ACTA ARITHMETICA
187.2 (2019)

Finding exact formulas for the L_2 discrepancy of digital $(0, n, 2)$-nets via Haar functions

by

RALPH KRITZINGER (Linz)

1. Introduction and main results. In this paper, we study the L_p discrepancy of special digital $(0, n, 2)$-nets with the main focus on precise computation of L_2 discrepancy.

Discrepancy theory treats the irregularities of point distributions, often in the d-dimensional unit cube $[0, 1)^d$ (see e.g. [13]). We study point sets $\mathcal{P}$ with N elements in the unit square $[0, 1)^2$. We define the *discrepancy function* of such a point set by

$$\Delta(\boldsymbol{t}, \mathcal{P}) = \frac{1}{N} \sum_{\boldsymbol{z} \in \mathcal{P}} \mathbf{1}_{[\mathbf{0}, \boldsymbol{t})}(\boldsymbol{z}) - t_1 t_2,$$

where for $\boldsymbol{t} = (t_1, t_2) \in [0, 1]^2$ we set $[\mathbf{0}, \boldsymbol{t}) = [0, t_1) \times [0, t_2)$ with volume $t_1 t_2$ and denote by $\mathbf{1}_{[\mathbf{0}, \boldsymbol{t})}$ the indicator function of this interval. The L_p *discrepancy* of $\mathcal{P}$ for $p \in [1, \infty)$ is given by

$$L_p(\mathcal{P}) := \|\Delta(\cdot, \mathcal{P})\|_{L_p([0,1)^2)} = \Big(\int_{[0,1]^2} |\Delta(\boldsymbol{t}, \mathcal{P})|^p \, \mathrm{d}\boldsymbol{t} \Big)^{1/p},$$

and the *star discrepancy* of $\mathcal{P}$ is defined as

$$L_\infty(\mathcal{P}) := \|\Delta(\cdot, \mathcal{P})\|_{L_\infty([0,1)^2)} = \sup_{\boldsymbol{t} \in [0,1]^2} |\Delta(\boldsymbol{t}, \mathcal{P})|.$$

Throughout this paper, for functions $f, g : \mathbb{N} \to \mathbb{R}^+$, we write $g(N) \lesssim f(N)$ and $g(N) \gtrsim f(N)$ if there exists a $C > 0$ such that $g(N) \le C f(N)$ or $g(N) \ge C f(N)$ for all $N \in \mathbb{N}$, $N \ge 2$, respectively. This constant C is independent of N, but might depend on several other parameters $\alpha_1, \ldots, \alpha_i$, which we sometimes emphasize by writing $\lesssim_{\alpha_1,\ldots,\alpha_i}$ and $\gtrsim_{\alpha_1,\ldots,\alpha_i}$, respectively. Further, we write $f(N) \asymp g(N)$ if $g(N) \lesssim f(N)$ and $g(N) \gtrsim f(N)$.

2010 *Mathematics Subject Classification*: Primary 11K06, 11K38; Secondary 42C10.
Key words and phrases: L_2 discrepancy, digital nets, Haar functions.
Received 16 November 2017; revised 23 February 2018.
Published online 10 December 2018.

DOI: 10.4064/aa171116-26-3

It is well known that for every $p \in [1,\infty)$ the L_p discrepancy of any point set $\mathcal{P}$ consisting of N points in $[0,1)^2$ satisfies

$$L_p(\mathcal{P}) \gtrsim_p \frac{\sqrt{\log N}}{N}, \tag{1}$$

where log denotes the natural logarithm. This was first shown by Roth [18] for $p = 2$ and hence for all $p \in [2,\infty)$, and later by Schmidt [20] for all $p \in (1,2)$. The case $p = 1$ was added by Halász [6]. For the star discrepancy of such a $\mathcal{P}$ we have the best possible lower bound

$$L_\infty(\mathcal{P}) \gtrsim \frac{\log N}{N}, \tag{2}$$

which is due to Schmidt [19].

An important class of point sets with low star discrepancy is formed by the digital nets (see e.g. [17, 4]). A *digital net* in base 2 is a point set $\{\boldsymbol{x}_0,\dots,\boldsymbol{x}_{2^n-1}\}$ in the d-dimensional unit interval $[0,1)^d$, which is generated by d matrices of size $n\times n$. Hence we need two matrices to generate a digital net in the unit square. The procedure is as follows. Let $n \geq 1$ be an integer.

- Choose a bijection $\varphi : \{0,1\} \to \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the field with two elements.
- Choose $n\times n$ matrices C_1 and C_2 over $\mathbb{Z}_2$.
- For $r \in \{0,1,\dots,2^n-1\}$ let $r = r_0 + 2r_1 + \dots + 2^{n-1}r_{n-1}$ with $r_i \in \{0,1\}$ for all $i \in \{0,\dots,n-1\}$ be the dyadic expansion of r. Map r to the vector $\vec{r} = (\varphi(r_0),\dots,\varphi(r_{n-1}))^\top$.
- Compute $C_j\vec{r} =: (y_{r,1}^{(j)},\dots,y_{r,n}^{(j)})^\top$ for $j = 1,2$.
- Compute $x_r^{(j)} = \varphi^{-1}(y_{r,1}^{(j)})/2 + \dots + \varphi^{-1}(y_{r,n}^{(j)})/2^n$ for $j = 1,2$.
- Set $\boldsymbol{x}_r = (x_r^{(1)}, x_r^{(2)})$.
- Repeat steps 3 to 6 for all $r \in \{0,1,\dots,2^n-1\}$ and set $\mathcal{P} := \{\boldsymbol{x}_1,\dots,\boldsymbol{x}_{2^n-1}\}$. We call $\mathcal{P}$ the *digital net generated by C_1 and C_2*.

A point set $\mathcal{P}$ in the unit square is called a $(0,n,2)$-*net* in base 2 if every dyadic box

$$\left[\frac{m_1}{2^{j_1}}, \frac{m_1+1}{2^{j_1}}\right) \times \left[\frac{m_2}{2^{j_2}}, \frac{m_2+1}{2^{j_2}}\right),$$

where $j_1, j_2 \in \mathbb{N}_0$ and $m_1 \in \{0,1,\dots,2^{j_1}-1\}$ and $m_2 \in \{0,1,\dots,2^{j_2}-1\}$ with volume 2^{-n}, i.e. with $j_1 + j_2 = n$, contains exactly one element of $\mathcal{P}$. It is well known that a digital net is a $(0,n,2)$-net if and only if the following condition holds: for any $d_1, d_2 \in \mathbb{N}_0$ with $d_1 + d_2 = n$ the first d_1 rows of C_1 and the first d_2 rows of C_2 are linearly independent. By Niederreiter [17], the star discrepancy of any $(0,n,2)$-net in base 2 is of best possible order $(\log N)/N$. In particular, by [15] we have the general upper bound

$$2^n L_\infty(\mathcal{P}) \leq n/3 + 19/3$$

for every digital $(0,n,2)$-net.

The situation is less clear for the L_2 discrepancy of digital $(0,n,2)$-nets. Classical nets like the Hammersley point set (see Example 1.3) fail to achieve the optimal order $\sqrt{\log N}/N$ of L_2 discrepancy. To reduce the L_2 discrepancy of digital nets, digital shifts have been applied to such nets in many previous papers [7, 4, 11]. A digital shift $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)^\top$ is an element of $\mathbb{Z}_2^n$. We obtain a shifted digital net by altering the fourth step in the construction scheme of digital nets above to $C_2\vec{r} + \boldsymbol{\sigma} =: (y_{r,1}^{(2)}, \dots, y_{r,n}^{(2)})$; hence after multiplication of the matrix C_2 by the vector $\vec{r}$ we also add the digital shift, before transforming the vector back to a number in $[0,1)$. Note that by [10, Lemma 2.2], without loss of generality we can apply the shift only to the second component.

We consider the following $n \times n$ matrices over $\mathbb{Z}_2$:

(3)

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & a_1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & a_2 \\ 0 & 0 & 1 & \cdots & 0 & 0 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & a_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 1 & a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

We study the discrepancy of the digital net $\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ with $\boldsymbol{a} = (a_1, \dots, a_{n-1})^T$, generated by A_1 and A_2 and shifted by $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)^T$. We simply write $\mathcal{P}_{\boldsymbol{a}}$ if we do not apply a shift. The set $\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ can be written as

$$\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}) = \left\{ \left(\frac{t_n}{2} + \dots + \frac{t_1}{2^n}, \frac{b_1}{2} + \dots + \frac{b_n}{2^n} \right) : t_1, \dots, t_n \in \{0,1\} \right\},$$

where $b_k = t_k \oplus a_k t_n \oplus \sigma_n$ for $k \in \{1, \dots, n-1\}$ and $b_n = t_n \oplus \sigma_n$. Here $\oplus$ denotes addition modulo 2.

We also consider symmetrized versions of shifted digital nets. It is convenient to define $\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma}) = \mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}) \cup \mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}^*)$, where $\boldsymbol{\sigma}^* = (\sigma_1 \oplus 1, \dots, \sigma_n \oplus 1)^T$. Note that $\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ can also be written in the form

$$\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma}) = \mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}) \cup \{(x, 1 - 2^{-n} - y) : (x,y) \in \mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})\},$$

which justifies the term "symmetrized digital net". Symmetrization can often reduce L_2 discrepancy to the best possible order (1) (see e.g. [3, 14, 2]). We will discuss this phenomenon in more detail in Section 5.

Theorem 1.1 gives an exact formula for the L_2 discrepancy of the class $\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ of shifted digital $(0,n,2)$-nets.

Theorem 1.1. *Let* $L = \sum_{i=1}^{n-1} a_i(1-2\sigma_i)$ *and* $\ell = \sum_{i=1}^{n}(1-2\sigma_i)$. *Then*

$$(2^n L_2(\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})))^2 = \frac{1}{64}\left((\ell-L)^2 + L^2 + 8\ell - 10L + \frac{5}{3}n\right) + \frac{1}{2^{n+4}}(2\sigma_n L - \ell + 4) + \frac{3}{8} - \frac{1}{9}\frac{1}{2^{2n+3}}.$$

Hence $L_2(\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})) \lesssim \sqrt{\log N}/N$ *if and only if* $|\ell - L| \lesssim \sqrt{n}$ *and* $|L| \lesssim \sqrt{n}$.

Remark 1.2. For a fixed $\boldsymbol{a} \in \mathbb{Z}_2^{n-1}$, how can we construct a shift $\boldsymbol{\sigma} \in \mathbb{Z}_2^n$ which satisfies $|\ell - L| \lesssim \sqrt{n}$ and $|L| \lesssim \sqrt{n}$? Put $I_0 := \{i \in \{1,\dots,n-1\} : a_i = 0\}$ and $I_1 := \{i \in \{1,\dots,n-1\} : a_i = 1\}$, and further $\ell_0 := |\{i \in I_0 : \sigma_i = 0\}|$ and $\bar{\ell}_0 := |\{i \in I_1 : \sigma_i = 0\}|$. Choose $\boldsymbol{\sigma}$ such that $\big||I_0| - 2\ell_0\big| \lesssim \sqrt{n}$ and $\big||I_1| - 2\bar{\ell}_0\big| \lesssim \sqrt{n}$; hence the number of zeros and ones in the components of the shifts whose indices belong to I_0 or I_1 respectively has to be balanced.

Example 1.3. We study a special instance of our nets, namely $\mathcal{P}_{\mathbf{0}}(\boldsymbol{\sigma})$, where $\mathbf{0} = (0,\dots,0) \in \mathbb{Z}_2^{n-1}$. This is the (digit shifted) *Hammersley point set* in base 2 (also known as the *van der Corput set* or *Roth net*). For $\boldsymbol{a} = \mathbf{0}$ we have $L = 0$ and $\ell = \sum_{i=1}^{n}(1-2\sigma_i) = \sum_{i=1}^{n}(2(1-\sigma_i)-1) = 2z - n$, where z denotes the number of zero digits in the digital shift $\boldsymbol{\sigma}$. We insert these values into Theorem 1.1 to find

$$(L_2(\mathcal{P}_{\mathbf{0}}(\boldsymbol{\sigma})))^2 = \frac{n^2}{64} + \frac{z^2}{16} - \frac{zn}{16} - \frac{19n}{192} + \frac{z}{4} + \frac{n}{2^{n+4}} - \frac{z}{2^{n+3}} + \frac{1}{2^{n+2}} + \frac{3}{8} - \frac{1}{9\cdot 2^{2n+2}}.$$

This formula was already obtained by Kritzer and Pillichshammer [11, Theorem 1] in 2006. Their proof is different from ours, since they used an explicit formula for the discrepancy function of the digit shifted Hammersley point set, found by Larcher and Pillichshammer [15, Example 2] in 2001 by an approach via Walsh functions. Like Haar functions, which will be the central tool used in this paper, the Walsh functions are also an orthonormal basis of $L_2([0,1)^2)$ and are useful in studying the L_2 discrepancy of digital nets. For more details on Walsh functions we refer to [5, Appendix A].

As an immediate corollary of Theorem 1.1 we compute the L_2 discrepancy of unshifted nets. Surprisingly, the L_2 discrepancy only depends on the number of zeros and ones in $\boldsymbol{a}$, but not on their positions. The result follows from Theorem 1.1 by setting $\sigma_i = 0$ for all $i = 1,\dots,n$, which yields $L = \sum_{i=1}^{n-1} a_i$ and $\ell = n$.

Corollary 1.4. *Let* $|\boldsymbol{a}| = \sum_{i=1}^{n-1} a_i$. *Then*

$$(2^n L_2(\mathcal{P}_{\boldsymbol{a}}))^2 = \frac{1}{64}\left((n-|\boldsymbol{a}|)^2 + |\boldsymbol{a}|^2 - 10|\boldsymbol{a}| + \frac{29}{3}n\right) + \frac{3}{8} - \frac{n-4}{2^{n+4}} - \frac{1}{9}\frac{1}{2^{2n+3}}.$$

Hence $L_2(\mathcal{P}_{\boldsymbol{a}}) \gtrsim (\log N)/N$ *for all* $\boldsymbol{a} \in \mathbb{Z}_2^{n-1}$.

Now we fix $\boldsymbol{a}$ and ask how large the L_2 discrepancy of the shifted nets $\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ is on average. In other words, we compute the mean of $(2^n L_2(\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})))^2$ over all possible shifts $\boldsymbol{\sigma} \in \mathbb{Z}_2^n$.

COROLLARY 1.5. *Let $\boldsymbol{a} \in \mathbb{Z}_2^{n-1}$ be fixed. Then*

$$\frac{1}{2^n} \sum_{\boldsymbol{\sigma} \in \mathbb{Z}_2^n} (2^n L_2(\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})))^2 = \frac{n}{24} + \frac{3}{8} + \frac{1}{2^{n+2}} - \frac{1}{9 \cdot 2^{2n+3}}.$$

Hence the mean of the squared L_2 discrepancy of $\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ over all possible shifts $\boldsymbol{\sigma} \in \mathbb{Z}_2^n$ is the same for all $\boldsymbol{a} \in \mathbb{Z}_2^{n-1}$ and of best possible order according to (1).

Proof. It is not difficult to verify $2^{-n} \sum_{\boldsymbol{\sigma} \in \mathbb{Z}_2^n} (\ell - L)^2 = n - |\boldsymbol{a}|$ and $2^{-n} \sum_{\boldsymbol{\sigma} \in \mathbb{Z}_2^n} L^2 = |\boldsymbol{a}|$ as well as $2^{-n} \sum_{\boldsymbol{\sigma} \in \mathbb{Z}_2^n} \ell = 2^{-n} \sum_{\boldsymbol{\sigma} \in \mathbb{Z}_2^n} L = 0$, which yields the result. ∎

REMARK 1.6. Dick and Pillichshammer [4] studied the problem of the mean squared L_2 discrepancy of digital nets. They did not only apply a shift $\boldsymbol{\sigma} \in \mathbb{Z}_2^n$ to the first n digits of the coordinates as in this paper, but also added random numbers from $[0, 2^{-n})$ to each component of all elements of the digital net after the shifting process. Then they computed the mean over all shifts and obtained the same result for every digital $(0,n,2)$-net. They also studied the problem in higher dimensions. With the methods used in [4] one can show that Corollary 1.5 actually holds for all digital $(0,n,2)$-nets.

We will prove the following exact result concerning the L_2 discrepancy of the symmetrized nets $\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma})$. This formula demonstrates that the L_2 discrepancy depends on $\boldsymbol{a}$ and on $\boldsymbol{\sigma}$, but only to a minor extent.

THEOREM 1.7. *Let $\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ have 2^{n+1} elements. Then*

$$(2^{n+1} L_2(\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma})))^2 = \frac{n}{24} + \frac{11}{8} + \frac{1}{2^n} - \frac{1}{9 \cdot 2^{2n+1}} - \frac{(-1)^{\sigma_n}}{2^{n+2}} L.$$

Hence the point sets $\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ achieve the optimal order of L_2 discrepancy without any conditions on $\boldsymbol{a}$ and $\boldsymbol{\sigma}$.

