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On certain vector valued Siegel modular forms of type $(k, 2)$ over $\mathbb{Z}_{(p)}$

by

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1. Introduction. For the graded ring M_*^{ev} of Siegel modular forms of even weight, it is known that the module of vector valued Siegel modular forms of type $(k, 2)$ with an even integer k and degree 2 is finitely generated over M_*^{ev}, and its generators were explicitly given by Satoh [10]. Let p be a prime number and $\mathbb{Z}_{(p)}$ the local ring of p-integral rational numbers. In this paper we study the module over $M_*^{\mathrm{ev}}(\mathbb{Z}_{(p)})$ of vector valued Siegel modular forms of type $(k, 2)$ and degree 2 such that all Fourier coefficients lie in $\mathrm{Sym}_2^*(\mathbb{Z}_{(p)}) := \{T = (t_{ij}) \in \mathrm{Sym}_2(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}_{(p)}\}$, where $M_*^{\mathrm{ev}}(\mathbb{Z}_{(p)})$ is the graded ring over $\mathbb{Z}_{(p)}$ of Siegel modular forms of degree 2 with even weight whose Fourier coefficients lie in $\mathbb{Z}_{(p)}$. Specifically, we give generators over $M_*^{\mathrm{ev}}(\mathbb{Z}_{(p)})$ of that module for an even integer k when $p \geq 5$.

We will now state our result more precisely. A Siegel modular form of type $(k, 2)$ is a holomorphic function f on the Siegel upper half-plane $\mathbb{H}_2$ with values in $\mathrm{Sym}_2(\mathbb{C})$, satisfying

$$f(M\langle Z\rangle) = \det(CZ + D)^k (CZ + D) f(Z)^t (CZ + D)$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in the Siegel modular group $\Gamma_2 = \mathrm{Sp}_2(\mathbb{Z})$ and for all $Z \in \mathbb{H}_2$. Here $(k, 2)$ comes from the fact that the automorphy factor is the one of representatives in the equivalence class of the representation $\det^k \otimes \mathrm{Sym}(2)$. We denote by $M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ the module consisting of all such f whose Fourier coefficients are in $\mathrm{Sym}_2^*(\mathbb{Z}_{(p)})$ and by $M_k(\Gamma_2)_{\mathbb{Z}_{(p)}}$ the module consisting of all scalar valued modular forms of degree 2 whose Fourier coefficients are in $\mathbb{Z}_{(p)}$. Let φ_6, X_{10}, X_{12} be Igusa's generators over $\mathbb{Z}$ of weight 6, 10, 12 respectively given in [6]. The following theorem is our main result.

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Theorem 1. *For each even integer k and each prime $p \geq 5$, there exist six generators over $M_*^{\mathrm{ev}}(\mathbb{Z}_{(p)})$ of the module $M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ whose determinant weights are* $10, 14, 16, 16, 18, 22$. *If we write them as $\Phi_k \in M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ $(k = 10, 14, 16, 18, 22)$ and $\Psi_{16} \in M_{16,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$, then we have (as a $\mathbb{Z}_{(p)}$-module)*

$$\begin{aligned} M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}} = M_{k-10}(\Gamma_2)_{\mathbb{Z}_{(p)}}\Phi_{10} \oplus M_{k-14}(\Gamma_2)_{\mathbb{Z}_{(p)}}\Phi_{14} \\ \oplus M_{k-16}(\Gamma_2)_{\mathbb{Z}_{(p)}}\Phi_{16} \oplus V_{k-16}(\Gamma_2)_{\mathbb{Z}_{(p)}}\Psi_{16} \\ \oplus V_{k-18}(\Gamma_2)_{\mathbb{Z}_{(p)}}\Phi_{18} \oplus W_{k-22}(\Gamma_2)_{\mathbb{Z}_{(p)}}\Phi_{22}, \end{aligned}$$

where

$$\begin{aligned} V_k(\Gamma_2)_{\mathbb{Z}_{(p)}} &= M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \cap \mathbb{Z}_{(p)}[\varphi_6, X_{10}, X_{12}], \\ W_k(\Gamma_2)_{\mathbb{Z}_{(p)}} &= M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \cap \mathbb{Z}_{(p)}[X_{10}, X_{12}]. \end{aligned}$$

We will construct Φ_k and Ψ_k explicitly by taking constant multiples of Satoh's generators given in [10]. The proof of the theorem is by induction on the determinant weight k and our main tool is the Witt operator.

2. Preliminaries

2.1. Siegel modular forms of type $(k, 2)$ and degree 2. The Siegel upper half-space of degree 2 is defined as

$$\mathbb{H}_2 := \{Z = X + iY \in \mathrm{Sym}_2(\mathbb{C}) \mid Y > 0 \text{ (positive definite)}\}.$$

The real symplectic group $\mathrm{Sp}_2(\mathbb{R})$ acts on $\mathbb{H}_2$ in the following way:

$$Z \mapsto M\langle Z\rangle := (AZ + B)(CZ + D)^{-1},$$
$$Z \in \mathbb{H}_2, \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R}).$$

A *Siegel modular form* of type $(k, 2)$ on Γ_2 with character ν is a holomorphic function f on $\mathbb{H}_2$ with values in $\mathrm{Sym}_2(\mathbb{C})$, satisfying

$$f(M\langle Z\rangle) = \nu(M)\det(CZ + D)^k (CZ + D) f(Z)\, {}^t(CZ + D)$$

for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2$ and for all $Z \in \mathbb{H}_2$. Here $(k, 2)$ comes from the fact that the automorphy factor is the one of representatives in the equivalence class of the representation $\det^k \otimes \mathrm{Sym}(2)$.

We denote by $M_{k,2}(\Gamma_2, \nu)$ (resp. $S_{k,2}(\Gamma_2, \nu)$) the $\mathbb{C}$-vector space of Siegel modular forms (resp. cusp forms) of type $(k, 2)$ on Γ_2 with character ν.

2.2. Fourier expansions. Any $F(Z) \in M_{k,2}(\Gamma_2, \nu)$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \leq T \in \frac{1}{2}\Lambda_2} a(T; F)\exp(2\pi i \operatorname{tr}(TZ)), \quad a(T; F) \in \mathrm{Sym}_2(\mathbb{C}),$$

where T runs over all positive semi-definite elements of $\frac{1}{2}\Lambda_2$ defined as

$$\Lambda_2 := \{T = (t_{ij}) \in \mathrm{Sym}_2(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}\}.$$

Taking $q_\tau := \exp(2\pi i\tau)$, $q_\omega := \exp(2\pi i\omega)$ and $q_{\tau'} := \exp(2\pi i\tau')$ for $Z = \left(\begin{smallmatrix}\tau & \omega\\ \omega & \tau'\end{smallmatrix}\right) \in \mathbb{H}_2$, we can write

$$q^T := \exp(2\pi i\,\mathrm{tr}(TZ)) = q_\omega^{2t_{12}} q_\tau^{t_{11}} q_{\tau'}^{t_{22}}.$$

