 below.

DEFINITION 1.1. We say that $V \subset \mathbb{F}_q^d$ is a *direction set* if given $x \in \mathbb{F}_q^d$, $x \neq \vec{0}$, there exist $v \in V$ and $t \in \mathbb{F}_q^*$ such that $x = tv$.

It is very convenient to work with a "canonical" direction set provided by the following simple observation.

LEMMA 1.2. *Let $S_t = \{x \in \mathbb{F}_q^d : \|x\| = t\}$. Let $\gamma \in \mathbb{F}_q^*$ be a non-square. Define $V_\gamma = S_0 \cup S_1 \cup S_\gamma$. Then V_γ is a direction set.*

Proof. Choose x such that $\|x\| = 0$. Then $x \in S_0$. Now choose x such that $\|x\| = t^2$ for some $t \neq 0$. Then $(x_1/t)^2 + (x_2/t)^2 = 1$, so $x = tv$ with $v \in S_1$. Finally, suppose that $\|x\| = u$ where u is not a square in $\mathbb{F}_q^*$. To see that $x = tv$ for some $v \in S_\gamma$, it is enough to check that $u\gamma^{-1}$ is a square in $\mathbb{F}_q$. Moreover, it is enough to prove that a product of two non-squares is a square. To see this, let $\phi : \mathbb{F}_q^* \to \mathbb{F}_q^*$ be given by $\phi(x) = ux$, where u is a non-square. The image of a square is certainly a non-square since otherwise u would be forced to be a square. It follows that an image of a non-square is a square since exactly half the elements of $\mathbb{F}_q^*$ are squares. ∎

Our main result is the following.

THEOREM 1.3. *Let $E \subset \mathbb{F}_q^d$, $d \geq 2$, and F be a subset of non-degenerate planes in* $\mathrm{Graff}(d-1, d)$. *Let γ be a non-square in $\mathbb{F}_q$. Suppose that $|E|\,|F| > q^{d+1}$. Then $|\Delta(E,F)| \gg q$. More precisely,*

$$|\Delta(E,F)| \geq \frac{|E|^2|F|^2}{2|E|^2|F|^2q^{-1} + 2q^{d-1}|E|\,|F| \cdot \max_{\|v\|=1,\gamma} \sum_t F(v,t)}.$$

When $d = 2$, a better exponent was obtained by Pham, Phuong, Sang, Valculescu and Vinh [5]. They proved that the conclusion of Theorem 1.3 holds in $\mathbb{F}_q^2$ if $|E|\,|F| > Cq^{8/3}$.

It is not clear if it is possible to weaken the $|E|\,|F| > q^{d+1}$ assumption in higher dimensions. It is not difficult to see that we cannot do better than assuming $|E|\,|F| > q^d$. To see this, take $q = p^2$ with p prime, let

$E = \mathbb{F}_p^d$ and F be the set of all $(d-1)$-dimensional affine planes in $\mathbb{F}_p^d$. Then $|E| \approx |F| \approx q^{d/2}$ while $\Delta(E,F) = p$.

2. Proof of Theorem 1.3. We begin with simple algebraic observations that make working with $\Delta(E,F)$ much easier. Parameterize each plane in $\mathrm{Graff}(d-1,d)$ by $(v,t) \in V_\gamma \times \mathbb{F}_q$, where V_γ is as in Lemma 1.2. Given $F \subset \mathrm{Graff}(d-1,d)$, we write the indicator function of F in the form $F(v,t)$. For a point $x \in E$ and a plane $F(v,t) \in F$, the distance function between them, denoted by $d[x, F(v,t)]$, is defined by

$$d[x, F(v,t)] := \frac{(x \cdot v - t)^2}{\|v\|}.$$

In the following lemma, we show that the size of $\Delta(E,F)$ is at least the number of distinct non-zero distances between points in E and planes in F.