REMARK 1.8. Again, the L_2 discrepancy of unshifted symmetrized nets depends only on the parameter $|\boldsymbol{a}|$, since

$$(2^{n+1} L_2(\widetilde{\mathcal{P}}_{\boldsymbol{a}}))^2 = \frac{n}{24} + \frac{11}{8} + \frac{1}{2^n} - \frac{1}{9 \cdot 2^{2n+1}} - \frac{1}{2^{n+2}} |\boldsymbol{a}|.$$

For the symmetrized shifted Hammersley point set $\widetilde{\mathcal{P}}_{\boldsymbol{0}}(\boldsymbol{\sigma})$ we obtain

$$(2^{n+1} L_2(\widetilde{\mathcal{P}}_{\boldsymbol{0}}(\boldsymbol{\sigma})))^2 = \frac{n}{24} + \frac{11}{8} + \frac{1}{2^n} - \frac{1}{9 \cdot 2^{2n+1}},$$

and so the L_2 discrepancy is independent of the shift $\boldsymbol{\sigma}$. This result has previously been obtained by the author [12] with the methods used in [15] and [11]. Further, we immediately obtain for every $\boldsymbol{a} \in \mathbb{Z}_2^{n-1}$ the average result

$$\frac{1}{2^n} \sum_{\boldsymbol{\sigma} \in \mathbb{Z}_2^n} (2^n L_2(\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma})))^2 = \frac{n}{24} + \frac{11}{8} + \frac{1}{2^n} - \frac{1}{9 \cdot 2^{2n+1}}.$$

Note that the fact that the nets $\widetilde{\mathcal{P}}_{\boldsymbol{a}}$ achieve the optimal order of L_2 discrepancy independently of $\boldsymbol{a}$ follows already from [12, Theorem 2].

Remark 1.9. Since the proofs of Theorems 1.1 and 1.7 as presented in Sections 3 and 4 are very technical and prone to mistakes, we tested the correctness of our formulas using Warnock's formula [21]. It states that for a point set $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ in the unit square with $\boldsymbol{x}_k = (x_{k,1}, x_{k,2})$ for $k = 0, \dots, N-1$ we have

$$(NL_{2,N}(\mathcal{P}))^2 = \frac{N^2}{9} - \frac{N}{2} \sum_{k=0}^{N-1} \prod_{i=1}^{2} (1 - x_{k,i}^2) + \sum_{k,l=0}^{N-1} \prod_{i=1}^{2} (1 - \max\{x_{k,i}, x_{l,i}\}).$$

This formula allows us to compute the L_2 discrepancy of $\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ exactly, provided that the number of points $N = 2^n$ is small (e.g. $n = 10$). Then we can compare the results of Warnock's formula with the output of our formulas for different choices of n, $\boldsymbol{a}$ and $\boldsymbol{\sigma}$. Note that Warnock's formula requires $\mathcal{O}(N \log N)$ operations to compute the L_2 discrepancy of a given point set (see e.g. [16, Section 2.4, Exercises 11, 12]), whereas our formulas allow a very fast computation of this quantity for $\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ and $\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma})$.

We close this introduction by pointing out three papers which heavily influenced the current paper. The first one is [11] by Kritzer and Pillichshammer, who obtained the exact result for the L_2 discrepancy of the shifted Hammersley point set and discovered the beautiful fact that it only depends on the number of zeroes in the shift $\boldsymbol{\sigma}$ but not on their position. It is a natural question whether this result can also be obtained with reasonable effort by using Haar functions, as Hinrichs [8] computed the Haar coefficients of the corresponding discrepancy function exactly in almost all cases. However, the aim of his paper was to estimate the Besov norm of the discrepancy function, and therefore in certain cases he was content with upper bounds rather than exact formulas. We apply the notation of [8] and use some of its results and ideas. The third paper which inspired this work is by Bilyk, Temlyakov and Yu [2], who computed the Fourier coefficients of the discrepancy function of the symmetrized Fibonacci lattice exactly in order to find an exact formula for its L_2 discrepancy. We do the same for a class of digital $(0, n, 2)$-net with the difference that we compute the Haar coefficients instead of the Fourier coefficients, since Haar functions fit the structure of digital nets much better than harmonic functions.

The outline of this paper is as follows. In Section 2 we introduce the Haar function system and present general formulas for the Haar coefficients of the discrepancy function of arbitrary point sets in the unit square. Section 3 is the longest and most technical one; there we compute all the Haar coefficients of $\Delta(\cdot, \mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}))$ exactly and insert them into Parseval's identity in order to prove Theorem 1.1. In Section 4 we do the same for the discrepancy function of symmetrized nets, but we omit all the technical details. In Section 5 we comment on the results for Haar coefficients in the previous sections. In particular, we point out which Haar coefficients cause a large L_2 discrepancy of (symmetrized) digital nets. We disprove a conjecture by Bilyk and give a new proof of a result by Larcher and Pillichshammer on symmetrized nets. In Section 6 we consider a different class of digital nets, for the L_2 discrepancy of which we can also find an exact formula with the same method as in Section 3. We therefore omit technicalities again. In Section 7 we discuss the L_p discrepancy of digital nets with the aid of a Littlewood–Paley inequality, and in the final Section 8 we mention several problems for future research.

2. The Haar expansion of the discrepancy function. A *dyadic interval* of length 2^{-j}, $j \in \mathbb{N}_0$, in $[0, 1)$ is an interval of the form

$$I = I_{j,m} := \left[\frac{m}{2^j}, \frac{m+1}{2^j}\right) \quad \text{for } m = 0, 1, \dots, 2^j - 1.$$

The left and right halves of $I_{j,m}$ are the dyadic intervals $I_{j+1,2m}$ and $I_{j+1,2m+1}$, respectively. The *Haar function* $h_{j,m}$ is the function on $[0, 1)$ which is $+1$ on the left half of $I_{j,m}$, -1 on the right half, and 0 outside of $I_{j,m}$. The L_∞-normalized Haar system consists of all Haar functions $h_{j,m}$ with $j \in \mathbb{N}_0$ and $m = 0, 1, \dots, 2^j - 1$ together with the indicator function $h_{-1,0}$ of $[0, 1)$. After normalization in $L_2([0, 1))$ we obtain the orthonormal Haar basis of $L_2([0, 1))$.

Let $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$ and define $\mathbb{D}_j = \{0, 1, \dots, 2^j - 1\}$ for $j \in \mathbb{N}_0$ and $\mathbb{D}_{-1} = \{0\}$. For $\boldsymbol{j} = (j_1, j_2) \in \mathbb{N}_{-1}^2$ and $\boldsymbol{m} = (m_1, m_2) \in \mathbb{D}_{\boldsymbol{j}} := \mathbb{D}_{j_1} \times \mathbb{D}_{j_2}$, the Haar function $h_{\boldsymbol{j},\boldsymbol{m}}$ is given as the tensor product

$$h_{\boldsymbol{j},\boldsymbol{m}}(\boldsymbol{t}) = h_{j_1,m_1}(t_1) h_{j_2,m_2}(t_2) \quad \text{for } \boldsymbol{t} = (t_1, t_2) \in [0, 1)^2.$$

We call $I_{\boldsymbol{j},\boldsymbol{m}} = I_{j_1,m_1} \times I_{j_2,m_2}$ *dyadic boxes* with level $|\boldsymbol{j}| = \max\{0, j_1\} + \max\{0, j_2\}$, where we set $I_{-1,0} = \mathbf{1}_{[0,1)}$. The system

$$\{2^{|\boldsymbol{j}|/2} h_{\boldsymbol{j},\boldsymbol{m}} : \boldsymbol{j} \in \mathbb{N}_{-1}^2,\, \boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}\}$$

is an orthonormal basis of $L_2([0, 1)^2)$, and Parseval's identity states that for every function $f \in L_2([0, 1)^2)$ we have

$$\|f\|^2_{L_2([0,1)^2)} = \sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^2} 2^{|\boldsymbol{j}|} \sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2, \tag{4}$$

where the numbers

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = \mu_{\boldsymbol{j},\boldsymbol{m}}(f) = \langle f, h_{\boldsymbol{j},\boldsymbol{m}} \rangle = \int\limits_{[0,1)^2} f(\boldsymbol{t}) h_{\boldsymbol{j},\boldsymbol{m}}(\boldsymbol{t}) \, \mathrm{d}\boldsymbol{t}$$

are the *Haar coefficients* of f.

Let $\mathcal{P}$ be an arbitrary 2^n-element point set in the unit square. The Haar coefficients of its discrepancy function $\Delta(\cdot, \mathcal{P})$ are as follows (see [8]). By $\boldsymbol{z} \in I_{\boldsymbol{j},\boldsymbol{m}}$ we actually mean $\boldsymbol{z} = (z_1, z_2) \in I_{\boldsymbol{j},\boldsymbol{m}} \cap \mathcal{P}$.

- If $\boldsymbol{j} = (-1, -1)$, then

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{-n} \sum_{\boldsymbol{z} \in \mathcal{P}} (1 - z_1)(1 - z_2) - \frac{1}{4}. \tag{5}$$

- If $\boldsymbol{j} = (j_1, -1)$ with $j_1 \in \mathbb{N}_0$, then

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = -2^{-n-j_1-1} \sum_{\boldsymbol{z} \in I_{\boldsymbol{j},\boldsymbol{m}}} (1 - |2m_1 + 1 - 2^{j_1+1} z_1|)(1 - z_2) + 2^{-2j_1-3}. \tag{6}$$

- If $\boldsymbol{j} = (-1, j_2)$ with $j_2 \in \mathbb{N}_0$, then

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = -2^{-n-j_2-1} \sum_{\boldsymbol{z} \in I_{\boldsymbol{j},\boldsymbol{m}}} (1 - |2m_2 + 1 - 2^{j_2+1} z_2|)(1 - z_1) + 2^{-2j_2-3}. \tag{7}$$

- If $\boldsymbol{j} = (j_1, j_2)$ with $j_1, j_2 \in \mathbb{N}_0$, then

$$\begin{aligned} \mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{-n-j_1-j_2-2} \sum_{\boldsymbol{z} \in I_{\boldsymbol{j},\boldsymbol{m}}} &(1 - |2m_1 + 1 - 2^{j_1+1} z_1|)(1 - |2m_2 + 1 - 2^{j_2+1} z_2|) \\ &- 2^{-2j_1-2j_2-4}. \end{aligned} \tag{8}$$

Note that we could also write $\boldsymbol{z} \in \mathring{I}_{\boldsymbol{j},\boldsymbol{m}}$, where $\mathring{I}_{\boldsymbol{j},\boldsymbol{m}}$ denotes the interior of $I_{\boldsymbol{j},\boldsymbol{m}}$, since the summands in (6)–(8) vanish if $\boldsymbol{z}$ lies on the boundary of the dyadic box. Hence, in order to compute the Haar coefficients of the discrepancy function, we have to deal with the sums over $\boldsymbol{z}$ above and to determine which points $\boldsymbol{z} = (z_1, z_2) \in \mathcal{P}$ lie in $I_{\boldsymbol{j},\boldsymbol{m}}$ with $\boldsymbol{j} \in \mathbb{N}_{-1}^2$ and $\boldsymbol{m} = (m_1, m_2) \in \mathbb{D}_{\boldsymbol{j}}$. If m_1 and m_2 are nonnegative integers, then they have dyadic expansions

$$m_1 = 2^{j_1-1} r_1 + \cdots + r_{j_1} \quad \text{and} \quad m_2 = 2^{j_2-1} s_1 + \cdots + s_{j_2} \tag{9}$$

with $r_{i_1}, s_{i_2} \in \{0, 1\}$ for all $i_1 \in \{1, \ldots, j_1\}$ and $i_2 \in \{1, \ldots, j_2\}$. Let

$$\boldsymbol{z} = (z_1, z_2) = \left(\frac{t_n}{2} + \cdots + \frac{t_1}{2^n}, \frac{b_1}{2} + \cdots + \frac{b_n}{2^n} \right) \in \mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}).$$

Then $\boldsymbol{z} \in I_{\boldsymbol{j},\boldsymbol{m}}$ if and only if

$$t_{n+1-k} = r_k \text{ for all } k \in \{1, \ldots, j_1\} \text{ and } b_k = s_k \text{ for all } k \in \{1, \ldots, j_2\}. \tag{10}$$

Further, for such a point $\boldsymbol{z}=(z_1,z_2)\in I_{\boldsymbol{j},\boldsymbol{m}}$ we have

$$2m_1+1-2^{j_1+1}z_1=1-t_{n-j_1}-2^{-1}t_{n-j_1-1}-\cdots-2^{j_1-n+1}t_1, \tag{11}$$

$$2m_2+1-2^{j_2+1}z_2=1-b_{j_2+1}-2^{-1}b_{j_2+2}-\cdots-2^{j_2-n+1}b_n. \tag{12}$$

These observations will be the starting point of all proofs in the following section.

3. The Haar coefficients of the discrepancy function of $\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$.

Recall the definitions of ℓ and L from Theorem 1.1. Throughout the whole section, σ_j' for $j\in\{1,\dots,n-1\}$ will always mean $\sigma_j\oplus a_j$. The idea of the proof of Theorem 1.1 is as follows: We partition the set $\mathbb{N}_{-1}^2$ into 13 smaller sets $\mathcal{J}_i$, $i=1,\dots,13$. Then we compute the Haar coefficients $\mu_{\boldsymbol{j},\boldsymbol{m}}$ of $\Delta(\cdot,\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}))$ for all $\boldsymbol{j}\in\mathcal{J}_i$ and further $\sum_{\boldsymbol{j}\in\mathcal{J}_i}2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m}}|^2$. Then Theorem 1.1 follows via Parseval from

$$(2^n\,L_2(\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})))^2=\sum_{i=1}^{13}\sum_{\boldsymbol{j}\in\mathcal{J}_i}2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m}}|^2.$$

Case 1: $\boldsymbol{j}\in\mathcal{J}_1:=\{(-1,-1)\}$

PROPOSITION 3.1. *Let* $\boldsymbol{j}\in\mathcal{J}_1$ *and* $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$. *Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m}}=\frac{1}{2^{n+1}}+\frac{1}{2^{2n+2}}+\frac{1}{2^{n+3}}(\ell-L).$$

Proof. By (5) we have

$$\begin{aligned}\mu_{\boldsymbol{j},\boldsymbol{m}}&=2^{-n}\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})}(1-z_1)(1-z_2)-\frac14\\&=1-2^{-n}\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})}z_1-2^{-n}\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})}z_2+2^{-n}\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})}z_1z_2-\frac14\\&=-\frac14+2^{-n}+2^{-n}\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})}z_1z_2,\end{aligned}$$

where we have applied

$$\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})}z_1=\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})}z_2=\sum_{l=0}^{2^n-1}\frac{l}{2}^{n}=2^{n-1}-2^{-1}$$

in the last step. We write $u=2^{-1}t_{n-1}+\cdots+2^{-n+1}t_1$, $v_1=2^{-1}(t_1\oplus\sigma_1)+\cdots+2^{n-1}(t_{n-1}\oplus\sigma_{n-1})$ and $v_2=2^{-1}(t_1\oplus\sigma_1')+\cdots+2^{n-1}(t_{n-1}\oplus\sigma_{n-1}')$ and consider

$$\sum_{\boldsymbol{z}\in\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma})} z_1z_2 = \sum_{t_1,\dots,t_n=0}^{1}\Big(\frac{t_n}{2}+\dots+\frac{t_1}{2^n}\Big)\Big(\frac{t_1\oplus a_1t_n\oplus\sigma_1}{2}+\dots+\frac{t_n\oplus\sigma_n}{2^n}\Big)$$
$$= \sum_{t_1,\dots,t_{n-1}=0}^{1}\Big(\frac{u}{2}\Big(v_1+\frac{\sigma_n}{2^n}\Big)+\Big(\frac{1}{2}+\frac{u}{2}\Big)\Big(v_2+\frac{\sigma_n\oplus 1}{2^n}\Big)\Big)$$
$$= \sum_{t_1,\dots,t_{n-1}=0}^{1}\Big(2^{-n-1}-2^{-n-1}\sigma_n+2^{-n-1}u+\frac{v_2}{2}+\frac{1}{2}(uv_1+uv_2)\Big)$$
$$= 2^{n-1}(2^{-n-1}-2^{-n-1}\sigma_n)+(2^{n-2}-2^{-1})(2^{-n-1}+2^{-1})$$
$$+\frac{1}{2}\sum_{t_1,\dots,t_{n-1}=0}^{1}(uv_1+uv_2),$$

where we have used

$$\sum_{t_1,\dots,t_{n-1}=0}^{1} u = \sum_{t_1,\dots,t_{n-1}=0}^{1} v_2 = \sum_{l=0}^{2^{n-1}-1} l/2^{n-1} = 2^{n-2}-2^{-1}$$

in the last step. We have

$$\sum_{t_1,\dots,t_{n-1}=0}^{1} uv_1 \tag{13}$$
$$= \sum_{t_1,\dots,t_{n-1}=0}^{1}\Big(\frac{t_{n-1}}{2}+\dots+\frac{t_1}{2^{n-1}}\Big)\Big(\frac{t_1\oplus\sigma_1}{2}+\dots+\frac{t_{n-1}\oplus\sigma_{n-1}}{2^n}\Big)$$
$$= \sum_{t_1,\dots,t_{n-1}=0}^{1}\Big(\sum_{k=1}^{n-1}\frac{t_k(t_k\oplus\sigma_k)}{2^{n-k}2^k}+\sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{n-1}\frac{t_{k_1}(t_{k_2}\oplus\sigma_{k_2})}{2^{n-k_1}2^{k_2}}\Big)$$
$$= \frac{1}{2^n}\sum_{k=1}^{n-1}2^{n-2}\sum_{t_k=0}^{1}t_k(t_k\oplus\sigma_k)+\frac{1}{2^n}\sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{n-1}2^{k_1-k_2}2^{n-3}\sum_{t_{k_1},t_{k_2}=0}^{1}t_{k_1}(t_{k_2}\oplus\sigma_{k_2})$$
$$= \frac{1}{4}\sum_{k=1}^{n-1}(1\oplus\sigma_k)+\frac{1}{8}\sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{n-1}2^{k_1-k_2}.$$