Using this notation, we have the generalized q-expansion

$$\begin{aligned} F &= \sum_{0\le T\in\frac{1}{2}\Lambda_2} a(T;F)q^T \\ &= \sum_{0\le(t_{ij})\in\frac{1}{2}\Lambda_2} (a(T;F)q_\omega^{2t_{12}})q_\tau^{t_{11}} q_{\tau'}^{t_{22}} \in \mathrm{Sym}_2(\mathbb{C})[q_\omega^{-1/2}, q_\omega^{1/2}][\![q_\tau^{1/2}, q_{\tau'}^{1/2}]\!]. \end{aligned}$$

For any subring R of $\mathbb{C}$, we denote by $M_{k,2}(\Gamma_2,\nu)_R$ the R-module consisting of those F in $M_{k,2}(\Gamma_2,\nu)$ for which $a(T;F)$ is in $\mathrm{Sym}_2^*(R)$ for every $T \in \frac{1}{2}\Lambda_2$ where

$$\mathrm{Sym}_2^*(R) := \{T = (t_{ij}) \in \mathrm{Sym}_2(\mathbb{C}) \mid t_{ii}, 2t_{ij} \in R\}.$$

From this, any element F in $M_{k,2}(\Gamma_2,\nu)_R$ can be regarded as an element of the ring of formal power series $\mathrm{Sym}_2^*(R)[q_\omega^{-1/2}, q_\omega^{1/2}][\![q_\tau^{1/2}, q_{\tau'}^{1/2}]\!]$.

2.3. Generators of scalar valued Siegel modular forms. Let φ_4, φ_6, X_{10}, X_{12} be Igusa's generators over $\mathbb{Z}$ of weight 4, 6, 10, 12 respectively given in [6]. Let $M_k(\Gamma_2,\nu)$ (resp. $S_k(\Gamma_2,\nu)$) be the $\mathbb{C}$-vector space consisting of the scalar valued Siegel modular forms (resp. cusp forms) of weight k on Γ_2 with character ν. We denote by $M_k(\Gamma_2,\nu)_{\mathbb{Z}_{(p)}}$ (resp. $S_k(\Gamma_2,\nu)_{\mathbb{Z}_{(p)}}$) the $\mathbb{Z}_{(p)}$-module consisting of the scalar valued Siegel modular forms in $M_{k,2}(\Gamma_2,\nu)$ (resp. cusp forms in $S_{k,2}(\Gamma_2,\nu)$) for which $a(T;F)$ is in $\mathbb{Z}_{(p)}$ for every $T \in \frac{1}{2}\Lambda_2$. By the result of Nagaoka [9], we have

$$\begin{aligned} M_*^{\mathrm{ev}}(\mathbb{Z}_{(p)}) &:= \bigoplus_{k\in2\mathbb{Z}} M_k(\Gamma_2)_{\mathbb{Z}_{(p)}} \\ &= \mathbb{Z}_{(p)}[\varphi_4,\varphi_6,X_{10},X_{12}] \quad \text{if } p\ge 5. \end{aligned}$$

Let $\mathcal{C}\mathrm{Sp}_2(\mathbb{Z})$ be the commutator subgroup of Γ_2. Let $\chi : \Gamma_2 \to \{\pm1\}$ be a non-trivial abelian character, which is basically a character of $\mathrm{Sp}_2(\mathbb{Z}/2\mathbb{Z}) \cong \Sigma_6$, the symmetric group on six letters. Any Siegel modular form of weight k on $\mathcal{C}\mathrm{Sp}_2(\mathbb{Z})$ also has a Fourier expansion of the form

$$\sum_{0\le T\in\frac{1}{2}\Lambda_2} b(T;F)\exp(2\pi i\,\mathrm{tr}(TZ)), \qquad b(T;F)\in\mathbb{C}.$$

In this case Igusa [5] showed that

$$M_*(\mathcal{C}\mathrm{Sp}_2(\mathbb{Z})) := \bigoplus_k M_k(\mathcal{C}\mathrm{Sp}_2(\mathbb{Z})) = \mathbb{C}[\varphi_4, \Delta_5, \varphi_6, X_{12}, \Delta_{30}],$$

where $\Delta_k \in M_k(\mathcal{C}\mathrm{Sp}_2(\mathbb{Z})) = M_k(\Gamma_2, \chi)$ is constructed by theta series with Fourier coefficients

$$b\left(\begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}; \Delta_5\right) = b\left(\begin{pmatrix} 3/2 & 1/4 \\ 1/4 & 5/2 \end{pmatrix}; \Delta_{30}\right) = 1.$$

The modular forms $\varphi_4, \Delta_5, \varphi_6, X_{12}$ are algebraically independent and they are Maass lifts (cf. [2], [8]). Moreover Δ_5 and Δ_{30} are Borcherds products (cf. [3]) and $\Delta_5^2 = X_{10}$. We remark that there exists a unique relation among the generators:

$$\Delta_{30}^2 \in \mathbb{C}[\varphi_4, \Delta_5, \varphi_6, X_{12}].$$

2.4. p-order of modular forms. We shall define the p-order of modular forms. Let p be a prime with $p \geq 5$ and ν_p the additive valuation on $\mathbb{Q}$ normalized as $\nu_p(p) = 1$.

Let F be a formal power series with bounded denominators of the form

$$F = \sum_{T \in \frac{1}{N}\Lambda_2} a(T; F) q^T, \qquad a(T; F) \in \mathrm{Sym}_2(\mathbb{Q}).$$

In the scalar valued case, let ν_p be just as in Böcherer–Nagaoka [1] and elsewhere. Define a value ν_p for F with $a(T; F) \in \mathrm{Sym}_2(\mathbb{Q})$ as

$$\nu_p(F) := \inf\left\{ \nu_p(a(T; F)) \;\middle|\; T \in \frac{1}{N}\Lambda_2 \right\},$$

where $\nu_p\big(\frac{a'}{a}, \frac{b'}{b}, \frac{c'}{c}\big) := \nu_p(\gcd(a', b', c')/\gcd(a, b, c))$ for a, a', b, b', c, $c' \in \mathbb{Z}$. Moreover, we define an order "$\succ$" for two elements of $\frac{1}{N}\Lambda_2$ following [7]. The following statement and its proof are due to Kikuta:

Lemma 1.

(1) *For $f = \sum_{T \in \frac{1}{N}\Lambda_2} a(T; f) q^T$ and $g = \sum_{T \in \frac{1}{N}\Lambda_2} a(T; g) q^T$ with $a(T; f)$, $a(T; g) \in \mathbb{Q}$, we have $\nu_p(fg) = \nu_p(f) + \nu_p(g)$.*

(2) *Let $F = \sum_{T \in \frac{1}{N}\Lambda_2} a(T; F) q^T$ with $a(T; F) \in \mathrm{Sym}_2(\mathbb{Q})$ and g be as in (1). Then $\nu_p(Fg) = \nu_p(F) + \nu_p(g)$.*

Proof. Since the proofs are similar, we prove only (2). We can take the minimum S in Λ_2 such that $\nu_p(a(S; F)) = \nu_p(F)$, and denote it S_0. That is, $\nu_p(a(S; F)) \geq \nu_p(F)$ for any $S \succ S_0$, and $\nu_p(a(S; F)) > \nu_p(F)$ for any $S \prec S_0$. Similarly, we can find $T_0 \in \frac{1}{N}\Lambda_2$ such that $\nu_p(a(T_0; g)) = \nu_p(g)$, $\nu_p(a(T; g)) \geq \nu_p(g)$ for any $T \succ T_0$, and $\nu_p(a(T; g)) > \nu_p(g)$ for any $T \prec T_0$.