LEMMA 2.1. *Let $F \subset \mathrm{Graff}(d-1,d)$ be parameterized as above, with coordinates $(v,t) \in V_\gamma \times \mathbb{F}_q$, where $\|v\| \neq 0$. Then*

$$|\Delta(E,F)| \geq \#\left\{\frac{(x \cdot v - t)^2}{\|v\|} \neq 0 : x \in E,\ (v,t) \in F\right\}.$$

Proof. It is enough to show that for $x, x' \in E$ and $(v,t), (v',t') \in F$, if $d[x, F(v,t)] = d[x', F(v',t')]$, then there is a rotation θ such that the translation from x to x' followed by θ takes the plane $F(v,t)$ to $F(v',t')$. Indeed, since $d[x, F(v,t)] = d[x', F(v',t')]$, we have

$$\frac{(x \cdot v - t)^2}{\|v\|} = \frac{(x' \cdot v' - t')^2}{\|v'\|}. \tag{2.1}$$

This implies that $\|v\|/\|v'\|$ is a square. From this we deduce, just as in the proof of Lemma 1.2 above, that either both $\|v\|$ and $\|v'\|$ are squares or both are non-squares. Since we are only considering $\|v\|$ and $\|v'\|$ that are equal to 1 or γ, we conclude that $\|v\| = \|v'\|$. From (2.1), we have $x \cdot v - t = \pm(x' \cdot v' - t')$. Without loss of generality, we assume that $x' = 0$. Since $\|v\| = \|v'\| \neq 0$, there exists a rotation $\theta \in O_d(\mathbb{F}_q)$ such that $\theta v = \pm v'$. Thus

$$\begin{aligned}\{\theta(y-x) : y \cdot v = t\} &= \{z : (\theta^{-1}z + x) \cdot v = t\} = \{z : z \cdot \theta v = t - x \cdot v\} \\ &= \{z : \pm z \cdot v' = \pm t'\} = \{z : z \cdot v' = t'\}.\end{aligned}$$

In other words, the translation from x to x' followed by the rotation θ about x' takes the plane $F(v,t)$ to $F(v',t')$. ∎

Before proving Theorem 1.3, we need to review the Fourier transform of functions on $\mathbb{F}_q^d$. Let χ be a non-trivial additive character on $\mathbb{F}_q$. For a function $f : \mathbb{F}_q \to \mathbb{C}$, we define

$$\widehat{f}(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d} \chi(-x \cdot m) f(x).$$

It is clear that

$$f(x) = \sum_{m \in \mathbb{F}_q^d} \chi(x \cdot m) \widehat{f}(m) \quad \text{and} \quad \sum_{m \in \mathbb{F}_q^d} |\widehat{f}(m)|^2 = q^{-d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2.$$

Proof of Theorem 1.3. In view of Lemma 2.1 it suffices to prove that

$$\#\left\{ \frac{(x \cdot v - t)^2}{\|v\|} : x \in E,\ (v,t) \in F \right\} \\ \geq \frac{|E|^2|F|^2}{2q^{-1}|F|^2|E|^2 + 2q^{d-1} \max_{v \in V_\gamma} F(v,t) \cdot |E|\,|F|}.$$

For $r \in \mathbb{F}_q$, let

$$\nu(r) := \sum_{(x \cdot v - t)^2 = r\|v\|} E(x) F(v,t).$$

By the Cauchy–Schwarz inequality,

$$|E|^2|F|^2 = \Big(\sum_r \nu(r) \Big)^2 \leq \sum_{r \in \mathbb{F}_q} \nu(r)^2 \cdot \#\left\{ \frac{(x \cdot v - t)^2}{\|v\|} : x \in E,\ (v,t) \in F \right\}.$$

This implies that

$$\#\left\{ \frac{(x \cdot v - t)^2}{\|v\|} : x \in E,\ (v,t) \in F \right\} \geq \frac{|E|^2|F|^2}{\sum_{r \in \mathbb{F}_q} \nu(r)^2}.$$

We are now going to show that

$$\sum_{r \in \mathbb{F}_q} \nu(r)^2 \leq 2q^{-1}|F|^2|E|^2 + 2q^{d-1}|F|\,|E| \cdot \max_{v \in V} \sum_t F(v,t).$$