Analogously,

$$\sum_{t_1,\dots,t_{n-1}=0}^{1} uv_2 = \frac{1}{4}\sum_{k=1}^{n-1}(1\oplus\sigma_k')+\frac{1}{8}\sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{n-1}2^{k_1-k_2} \tag{14}$$

and therefore

$$\sum_{t_1,\dots,t_{n-1}=0}^{1} (uv_1 + uv_2) = \frac{1}{4}\sum_{k=1}^{n-1}(1 \oplus \sigma_k + 1 \oplus \sigma'_k) + \frac{1}{4}\sum_{\substack{k_1,k_2=1\\ k_1\neq k_2}}^{n-1} 2^{k_1-k_2}.$$

If $a_k = 0$ then $1\oplus\sigma_k+1\oplus\sigma'_k = 2-2\sigma_k$, and if $a_k = 1$ then $1\oplus\sigma_k+1\oplus\sigma'_k = 1$; hence $1 \oplus \sigma_k + 1 \oplus \sigma'_k = (1 - a_k)(1 - 2\sigma_k) + 1$ and

$$\sum_{k=1}^{n-1}(1 \oplus \sigma_k + 1 \oplus \sigma'_k) = \ell - (1 - 2\sigma_n) - L + n - 1.$$

Further, a direct calculation yields

$$\sum_{\substack{k_1,k_2=1\\ k_1\neq k_2}}^{n-1} 2^{k_1-k_2} = \sum_{k_1,k_2=1}^{n-1} 2^{k_1-k_2} - \sum_{k=1}^{n-1} 1 = 2^n - n - 3 + 2^{-n+2}.$$

Now we put everything together to arrive at the desired result. ■

The following consequence is immediate.

Lemma 3.2. *We have*

$$\sum_{\boldsymbol{j}\in\mathcal{J}_1} 2^{|\boldsymbol{j}|} \sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 = \left(\frac{1}{2^{n+1}} + \frac{1}{2^{2n+2}} + \frac{1}{2^{n+3}}(\ell - L)\right)^2.$$

Case 2: $\boldsymbol{j} \in \mathcal{J}_2 := \{(-1, j_2) : 0 \le j_2 \le n - 2\}$

Proposition 3.3. *Let* $\boldsymbol{j} \in \mathcal{J}_2$ *and* $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. *Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{-2n-2} - 2^{-n-j_2-3} - 2^{-2n-1}(\sigma_{j_2+1} \oplus a_{j_2+1}\sigma_n) + 2^{-2j_2-3}\sum_{k=1}^{j_2} \frac{s_k \oplus \sigma_k + s_k \oplus \sigma'_k}{2^{n+1-k}}.$$

Proof. For $\boldsymbol{z} \in I_{\boldsymbol{j},\boldsymbol{m}}$ we have $b_k = s_k$ for all $k \in \{1, \dots, j_2\}$ and therefore

$$\begin{aligned} 1 - z_1 &= 1 - \frac{t_n}{2} - \dots - \frac{t_1}{2^n} \\ &= 1 - \frac{t_n}{2} - \dots - \frac{t_{j_2+1}}{2^{n-j_2}} - \frac{s_{j_2} \oplus a_{j_2}t_n \oplus \sigma_{j_2}}{2^{n-j_2+1}} - \dots - \frac{s_1 \oplus a_1t_n \oplus \sigma_1}{2^n} \\ &= 1 - u - \frac{t_{j_2+1}}{2^{n-j_2}} - \varepsilon(m_2, t_n), \end{aligned}$$

where $u := \frac{t_n}{2} - \dots - \frac{t_{j_2+2}}{2^{n-j_2-1}}$ and $\varepsilon := \frac{s_{j_2}\oplus a_{j_2}t_n\oplus\sigma_{j_2}}{2^{n-j_2+1}} + \dots + \frac{s_1\oplus a_1t_n\oplus\sigma_1}{2^n}$.

Further,

$$
\begin{aligned}
1-|2m_2+1-2^{j_2+1}z_2| &= 1-|1-b_{j_2+1}-\cdots-2^{j_2-n+1}b_n| \\
&= \begin{cases} v & \text{if } b_{j_2+1}=0, \text{ i.e. } t_{j_2+1}=a_{j_2+1}t_n\oplus\sigma_{j_2+1}, \\ 1-v & \text{if } b_{j_2+1}=1, \text{ i.e. } t_{j_2+1}=a_{j_2+1}t_n\oplus\sigma_{j_2+1}\oplus 1, \end{cases}
\end{aligned}
$$

where $v=v(t_n)=2^{-1}b_{j_2+2}+\cdots+2^{j_2-n+1}b_n$. We fix $t_{j_2+2},\ldots,t_n$; hence $\varepsilon(m_2,t_n)$ depends on m_2, and u and v are fixed as well. Then

$$
\begin{aligned}
&\sum_{t_{j_2+1}=0}^{1}(1-z_1)(1-|2m_2+1-2^{j_2+1}z_2|) \\
&\quad=\left(1-u-\frac{a_{j_2+1}t_n\oplus\sigma_{j_2+1}}{2^{n-j_2}}-\varepsilon(m_2,t_n)\right)v \\
&\qquad+\left(1-u-\frac{a_{j_2+1}t_n\oplus\sigma_{j_2+1}\oplus 1}{2^{n-j_2}}-\varepsilon(m_2,t_n)\right)(1-v) \\
&\quad=1-2^{-n+j_2}-\varepsilon(m_1,t_n)-u+2^{-n+j_2}v-2^{-n+j_2}(a_{j_2+1}t_n\oplus\sigma_{j_2+1})(2v-1).
\end{aligned}
$$

We sum the last expression over the remaining digits $t_{j_2+2},\ldots,t_n$ and observe that

$$
\sum_{t_{j_2+2},\ldots,t_n=0}^{1} v=\sum_{t_{j_2+2},\ldots,t_n=0}^{1} u=\sum_{l=0}^{2^{n-j_2-1}-1}\frac{l}{2^{n-j_2-1}}=2^{n-j_2-2}-2^{-1}.
$$

Hence

$$
\begin{aligned}
&\sum_{\boldsymbol{z}\in I_{\boldsymbol{j},\boldsymbol{m}}}(1-z_1)(1-|2m_2+1-2^{j_2+1}z_2|) \\
&\quad=\frac{1}{4}(2^{n-j_2}-2^{-n+j_2+1}+1)-2^{-n+j_2}\sum_{t_{j_2+2},\ldots,t_n=0}^{1}(a_{j_2+1}t_n\oplus\sigma_{j_2+1})(2v-1) \\
&\qquad-\sum_{t_{j_2+2},\ldots,t_n=0}^{1}\varepsilon(m_1,t_n).
\end{aligned}
$$

From the definition of $\varepsilon(m_2,t_n)$ it is easy to see that

$$
\sum_{t_{j_2+2},\ldots,t_n=0}^{1}\varepsilon(m_1,t_n)=2^{n-j_2-2}\sum_{k=1}^{j_2}\frac{s_k\oplus\sigma_k+s_k\oplus\sigma_k'}{2^{n+1-k}}.
$$

We compute $\sum_{t_{j_2+2},\ldots,t_n=0}^{1}(a_{j_2+1}t_n\oplus\sigma_{j_2+1})(2v-1)$, distinguishing two cases.

If $a_{j_2+1}=0$, we obtain

$$\begin{aligned}
&\sum_{t_{j_2+2},\dots,t_n=0}^{1}(a_{j_2+1}t_n\oplus\sigma_{j_2+1})(2v-1)\\
&=\sum_{t_{j_2+2},\dots,t_n=0}^{1}\sigma_{j_2+1}\biggl(2\biggl(\sum_{k=j_2+2}^{n-1}\frac{t_k\oplus a_kt_n\oplus\sigma_k}{2^{k-j_2-1}}+\frac{t_n\oplus\sigma_n}{2^{n-j_2-1}}\biggr)-1\biggr)\\
&=\sum_{t_{j_2+2},\dots,t_{n-1}=0}^{1}\sigma_{j_2+1}\biggl\{\biggl(2\biggl(\sum_{k=j_2+2}^{n-1}\frac{t_k\oplus\sigma_k}{2^{k-j_2-1}}+\frac{\sigma_n}{2^{n-j_2-1}}\biggr)-1\biggr)\\
&\qquad+\biggl(2\biggl(\sum_{k=j_2+2}^{n-1}\frac{t_k\oplus\sigma'_k}{2^{k-j_2-1}}+\frac{1\oplus\sigma_n}{2^{n-j_2-1}}\biggr)-1\biggr)\biggr\}\\
&=\sigma_{j_2+1}\sum_{l=0}^{2^{n-j_2-2}-1}\biggl\{2\biggl(\frac{l}{2^{n-j_2-2}}+\frac{\sigma_n}{2^{n-j_2-1}}\biggr)-1+2\biggl(\frac{l}{2^{n-j_2-2}}+\frac{1-\sigma_n}{2^{n-j_2-1}}\biggr)-1\biggr\}\\
&=-\sigma_{j_2+1}=-\sigma_{j_2+1}\oplus a_{j_2+1}\sigma_n.
\end{aligned}$$

If $a_{j_2+1}=1$, then

$$\begin{aligned}
&\sum_{t_{j_2+2},\dots,t_n=0}^{1}(a_{j_2+1}t_n\oplus\sigma_{j_2+1})(2v-1)\\
&=\sum_{t_{j_2+2},\dots,t_n=0}^{1}(t_n\oplus\sigma_{j_2+1})\biggl(2\biggl(\sum_{k=j_2+2}^{n-1}\frac{t_k\oplus a_kt_n\oplus\sigma_k}{2^{k-j_2-1}}+\frac{t_n\oplus\sigma_n}{2^{n-j_2-1}}\biggr)-1\biggr)\\
&=\sum_{t_{j_2+2},\dots,t_{n-1}=0}^{1}\biggl(2\biggl(\sum_{k=j_2+2}^{n-1}\frac{t_k\oplus a_k(\sigma_{j_2+1}\oplus 1)\oplus\sigma_k}{2^{k-j_2-1}}+\frac{\sigma_{j_2+1}\oplus 1\oplus\sigma_n}{2^{n-j_2-1}}\biggr)-1\biggr)\\
&=\sum_{l=0}^{2^{n-j_2-2}-1}\biggl(2\biggl(\frac{l}{2^{n-j_2-2}}+\frac{\sigma_{j_1+1}\oplus 1\oplus\sigma_n}{2^{n-j_2-2}}\biggr)-1\biggr)\\
&=\sigma_{j_2+1}\oplus 1\oplus\sigma_n-1=-\sigma_{j_2+1}\oplus\sigma_n=-\sigma_{j_2+1}\oplus a_{j_2+1}\sigma_n.
\end{aligned}$$

Thus, in any case $\sum_{t_{j_2+2},\dots,t_n=0}^{1}(a_{j_2+1}t_n\oplus\sigma_{j_2+1})(2v-1)=-\sigma_{j_2+1}\oplus a_{j_2+1}\sigma_n$ and we arrive at

$$\begin{aligned}
\sum_{\boldsymbol{z}\in I_{\boldsymbol{j},\boldsymbol{m}}}(1-z_1)(1-|2m_2+1-2^{j_2+1}z_2|)&=\frac14(2^{n-j_2}-2^{-n+j_2+1}+1)\\
&\quad+2^{-n+j_2}(\sigma_{j_2+1}\oplus a_{j_2+1}\sigma_n)-2^{n-j_2-2}\sum_{k=1}^{j_2}\frac{s_k\oplus\sigma_k+s_k\oplus\sigma'_k}{2^{n+1-k}}.
\end{aligned}$$

The rest follows from (7). ∎

LEMMA 3.4. *We have*

$$\sum_{\boldsymbol{j}\in\mathcal{J}_2} 2^{|\boldsymbol{j}|} \sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 = \frac{1}{9}2^{-4n-6}\big(2n2^{2n} - 9(n-1)2^{n+2} + 2^{2n+3} - 44\big)$$
$$+ 2^{-3n-3}\Big(\sum_{i=1}^{n-1}\sigma_i + \sigma_n L\Big) - 2^{-2n-8}\sum_{i=0}^{n-2} 2^{-2i}\sum_{k=1}^{i} a_k 2^{2k}.$$

Proof. We write $S(m_2) := \sum_{k=1}^{j_2} \frac{s_k\oplus\sigma_k + s_k\oplus\sigma_k'}{2^{n+1-k}}$. Then

$$\sum_{m_2\in\mathbb{D}_{j_2}} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 = \sum_{s_1,\dots,s_{j_2}=0}^{1} \big\{(2^{-2n-2} - 2^{-n-j_2-3} - 2^{-2n-1}(\sigma_{j_2+1}\oplus a_{j_2+1}\sigma_n))^2$$
$$+ 2^{-2j_2-2}\big(2^{-2n-2} - 2^{-n-j_2-3} - 2^{-2n-1}(\sigma_{j_2+1}\oplus a_{j_2+1}\sigma_n)\big)S(m_2)$$
$$+ 2^{-4j_2-6}S(m_2)^2\big\}.$$

Since

$$\sum_{m_2\in\mathbb{D}_{j_2}} S(m_2) = \sum_{s_1,\dots,s_{j_2}=0}^{1}\sum_{k=1}^{j_2}\frac{s_k\oplus\sigma_k + s_k\oplus\sigma_k'}{2^{n+1-k}}$$
$$= \sum_{k=1}^{j_2} 2^{j_2-1}\sum_{s_k=0}^{1}\frac{s_k\oplus\sigma_k + s_k\oplus\sigma_k\oplus a_k}{2^{n+1-k}}$$
$$= \sum_{k=1}^{j_2} 2^{j_2-1}\frac{2}{2^{n+1-k}} = 2^{2j_2-n} - 2^{j_2-n}$$

and

$$\sum_{m_2\in\mathbb{D}_{j_2}} S(m_2)^2$$
$$= \sum_{s_1,\dots,s_{j_2}=0}^{1}\Big\{\sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{j_2}\frac{(s_{k_1}\oplus\sigma_{k_1} + s_{k_1}\oplus\sigma_{k_1}')(s_{k_2}\oplus\sigma_{k_2} + s_{k_2}\oplus\sigma_{k_2}')}{2^{n+1-k_1}2^{n+1-k_2}}$$
$$+ \sum_{k=1}^{j_2}\frac{(s_k\oplus\sigma_k + s_k\oplus\sigma_k')^2}{2^{2n+2-2k}}\Big\}$$
$$= \sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{j_2} 2^{j_2-2}\frac{4}{2^{n+1-k_1}2^{n+1-k_2}} + \sum_{k=1}^{j_2} 2^{j_2-1}\frac{a_k^2 + (1+a_k\oplus 1)^2}{2^{2n+2-2k}}$$
$$= \frac{1}{3}2^{-2n+j_2+2} + \frac{1}{3}2^{-2n+3j_2+1} - 2^{-2n+2j_2+1} + \sum_{k=1}^{j_2} 2^{j_2-1}\frac{4-2a_k}{2^{2n+2-2k}},$$

we obtain the claimed result by combining all these expressions, summing $2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m}}|^2$ over all $\boldsymbol{j}\in\mathcal{J}_2$ and using the fact that

$$\sum_{j_2=0}^{n-2}(\sigma_{j_2+1}\oplus a_{j_2+1}\sigma_n)=\sum_{i=1}^{n-1}(\sigma_i\oplus a_i\sigma_n)=\sum_{i=1}^{n-1}(\sigma_i-a_i\sigma_n)^2$$
$$=\sum_{i=1}^{n-1}(\sigma_i-2a_i\sigma_i\sigma_n+a_i\sigma_n)=\sum_{i=1}^{n-1}\sigma_i+\sigma_n L. \ \blacksquare$$