For $R \in \frac{1}{N}\Lambda_2$, the coefficient $a(R;Fg)$ of Fg is given by

$$a(R;Fg) = \sum_{\substack{S+T=R \\ S,T\in\frac{1}{N}\Lambda_2}} a(S;F)a(T;g).$$

We first consider the case $R \neq S_0 + T_0$. Then $S \neq S_0$ or $T \neq T_0$. In both cases, $\nu_p(a(S;F)) \geq \nu_p(F)$ and $\nu_p(a(T;g)) \geq \nu_p(g)$. This implies

$$\begin{aligned}\nu_p(a(S;F)a(T;g)) &= \nu_p(a(S;F)) + \nu_p(a(T;g)) \\ &\geq \nu_p(a(S_0;F)) + \nu_p(a(T_0;g)) = \nu_p(F) + \nu_p(g).\end{aligned}$$

If $R = S_0 + T_0$, then we have the cases (i) $S \prec S_0$ or $T \prec T_0$, and (ii) $S = S_0$ and $T = T_0$. In case (i), we obtain (a) $\nu_p(a(S;F)) > \nu_p(F)$ and $\nu_p(a(T;g)) \geq \nu_p(g)$, or (b) $\nu_p(a(S;F)) \geq \nu_p(F)$ and $\nu_p(a(T;g)) > \nu_p(g)$. Then

$$\begin{aligned}\nu_p(a(S;F)a(T;g)) &= \nu_p(a(S;F)) + \nu_p(a(T;g)) \\ &> \nu_p(a(S_0;F)) + \nu_p(a(T_0;g)) = \nu_p(F) + \nu_p(g)\end{aligned}$$

In case (ii), we obtain $\nu_p(a(S;F)) = \nu_p(F)$ and $\nu_p(a(T;g)) = \nu_p(g)$. These show that

$$\begin{aligned}&\nu_p(a(R;Fg)) \geq \nu_p(F) + \nu_p(g) \quad \text{for any } R \neq S_0 + T_0, \\ &\nu_p(a(S_0+T_0;Fg)) = \nu_p(F) + \nu_p(g),\end{aligned}$$

and hence $\nu_p(Fg) = \nu_p(F) + \nu_p(g)$. ∎

We remark that, for a formal power series of the form

$$F = \sum_{T\in\frac{1}{N}\Lambda_2} a(T;F)q^T, \quad a(T;F) \in \mathrm{Sym}_2(\mathbb{Q}),$$

we have $a(T;F) \in \mathrm{Sym}_2^*(\mathbb{Z}_{(p)})$ for all $T \in \frac{1}{N}\Lambda_2$ if and only if $\nu_p(F) \geq 0$.

2.5. Generators of vector valued Siegel modular forms. Let R be a subring of $\mathbb{C}$, and N be 1 or 2. For a formal power series f of the form

$$f = \sum_{T\in\frac{1}{N}\Lambda_2} a(T;f)q^T \in R[q_\omega^{-1/N}, q_\omega^{1/N}][\![q_\tau^{1/N}, q_{\tau'}^{1/N}]\!],$$

the theta operator $\Theta^{[1]}$ is defined by

$$\Theta^{[1]}(f) = \sum_{T\in\frac{1}{N}\Lambda_2} T\cdot a(T;f)q^T \in \mathrm{Sym}_2^*(R)[q_\omega^{-1/N}, q_\omega^{1/N}][\![q_\tau^{1/N}, q_{\tau'}^{1/N}]\!].$$

Let Γ be either Γ_2 or $\mathcal{C}\mathrm{Sp}_2(\mathbb{Z})$, and $f \in M_k(\Gamma)$ and $g \in M_j(\Gamma)$. We put

$$[f,g] := \frac{1}{j}f\Theta^{[1]}(g) - \frac{1}{k}g\Theta^{[1]}(f).$$

Then a result of Satoh [10] states that $[f,g] \in M_{k+j,2}(\Gamma)$.

Let φ_4, φ_6, X_{10}, X_{12} be Igusa's generators over $\mathbb{Z}$ of weight 4, 6, 10, 12 respectively, given in [6]. It is known that the $M_*^{\mathrm{ev}}(\Gamma_2)$-module of Siegel modular forms of type $(k,2)$ has six generators:

Theorem 2 (Satoh [10]). *For each even integer k, we have (as a $\mathbb{C}$-vector space)*

$$\begin{aligned} M_{k,2}(\Gamma_2) = {} & M_{k-10}(\Gamma_2)[\varphi_4,\varphi_6] \oplus M_{k-14}(\Gamma_2)[\varphi_4, X_{10}] \\ & \oplus M_{k-16}(\Gamma_2)[\varphi_4, X_{12}] \oplus V_{k-16}(\Gamma_2)[\varphi_6, X_{10}] \\ & \oplus V_{k-18}(\Gamma_2)[\varphi_6, X_{12}] \oplus W_{k-22}(\Gamma_2)[X_{10}, X_{12}], \end{aligned}$$

where

$$V_k(\Gamma_2) = M_k(\Gamma_2) \cap \mathbb{C}[\varphi_6, X_{10}, X_{12}], \qquad W_k(\Gamma_2) = M_k(\Gamma_2) \cap \mathbb{C}[X_{10}, X_{12}].$$

We construct Φ_k (k = 10, 14, 16, 18, 22) and Ψ_{16} by taking constant multiples of these generators:

$$\begin{aligned} &\Phi_{10} = -\tfrac{1}{144}[\varphi_4,\varphi_6], \quad && \Phi_{14} = 10[\varphi_4, X_{10}], \quad && \Phi_{16} = 12[\varphi_4, X_{12}], \\ &\Psi_{16} = 10[\varphi_6, X_{10}], && \Phi_{18} = 12[\varphi_6, X_{12}], && \Phi_{22} = -120[X_{10}, X_{12}]. \end{aligned}$$

Then

$$\begin{aligned} &a\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \Phi_{10}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, && a\left(\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}; \Psi_{16}\right) = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \\ &a\left(\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}; \Phi_{14}\right) = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, && a\left(\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}; \Phi_{18}\right) = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \\ &a\left(\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}; \Phi_{16}\right) = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, && a\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}; \Phi_{22}\right) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \end{aligned}$$

Moreover, we put $\Phi_9 := 10[\varphi_4, \Delta_5]$, $\Phi_{11} := 10[\varphi_6, \Delta_5]$, $\Phi_{17} := -120[\Delta_5, X_{12}]$. We will use them in the proof of the main theorem.