Indeed, applying the Cauchy–Schwarz inequality again gives us

$$\begin{aligned} \sum_{r \in \mathbb{F}_q} \nu(r)^2 &\leq |F| \sum_{\substack{x, x', v, t \\ d[x, F(v,t)] = d[x', F(v,t]}} E(x) E(x') F(v,t) \\ &= |F| \Big(\sum_{x \cdot v - x' \cdot v = 0} F(v,t) E(x) E(x') + \sum_{x \cdot v + x' \cdot v - 2t = 0} F(v,t) E(x) E(x') \Big) \\ &= |F| (\mathrm{I} + \mathrm{II}). \end{aligned}$$

We now bound I and II as follows:

$$\begin{aligned}
(2.2)\qquad \mathrm{I} &= \sum_{x\cdot v - x'\cdot v = 0} F(v,t)E(x)E(x') \\
&= q^{-1}|F|\,|E|^2 + q^{-1}\sum_{s\neq 0}\sum_{v,t,x,x'} \chi(sv\cdot(x-x'))F(v,t)E(x)E(x') \\
&= q^{-1}|F|\,|E|^2 + q^{-1}\sum_{s\neq 0}\sum_{v,t,x,x'} \chi(sv\cdot(x-x'))F(v,t)E(x)E(x') \\
&= q^{-1}|F|\,|E|^2 + q^{3}\sum_{s\neq 0}\sum_{v,t} |\widehat{E}(sv)|^2 F(v,t) \\
&\leq q^{-1}|F|\,|E|^2 + q^{2d-1}\cdot \max_{v\in V}\sum_{t} F(v,t)\cdot \sum_{z\in\mathbb{F}_q^d} |\widehat{E}(z)|^2 \\
&= q^{-1}|F|\,|E|^2 + q^{d-1}|E|\cdot \max_{v\in V}\sum_{t} F(v,t),
\end{aligned}$$

where we have used $\sum_{z\in\mathbb{F}_q^d} |\widehat{E}(z)|^2 = q^{-d}|E|$, and

$$\begin{aligned}
(2.3)\qquad \mathrm{II} &= \sum_{x\cdot v - x'\cdot v = 2t} F(v,t)E(x)E(x') \\
&= q^{-1}|F|\,|E|^2 \\
&\quad + q^{-1}\sum_{s\neq 0}\sum_{v,t,x,x'} \chi(sv\cdot(x+x'))\chi(2st)F(v,t)E(x)E(x') \\
&= q^{-1}|F|\,|E|^2 + q^{3}\sum_{s\neq 0}\sum_{v,t} \widehat{E}(sv)\widehat{E}(sv)\chi(st+st)F(v,t) \\
&\leq q^{-1}|F|\,|E|^2 + q^{2d-1}\sum_{s\neq 0}\sum_{v,t} |\widehat{E}(sv)|^2 F(v,t) \\
&\leq q^{-1}|F|\,|E|^2 + q^{2d-1}\cdot \max_{v\in V}\sum_{t} F(v,t)\cdot \sum_{z\in\mathbb{F}_q^d} |\widehat{E}(z)|^2 \\
&= q^{-1}|F|\,|E|^2 + q^{d-1}|E|\cdot \max_{v\in V}\sum_{t} F(v,t).
\end{aligned}$$

Putting (2.2) and (2.3) together, we obtain

$$\sum_{r\in\mathbb{F}_q} \nu(r)^2 \leq 2q^{-1}|F|^2|E|^2 + 2q^{d-1}|F|\,|E|\cdot \max_{v\in V}\sum_{t} F(v,t).$$

We conclude that

$$\#\left\{\frac{(x\cdot v - t)^2}{\|v\|} : x\in E,\ (v,t)\in F\right\} \geq \frac{|E|^2|F|^2}{2q^{-1}|F|^2|E|^2 + 2q^{d-1}\max_{v\in V_\gamma} F(v,t)\cdot |E|\,|F|}.$$

Hence,

$$|\Delta(E,F)| \geq \frac{|E|^2|F|^2}{2q^{-1}|F|^2|E|^2 + 2q^{d-1}\max_{v\in V_\gamma} F(v,t)\cdot |E|\,|F|}.$$

This concludes the proof once we note that

$$\max_{v\in V_\gamma} F(v,t) \leq q. \blacksquare$$

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