Case 3: $\boldsymbol{j}\in\mathcal{J}_3:=\{(-1,n-1)\}$

PROPOSITION 3.5. *Let $\boldsymbol{j}\in\mathcal{J}_3$ and $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$. Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m}}=2^{-2n-1}\Big(-\sigma_n+\sum_{k=1}^{n-1}\frac{s_k\oplus a_k(\sigma_n\oplus 1)\oplus\sigma_k}{2^{n-k}}\Big).$$

Proof. For $j_2=n-1$ we have $1-|2m_2+1-2^{j_2+1}z_2|=1-|1-b_n|=b_n=t_n\oplus\sigma_n$. Writing

$$\varepsilon(t_n,m_2):=\sum_{k=1}^{n-1}\frac{s_k\oplus a_k t_n\oplus\sigma_k}{2^{n+1-k}},$$

we get

$$\sum_{\boldsymbol{z}\in I_{\boldsymbol{j},\boldsymbol{m}}}(1-z_1)(1-|2m_2+1-2^{j_2+1}|)=\sum_{t_n=0}^{1}\Big(1-\frac{t_n}{2}-\varepsilon(t_n,m_2)\Big)(t_n\oplus\sigma_n)$$
$$=1-\frac{\sigma_n\oplus 1}{2}-\varepsilon(\sigma_n\oplus 1,m_1),$$

which leads to $\mu_{\boldsymbol{j},\boldsymbol{m}}=2^{-2n-1}(\sigma_n\oplus 1+2\varepsilon(\sigma_n\oplus 1,m_1)-1)$ via (7) and hence to the result. ∎

LEMMA 3.6. *We have*

$$\sum_{\boldsymbol{j}\in\mathcal{J}_3}2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m}}|^2=\frac{1}{3}2^{-4n-4}(2^{2n}-3\cdot 2^n+2+3\sigma_n 2^{n+1}).$$

Proof. We have

$$\sum_{\boldsymbol{j}\in\mathcal{J}_3}2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m}}|^2$$
$$=2^{n-1}\sum_{s_1,\dots,s_{n-1}=0}^{1}\Big(2^{-2n-1}\Big(-\sigma_n+\sum_{k=1}^{n-1}\frac{s_k\oplus a_k(\sigma_n\oplus 1)\oplus\sigma_k}{2^{n-k}}\Big)\Big)^2$$
$$=2^{n-1}\sum_{l=0}^{2^{n-1}-1}\Big(2^{-2n-1}\Big(-\sigma_n+\frac{l}{2^{n-1}}\Big)\Big)^2,$$

which leads to the claimed result. ∎

Case 4: $\boldsymbol{j} \in \mathcal{J}_4 := \{(-1, j_2) : j_2 \geq n\}$

Proposition 3.7. *Let $\boldsymbol{j} \in \mathcal{J}_4$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{-2j_2-3}.$$

Proof. If $j_2 \geq n$, no point of $\mathcal{P}$ is contained in the interior of $I_{\boldsymbol{j},\boldsymbol{m}}$ and therefore only the linear part $-t_1t_2$ contributes to the Haar coefficient of the discrepancy function in this case. Hence, the given formula is an immediate consequence of (7). ∎

Lemma 3.8. *We have*

$$\sum_{\boldsymbol{j}\in\mathcal{J}_4} 2^{|\boldsymbol{j}|} \sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 = \frac{1}{48}2^{-2n}.$$

Proof. It is easy to compute

$$\sum_{\boldsymbol{j}\in\mathcal{J}_4} 2^{|\boldsymbol{j}|} \sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 = \sum_{j_2=n}^{\infty} 2^{2j_2}2^{-4j_2-6} = \frac{1}{48}2^{-2n}. \quad ∎$$

Case 5: $\boldsymbol{j} \in \mathcal{J}_5 := \{(0, -1)\}$

Proposition 3.9. *Let $\boldsymbol{j} \in \mathcal{J}_5$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = \frac{1}{2^{2n+2}} - \frac{1}{2^{n+3}} + \frac{1}{2^{n+3}}L - \frac{1}{2^{2n+1}}\sigma_n.$$

Proof. For $\boldsymbol{z} \in \mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ we have

$$\begin{aligned} 1 - z_2 &= 1 - \frac{b_1}{2} - \cdots - \frac{b_n}{2^n} \\ &= 1 - \frac{t_1 \oplus a_1 t_n \oplus \sigma_1}{2} - \cdots - \frac{t_{n-1} \oplus a_{n-1}t_n \oplus \sigma_{n-1}}{2^{n-1}} - \frac{t_n \oplus \sigma_n}{2^n} \end{aligned}$$

and

$$1 - |2m_1 + 1 - 2z_1| = 1 - |1 - 2z_1| = 1 - \left|1 - t_n - \frac{t_{n-1}}{2} - \cdots - \frac{t_1}{2^{n-1}}\right|.$$

Hence, writing $u = 2^{-1}t_{n-1} + \cdots + 2^{-n+1}t_1$, $v_1 = 2^{-1}(t_1 \oplus \sigma_1) + \cdots + 2^{-n+1}(t_{n-1} \oplus \sigma_{n-1})$ and $v_2 = 2^{-1}(t_1 \oplus \sigma_1') + \cdots + 2^{-n+1}(t_{n-1} \oplus \sigma_{n-1}')$, we have

$$\begin{aligned}
&\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})}(1-|2m_1+1-2z_1|)(1-z_2)\\
&\quad=\sum_{t_1,\dots,t_n=0}^{1}\left(1-\left|1-t_n-\frac{t_{n-1}}{2}-\cdots-\frac{t_1}{2^{n-1}}\right|\right)\\
&\qquad\times\left(1-\frac{t_1\oplus a_1t_n\oplus\sigma_1}{2}-\cdots-\frac{t_{n-1}\oplus a_{n-1}t_n\oplus\sigma_{n-1}}{2^{n-1}}-\frac{t_n\oplus\sigma_n}{2^n}\right)\\
&\quad=\sum_{t_1,\dots,t_{n-1}=0}^{1}\left\{u\left(1-v_1-\frac{\sigma_n}{2^n}\right)+(1-u)\left(1-v_2-\frac{\sigma_n\oplus 1}{2^n}\right)\right\}\\
&\quad=\sum_{t_1,\dots,t_{n-1}=0}^{1}\{1-2^{-n}+2^{-n}\sigma_n-v_2+u(2^{-n}-2^{-n+1}\sigma_n)+u(v_2-v_1)\}\\
&\quad=2^{n-1}(1-2^{-n}+2^{-n}\sigma_n)+(2^{n-2}-2^{-1})(-1+2^{-n}-2^{-n+1}\sigma_n)\\
&\qquad+\sum_{t_1,\dots,t_{n-1}=0}^{1}u(v_2-v_1),
\end{aligned}$$

where we have used $\sum_{t_1,\dots,t_{n-1}=0}^{1}u=\sum_{t_1,\dots,t_{n-1}=0}^{1}v_2=2^{n-2}-2^{-1}$ in the last step. By (13) and (14) we find

$$\sum_{t_1,\dots,t_{n-1}=0}^{1}u(v_2-v_1)=\frac{1}{4}\sum_{k=1}^{n-1}(\sigma_k'\oplus 1-\sigma_k\oplus 1)=-\frac{1}{4}\sum_{k=1}^{n-1}a_k(1-2\sigma_k)=-\frac{L}{4},$$

and therefore

$$\sum_{\boldsymbol{z}\in\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})}(1-|2m_1+1-2z_1|)(1-z_2)=\frac{1}{4}-2^{-n-1}+2^{n-2}+2^{-n}\sigma_n-\frac{L}{4}.$$

The rest follows from (6). ■

Lemma 3.10. *We have*

$$\sum_{\boldsymbol{j}\in\mathcal{J}_5}2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m}}|^2=\left(\frac{1}{2^{2n+2}}-\frac{1}{2^{n+3}}+\frac{1}{2^{n+3}}L-\frac{1}{2^{2n+1}}\sigma_n\right)^2.$$

Case 6: $\boldsymbol{j}\in\mathcal{J}_6:=\{(j_1,-1):1\le j_1\le n-1\}$

Proposition 3.11. *Let* $\boldsymbol{j}\in\mathcal{J}_6$ *and* $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$. *Then*

$\mu_{\boldsymbol{j},\boldsymbol{m}}=2^{-2n-2j_1-3}\big(2^{2j_1+1}-2^{j_1+n}+2^{2n+1}\varepsilon(m_1)-2^{2j_1+2}(a_{n-j_1}r_1\oplus\sigma_{n-j_1})\big)$,

where $\varepsilon(m_1)=\frac{r_1\oplus\sigma_n}{2^n}+\sum_{k=2}^{j_1}\frac{r_k\oplus a_{n+1-k}r_1\oplus\sigma_{n+1-k}}{2^{n+1-k}}$.

Proof. Similar to the proof of Proposition 2 we write

$$1-z_2=1-u-\frac{b_{n-j_1}}{2^{n-j_1}}-\varepsilon(m_1)$$

with $u = 2^{-1}b_1 + \cdots + 2^{-n+j_1+1}b_{n-j_1-1}$ and

$$\begin{aligned}\varepsilon(m_1) &= 2^{-n+j_1-1}b_{n-j_1+1} + \cdots + 2^{-n}b_n \\ &= \frac{r_1 \oplus \sigma_n}{2^n} + \sum_{k=2}^{j_1} \frac{r_k \oplus a_{n+1-k}r_1 \oplus \sigma_{n+1-k}}{2^{n+1-k}}.\end{aligned}$$

We also have

$$\begin{aligned}1 - |2m_1 + 1 - 2^{j_1+1}z_1| &= 1 - |1 - t_{n-j_1} - \cdots - 2^{j_1-n+1}t_1| \\ &= \begin{cases} v & \text{if } t_{n-j_1} = 0, \\ 1-v & \text{if } t_{n-j_1} = 1, \end{cases}\end{aligned}$$

where $v = 2^{-1}t_{n-j_1-1} + \cdots + 2^{j_1-n+1}t_1$. Let us first fix $t_1, \ldots, t_{n-j_1-1}$ and hence u and v. Then we have

$$\begin{aligned}&\sum_{t_{n-j_1}=0}^{1} \left(1 - u - \frac{b_{n-j_1}}{2^{n-j_1}} - \varepsilon(m_1)\right)(1 - |1 - t_{n-j_1} - \cdots - 2^{j_1-n+1}t_1|) \\ &\quad = \left(1 - u - \frac{a_{n-j_1}r_1 \oplus \sigma_{n-j_1}}{2^{n-j_1}} - \varepsilon(m_1)\right)v \\ &\qquad + \left(1 - u - \frac{1 \oplus a_{n-j_1}r_1 \oplus \sigma_{n-j_1}}{2^{n-j_1}} - \varepsilon(m_1)\right)(1-v) \\ &\quad = 1 - 2^{-n+j_1} - \varepsilon - u + 2^{-n+j_1}v - 2^{-n+j_1+1}(a_{n-j_1}r_1 \oplus \sigma_{n-j_1})v \\ &\qquad + 2^{-n+j_1}(a_{n-j_1}r_1 \oplus \sigma_{n-j_1}).\end{aligned}$$

Since

$$\sum_{t_1,\ldots,t_{n-j_1-1}=0}^{1} u = \sum_{t_1,\ldots,t_{n-j_1-1}=0}^{1} v = \sum_{l=0}^{2^{n-j_1-1}-1} \frac{l}{2^{n-j_1-1}} = 2^{n-j_1-2} - \frac{1}{2},$$

we find

$$\begin{aligned}&\sum_{\boldsymbol{z} \in I_{\boldsymbol{j},\boldsymbol{m}}} (1 - z_2)(1 - |2m_1 + 1 - 2^{j_1+1}z_1|) \\ &= 2^{-n-j_1-2}\big(2^{2n} + 2^{j_1+n} - 2^{2j_1+1} - 2^{2n+1}\varepsilon(m_1) + 2^{2j_1+2}(a_{n-j_1}r_1 \oplus \sigma_{n-j_1})\big).\end{aligned}$$

The rest follows from (6). ∎

Lemma 3.12. *We have*

$$\begin{aligned}&\sum_{\boldsymbol{j} \in \mathcal{J}_6} 2^{|\boldsymbol{j}|} \sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 \\ &\qquad = \frac{1}{9}4^{-2n-3}\big((3n+11)4^n - 56\big) - 2^{-3n-4}\Big(n - 1 - 2\sum_{i=1}^{n-1}\sigma_i - 2\sigma_n L\Big).\end{aligned}$$

Proof. To compute $\sum_{m_2\in\mathbb{D}_{j_2}}\mu_{\boldsymbol{j},\boldsymbol{m}}^2$, we first sum over r_1, write $\varepsilon=\varepsilon(r_1)$ and find

$$\begin{aligned}\sum_{r_1,\dots,r_{j_1}=0}^{1}\mu_{\boldsymbol{j},\boldsymbol{m}}^2\\ &=\sum_{r_2,\dots,r_{j_1}=0}^{1}\Big\{\big(2^{-2n-2j_1-3}(2^{2j_1+1}-2^{j_1+n}+2^{2n+1}\varepsilon(0)-2^{2j_1+2}\sigma_{n-j_1})\big)^2\\ &\quad+\big(2^{-2n-2j_1-3}(2^{2j_1+1}-2^{j_1+n}+2^{2n+1}\varepsilon(1)-2^{2j_1+2}\sigma'_{n-j_1})\big)^2\Big\}.\end{aligned}$$

We arrive at the claimed formula by writing $\varepsilon(0)=\frac{l}{2^{n-1}}+\frac{\sigma_n}{2^n}$ and $\varepsilon(1)=\frac{l}{2^{n-1}}+\frac{1-\sigma_n}{2^n}$ and by replacing the sum over $r_2,\dots,r_{j_1}$ by a sum over l running from 0 to $2^{j_1-1}-1$. ■

Case 7: $\boldsymbol{j}\in\mathcal{J}_7:=\{(j_1,-1):j_1\ge n\}$. This case is completely analogous to Case 4.