Proposition 1. *Let p be a prime with $p \geq 5$. Then $\nu_p(\Psi_{16}) \geq 0$ and $\nu_p(\Phi_k) \geq 0$ for $k = 9, 10, 11, 14, 16, 17, 18, 22$.*

Proof. Since $\nu_p(\varphi_k) \geq 0$ ($k = 4, 6$) and $\nu_p(X_k) \geq 0$ ($k = 10, 12$), it follows that $\nu_p(\Theta^{[1]}(\varphi_k)) \geq 0$ ($k = 4, 6$) and $\nu_p(\Theta^{[1]}(X_k)) \geq 0$ ($k = 10, 12$). By a direct calculation, we see that p does not divide the denominators of the Fourier coefficients of Ψ_{16} and Φ_k, which yields the assertion. ■

2.6. The Witt operator. Let F be a holomorphic function on $\mathbb{H}_2$. Then the *Witt operator* is defined by

$$W(F)(\tau,\tau') := F\begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}, \qquad (\tau,\tau') \in \mathbb{H}_1 \times \mathbb{H}_1.$$

This operator was first introduced in Witt [11]. We extend the Witt operator to the case of vector valued forms. Let $G = \left(\begin{smallmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{smallmatrix}\right) \in M_{k,2}(\Gamma_2,\nu)$ be a vector valued Siegel modular form of type $(k, 2)$ on Γ_2 with character ν. Then we define

$$W(G)(\tau, \tau') := \begin{pmatrix} W(G_{11}) & W(G_{12}) \\ W(G_{12}) & W(G_{22}) \end{pmatrix}, \qquad (\tau, \tau') \in \mathbb{H}_1 \times \mathbb{H}_1.$$

For later use, we list some examples:

$$W(\varphi_4)(\tau, \tau') = E_4(\tau)E_4(\tau'), \qquad W(\varphi_6)(\tau, \tau') = E_6(\tau)E_6(\tau'),$$
$$W(X_{10})(\tau, \tau') \equiv 0, \qquad W(X_{12})(\tau, \tau') = 12\Delta(\tau)\Delta(\tau'),$$
$$W(\Delta_5)(\tau, \tau') \equiv 0,$$
$$W(\Phi_{10}) = -\begin{pmatrix} \Delta(\tau)E_4(\tau')E_6(\tau') & 0 \\ 0 & E_4(\tau)E_6(\tau)\Delta(\tau') \end{pmatrix},$$
$$W(\Phi_{16}) = -12\begin{pmatrix} E_6(\tau)\Delta(\tau)E_4(\tau')\Delta(\tau') & 0 \\ 0 & E_4(\tau)\Delta(\tau)E_6(\tau')\Delta(\tau') \end{pmatrix},$$
$$W(\Phi_{18}) = -12\begin{pmatrix} E_4(\tau)^2\Delta(\tau)E_6(\tau')\Delta(\tau') & 0 \\ 0 & E_6(\tau)\Delta(\tau)E_4(\tau')^2\Delta(\tau') \end{pmatrix},$$
$$W(\Phi_{14}) = W(\Psi_{16}) = W(\Phi_{22}) \equiv 0,$$
$$W(\Phi_9) = -2E_4(\tau)E_4(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$W(\Phi_{11}) = -2E_6(\tau)E_6(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$W(\Phi_{17}) = -288\Delta(\tau)\Delta(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where η is the Dedekind eta function defined as $\eta(\tau) = q_\tau^{1/24}\prod_{m=1}^{\infty}(1 - q_\tau^m)$.

3. Proof of the main theorem (Theorem 1). By Proposition 1, the inclusion "$\supset$" is clear.

To prove "$\subset$", we use induction on the determinant weight. By Theorem 2 (Subsection 2.5), any $F \in M_{k,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}$ can be written in the form

$$\begin{aligned} F = (P_1 + X_{10}Q_1)\Phi_{10} + (P_2 + X_{10}Q_2)\Phi_{14} + (P_3 + X_{10}Q_3)\Phi_{16} \\ + (P_4 + X_{10}Q_4)\Psi_{16} + (P_5 + X_{10}Q_5)\Phi_{18} + (P_6 + X_{10}Q_6)\Phi_{22}, \end{aligned}$$

where $P_1 \in M_{k-10}(\Gamma_2) \cap \mathbb{C}[\varphi_4, \varphi_6, X_{12}]$, $Q_1 \in M_{k-20}(\Gamma_2)$, $P_2 \in M_{k-14}(\Gamma_2) \cap \mathbb{C}[\varphi_4, \varphi_6, X_{12}]$, $Q_2 \in M_{k-24}(\Gamma_2)$, $P_3 \in M_{k-16}(\Gamma_2) \cap \mathbb{C}[\varphi_4, \varphi_6, X_{12}]$, $Q_3 \in M_{k-26}(\Gamma_2)$, $P_4 \in V_{k-16}(\Gamma_2) \cap \mathbb{C}[\varphi_6, X_{12}]$, $Q_4 \in V_{k-26}(\Gamma_2)$, $P_5 \in V_{k-18}(\Gamma_2) \cap \mathbb{C}[\varphi_6, X_{12}]$, $Q_5 \in V_{k-28}(\Gamma_2)$, $P_6 \in W_{k-22}(\Gamma_2) \cap \mathbb{C}[X_{12}]$, $Q_6 \in W_{k-32}(\Gamma_2)$.

Here we regard P_i as polynomials (with coefficients in $\mathbb{C}$): $P_1 = P_1(\varphi_4, \varphi_6, X_{12})$, $P_2 = P_2(\varphi_4, \varphi_6, X_{12})$, $P_3 = P_3(\varphi_4, \varphi_6, X_{12})$, $P_4 = P_4(\varphi_6, X_{12})$, $P_5 = P_5(\varphi_6, X_{12})$, $P_6 = P_6(X_{12})$.

We start with the following lemma:

LEMMA 2. *$\nu_p(P_i) \geq 0$ for $i = 1, 3, 5$.*

Proof. We apply the Witt operator to F. Using the fact that $W(\varDelta_5) = 0$, we get

$$\begin{aligned}
W(F) &= W(P_1)W(\varPhi_{10}) + W(P_3)W(\varPhi_{16}) + W(P_5)W(\varPhi_{18}) \\
&= -W(P_1)\begin{pmatrix} \varDelta(\tau)E_4(\tau')E_6(\tau') & 0 \\ 0 & E_4(\tau)E_6(\tau)\varDelta(\tau') \end{pmatrix} \\
&\quad - 12W(P_3)\begin{pmatrix} E_6(\tau)\varDelta(\tau)E_4(\tau')\varDelta(\tau') & 0 \\ 0 & E_4(\tau)\varDelta(\tau)E_6(\tau')\varDelta(\tau') \end{pmatrix} \\
&\quad - 12W(P_5)\begin{pmatrix} E_4(\tau)^2\varDelta(\tau)E_6(\tau')\varDelta(\tau') & 0 \\ 0 & E_6(\tau)\varDelta(\tau)E_4(\tau')^2\varDelta(\tau') \end{pmatrix} \\
&= \left\{ -W(P_1)\begin{pmatrix} E_4(\tau')E_6(\tau') & 0 \\ 0 & E_4(\tau)E_6(\tau) \end{pmatrix} \right. \\
&\quad - 12W(P_3)\begin{pmatrix} E_6(\tau)E_4(\tau')\varDelta(\tau') & 0 \\ 0 & E_4(\tau)\varDelta(\tau)E_6(\tau') \end{pmatrix} \\
&\quad \left. - 12W(P_5)\begin{pmatrix} E_4(\tau)^2E_6(\tau')\varDelta(\tau') & 0 \\ 0 & E_6(\tau)\varDelta(\tau)E_4(\tau')^2 \end{pmatrix} \right\} \begin{pmatrix} \varDelta(\tau) & 0 \\ 0 & \varDelta(\tau') \end{pmatrix} \\
&= \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix} \begin{pmatrix} \varDelta(\tau) & 0 \\ 0 & \varDelta(\tau') \end{pmatrix}.
\end{aligned}$$