PROPOSITION 3.13. *Let $\boldsymbol{j}\in\mathcal{J}_7$ and $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$. Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m}}=-2^{-2j_1-3}.$$

LEMMA 3.14. *We have*

$$\sum_{\boldsymbol{j}\in\mathcal{J}_7}2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m}}|^2=\frac{1}{48}2^{-2n}.$$

Case 8: $\boldsymbol{j}\in\mathcal{J}_8:=\{(0,j_2):0\le j_2\le n-2\}$

PROPOSITION 3.15. *Let $\boldsymbol{j}\in\mathcal{J}_8$ and $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$. Then*

$$\begin{aligned}\mu_{\boldsymbol{j},\boldsymbol{m}}&=2^{-2j_2-4}\sum_{k=1}^{j_2}a_k\frac{2(s_k\oplus\sigma_k)-1}{2^{n-k}}\\ &\quad+2^{-2n-2}\big(1+2\sigma_{j_2+1}(\sigma_n-1)+2\sigma_n(\sigma'_{j_2+1}-1)\big).\end{aligned}$$

Proof. In this case, the condition $\boldsymbol{z}\in I_{\boldsymbol{j},\boldsymbol{m}}$ results in $b_k=s_k$ for all $k\in\{1,\dots,j_2\}$ and

$$\begin{aligned}2m_1+1-2^{j_1+1}z_1&=1-2z_1=1-t_n-\dots-2^{-n+1}t_1\\ &=1-t_n-u-2^{-n+j_2+1}t_{j_2+1}-\varepsilon(m_2,t_n),\end{aligned}$$

where $u=2^{-1}t_{n-1}+\dots+2^{-n+j_2+2}t_{j_2+2}$ and $\varepsilon(m_2,t_n)=\sum_{k=1}^{j_2}\frac{s_k\oplus a_kt_n\oplus\sigma_k}{2^{n-k}}$. Further, we have

$$\begin{aligned}2m_2+1-2^{j_2+1}z_2&=1-b_{j_2+1}-\dots-2^{-n+j_2+1}b_n\\ &=1-t_{j_2+1}\oplus a_{j_2+1}t_n\oplus\sigma_{j_2+1}-v(t_n)-2^{-n+j_2+1}t_n\oplus\sigma_n\end{aligned}$$

with $v(t_n)=\sum_{k=j_2+2}^{n-1}2^{j_2+1-k}(s_k\oplus a_kt_n\oplus\sigma_k)$. Therefore

$$\begin{aligned}
&\sum_{\boldsymbol{z}\in I_{\boldsymbol{j},\boldsymbol{m}}}(1-|2m_1+1-2^{j_1+1}z_1|)(1-|2m_2+1-2^{j_2+1}z_2|)\\
&\quad=\sum_{t_{j_2+1},\dots,t_n=0}^{1}(1-|1-t_n-u-2^{-n+j_2+1}t_{j_2+1}-\varepsilon(m_2,t_n)|)\\
&\qquad\times(1-|1-t_{j_2+1}\oplus a_{j_2+1}t_n\oplus\sigma_{j_2+1}-v(t_n)-2^{-n+j_2+1}(t_n\oplus\sigma_n)|)\\
&\quad=\sum_{t_{j_2+2},\dots,t_{n-1}=0}^{1}\big\{\big(u+2^{-n+j_2+1}\sigma_{j_2+1}+\varepsilon(m_2,0)\big)\big(v(0)+2^{-n+j_2+1}\sigma_n\big)\\
&\qquad+\big(u+2^{-n+j_2+1}(\sigma_{j_2+1}\oplus 1)+\varepsilon(m_2,0)\big)\big(1-v(0)-2^{-n+j_2+1}\sigma_n\big)\\
&\qquad+\big(1-u-2^{-n+j_2+1}\sigma'_{j_2+1}-\varepsilon(m_2,1)\big)\big(v(1)+2^{-n+j_2+1}(\sigma_n\oplus 1)\big)\\
&\quad+\big(1-u-2^{-n+j_2+1}(\sigma'_{j_2+1}\oplus 1)-\varepsilon(m_2,1)\big)\big(1-v(1)-2^{-n+j_2+1}(\sigma_n\oplus 1)\big)\big\}\\
&\quad=\sum_{t_{j_2+2},\dots,t_{n-1}=0}^{1}4^{-n}\big\{4^{j_2+1}\big(2\sigma_n(\sigma_{j_2+1}+\sigma'_{j_2+1}-1)-2\sigma'_{j_2+1}+1\big)\\
&\qquad+4^n\big(1+\varepsilon(m_2,0)-\varepsilon(m_2,1)+2^{n+j_2+1}(\sigma'_{j_2+1}-\sigma_{j_2+1})\big)\\
&\qquad+2^{n+j_2+1}\big((2\sigma_{j_2+1}-1)v(0)-(2\sigma'_{j_2+1}-1)v(1)\big)\big\}.
\end{aligned}$$

In the last expression, only $v(0)$ and $v(1)$ depend on the digits $t_{j_2+2},\dots,t_{n-1}$ and we have

$$\sum_{t_{j_2+2},\dots,t_{n-1}=0}^{1}v(0)=\sum_{t_{j_2+2},\dots,t_{n-1}=0}^{1}v(1)=\sum_{l=0}^{2^{n-j_2-2}-1}\frac{l}{2^{n-j_2-2}}=2^{n-j_2-3}-\frac{1}{2}.$$

Hence, we can compute $\sum_{\boldsymbol{z}\in I_{\boldsymbol{j},\boldsymbol{m}}}(1-|2m_1+1-2^{j_1+1}z_1|)(1-|2m_2+1-2^{j_2+1}z_2|)$ and the Haar coefficients via (8). Note that

$$\varepsilon(m_2,0)-\varepsilon(m_2,1)=\sum_{k=1}^{j_2}\frac{s_k\oplus\sigma_k-s_k\oplus a_k\oplus\sigma_k}{2^{n-k}}=\sum_{k=1}^{j_2}a_k\frac{2(s_k\oplus\sigma_k)-1}{2^{n-k}},$$

where the relation $s_k\oplus\sigma_k-s_k\oplus a_k\oplus\sigma_k=a_k(2(s_k\oplus\sigma_k)-1)$ can be seen easily. ∎

Lemma 3.16. *We have*

$$\sum_{\boldsymbol{j}\in\mathcal{J}_8}2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m}}|^2=\frac{1}{3}4^{-2n-3}(4^n-4)+2^{-2n-8}\sum_{i=0}^{n-2}2^{-2i}\sum_{k=1}^{i}a_k2^{2k}.$$

Proof. For the sake of brevity we write

$$f:=2^{-2n-2}\big(1+2\sigma_{j_2+1}(\sigma_n-1)+2\sigma_n(\sigma'_{j_2+1}-1)\big).$$

Note that f does not depend on m_2. Then $\sum_{m_2\in\mathbb{D}_{j_2}} \mu_{\boldsymbol{j},\boldsymbol{m}}^2$ equals

$$\sum_{m_2\in\mathbb{D}_{j_2}} \left\{2^{-4j_2-8}\Big(\sum_{k=1}^{j_2} a_k \frac{2(s_k\oplus\sigma_k)-1}{2^{n-k}}\Big)^2 + 2^{-2j_2-3} f \sum_{k=1}^{j_2} a_k \frac{2(s_k\oplus\sigma_k)-1}{2^{n-k}} + f^2\right\}.$$

Since

$$\sum_{s_1,\dots,s_{j_2}=0}^{1}\sum_{k=1}^{j_2} a_k \frac{2(s_k\oplus\sigma_k)-1}{2^{n-k}} = \sum_{k=1}^{j_2} \frac{a_k}{2^{n-k}} 2^{j_2-1}\sum_{s_k=0}^{1}(2(s_k\oplus\sigma_k)-1)=0$$

and

$$\begin{aligned}
&\sum_{s_1,\dots,s_{j_2}=0}^{1}\Big(\sum_{k=1}^{j_2} a_k \frac{2(s_k\oplus\sigma_k)-1}{2^{n-k}}\Big)^2\\
&= \sum_{s_1,\dots,s_{j_2}=0}^{1}\Bigg(\sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{j_2} a_{k_1}a_{k_2}\frac{(2(s_{k_1}\oplus\sigma_{k_1})-1)(2(s_{k_2}\oplus\sigma_{k_2})-1)}{2^{n-k_1}2^{n-k_2}} + \sum_{k=1}^{j_2} a_k \frac{(2(s_k\oplus\sigma_k)-1)^2}{2^{2n-2k}}\Bigg)\\
&= \sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{j_2} \frac{a_{k_1}a_{k_2}}{2^{n-k_1}2^{n-k_2}} 2^{j_2-2}\sum_{s_{k_1},s_{k_2}=0}^{1}(2(s_{k_1}\oplus\sigma_{k_1})-1)(2(s_{k_2}\oplus\sigma_{k_2})-1)\\
&\quad+\sum_{k=1}^{j_2}\frac{a_k}{2^{2n-2k}}2^{j_2-1}\sum_{s_k=0}^{1}(2(s_k\oplus\sigma_k)-1)^2\\
&= 2^{j_2-2n}\sum_{k=1}^{j_2} a_k 2^{2k},
\end{aligned}$$

this yields

$$\sum_{m_2\in\mathbb{D}_{j_2}} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 = 2^{-3j_2-2n-8}\sum_{k=1}^{j_2} a_k 2^{2k} + 2^{j_2} f^2.$$

Note that $1+2\sigma_{j_2+1}(\sigma_n-1)+2\sigma_n(\sigma'_{j_2+1}-1) = 1-2\sigma_{j_2+1}$ if $\sigma_n=0$, and $1+2\sigma_{j_2+1}(\sigma_n-1)+2\sigma_n(\sigma'_{j_2+1}-1) = 2\sigma'_{j_2+1}-1$ if $\sigma_n=1$; thus $(1+2\sigma_{j_2+1}(\sigma_n-1)+2\sigma_n(\sigma'_{j_2+1}-1))^2=1$ in any case and $f^2=2^{-4n-4}$. After summation over j_2 we obtain the result. ∎

Case 9: $\boldsymbol{j}\in\mathcal{J}_9 := \{(j_1,j_2)\in\mathbb{N}_0^2 : j_1+j_2\le n-2,\ j_1\ge 1\}$

PROPOSITION 3.17. *Let $\boldsymbol{j}\in\mathcal{J}_9$ and $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$. Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{-2n-2}\big(2(a_{n-j_1}r_1\oplus\sigma_{j_2+1})-1\big)\big(2(a_{j_2+1}r_1\oplus\sigma_{n-j_1})-1\big).$$

Proof. By (10)–(12), the condition $\boldsymbol{z} \in I_{\boldsymbol{j},\boldsymbol{m}}$ yields

$$2m_1 + 1 - 2^{j_1+1} z_1 = 1 - t_{n-j_1} - u - 2^{j_1+j_2-n+1} t_{j_2+1} - \varepsilon_1$$

with $u = 2^{-1} t_{n-j_1-1} + \cdots + 2^{j_1+j_2-n+2} t_{j_2+2}$ and

$$\varepsilon_1 = 2^{j_1-n+1} \sum_{k=1}^{j_2} 2^{k-1} t_k = 2^{j_1-n+1} \sum_{k=1}^{j_2} 2^{k-1} (s_k \oplus a_k r_1 \oplus \sigma_k) = \varepsilon_1(\boldsymbol{m}).$$

Similarly, we write

$$2m_2 + 1 - 2^{j_2+1} z_2 = 1 - b_{j_2+1} - v - 2^{j_1+j_2-n+1} b_{n-j_1} - \varepsilon_2$$

with $v = 2^{-1} b_{j_2+2} + \cdots + 2^{j_1+j_2-n+2} b_{n-j_1-1}$ and

$$\begin{aligned}\varepsilon_2 &= 2^{j_2-n+1} \sum_{k=1}^{j_1} 2^{k-1} b_{n+1-k} \\ &= 2^{j_2-n+1} \Big(r_1 \oplus \sigma_n + \sum_{k=1}^{j_2} 2^{k-1} (r_k \oplus a_{n+1-k} r_1 \oplus \sigma_{n+1-k}) \Big) = \varepsilon_2(\boldsymbol{m}).\end{aligned}$$

We fix the digits $t_{j_2+2}, \ldots, t_{n-j_1-1}$; then u and v are also fixed. We sum

$$(1 - |2m_1 + 1 - 2^{j_1+1} z_1|)(1 - |2m_2 + 1 - 2^{j_2+1} z_2|)$$

over $t_{n-j_1} \in \{0,1\}$ and $t_{j_2+1} \in \{0,1\} = \{a_{j_2+1} r_1 \oplus \sigma_{j_2+1}, a_{j_2+1} r_1 \oplus \sigma_{j_2+1} \oplus 1\}$ and find after lengthy calculations

$$\begin{aligned}\sum_{t_{j_2+1}, t_{n-j_1}=0}^{1} & (1 - |1 - t_{n-j_1} - u - 2^{j_1+j_2-n+1} t_{j_2+1} - \varepsilon_1|) \\ & \times (1 - |1 - b_{j_2+1} - v - 2^{j_1+j_2-n+1} b_{n-j_1} - \varepsilon_2|) \\ &= 1 + 4^{-n+j_1+j_2+1} \big(2(a_{n-j_1} r_1 \oplus \sigma_{j_2+1}) - 1\big)\big(2(a_{j_2+1} r_1 \oplus \sigma_{n-j_1}) - 1\big).\end{aligned}$$

Summation over the remaining digits $t_{j_2+2}, \ldots, t_{n-j_1-1}$ yields

$$\begin{aligned}&\sum_{\boldsymbol{z} \in I_{\boldsymbol{j},\boldsymbol{m}}} (1 - |2m_1 + 1 - 2^{j_1+1} z_1|)(1 - |2m_2 + 1 - 2^{j_2+1} z_2|) \\ &= 2^{n-j_1-j_2-2} + 2^{-n+j_1+j_2} \big(2(a_{n-j_1} r_1 \oplus \sigma_{j_2+1}) - 1\big)\big(2(a_{j_2+1} r_1 \oplus \sigma_{n-j_1}) - 1\big),\end{aligned}$$

and the result follows from (8). ∎

Since $\mu_{\boldsymbol{j},\boldsymbol{m}}^2 = 2^{-4n-4}$ is independent of $\boldsymbol{j}$ and $\boldsymbol{m}$ in this case, the following consequence is straightforward.

LEMMA 3.18. *We have*

$$\sum_{\boldsymbol{j} \in \mathcal{J}_9} 2^{|\boldsymbol{j}|} \sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 = \frac{1}{9} 4^{-2n-3} (3n4^n - 7 \cdot 4^n + 16).$$

Case 10: $\boldsymbol{j} \in \mathcal{J}_{10} := \{(0,n-1)\}$

PROPOSITION 3.19. *Let $\boldsymbol{j} \in \mathcal{J}_{10}$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = \frac{1}{2^{2n+2}}\bigg(1 - 2\bigg|\sigma_n - \sum_{k=1}^{n-1} \frac{s_k \oplus a_k(\sigma_n \oplus 1) \oplus \sigma_k}{2^{n-k}}\bigg|\bigg).$$

Proof. We have $1 - |2m_1 + 1 - 2z_1| = 1 - \big|1 - t_n - \sum_{k=1}^{n-1} \frac{s_k \oplus a_k t_n \oplus \sigma_k}{2^{n-k}}\big|$ and $1 - |2m_2 + 1 - 2^n z_2| = 1 - |1 - b_n| = b_n$, and therefore

$$\begin{aligned}\sum_{\boldsymbol{z}\in I_{\boldsymbol{j},\boldsymbol{m}}} &(1 - |2m_1 + 1 - 2z_1|)(1 - |2m_2 + 1 - 2^n z_2|) \\ &= \sum_{t_n=0}^{1}\bigg(1 - \bigg|1 - t_n - \sum_{k=1}^{n-1} \frac{s_k \oplus a_k t_n \oplus \sigma_k}{2^{n-k}}\bigg|\bigg)(t_n \oplus \sigma_n) \\ &= 1 - \bigg|1 - \sigma_n \oplus 1 - \sum_{k=1}^{n-1} \frac{s_k \oplus a_k(\sigma_n \oplus 1) \oplus \sigma_k}{2^{n-k}}\bigg|;\end{aligned}$$

the rest of the proof is straightforward by (8). ∎

LEMMA 3.20. *We have*

$$\sum_{\boldsymbol{j}\in\mathcal{J}_{10}} 2^{|\boldsymbol{j}|} \sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 = \frac{1}{3}2^{-4n-6}(2^{2n}+8).$$

Proof. In both cases $\sigma_n = 0$ and $\sigma_n = 1$ we find

$$2^{n-1}\sum_{m_2=0}^{2^{n-1}-1} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 = 2^{n-1}\frac{1}{2^{4n+4}}\sum_{l=0}^{2^{n-1}-1}\bigg(1 - 2\frac{l}{2^{n-1}}\bigg)^2,$$

which yields the claim. ∎

Case 11: $\boldsymbol{j} \in \mathcal{J}_{11} := \{(j_1,j_2) \in \mathbb{N}_0^2 : j_1 + j_2 = n-1,\ j_1 \geq 1\}$

PROPOSITION 3.21. *Let $\boldsymbol{j} \in \mathcal{J}_{11}$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. Then*

$$\begin{aligned}&\mu_{\boldsymbol{j},\boldsymbol{m}} = \\ &2^{-2n-1}\bigg\{\bigg(1 - \bigg|1 - a_{j_2+1}r_1 \oplus \sigma_{j_2+1} - \sum_{k=2}^{j_1} \frac{r_k \oplus a_{n+1-k}r_1 \oplus \sigma_{n+1-k}}{2^{j_1-k+1}} - \frac{r_1 \oplus \sigma_n}{2^{j_1}}\bigg|\bigg) \\ &\qquad\qquad\qquad\qquad \times \bigg(\sum_{k=1}^{j_2} \frac{s_k \oplus a_k r_1 \oplus \sigma_k}{2^{j_2-k+1}}\bigg)\end{aligned}$$

$$+\left(1-\left|1-1\oplus a_{j_2+1}r_1\oplus\sigma_{j_2+1}-\sum_{k=2}^{j_1}\frac{r_k\oplus a_{n+1-k}r_1\oplus\sigma_{n+1-k}}{2^{j_1-k+1}}-\frac{r_1\oplus\sigma_n}{2^{j_1}}\right|\right)$$
$$\times\left(1-\sum_{k=1}^{j_2}\frac{s_k\oplus a_kr_1\oplus\sigma_k}{2^{j_2-k+1}}\right)\Bigg\}-2^{-2n-2}.$$

Proof. By the condition $\boldsymbol{z}\in I_{\boldsymbol{j},\boldsymbol{m}}$ all digits but $t_{j_2+1}=t_{n-j_1}$ are fixed. Hence, we get the result by summing $(1-|2m_1+1-2^{j_1+1}z_1|)(1-|2m_2+1-2^{j_2+1}z_2|)$ over the two possibilities $t_{j_2+1}=0,1$ and expressing the other digits of z_1 and z_2 in terms of the digits r_{i_1} and s_{i_2} of m_1 and m_2 according to (10). ∎

Lemma 3.22. *We have*

$$\sum_{\boldsymbol{j}\in\mathcal{J}_{11}}2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m}}|^2=\frac{1}{27}2^{-4n-6}(3n2^{2n}+7\cdot 2^{2n}+48n-88).$$

Proof. As usual, we first investigate $\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}\mu_{\boldsymbol{j},\boldsymbol{m}}^2$. We sum over r_1 to obtain

$$\sum_{r_2,\dots,r_{j_1}=0}^{1}\sum_{s_1,\dots,s_{j_2}=0}^{1}$$
$$\Bigg\{2^{-2n-1}\left(1-\left|1-1\oplus\sigma_{j_2+1}-\sum_{k=2}^{j_1}2^{k-1-j_1}(r_k\oplus\sigma_{n+1-k})-2^{-j_1}\sigma_n\right|\right)$$
$$\times\left(\sum_{k=1}^{j_2}2^{k-1-j_2}(s_k\oplus\sigma_k)\right)$$
$$+2^{-2n-1}\left(1-\left|1-1\oplus\sigma_{j_2+1}-\sum_{k=2}^{j_1}2^{k-1-j_1}(r_k\oplus\sigma_{n+1-k})-2^{-j_1}\sigma_n\right|\right)$$
$$\times\left(1-\sum_{k=1}^{j_2}2^{k-1-j_2}(s_k\oplus\sigma_k)\right)-2^{-2n-2}\Bigg\}^2$$
$$+\sum_{r_2,\dots,r_{j_1}=0}^{1}\sum_{s_1,\dots,s_{j_2}=0}^{1}$$
$$\Bigg\{2^{-2n-1}\left(1-\left|1-1\oplus\sigma'_{j_2+1}-\sum_{k=2}^{j_1}2^{k-1-j_1}(r_k\oplus\sigma'_{n+1-k})-2^{-j_1}(\sigma_n\oplus 1)\right|\right)$$
$$\times\left(\sum_{k=1}^{j_2}2^{k-1-j_2}(s_k\oplus\sigma'_k)\right)$$

$$+2^{-2n-1}\Big(1-\Big|1-1\oplus\sigma'_{j_2+1}-\sum_{k=2}^{j_1}2^{k-1-j_1}(r_k\oplus\sigma'_{n+1-k})-2^{-j_1}(\sigma_n\oplus 1)\Big|\Big)$$
$$\times\Big(1-\sum_{k=1}^{j_2}2^{k-1-j_2}(s_k\oplus\sigma'_k)\Big)-2^{-2n-2}\Big\}^2=:M_1(\sigma_{j_2+1})+M_2(\sigma'_{j_2+1}).$$

We can compute $M_1(0)$ via

$$\sum_{l_1=0}^{2^{j_1-1}-1}\sum_{l_2=0}^{2^{j_2}-1}\Big[2^{-2n-1}\Big\{\frac{l_2}{2^{j_2}}\Big(\frac{l_1}{2^{j_1-1}}+\frac{\sigma_n}{2^{j_1}}\Big)$$
$$+\Big(1-\frac{l_2}{2^{j_2}}\Big)\Big(1-\frac{l_1}{2^{j_1-1}}-\frac{\sigma_n}{2^{j_1}}\Big)\Big\}-2^{-2n-2}\Big]^2.$$

Similarly, one calculates $M_1(1)$ and finds $M_1(1)=M_1(0)$. We can compute $M_2(0)$ with the same formula as for $M_1(0)$—we just have to replace σ_n by $1-\sigma_n$. Again we have $M_2(1)=M_2(0)$ and therefore $\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}\mu_{\boldsymbol{j},\boldsymbol{m}}^2=M_1(0)+M_2(0)$. The rest follows by a straightforward summation of $2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}\mu_{\boldsymbol{j},\boldsymbol{m}}^2$ over all $\boldsymbol{j}\in\mathcal{J}_{11}$. ∎

Case 12: $\boldsymbol{j}\in\mathcal{J}_{12}:=\{(j_1,j_2)\in\mathbb{N}_0^2: j_1+j_2\geq n,\ 1\leq j_1,j_2\leq n-1\}$

PROPOSITION 3.23. *Let $\boldsymbol{j}\in\mathcal{J}_{12}$ and $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$. Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m}}=2^{-n-j_1-j_2-2}\Big(1-\Big|1-\sum_{k=1}^{n-j_1}\frac{s_k\oplus a_kr_1\oplus\sigma_k}{2^{n-j_1-k}}\Big|\Big)$$
$$\times\Big(1-\Big|1-\sum_{k=2}^{n-j_2}\frac{r_k\oplus a_{n+1-k}r_1\oplus\sigma_{n+1-k}}{2^{n-j_2-k}}-\frac{r_1\oplus\sigma_n}{2^{n-j_2-1}}\Big|\Big)-2^{-2j_1-2j_2-4}$$

if $s_\mu\oplus a_\mu r_1\oplus\sigma_\mu=r_{n+1-\mu}$ for all $\mu\in\{n+1-j_1,\ldots,j_2\}$, and $\mu_{\boldsymbol{j},\boldsymbol{m}}=-2^{-2j_1-2j_2-4}$ otherwise.

Proof. Again, the condition $\boldsymbol{z}\in I_{\boldsymbol{j},\boldsymbol{m}}$ implies, by (10), that $t_{n+1-k}=r_k$ for all $k\in\{1,\ldots,j_1\}$ and $b_k=s_k$ for all $k\in\{1,\ldots,j_2\}$. As a result, for $\mu\in\{n+1-j_1,\ldots,j_2\}$ we must have

$$r_{n+1-\mu}=b_\mu\oplus a_\mu t_n\oplus\sigma_\mu=s_\mu\oplus a_\mu r_1\oplus\sigma_\mu \tag{15}$$

in order to have a point of $\mathcal{P}$ in the dyadic box $I_{\boldsymbol{j},\boldsymbol{m}}$. Hence, if (15) is not satisfied, then only the linear part of the discrepancy function contributes to the Haar coefficient and hence $\mu_{\boldsymbol{j},\boldsymbol{m}}=-2^{-2j_1-2j_2-4}$.

Assume now that (15) is satisfied and let $\boldsymbol{z}=(z_1,z_2)$ be the single point in $I_{\boldsymbol{j},\boldsymbol{m}}$. Then by (11) and (12) we obtain

$$\mu_{\boldsymbol{j},\boldsymbol{m}}=2^{-n-j_1-j_2-2}(1-|1-t_{n-j_1}-\cdots-2^{j_1-n+1}t_1|)$$
$$\times(1-|1-b_{j_2+1}-\cdots-2^{j_2-n+1}b_n|)-2^{-2j_1-2j_2-4},$$

where the above conditions on the digits give $t_k = s_k \oplus a_k r_1 \oplus \sigma_k$ for $k = 1, \dots, n - j_1$ and $b_{n+1-k} = r_k \oplus a_{n+1-k} r_1 \oplus \sigma_{n+1-k}$ for $k = 2, \dots, n - j_2$ as well as $b_n = r_1 \oplus \sigma_n$. Hence the result follows. ■

Lemma 3.24. *We have*

$$\sum_{\boldsymbol{j} \in \mathcal{J}_{12}} 2^{|\boldsymbol{j}|} \sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 = \frac{1}{27} 4^{-2n-2} - \frac{1}{27} 4^{-n-2} - \frac{1}{9} n 4^{-2n-1} + \frac{5}{9} n 4^{-n-3}.$$

Proof. We write

$$\begin{aligned}\sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 &= \sum_{m_1=0}^{2^{j_1}-1} \Big(\sum_{\substack{m_2=0 \\ (15)\text{ satisfied}}}^{2^{j_2}-1} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 + \sum_{\substack{m_2=0 \\ (15)\text{ not satisfied}}}^{2^{j_2}-1} (-2^{-2j_2-2j_2-4})^2 \Big) \\ &= \sum_{m_1=0}^{2^{j_1}-1} \sum_{\substack{m_2=0 \\ (15)\text{ satisfied}}}^{2^{j_2}-1} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 + 2^{j_1}(2^{j_2} - 2^{n-j_1}) 2^{-4j_1-4j_2-8}.\end{aligned}$$

Note that for a fixed $m_1 \in \mathbb{D}_{j_1}$ the system (15) fixes the digits $s_{n-j_1+1}, \dots, s_{j_2}$ and thus the digits $s_1, \dots, s_{n-j_1}$ remain free. This means that there are 2^{n-j_1} elements in $\mathbb{D}_{j_2}$ which satisfy (15), whereas the remaining $2^{j_2} - 2^{n-j_1}$ elements do not. It is where the factor $2^{j_2} - 2^{n-j_1}$ in the last expression comes from. Let us study

$$\sum_{m_1=0}^{2^{j_1}-1} \sum_{\substack{m_2=0 \\ (15)\text{ satisfied}}}^{2^{j_2}-1} \mu_{\boldsymbol{j},\boldsymbol{m}}^2.$$

It equals

$$\begin{aligned}&\sum_{r_2,\dots,r_{j_1}=0}^{1} \sum_{s_1,\dots,s_{n-j_1}}^{1} \\ &\qquad \big(2^{-n-j_1-j_2-2}(1 - |1 - s_{n-j_1} \oplus \sigma_{n-j_1} - \dots - 2^{j_1-n+1}(s_1 \oplus \sigma_1)|) \\ &\qquad \times (1 - |1 - r_{n-j_2} \oplus \sigma_{j_2+1} - \dots - 2^{j_2-n+1} \sigma_n|) - 2^{-2j_1-2j_2-4}\big)^2 \\ &\quad + \sum_{r_2,\dots,r_{j_1}=0}^{1} \sum_{s_1,\dots,s_{n-j_1}}^{1} \\ &\qquad \big(2^{-n-j_1-j_2-2}(1 - |1 - s_{n-j_1} \oplus \sigma'_{n-j_1} - \dots - 2^{j_1-n+1}(s_1 \oplus \sigma'_1)|) \\ &\qquad \times (1 - |1 - r_{n-j_2} \oplus \sigma'_{j_2+1} - \dots - 2^{j_2-n+1}(\sigma_n \oplus 1)|) - 2^{-2j_1-2j_2-4}\big)^2 \\ &=: S_1 + S_2,\end{aligned}$$

where we have already summed over r_1. The sums S_1 and S_2 can be computed similarly. Note that the summands in S_1 do not depend on the digits

$r_{n-j_2+1},\dots,r_{j_1}$. Summation over r_{n-j_2} and s_{n-j_1} leads to

$$S_1 = 2^{j_1+j_2-n} \sum_{r_2,\dots,r_{n-j_2-1}=0}^{1} \sum_{s_1,\dots,s_{n-j_1-1}=0}^{1} \Big\{\big(2^{-n-j_1-j_2-2}u(v+2^{j_2-n+1}\sigma_n) - 2^{-2j_2-2j_2-4}\big)^2 + \big(2^{-n-j_1-j_2-2}u(1-v-2^{j_2-n+1}\sigma_n) - 2^{-2j_2-2j_2-4}\big)^2 + \big(2^{-n-j_1-j_2-2}(1-u)(v+2^{j_2-n+1}\sigma_n) - 2^{-2j_2-2j_2-4}\big)^2 + \big(2^{-n-j_1-j_2-2}(1-u)(1-v-2^{j_2-n+1}\sigma_n) - 2^{-2j_2-2j_2-4}\big)^2\Big\},$$

where $u=2^{-1}(s_{n-j_1-1}\oplus\sigma_{n-j_1-1})+\dots+2^{j_1-n+1}(s_1\oplus\sigma_1)$ and $v=2^{-1}(r_{n-j_2-1}\oplus\sigma_{j_2+2})+\dots+2^{j_2-n+2}(r_2\oplus\sigma_{n-1})$. To compute the sum over the remaining digits, we replace u by $2^{-n+j_1+1}l_1$ and v by $2^{-n+j_2+2}l_2$, and let l_1 run from 0 to $2^{n-j_1-1}-1$ and l_2 run from 0 to $2^{n-j_2-2}-1$, respectively. This yields

$$S_1 = -2^{-3j_1-3j_2-8} + \frac{1}{9}2^{-5n-1} + \frac{1}{9}2^{-3n-2j_1-2} + \frac{1}{9}2^{-3n-2j_2-4} + \frac{1}{9}2^{-n-2j_1-2j_2-5} + 2^{n-4j_1-4j_2-9} - \sigma_n\left(\frac{1}{3}2^{-5n-2} + \frac{1}{3}2^{-3n-2j_1-3}\right).$$

We obtain a similar result for S_2 with the only difference that σ_n is replaced by $1-\sigma_n$. Altogether, we find

$$\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 = -2^{-3j_1-3j_2-7} + \frac{1}{9}2^{-5n-2} + \frac{1}{9}2^{-3n-2j_1-3} + \frac{1}{9}2^{-3n-2j_2-3} + \frac{1}{9}2^{-n-2j_1-2j_2-4} + 2^{n-4j_1-4j_2-8} + 2^{j_1}(2^{j_2}-2^{n-j_1})2^{-4j_1-4j_2-8}.$$

The rest follows by a straightforward summation of $2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}\mu_{\boldsymbol{j},\boldsymbol{m}}^2$ over all $\boldsymbol{j}\in\mathcal{J}_{11}$. ∎

Case 13: $\boldsymbol{j}\in\mathcal{J}_{13} := \{(j_1,j_2)\in\mathbb{N}_0^2 : j_1\ge n$ **or** $j_2\ge n\}$

PROPOSITION 3.25. *Let $\boldsymbol{j}\in\mathcal{J}_{13}$ and $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$. Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m}} = -2^{-2j_1-2j_2-4}.$$

Proof. No point lies in the interior of $I_{\boldsymbol{j},\boldsymbol{m}}$ if $j_1\ge n$ or $j_2\ge n$, and hence the result follows directly from (8). ∎

Since the Haar coefficients in this case are independent of $\boldsymbol{m}$, the following consequence is easy to verify.

LEMMA 3.26. *We have*

$$\sum_{\boldsymbol{j}\in\mathcal{J}_{13}} 2^{|\boldsymbol{j}|} \sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 = \frac{1}{9}2^{-4n-4}(2^{2n+1}-1).$$

4. The Haar coefficients of symmetrized digital nets. From the construction $\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma})=\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})\cup\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}^*)$, it is easy to see that for the Haar coefficients $\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}$ of $\Delta(\cdot,\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma}))$ we have $\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}=\frac{1}{2}(\mu_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{\sigma}}+\mu_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{\sigma}^*})$ (compare [9, proof of Lemma 3]). Here, $\mu_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{\sigma}}$ denote the Haar coefficients of $\Delta(\cdot,\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}))$ and $\mu_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{\sigma}^*}$ those of $\Delta(\cdot,\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}^*))$. Hence, it is easy to derive $\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}$ from our previous results.