Since $\nu_p(F) \geq 0$, we have $\nu_p(W(F)) \geq 0$. The $(1,1)$-component and $(2,2)$-component of $W(F)$ are

$$\begin{aligned}
f_{11}\varDelta(\tau) = \big(&-W(P_1)E_4(\tau')E_6(\tau') - 12W(P_3)E_6(\tau)E_4(\tau')\varDelta(\tau') \\
&- 12W(P_5)E_4(\tau)^2E_6(\tau')\varDelta(\tau')\big)\varDelta(\tau), \\
f_{22}\varDelta(\tau') = \big(&-W(P_1)E_4(\tau)E_6(\tau) - 12W(P_3)E_4(\tau)\varDelta(\tau)E_6(\tau') \\
&- 12W(P_5)E_6(\tau)\varDelta(\tau)E_4(\tau')^2\big)\varDelta(\tau').
\end{aligned}$$

Then we see that

$$\begin{aligned}
E_4(\tau)\varDelta(\tau)E_6(\tau')f_{11} &- E_6(\tau)E_4(\tau')\varDelta(\tau')f_{22} \\
&= \big(E_4(\tau)^3\varDelta(\tau') - \varDelta(\tau)E_4(\tau')^3\big) \\
&\quad \times \big(E_4(\tau)E_4(\tau')W(P_1) + 2^8 \cdot 3^4\varDelta(\tau)\varDelta(\tau')W(P_5)\big),
\end{aligned}$$

$$-E_4(\tau)E_6(\tau)f_{11} + E_4(\tau')E_6(\tau')f_{22} = 2^2 \cdot 3(E_4(\tau)^3\Delta(\tau') - \Delta(\tau)E_4(\tau')^3) \times \big(E_4(\tau)E_4(\tau')W(P_3) + E_6(\tau)E_6(\tau')W(P_5)\big).$$

Since $\nu_p(\mathrm{LHS}) \geq 0$, we have $\nu_p(\mathrm{RHS}) \geq 0$ for both formulas above. Moreover, the Fourier expansion

$$E_4(\tau)^3\Delta(\tau') - \Delta(\tau)E_4(\tau')^3 = q_{\tau'} - q_\tau + \cdots \in \mathbb{Z}[\![q_\tau, q_{\tau'}]\!]$$

implies

$$\nu_p\big(E_4(\tau)^3\Delta(\tau') - \Delta(\tau)E_4(\tau')^3\big) = 0.$$

Applying Lemma 1, we obtain

$$\nu_p\big(E_4(\tau)E_4(\tau')W(P_1) + 2^8 \cdot 3^4\Delta(\tau)\Delta(\tau')W(P_5)\big) \geq 0, \tag{3.1}$$

$$\nu_p\big(E_4(\tau)E_4(\tau')W(P_3) + E_6(\tau)E_6(\tau')W(P_5)\big) \geq 0. \tag{3.2}$$

We separate the argument into several cases:

CASE $k \not\equiv 0 \pmod 6$: In this case we have $P_5 = 0$ as a polynomial. Hence $P_1, P_3 \in \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{12}]$, and therefore $\nu_p(P_1) \geq 0$ and $\nu_p(P_3) \geq 0$.

CASE $k \equiv 0 \pmod{12}$ ($k \equiv 0 \pmod 4$ and $k \equiv 0 \pmod 6$): We can write

$$W(P_1) = E_6(\tau)E_6(\tau') \sum_{\substack{a\equiv 2 \,(\mathrm{mod}\, 3)\\ 4a+12b+12c=k-16}} \gamma_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c,$$

$$W(P_3) = \sum_{\substack{a\equiv 2 \,(\mathrm{mod}\, 3)\\ 4a+12b+12c=k-16}} \gamma'_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c,$$

$$W(P_5) = E_6(\tau)E_6(\tau') \sum_{12b+12c=k-24} \gamma''_{bc} W(\varphi_6^2)^b W(X_{12})^c.$$

Using these formulas, we can write

$$\begin{aligned}
&E_4(\tau)E_4(\tau')W(P_1) + 2^8 \cdot 3^4\Delta(\tau)\Delta(\tau')W(P_5)\\
&\quad = E_6(\tau)E_6(\tau') \sum_{\substack{a\equiv 0 \,(\mathrm{mod}\, 3),\ a\geq 3\\ 4a+12b+12c=k-12}} \gamma_{(a-1)bc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c\\
&\qquad + 2^6 \cdot 3^3 E_6(\tau)E_6(\tau') \sum_{\substack{c\geq 1\\ 12b+12c=k-12}} \gamma''_{b(c-1)} W(\varphi_6^2)^b W(X_{12})^c\\
&\quad = E_6(\tau)E_6(\tau')\Big\{ \sum_{\substack{a\equiv 0 \ (\mathrm{mod}\ 3),\ a\geq 3\\ 4a+12b+12c=k-12}} \gamma_{(a-1)bc} W(\varphi_4)^a W(\varphi_6)^{2b} W(X_{12})^c\\
&\qquad + 2^6 \cdot 3^3 \sum_{\substack{c\geq 1\\ 12b+12c=k-12}} \gamma''_{b(c-1)} W(\varphi_6)^{2b} W(X_{12})^c \Big\}.
\end{aligned}$$

Since $\nu_p(\mathrm{LHS}) \geq 0$, we have $\nu_p(\mathrm{RHS}) \geq 0$ for both formulas above. By the same argument as in [9, p. 416], it follows that $\gamma_{(a-1)bc} \in \mathbb{Z}_{(p)}$ and $2^6 \cdot 3^3 \gamma''_{b(c-1)} \in \mathbb{Z}_{(p)}$. Hence $P_1 \in \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{12}]$ and $P_5 \in \mathbb{Z}_{(p)}[\varphi_4, \varphi_6]$. These mean that $\nu_p(P_1) \geq 0$ and $\nu_p(P_5) \geq 0$. From (3.2), we have $\nu_p(W(P_3)) \geq 0$. Hence $P_3 \in \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{12}]$, so $\nu_p(P_3) \geq 0$.