PROPOSITION 4.1. *Let $\boldsymbol{j}\in\mathbb{N}_{-1}^2$ and $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$. Then $\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}$ equals:*

- $\frac{1}{2^{n+1}}+\frac{1}{2^{2n+2}}$ *if $\boldsymbol{j}\in\mathcal{J}_1$.*
- $\frac{1}{2^{2n+3}}\big(2-\frac{1}{2^{2j_2-n}}\big)-\frac{1+a_{j_2+1}(2(\sigma_{j_2+1}\oplus\sigma_n)-1)}{2^{2n+2}}$ *if $\boldsymbol{j}\in\mathcal{J}_2$.*
- $-\frac{1}{2^{3n+1}}+\frac{1}{2^{2n+2}}\sum_{k=1}^{n-1}\frac{a_k(1-s_k\oplus\sigma_k\oplus\sigma_n)}{2^{n-k}}$ *if $\boldsymbol{j}\in\mathcal{J}_3$.*
- -2^{-2j_i-3} *with $i=1$ or $i=2$ if $\boldsymbol{j}\in\mathcal{J}_4$ or $\boldsymbol{j}\in\mathcal{J}_7$, respectively.*
- $-\frac{1}{2^{n+3}}$ *if $\boldsymbol{j}\in\mathcal{J}_5$.*
- $-\frac{1}{2^{n+2j_1+3}}$ *if $\boldsymbol{j}\in\mathcal{J}_6$.*
- $\frac{1}{2^{2n+2}}(\sigma_{j_2+1}+\sigma'_{j_2+1}-1)(2\sigma_n-1)$ *if $\boldsymbol{j}\in\mathcal{J}_8$.*
- $\frac{1}{2^{2n+2}}(2(a_{n-j_1}r_1\oplus\sigma_{j_2+1})-1)(2(a_{j_2+1}r_1\oplus\sigma_{n-j_1})-1)$ *if $\boldsymbol{j}\in\mathcal{J}_9$.*
- $-(-1)^{\sigma_n}2^{-2n-2}\sum_{k=1}^{n-1}\frac{(1-a_k)(2(s_k\oplus\sigma_k)-1)}{2^{n-k}}$ *if $\boldsymbol{j}\in\mathcal{J}_{10}$.*
- $$\begin{aligned}&2^{-2n-2}\big\{\big(1-\big|1-a_{j_2+1}r_1\oplus\sigma_{j_2+1}-u-2^{-j_1}(r_1\oplus\sigma_n)\big|\big)v\\&+\big(1-\big|1-a_{j_2+1}r_1\oplus\sigma_{j_2+1}\oplus 1-u-2^{-j_1}(r_1\oplus\sigma_n)\big|\big)(1-v)\big\}\\&+2^{-2n-2}\big\{\big(1-\big|1-a_{j_2+1}r_1\oplus\sigma_{j_2+1}\oplus 1-u'-2^{-j_1}(r_1\oplus\sigma_n\oplus 1)\big|\big)v\\&+\big(1-\big|1-a_{j_2+1}r_1\oplus\sigma_{j_2+1}-u'-2^{-j_1}(r_1\oplus\sigma_n\oplus 1)\big|\big)(1-v')\big\}-2^{-2n-2}\end{aligned}$$
  if $\boldsymbol{j}\in\mathcal{J}_{11}$, where $u=\sum_{k=2}^{j_1}2^{k-1-j_1}(r_k\oplus a_{n+1-k}r_1\oplus\sigma_{n+1-k})$, $u'=\sum_{k=2}^{j_1}2^{k-1-j_1}-u$, $v=\sum_{k=1}^{j_2}2^{k-1-j_2}(s_k\oplus a_kr_1\oplus\sigma_k)$ and $v'=\sum_{k=1}^{j_2}2^{k-1-j_2}-v$.
- *For $\boldsymbol{j}\in\mathcal{J}_{12}$:*
  $$\begin{aligned}&2^{-n-j_1-j_2-3}\left(1-\left|1-\sum_{k=1}^{n-j_1}\frac{s_k\oplus a_kr_1\oplus\sigma_k}{2^{n-j_1-k}}\right|\right)\\&\qquad\times\left(1-\left|1-\sum_{k=2}^{n-j_2}\frac{r_k\oplus a_{n+1-k}r_1\oplus\sigma_{n+1-k}}{2^{n-j_2-k}}-\frac{r_1\oplus\sigma_n}{2^{n-j_2-1}}\right|\right)\\&\qquad+2^{-n-j_1-j_2-3}\left(1-\left|1-\sum_{k=1}^{n-j_1}\frac{s_k\oplus a_kr_1\oplus\sigma_k\oplus 1}{2^{n-j_1-k}}\right|\right)\\&\qquad\times\left(1-\left|1-\sum_{k=2}^{n-j_2}\frac{r_k\oplus a_{n+1-k}r_1\oplus\sigma_{n+1-k}\oplus 1}{2^{n-j_2-k}}-\frac{r_1\oplus\sigma_n\oplus 1}{2^{n-j_2-1}}\right|\right)\\&\qquad-2^{-2j_1-2j_2-4}\end{aligned}$$

if $j_1+j_2=n$;

$$2^{-n-j_1-j_2-3}\left(1-\left|1-\sum_{k=1}^{n-j_1}\frac{s_k\oplus a_kr_1\oplus\sigma_k}{2^{n-j_1-k}}\right|\right)$$
$$\times\left(1-\left|1-\sum_{k=2}^{n-j_2}\frac{r_k\oplus a_{n+1-k}r_1\oplus\sigma_{n+1-k}}{2^{n-j_2-k}}-\frac{r_1\oplus\sigma_n}{2^{n-j_2-1}}\right|\right)-2^{-2j_1-2j_2-4}$$

if $j_1+j_2\geq n+1$ *and* $s_\mu\oplus a_\mu r_1\oplus\sigma_\mu=r_{n+1-\mu}$ *for all* $\mu\in\{j_2,\dots,n+1-j_1\}$;

$$2^{-n-j_1-j_2-3}\left(1-\left|1-\sum_{k=1}^{n-j_1}\frac{s_k\oplus a_kr_1\oplus\sigma_k\oplus 1}{2^{n-j_1-k}}\right|\right)$$
$$\times\left(1-\left|1-\sum_{k=2}^{n-j_2}\frac{r_k\oplus a_{n+1-k}r_1\oplus\sigma_{n+1-k}\oplus 1}{2^{n-j_2-k}}-\frac{r_1\oplus\sigma_n\oplus 1}{2^{n-j_2-1}}\right|\right)-2^{-2j_1-2j_2-4}$$

if $j_1+j_2\geq n+1$ *and* $s_\mu\oplus a_\mu r_1\oplus\sigma_\mu\oplus 1=r_{n+1-\mu}$ *for all* $\mu\in\{j_2,\dots,n+1-j_1\}$; *and* $-2^{-2j_1-2j_2-4}$ *otherwise.*

- $-2^{-2j_1-2j_2-4}$ *if* $\boldsymbol{j}\in\mathcal{J}_{13}$.

Now we have to calculate

$$\Sigma_i:=\sum_{\boldsymbol{j}\in\mathcal{J}_i}2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}|^2$$

for all $i\in\{1,\dots,13\}$. In many cases this is easy, and the argument in the more difficult cases is very similar to what we did in the previous section. We therefore state the following results without proofs.

LEMMA 4.2. *Consider a symmetrized net* $\widetilde{\mathcal{P}_{\boldsymbol{a}}}(\boldsymbol{\sigma})$. *Let* $\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}$ *for* $\boldsymbol{j}\in\mathbb{N}_{-1}^2$ *and* $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$ *be the Haar coefficients of the corresponding discrepancy function. Then*

- $\Sigma_1=\left(\frac{1}{2^{n+1}}+\frac{1}{2^{2n+2}}\right)^2$.
- $\Sigma_2=\frac{1}{3\cdot 2^{4n+4}}(2^{2n}-4)-\frac{(-1)^{\sigma_n}}{2^{3n+4}}L+\frac{1}{2^{4n+6}}\sum_{i=1}^{n-1}a_i2^{2i}$.
- $\Sigma_3=\frac{1}{2^{4n+6}}\sum_{i=1}^{n-1}a_i2^{2i}+\frac{1}{2^{4n+4}}$.
- $\Sigma_4=\sum_{\boldsymbol{j}\in\mathcal{J}_7}2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}|^2=\frac{1}{48\cdot 2^{2n}}$.
- $\Sigma_5=\frac{1}{2^{2n+6}}$.
- $\Sigma_6=\frac{1}{3\cdot 2^{4n+6}}(2^{2n}-4)$.
- $\Sigma_7=\frac{1}{3\cdot 2^{4n+6}}(2^{2n}-4)-\frac{1}{2^{4n+6}}\sum_{i=1}^{n-1}a_i2^{2i}$.
- $\Sigma_8=\frac{1}{9\cdot 2^{4n+6}}(3n\cdot 2^{2n}-7\cdot 2^{2n}+16)$.
- $\Sigma_9=\frac{1}{3\cdot 2^{4n+6}}(2^{2n}-4)-\frac{1}{2^{4n+6}}\sum_{i=1}^{n-1}a_i2^{2i}$.

- $\Sigma_{10} = \frac{1}{9}2^{-4n-6}(5 \cdot 4^n + 4 - 24n)$.
- $\Sigma_{11} = \frac{1}{3 \cdot 2^{4n+6}}\big(n(2^{2n}+8) - 2(2^{2n}+2)\big)$.
- $\Sigma_{12} = \frac{1}{9 \cdot 2^{4n+4}}(2^{2n+1} - 1)$.

We obtain Theorem 2 via $(L_2(\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma})))^2 = \sum_{i=1}^{13} \Sigma_i$.

5. Why do (symmetrized) digital nets fail to have the optimal order of L_2 discrepancy? In many previous papers (e.g. [3, 14]) it has been observed that the reason that a point set fails to have the optimal order of L_2 discrepancy can often be found in the zeroth Fourier coefficient of the corresponding discrepancy function (which is the same as the Haar coefficient for $\boldsymbol{j} = (-1,-1)$). This recurring phenomenon led to the following conjecture by Bilyk [1]:

Whenever an N-element point set $\mathcal{P}$ in $[0,1)^2$ satisfies $L_\infty(\mathcal{P}) \lesssim (\log N)/N$ (i.e. its star discrepancy is of best possible order in N) and $L_2(\mathcal{P}) \gtrsim (\log N)/N$, then $\mathcal{P}$ should also satisfy

$$\Big| \int_{[0,1)^2} \Delta(\boldsymbol{t}, \mathcal{P}) \, \mathrm{d}\boldsymbol{t} \Big| \gtrsim \frac{\log N}{N}.$$

Our results imply that it is not true. Consider the point set $\mathcal{P}_{\mathbf{1}}$, where $\mathbf{1} = (1,\dots,1) \in \mathbb{Z}_2^{n-1}$. Then by Proposition 1 we have $\mu_{(-1,-1),(0,0)} = 2^{-2n-2} + 5 \cdot 2^{-n-3} \leq 1/N$, but $L_2(\mathcal{P}_{\mathbf{1}}) \gtrsim (\log N)/N$, which follows from Corollary 1.4. Note that $L_\infty(\mathcal{P}_{\mathbf{1}}) \lesssim (\log N)/N$, since $\mathcal{P}_{\mathbf{1}}$ is a $(0,n,2)$-net. Hence $\mathcal{P}_{\mathbf{1}}$ is a counterexample to Bilyk's conjecture. More generally, none of the nets $\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ achieves the optimal order of L_2 discrepancy. The reason is that for all $\boldsymbol{a}$ at least one of the inequalities $\mu_{(-1,-1),(0,0)} \gtrsim (\log N)/N$ or $\mu_{(0,-1),(0,0)} \gtrsim (\log N)/N$ holds; hence in some cases the Haar coefficient for $\boldsymbol{j} = (-1,-1)$ is not the one causing trouble.

We point out that an earlier counterexample to the above conjecture appears in [14]. To state it, we consider the digital $(0,n,2)$-net generated by the matrices $C_1 = A_1$ (see (3)) and the matrix

$$C_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

which we call $\mathcal{P}_c$. We denote its shifted version by $\mathcal{P}_c(\boldsymbol{\sigma})$. The following theorem of Larcher and Pillichshammer [14, Theorem 1] shows that not every symmetrized digital net achieves the optimal order of L_2 discrepancy. Their proof is based on a Walsh function analysis of the discrepancy function. Here we shall give a new proof based on Haar functions.

THEOREM 5.1 (Larcher and Pillichshammer). *The L_2 discrepancy of the symmetrized point set $\mathcal{P}_c^{\mathrm{sym}} := \mathcal{P}_c \cup \{(x, 1-y) : (x,y) \in \mathcal{P}_c\}$ with $N = 2^{n+1}$ elements satisfies*

$$L_2(\mathcal{P}_c^{\mathrm{sym}}) \gtrsim \frac{\log N}{N}.$$

(*Note that* $\mu_{(-1,-1),(0,0)}(\Delta(\cdot, \mathcal{P}_c^{\mathrm{sym}})) = 2^{-n-2}$ *and* $L_\infty(\mathcal{P}_c^{\mathrm{sym}}) \lesssim (\log N)/N$.)

Proof. Instead of $\mathcal{P}_c^{\mathrm{sym}}$ we investigate the L_2 discrepancy of $\widetilde{\mathcal{P}}_c(\boldsymbol{\sigma}) = \mathcal{P}_c(\boldsymbol{\sigma}) \cup \mathcal{P}_c(\boldsymbol{\sigma}^*)$, because the difference between $L_2(\mathcal{P}_c^{\mathrm{sym}})$ and $L_2(\widetilde{\mathcal{P}}_c(\mathbf{0}))$ is at most 2^{-n} (see [9, Lemma 4]). Let $\mu^{\boldsymbol{\sigma}}_{\boldsymbol{j},\boldsymbol{m}}$ denote the Haar coefficients of $\Delta(\cdot, \mathcal{P}_c(\boldsymbol{\sigma}))$, and $\tilde{\mu}^{\boldsymbol{\sigma}}_{\boldsymbol{j},\boldsymbol{m}}$ those of $\Delta(\cdot, \widetilde{\mathcal{P}}_c(\boldsymbol{\sigma}))$. The idea of the proof is as follows: By Parseval's identity we have $L_2(\widetilde{\mathcal{P}}_c(\mathbf{0})) \geq \tilde{\mu}^{\mathbf{0}}_{(-1,0),(0,0)}$. We will show $\tilde{\mu}^{\mathbf{0}}_{(-1,0),(0,0)} \gtrsim (\log N)/N$, which yields the result.

In order to calculate $\tilde{\mu}^{\boldsymbol{\sigma}}_{(-1,0),(0,0)}$, we first compute $\mu^{\boldsymbol{\sigma}}_{(-1,0),(0,0)}$ for an arbitrary shift. We write

$$\mathcal{P}_c(\boldsymbol{\sigma}) = \left\{ \left(\frac{t_n}{2} + \cdots + \frac{t_1}{2^n}, \frac{t_1 \oplus \sigma_1}{2} + \cdots + \frac{t_1 \oplus t_n \oplus \sigma_n}{2^n} \right) : t_1, \ldots, t_n \in \{0,1\} \right\}.$$

For $\boldsymbol{z} = (z_1, z_2) \in \mathcal{P}_c(\boldsymbol{\sigma})$ we have

$$\begin{aligned}
&\sum_{\boldsymbol{z} \in \mathcal{P}_c(\boldsymbol{\sigma})} (1 - z_1)(1 - |1 - 2z_2|) \\
&= \sum_{t_1,\ldots,t_n=0}^{1} \left(1 - \frac{t_n}{2} - \cdots - \frac{t_1}{2^n}\right) \\
&\qquad\qquad \times \left(1 - \left|1 - t_1 \oplus \sigma_1 - \frac{t_1 \oplus t_2 \oplus \sigma_2}{2} - \cdots - \frac{t_1 \oplus t_n \oplus \sigma_n}{2^{n-1}}\right|\right) \\
&= \sum_{t_2,\ldots,t_n=0}^{1} \left\{ \left(1 - u - \frac{\sigma_1}{2^n}\right) v(\sigma_1) + \left(1 - u - \frac{\sigma_1 \oplus 1}{2^n}\right)(1 - v(\sigma_1 \oplus 1)) \right\} \\
&= \sum_{t_2,\ldots,t_n=0}^{1} \{ -2^{-2n-2} + 2^{-n+1} + 2^{-2n+1}\sigma_1 - 2^{-n+1}u + 2v(\sigma_1) \\
&\qquad\qquad\qquad\qquad - 2^{-n}v(\sigma_1) - 2uv(\sigma_1) \},
\end{aligned}$$

where $u = 2^{-1}t_n + \cdots + 2^{-n+1}t_2$ and $v(t_1) = 2^{-1}(t_1 \oplus t_2 \oplus \sigma_2) + \cdots + 2^{-n+1}(t_1 \oplus t_n \oplus \sigma_n)$. In the last step we have used $v(\sigma_1 \oplus 1) = 1 - 2^{-n+1} - v(\sigma_1)$. We have $\sum_{t_2,\ldots,t_n=0}^{1} u = \sum_{t_2,\ldots,t_n=0}^{1} v(\sigma_1) = 2^{n-2} - 2^{-1}$; hence it remains to investigate $\sum_{t_2,\dots,t_n=0}^{1} uv(\sigma_1)$. We find that

$$\begin{aligned}\sum_{t_2,\dots,t_n=0}^{1} uv(\sigma_1) &= \sum_{k=2}^{n}\sum_{t_2,\dots,t_n=0}^{1}\frac{t_k(t_k\oplus\sigma_k\oplus\sigma_1)}{2^{n+1-k}2^{k-1}}\\ &\quad+\sum_{\substack{k_1,k_2=2\\k_1\neq k_2}}^{n}\sum_{t_2,\dots,t_n=0}^{1}\frac{t_{k_1}(t_{k_2}\oplus\sigma_{k_2}\oplus\sigma_1)}{2^{n+1-k_1}2^{k_2-1}}\\ &=\frac{1}{2^n}\sum_{k=2}^{n}2^{n-2}\sum_{t_k=0}^{1}t_k(t_k\oplus\sigma_k\oplus\sigma_1)\\ &\quad+\frac{1}{2^n}\sum_{\substack{k_1,k_2=2\\k_1\neq k_2}}^{n}2^{k_1-k_2}2^{n-3}\sum_{t_{k_1},t_{k_2}=0}^{1}t_{k_1}(t_{k_2}\oplus\sigma_{k_2}\oplus\sigma_1)\\ &=\frac{1}{4}\sum_{k=2}^{n}(1\oplus\sigma_k\oplus\sigma_1)+\frac{1}{8}\sum_{\substack{k_1,k_2=2\\k_1\neq k_2}}^{n}2^{k_1-k_2}.\end{aligned}$$

Combining our results with (7) yields

$$\mu_{(-1,0),(0,0)} = 2^{-2n-2}-2^{-n-3}n-2^{-2n-1}\sigma_1+2^{-n-2}\sum_{k=2}^{n}(1\oplus\sigma_k\oplus\sigma_1).$$

Since $\tilde{\mu}^{\boldsymbol{\sigma}}_{(-1,0),(0,0)}=\frac{1}{2}(\mu^{\boldsymbol{\sigma}}_{(-1,0),(0,0)}+\mu^{\boldsymbol{\sigma}^*}_{(-1,0),(0,0)})$, we derive

$$\tilde{\mu}^{\boldsymbol{\sigma}}_{(-1,0),(0,0)} = -2^{-n-3}n+2^{-n-2}\sum_{k=2}^{n}(1\oplus\sigma_k\oplus\sigma_1).$$

In particular, for $\boldsymbol{\sigma}=\mathbf{0}$ we find $\tilde{\mu}^{\mathbf{0}}_{(-1,0),(0,0)}=2^{-n-3}(n-2)\gtrsim(\log N)/N$, and we are done. ∎

6. Further results. Our method is not restricted to the class of digital nets $\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$. For instance, one could also study the nets $\mathcal{P}_{\boldsymbol{c}}(\boldsymbol{\sigma})$ generated by $C_1=A_1$ and

$$C_2=\begin{pmatrix}1&0&0&\cdots&0&0&0\\ c_2&1&0&\cdots&0&0&0\\ c_3&0&1&\cdots&0&0&0\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\ c_{n-2}&0&0&\cdots&1&0&0\\ c_{n-1}&0&0&\cdots&0&1&0\\ c_n&0&0&\cdots&0&0&1\end{pmatrix},$$

where we write $\boldsymbol{c}=(c_2,\dots,c_n)$ and again we apply a digital shift $\boldsymbol{\sigma}=$

$(\sigma_1,\dots,\sigma_n)$ to the second components of the relevant points. We simply write $\mathcal{P}_{\boldsymbol{c}}$ if we do not apply a shift. Further we put $\widetilde{\mathcal{P}}_{\boldsymbol{c}}(\boldsymbol{\sigma}) := \mathcal{P}_{\boldsymbol{c}}(\boldsymbol{\sigma}) \cup \mathcal{P}_{\boldsymbol{c}}(\boldsymbol{\sigma}^*)$. There are many parallel tracks in the computation of the Haar coefficients of $\Delta(\cdot,\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}))$ and $\Delta(\cdot,\mathcal{P}_{\boldsymbol{c}}(\boldsymbol{\sigma}))$. We leave it to the reader as a (tedious) exercise to show the following theorem with the method demonstrated in Section 3.

THEOREM 6.1. *Let $L=\sum_{i=2}^n c_i(1-2\sigma_i)$ and $\ell=\sum_{i=1}^n (1-2\sigma_i)$. Then*

$$(2^n\, L_2(\mathcal{P}_{\boldsymbol{c}}(\boldsymbol{\sigma})))^2 = \frac{1}{64}\left((\ell-L)^2+L^2+8\ell+2L(2\sigma_1-5)+\frac{5}{3}n\right) - \frac{1}{2^{n+4}}(\ell-4)+\frac{3}{8}-\frac{1}{9}\,\frac{1}{2^{2n+3}}.$$

For unshifted nets we find a result of the very same form as Corollary 1.4.

COROLLARY 6.2. *Let $|\boldsymbol{c}|=\sum_{i=2}^n c_i$. Then*

$$(2^n\, L_2(\mathcal{P}_{\boldsymbol{c}}))^2 = \frac{1}{64}\left((n-|\boldsymbol{c}|)^2+|\boldsymbol{c}|^2-10|\boldsymbol{c}|+\frac{29}{3}n\right)+\frac{3}{8}-\frac{n-4}{2^{n+4}}-\frac{1}{9}\,\frac{1}{2^{2n+3}}.$$

However, there are major differences between the L_2 discrepancies of the symmetrized nets $\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ and $\widetilde{\mathcal{P}}_{\boldsymbol{c}}(\boldsymbol{\sigma})$, as our next theorem demonstrates. Since exact computation of $\sum_{\boldsymbol{j}\in\mathcal{J}_i} 2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} |\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}|^2$ for $i\in\{11,12\}$ is very complicated, we do not calculate the L_2 discrepancy exactly. However, we can show that

$$\sum_{\boldsymbol{j}\in\mathbb{N}_{-1}^2\setminus\{(-1,0)\}} 2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} |\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}|^2 \lesssim n/2^{2n}$$

and $2^{|\boldsymbol{j}|}\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} |\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}|^2 = 2^{-2n-6}(L^2-2(1-2\sigma_1)L+1)$ for $\boldsymbol{j}=(-1,0)$. Therefore the following result is a consequence of Parseval's identity.

THEOREM 6.3. *Let L be as in Theorem* 6.1. *Then $L_2(\widetilde{\mathcal{P}}_{\boldsymbol{c}}(\boldsymbol{\sigma}))\lesssim\sqrt{\log N}$ if and only if $|L|\lesssim\sqrt{n}$. For unshifted symmetrized nets we have $L_2(\widetilde{\mathcal{P}}_{\boldsymbol{c}})\lesssim\sqrt{\log N}$ if and only if $|\boldsymbol{c}|\lesssim\sqrt{n}$.*

7. Results on L_p discrepancy. The calculation of the Haar coefficients of the discrepancy functions allows us to study not only the L_2 discrepancy of point sets, but also the L_p discrepancy for all $p\in(1,\infty)$. The key tool is the Littlewood–Paley inequality for Haar functions. It states that for every f in $L_p([0,1)^2)$ with $p\in(1,\infty)$ we have $\|f\|_{L_p([0,1)^2)}\asymp\|S(f)\|_{L_p([0,1)^2)}$, where

$$S(f) := \Big(\sum_{\boldsymbol{j}\in\mathbb{N}_{-1}^s,\,\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} 2^{2|\boldsymbol{j}|}|\mu_{\boldsymbol{j},\boldsymbol{m}}|^2\mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}}\Big)^{1/2}.$$

The Littlewood–Paley inequality enables us to give sufficient and necessary conditions for the point sets we are studying to achieve the optimal order of L_p discrepancy. It is not necessary to work with the exact Haar coefficients to show these conditions. The following upper bounds can be derived immediately from the propositions in Section 3.

Corollary 7.1. *Let $\mu_{\boldsymbol{j},\boldsymbol{m}}$ be the Haar coefficients of $\Delta(\cdot,\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}))$. Let $\boldsymbol{j}=(j_1,j_2)\in\mathbb{N}_0^2$.*

(i) *If $j_1=0$ and $0\le j_2\le n-2$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}|\lesssim 2^{-n-j_2}$.*
(ii) *If $j_1+j_2<n-1$ and $j_1\ge 1$, $j_2\ge 0$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}|=2^{-2n-2}$.*
(iii) *If $j_1+j_2\ge n-1$ and $0\le j_1,j_2\le n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}|\lesssim 2^{-n-j_1-j_2}$ and $|\mu_{\boldsymbol{j},\boldsymbol{m}}|=2^{-2j_1-2j_2-4}$ for all but at most 2^n coefficients $\mu_{\boldsymbol{j},\boldsymbol{m}}$ with $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$ (the equality occurs if there is no point of $\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma})$ in the interior of $I_{\boldsymbol{j},\boldsymbol{m}}$).*
(iv) *If $j_1\ge n$ or $j_2\ge n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}|=2^{-2j_1-2j_2-4}$.*

Now let $\boldsymbol{j}=(-1,j_2)$ with $j_2\in\mathbb{N}_0$.

(v) *If $j_2<n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}|\lesssim 2^{-n-j_2}$.*
(vi) *If $j_2\ge n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}|=2^{-2j_2-3}$.*

Next let $\boldsymbol{j}=(j_1,-1)$ with $j_1\in\mathbb{N}_0$.

(vii) *If $j_1=0$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}|=\frac{1}{2^{2n+2}}-\frac{1}{2^{n+3}}+\frac{1}{2^{n+3}}L-\frac{1}{2^{2n+1}}\sigma_n$.*
(viii) *If $1\le j_1<n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}|\lesssim 2^{-n-j_1}$.*
(ix) *If $j_1\ge n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}|=2^{-2j_1-3}$.*

Finally, if $\boldsymbol{j}=(-1,-1)$ then

(vi) *$\mu_{\boldsymbol{j},\boldsymbol{m}}=\frac{1}{2^{n+1}}+\frac{1}{2^{2n+2}}+\frac{1}{2^{n+3}}(\ell-L)$.*

We insert these bounds into the Littlewood–Paley inequality to show the following result. The proof is basically the same as in [9], where the result has been shown for the Hammersley point set. Of course we can do the same for the class $\mathcal{P}_{\boldsymbol{c}}(\boldsymbol{\sigma})$ of shifted nets.

Theorem 7.2. *Let ℓ and L be as in Theorem 1.1 and $p\in(1,\infty)$. Then*

$$L_p(\mathcal{P}_{\boldsymbol{a}}(\boldsymbol{\sigma}))\lesssim_p\frac{\sqrt{\log N}}{N}$$

if and only if $|\ell-L|\lesssim_p\sqrt{n}$ and $|L|\lesssim_p\sqrt{n}$. An analogous result holds for $\mathcal{P}_{\boldsymbol{c}}(\boldsymbol{\sigma})$.

For symmetrized nets we find the following conditions which ensure the optimal order of L_p discrepancy.

Theorem 7.3. *Let $p\in(1,\infty)$. Then*

$$L_p(\widetilde{\mathcal{P}}_{\boldsymbol{a}}(\boldsymbol{\sigma}))\lesssim\frac{\sqrt{\log N}}{N}$$

for all $\boldsymbol{a}\in\mathbb{Z}_2^{n-1}$ and all $\boldsymbol{\sigma}\in\mathbb{Z}_2^n$. Moreover,

$$L_p(\widetilde{\mathcal{P}}_{\boldsymbol{c}}(\boldsymbol{\sigma}))\lesssim\frac{\sqrt{\log N}}{N}\quad\text{if and only if}\quad|L|\lesssim\sqrt{n}.$$

8. Outlook. It would be increasingly difficult to obtain exact formulas for the L_2 discrepancy of more complicated digital nets. However, we could ask for conditions on matrices C_1 and C_2 such that the L_2 discrepancy of the digital net generated by them is of optimal order (1). Let us, for instance, consider the digital $(0,n,2)$-net $\mathcal{P}_{\mathrm{tri}}$ generated by $C_1 = A_1$ and

$$C_2 = \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ 0 & 1 & a_{2,3} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ 0 & 0 & 1 & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{n-2,n-1} & a_{n-2,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

We observed that either $\mu_{(-1,-1),(0,0)}(\Delta(\cdot,\mathcal{P}_{\boldsymbol{a}})) \gtrsim (\log N)/N$ or $\mu_{(0,-1),(0,0)}(\Delta(\cdot,\mathcal{P}_{\boldsymbol{a}})) \gtrsim (\log N)/N$. If we could show a similar result for $\Delta(\cdot,\mathcal{P}_{\mathrm{tri}})$, then we would know that the nets $\mathcal{P}_{\mathrm{tri}}$ fail to achieve the optimal order of L_2 discrepancy as well. However, this is not the case in general. We define several parameters to demonstrate this claim: For $\mu \in \{1,\dots,n\}$ put $l_\mu(\mu) := 1$, and for $k \in \{1,\dots,\mu-1\}$ put

$$l_\mu(k) := \begin{cases} 0 & \text{if } \exists i \in \{k+1,\dots,\mu\} : a_{k,i} = 1, \\ 1 & \text{if } \forall i \in \{k+1,\dots,\mu\} : a_{k,i} = 0. \end{cases}$$

Then a direct computation similar to the proofs of Propositions 3.1 and 3.9 yields

$$\mu_{(-1,-1),(0,0)}(\Delta(\boldsymbol{t},\mathcal{P}_{\mathrm{tri}})) = \frac{1}{2^{n+3}} \sum_{k=1}^{n} l_n(k) + \frac{1}{2^{n+1}} + \frac{1}{2^{2n+2}},$$

$$\mu_{(0,-1),(0,0)}(\Delta(\boldsymbol{t},\mathcal{P}_{\mathrm{tri}})) = \frac{1}{2^{n+3}} \Big(\sum_{k=1}^{n-1} l_{n-1}(k) - \sum_{k=1}^{n} l_n(k) \Big) + \frac{1}{2^{2n+2}}.$$

Hence both $\mu_{(-1,-1),(0,0)}(\Delta(\cdot,\mathcal{P}_{\mathrm{tri}})) \lesssim 1/N$ and $\mu_{(0,-1),(0,0)}(\Delta(\cdot,\mathcal{P}_{\mathrm{tri}})) \lesssim 1/N$ if we choose C_2 for instance of the form

$$\begin{pmatrix} 1 & a_{1,2} & a_{1,3} & \cdots & a_{1,n-2} & 1 & 1 \\ 0 & 1 & a_{2,3} & \cdots & a_{2,n-2} & 1 & 1 \\ 0 & 0 & 1 & \cdots & a_{3,n-2} & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & a_{1,3} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ 0 & 1 & 1 & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ 0 & 0 & 1 & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & a_{n-2,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

We assume that we achieve the lowest possible L_2 discrepancy for the net $\mathcal{P}_{\mathrm{tri}}$ if we fill the whole upper right triangle of C_2 with ones. We intend to investigate whether the corresponding digital net achieves the optimal order of L_2 discrepancy without shifting or symmetrization, and we hope to determine precise conditions on C_2 which ensure that.

References

[1] D. Bilyk, *The L^2-discrepancy of irrational lattices*, in: Monte Carlo and Quasi-Monte Carlo Methods 2012, J. Dick et al. (eds.), Springer, Berlin, 2013, 289–296.

[2] D. Bilyk, V. N. Temlyakov and R. Yu, *Fibonacci sets and symmetrization in discrepancy theory*, J. Complexity 28 (2012), 18–36.

[3] H. Davenport, *Note on irregularities of distribution*, Mathematika 3 (1956), 131–135.

[4] J. Dick and F. Pillichshammer, *On the mean squared L^2 discrepancy of randomized digital (t, m, s)-nets over $\mathbb{Z}_2$*, Acta Arith. 117 (2005), 371–403.

[5] J. Dick and F. Pillichshammer, *Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration*, Cambridge Univ. Press, Cambridge, 2010.

[6] G. Halász, *On Roth's method in the theory of irregularities of point distributions*, in: Recent Progress in Analytic Number Theory, Vol. 2, Academic Press, London, 1981, 79–94.

[7] J. H. Halton and S. K. Zaremba, *The extreme and L^2 discrepancies of some plane sets*, Monatsh. Math. 73 (1969), 316–328.

[8] A. Hinrichs, *Discrepancy of Hammersley points in Besov spaces of dominating mixed smoothness*, Math. Nachr. 283 (2010), 478–488.

[9] A. Hinrichs, R. Kritzinger and F. Pillichshammer, *Optimal order of L_p discrepancy of digit shifted Hammersley point sets in dimension 2*, Unif. Distrib. Theory 10 (2015), 115–133.

[10] P. Kritzer, *On some remarkable properties of the two-dimensional Hammersley point set in base 2*, J. Théor. Nombres Bordeaux 18 (2006), 203–221.

[11] P. Kritzer and F. Pillichshammer, *An exact formula for the L_2 discrepancy of the shifted Hammersley point set*, Unif. Distrib. Theory 1 (2006), 1–13.

[12] R. Kritzinger, *An exact formula for the L_2 discrepancy of the symmetrized Hammersley point set*, Math. Comput. Simulat. 143 (2018), 3–13.

[13] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York, 1974.

[14] G. Larcher and F. Pillichshammer, *Walsh series analysis of the L_2-discrepancy of symmetrisized point sets*, Monatsh. Math. 132 (2001), 1–18.

[15] G. Larcher and F. Pillichshammer, *Sums of distances to the nearest integer and the discrepancy of digital nets*, Acta Arith. 106 (2003), 379–408.

[16] J. Matoušek, *Geometric Discrepancy. An Illustrated Guide*, Algorithms Combin. 18, Springer, Berlin, 1999.

[17] H. Niederreiter, *Point sets and sequences with small discrepancy,* Monatsh. Math. 104 (1987), 273–337.

[18] K. F. Roth, *On irregularities of distribution*, Mathematika 1 (1954), 73–79.

[19] W. M. Schmidt, *Irregularities of distribution VII*, Acta Arith. 21 (1972), 45–50.

[20] W. M. Schmidt, *Irregularities of distribution. X*, in: Number Theory and Algebra, Academic Press, New York, 1977, 311–329.

[21] T. T. Warnock, *Computational investigations of low discrepancy point sets*, in: Applications of Number Theory to Numerical Analysis, S. K. Zaremba (ed.), Academic Press, New York, 1972, 319–343.

Ralph Kritzinger
Institute of Financial Mathematics and Applied Number Theory
Johannes Kepler University
4040 Linz, Austria
E-mail: ralph.kritzinger@jku.at