Case $k \equiv 6 \pmod{12}$ ($k \equiv 2 \pmod 4$ and $k \equiv 0 \pmod 6$): We argue similarly to the case of $k \equiv 0 \pmod{12}$. ∎

Next we prove the following lemma:

Lemma 3. *$\nu_p(P_i) \geq 0$ for $i = 2, 4, 6$.*

Proof. We put

$$\begin{aligned} G &:= F - (P_1 \Phi_{10} + P_3 \Phi_{16} + P_5 \Phi_{18}) \\ &= (\Delta_5)^2 (Q_1 \Phi_{10} + Q_3 \Phi_{16} + Q_5 \Phi_{18}) \\ &\quad + \Delta_5 \{(P_2 + X_{10} Q_2)\Phi_9 + (P_4 + X_{10} Q_4)\Phi_{11} + (P_6 + X_{10} Q_6)\Phi_{17}\}. \end{aligned}$$

Then it is clear that $W(G) = 0$. For the reason given in Ibukiyama–Wakatsuki [4, p. 198], G/Δ_5 is holomorphic and therefore

$$\begin{aligned} G/\Delta_5 &= \Delta_5 (Q_1 \Phi_{10} + Q_3 \Phi_{16} + Q_5 \Phi_{18}) + (P_2 + X_{10} Q_2)\Phi_9 \\ &\quad + (P_4 + X_{10} Q_4)\Phi_{11} + (P_6 + X_{10} Q_6)\Phi_{17} \\ &\in M_{k-5,2}(\Gamma_2, \chi). \end{aligned}$$

By Lemma 2, we have $\nu_p(G) \geq 0$. Moreover, the Fourier expansion

$$\Delta_5 = (-q_\omega^{-1/2} + q_\omega^{1/2}) q_\tau^{1/2} q_{\tau'}^{1/2} + \cdots \in \mathbb{Z}[q_\omega^{-1/2}, q_\omega^{1/2}][\![q_\tau^{1/2}, q_{\tau'}^{1/2}]\!]$$

indicates that

$$\nu_p(\Delta_5) = 0.$$

Applying Lemma 1, we have $\nu_p(G/\Delta_5) \geq 0$ and also $\nu_p(W(G/\Delta_5)) \geq 0$.

On the other hand,

$$\begin{aligned} W(\Phi_9) &= -2E_4(\tau)E_4(\tau')\eta(\tau)^{12}\eta(\tau')^{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ W(\Phi_{11}) &= -2E_6(\tau)E_6(\tau')\eta(\tau)^{12}\eta(\tau')^{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ W(\Phi_{17}) &= -288\Delta(\tau)\Delta(\tau')\eta(\tau)^{12}\eta(\tau')^{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

These imply

$$\begin{aligned} W(G/\Delta_5) &= W(P_2)W(\Phi_9) + W(P_4)W(\Phi_{11}) + W(P_6)W(\Phi_{17}) \\ &= W(P_2)\left(-2E_4(\tau)E_4(\tau')\eta(\tau)^{12}\eta(\tau')^{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \end{aligned}$$

$$+ W(P_4)\Big(-2E_6(\tau)E_6(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Big)$$
$$+ W(P_6)\Big(-288\Delta(\tau)\Delta(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Big).$$

CASE $k \not\equiv 4 \pmod 6$: In this case we have $P_4 = P_6 = 0$ as polynomials. Therefore $P_2 \in \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{12}]$ and hence $\nu_p(P_2) \geq 0$.

CASE $k \equiv 4 \pmod{12}$ ($k \equiv 0 \pmod 4$ and $k \equiv 4 \pmod 6$): We can write

$$W(P_2) = E_6(\tau)E_6(\tau') \sum_{\substack{a\equiv 2 \,(\mathrm{mod}\, 3)\\ 4a+12b+12c=k-20}} \gamma_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_4) = \sum_{12b+12c=k-16} \gamma'_{bc} W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_6) = 0.$$

Using these formulas, we can write

$$W(G/\Delta_5) = -2\Big(\sum_{\substack{a\equiv 0 \,(\mathrm{mod}\, 3),\ a\geq 3\\ 4a+12b+12c=k-16}} \gamma_{(a-1)bc} W(\varphi_4)^a W(\varphi_6)^{2b} W(X_{12})^c$$
$$+ \sum_{12b+12c=k-16} \gamma'_{bc} W(\varphi_6)^{2b} W(X_{12})^c\Big) E_6(\tau)E_6(\tau')\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Again by the same argument as in [9, p. 416], we get $\gamma_{(a-1)bc} \in \mathbb{Z}_{(p)}$ and $\gamma'_{bc} \in \mathbb{Z}_{(p)}$. Hence $P_2 \in \mathbb{Z}_{(p)}[\varphi_4, \varphi_6, X_{12}]$ and $P_4 \in \mathbb{Z}_{(p)}[\varphi_6, X_{12}]$. These mean that $\nu_p(P_2) \geq 0$ and $\nu_p(P_4) \geq 0$.

CASE $k \equiv 10 \pmod{12}$ ($k \equiv 2 \pmod 4$ and $k \equiv 4 \pmod 6$): We can write

$$W(P_2) = \sum_{\substack{a\equiv 2 \,(\mathrm{mod}\, 3)\\ 4a+12b+12c=k-14}} \gamma_{abc} W(\varphi_4)^a W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_4) = E_6(\tau)E_6(\tau') \sum_{12b+12c=k-22} \gamma'_{bc} W(\varphi_6^2)^b W(X_{12})^c,$$
$$W(P_6) = \gamma''_{(k-22)/12} W(X_{12})^{(k-22)/12}.$$

Using these formulas, we can write

$$W(G/\Delta_5) = \Big(-2 \sum_{\substack{a\equiv 0 \,(\mathrm{mod}\, 3),\ a\geq 3\\ 4a+12b+12c=k-10}} \gamma_{(a-3)bc} W(\varphi_4)^a W(\varphi_6)^{2b} W(X_{12})^c$$
$$- 2 \sum_{\substack{b\geq 1\\ 12b+12c=k-10}} \gamma'_{(b-1)c} W(\varphi_6)^{2b} W(X_{12})^c$$
$$- 24\gamma''_{(k-22)/12} W(X_{12})^{(k-10)/12}\Big)\eta(\tau)^{12}\eta(\tau')^{12}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Again by the same argument as in [9, p. 416], we obtain $-2\gamma_{(a-3)bc}\in\mathbb{Z}_{(p)}$, $-2\gamma'_{(b-1)c}\in\mathbb{Z}_{(p)}$ and $-24\gamma''_{(k-22)/12}\in\mathbb{Z}_{(p)}$. Hence $P_2\in\mathbb{Z}_{(p)}[\varphi_4,\varphi_6,X_{12}]$, $P_4\in\mathbb{Z}_{(p)}[\varphi_6,X_{12}]$ and $P_6\in\mathbb{Z}_{(p)}[X_{12}]$. These imply $\nu_p(P_2)\geq 0$, $\nu_p(P_4)\geq 0$ and $\nu_p(P_6)\geq 0$.

This completes the proof of Lemma 3. ■

We are now in a position to prove Theorem 1. Let G be the function appearing in the proof of Lemma 3, and set

$$\begin{aligned}H &:= G/\Delta_5-(P_2\Phi_9+P_4\Phi_{11}+P_6\Phi_{17})\\ &= \Delta_5(Q_1\Phi_{10}+Q_3\Phi_{16}+Q_5\Phi_{18})+X_{10}(Q_2\Phi_9+Q_4\Phi_{11}+Q_6\Phi_{17}).\end{aligned}$$

Then $\nu_p(H)\geq 0$. Since $\nu_p(\Delta_5)=0$, and by Lemma 1, we have

$$0\leq\nu_p(H/\Delta_5)=\nu_p(Q_1\Phi_{10}+Q_2\Phi_{14}+Q_3\Phi_{16}+Q_4\Psi_{16}+Q_5\Phi_{18}+Q_6\Phi_{22}).$$

This is of type $(k-10,2)$. By repeating this argument, the proof is reduced to the case of $k\leq 22$, which will be checked directly.

Case $k=10$: Clearly $M_{10,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}=\mathbb{Z}_{(p)}\Phi_{10}$.

Case $k=14$: Any $F=a_1\varphi_4\Phi_{10}+a_2\Phi_{14}\in M_{14,2}(\Gamma_2)$ satisfies

$$\Phi(F)=-a_1E_4\Delta\begin{pmatrix}1&0\\0&0\end{pmatrix},$$

where Φ is the Siegel Φ operator. Since $\Phi(F)\in M_{16}(\Gamma_1)_{\mathbb{Z}_{(p)}}$, we get $a_1\in\mathbb{Z}_{(p)}$. Hence $a_2\in\mathbb{Z}_{(p)}$, i.e.

$$M_{14,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}=\mathbb{Z}_{(p)}\varphi_4\Phi_{10}\oplus\mathbb{Z}_{(p)}\Phi_{14}.$$

Case $k=16$: Any $F=a_1\varphi_6\Phi_{10}+a_2\Phi_{16}+a_3\Psi_{16}$ satisfies

$$\Phi(F)=-a_1E_6\Delta\begin{pmatrix}1&0\\0&0\end{pmatrix}.$$

Since $\Phi(F)\in M_{18}(\Gamma_1)_{\mathbb{Z}_{(p)}}$, we get $a_1\in\mathbb{Z}_{(p)}$. Moreover,

$$\begin{aligned}&W(F-a_1\varphi_6\Phi_{10})\\ &\quad=a_2\cdot(-12)\begin{pmatrix}E_6(\tau)\Delta(\tau)E_4(\tau')\Delta(\tau')&0\\0&E_4(\tau)\Delta(\tau)E_6(\tau')\Delta(\tau')\end{pmatrix}.\end{aligned}$$

Since $\nu_p(W(F-a_1\varphi_6\Phi_{10}))\geq 0$, we get $a_2\in\mathbb{Z}_{(p)}$ and hence also $a_3\in\mathbb{Z}_{(p)}$. Thus

$$M_{16,2}(\Gamma_2)_{\mathbb{Z}_{(p)}}=\mathbb{Z}_{(p)}\varphi_6\Phi_{10}\oplus\mathbb{Z}_{(p)}\Phi_{16}\oplus\mathbb{Z}_{(p)}\Psi_{16}.$$

Case $k=18$: Any $F=a_1\varphi_4^2\Phi_{10}+a_2\varphi_4\Phi_{14}+a_3\Phi_{18}$ satisfies

$$\Phi(F)=-a_1E_4^2\Delta\begin{pmatrix}1&0\\0&0\end{pmatrix}.$$

Since $\Phi(F)\in M_{20}(\Gamma_1)_{\mathbb{Z}_{(p)}}$, we get $a_1\in\mathbb{Z}_{(p)}$. Moreover,

$$\begin{aligned}&W(F - a_1\varphi_4^2\Phi_{10})\\&= a_3\cdot(-12)\begin{pmatrix}E_4(\tau)^2\Delta(\tau)E_6(\tau')\Delta(\tau') & 0\\ 0 & E_6(\tau)\Delta(\tau)E_4(\tau')^2\Delta(\tau')\end{pmatrix}.\end{aligned}$$

Since $\nu_p(W(F - a_1\varphi_4^2\Phi_{10})) \geq 0$, we get $a_3 \in \mathbb{Z}_{(p)}$ and hence also $a_2 \in \mathbb{Z}_{(p)}$. Consequently,

$$M_{18,2}(\Gamma_2)_{\mathbb{Z}_{(p)}} = \mathbb{Z}_{(p)}\varphi_4^2\Phi_{10} \oplus \mathbb{Z}_{(p)}\varphi_4\Phi_{14} \oplus \mathbb{Z}_{(p)}\Phi_{18}.$$

Case $k = 20$: Any $F = (a_1\varphi_4\varphi_6 + a_2X_{10})\Phi_{10} + a_3\varphi_6\Phi_{14} + a_4\varphi_4\Phi_{16} \in M_{20,2}(\Gamma_2)$ satisfies

$$\Phi(F) = -a_1E_4E_6\Delta\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

Since $\Phi(F) \in M_{22}(\Gamma_1)_{\mathbb{Z}_{(p)}}$, we get $a_1 \in \mathbb{Z}_{(p)}$. Moreover,

$$\begin{aligned}&W(F - a_1\varphi_4\varphi_6\Phi_{10})\\&= a_4\cdot(-12)E_4(\tau)E_4(\tau')\begin{pmatrix}E_6(\tau)\Delta(\tau)E_4(\tau')\Delta(\tau') & 0\\ 0 & E_4(\tau)\Delta(\tau)E_6(\tau')\Delta(\tau')\end{pmatrix}.\end{aligned}$$

Since $\nu_p(W(F - a_1\varphi_4\varphi_6\Phi_{10})) \geq 0$, we find that $a_4 \in \mathbb{Z}_{(p)}$ and hence $\nu_p(F - a_1\varphi_4\varphi_6\Phi_{10} - a_4\varphi_4\Phi_{16}) \geq 0$. Since

$$\begin{aligned}&a_2X_{10}\Phi_{10} + a_3\varphi_6\Phi_{14}\\
&= q_\tau q_{\tau'}\left(\begin{pmatrix}2a_3 & 0\\ 0 & 2a_3\end{pmatrix} + \begin{pmatrix}-a_3 & \frac{1}{2}a_3\\ \frac{1}{2}a_3 & -a_3\end{pmatrix}q_\omega^{-1} + \begin{pmatrix}-a_3 & -\frac{1}{2}a_3\\ -\frac{1}{2}a_3 & -a_3\end{pmatrix}q_\omega\right)\\
&\quad + q_\tau q_{\tau'}^2\left(\begin{pmatrix}-564a_3 & 0\\ 0 & 2a_2 - 1800a_3\end{pmatrix} + \begin{pmatrix}280a_3 & -140a_3\\ -140a_3 & -a_2 + 896a_3\end{pmatrix}q_\omega^{-1}\right.\\
&\quad + \begin{pmatrix}280a_3 & 140a_3\\ 140a_3 & -a_2 + 896a_3\end{pmatrix}q_\omega + \begin{pmatrix}2a_3 & -2a_3\\ -2a_3 & 4a_3\end{pmatrix}q_\omega^{-2}\\
&\quad \left. + \begin{pmatrix}2a_3 & 2a_3\\ 2a_3 & 4a_3\end{pmatrix}q_\omega^2\right) + \cdots,\end{aligned}$$

we get a_2 and $a_3 \in \mathbb{Z}_{(p)}$. Therefore,

$$M_{20,2}(\Gamma_2)_{\mathbb{Z}_{(p)}} = \mathbb{Z}_{(p)}\varphi_4\varphi_6\Phi_{10} \oplus \mathbb{Z}_{(p)}X_{10}\Phi_{10} \oplus \mathbb{Z}_{(p)}\varphi_6\Phi_{14} \oplus \mathbb{Z}_{(p)}\varphi_4\Phi_{16}.$$

Case $k = 22$: Any $F = (a_1\varphi_4^3 + a_2\varphi_6^2 + a_3X_{12})\Phi_{10} + a_4\varphi_4^2\Phi_{14} + a_5\varphi_6\Phi_{16} + a_6\varphi_6\Psi_{16} + a_7\Phi_{22}$ satisfies

$$\begin{aligned}&\Phi(F) = -(a_1E_4^3 + a_2E_6^2)\Delta\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\\&= -\{(a_1+a_2)q_\tau + (696a_1 - 1032a_2)q_\tau^2 + (104652a_1 + 245196a_2)q_\tau^3 + \cdots\}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.\end{aligned}$$

Since $\Phi(F) \in M_{24}(\Gamma_1)_{\mathbb{Z}_{(p)}}$, we have

$$a_1 + a_2 \in \mathbb{Z}_{(p)}, \quad 696a_1 - 1032a_2 \in \mathbb{Z}_{(p)}.$$

These imply

$$\begin{aligned} a_1 &= \frac{1}{2^6 \cdot 3^3}\{1032(a_1 + a_2) + (696a_1 - 1032a_2)\} \in \mathbb{Z}_{(p)}, \\ a_2 &= \frac{1}{2^6 \cdot 3^3}\{696(a_1 + a_2) - (696a_1 - 1032a_2)\} \in \mathbb{Z}_{(p)}. \end{aligned}$$

Moreover,

$$\begin{aligned} &W\big(F - (a_1\varphi_4^3 + a_2\varphi_6^2)\Phi_{10}\big) \\ &= a_3 \cdot (-12)\Delta(\tau)\Delta(\tau')\begin{pmatrix} \Delta(\tau)E_4(\tau')E_6(\tau') & 0 \\ 0 & E_4(\tau)E_6(\tau)\Delta(\tau') \end{pmatrix} \\ &\quad + a_5 \cdot (-12)E_6(\tau)E_6(\tau')\begin{pmatrix} E_6(\tau)\Delta(\tau)E_4(\tau')\Delta(\tau') & 0 \\ 0 & E_4(\tau)\Delta(\tau)E_6(\tau')\Delta(\tau') \end{pmatrix} \\ &= -12\begin{pmatrix} a_5q_\tau q_{\tau'} + (a_3 - 1032a_5)q_\tau^2 q_{\tau'} + \cdots & 0 \\ 0 & a_5q_\tau q_{\tau'} + (a_3 - 1032a_5)q_\tau q_{\tau'}^2 + \cdots \end{pmatrix}. \end{aligned}$$

Since $\nu_p(W(F - (a_1\varphi_4^3 + a_2\varphi_6^2)\Phi_{10})) \geq 0$, we get a_3 and $a_5 \in \mathbb{Z}_{(p)}$ and hence $\nu_p(F - (a_1\varphi_4^3 + a_2\varphi_6^2 + a_3X_{12})\Phi_{10} - a_5\varphi_6\Phi_{16}) \geq 0$. Since

$$\begin{aligned} &a_4\varphi_4^2\Phi_{14} + a_6\varphi_6\Psi_{16} + a_7\Phi_{22} \\ &\quad = q_\tau q_{\tau'}\Bigg(\begin{pmatrix} 2a_4 + 2a_6 & 0 \\ 0 & 2a_4 + 2a_6 \end{pmatrix} + \begin{pmatrix} -a_4 - a_6 & \frac{1}{2}a_4 + \frac{1}{2}a_6 \\ \frac{1}{2}a_4 + \frac{1}{2}a_6 & -a_4 - a_6 \end{pmatrix}q_\omega^{-1} \\ &\qquad + \begin{pmatrix} -a_4 - a_6 & -\frac{1}{2}a_4 - \frac{1}{2}a_6 \\ -\frac{1}{2}a_4 - \frac{1}{2}a_6 & -a_4 - a_6 \end{pmatrix}q_\omega\Bigg) \\ &\qquad + q_\tau q_{\tau'}^2\Bigg(\begin{pmatrix} 1404a_4 - 2052a_6 & 0 \\ 0 & 168a_4 - 408a_6 \end{pmatrix} \\ &\qquad + \begin{pmatrix} -704a_4 + 1024a_6 & 352a_4 - 512a_6 \\ 352a_4 - 512a_6 & -88a_4 + 200a_6 \end{pmatrix}q_\omega^{-1} \\ &\qquad + \begin{pmatrix} -704a_4 + 1024a_6 & -352a_4 + 512a_6 \\ -352a_4 + 512a_6 & -88a_4 + 200a_6 \end{pmatrix}q_\omega \\ &\qquad + \begin{pmatrix} 2a_4 + 2a_6 & -2a_4 - 2a_6 \\ -2a_4 - 2a_6 & 4a_4 + 4a_6 \end{pmatrix}q_\omega^{-2} + \begin{pmatrix} 2a_4 + 2a_6 & 2a_4 + 2a_6 \\ 2a_4 + 2a_6 & 4a_4 + 4a_6 \end{pmatrix}q_\omega^2\Bigg) \\ &\qquad + q_\tau^2 q_{\tau'}^2\Bigg(\begin{pmatrix} 75776a_4 + 329216a_6 + 36a_7 & 0 \\ 0 & 75776a_4 + 329216a_6 + 36a_7 \end{pmatrix} \end{aligned}$$

$$+\begin{pmatrix} -33960a_4-144840a_6-16a_7 & 239160a_4+511608a_6+70a_7 \\ 239160a_4+511608a_6+70a_7 & -33960a_4-144840a_6-16a_7 \end{pmatrix} q_\omega^{-1}$$

$$+\begin{pmatrix} -33960a_4-144840a_6-16a_7 & -239160a_4-511608a_6-70a_7 \\ -239160a_4-511608a_6-70a_7 & -33960a_4-144840a_6-16a_7 \end{pmatrix} q_\omega$$

$$+\begin{pmatrix} -3840a_4-19968a_6-2a_7 & 1920a_4+9984a_6+a_7 \\ 1920a_4+9984a_6+a_7 & -3840a_4-19968a_6-2a_7 \end{pmatrix} q_\omega^{-2}$$

$$+\begin{pmatrix} -3840a_4-19968a_6-2a_7 & -1920a_4-9984a_6-a_7 \\ -1920a_4-9984a_6-a_7 & -3840a_4-19968a_6-2a_7 \end{pmatrix} q_\omega^{2}$$

$$+\begin{pmatrix} -88a_4+200a_6 & -264a_4+312a_6 \\ -264a_4+312a_6 & -88a_4+200a_6 \end{pmatrix} q_\omega^{-3}$$

$$+\begin{pmatrix} -88a_4+200a_6 & 264a_4-312a_6 \\ 264a_4-312a_6 & -88a_4+200a_6 \end{pmatrix} q_\omega^{3}\Bigg) + \cdots,$$

we have

$$a_4+a_6 \in \mathbb{Z}_{(p)}, \quad 352a_4-512a_6 \in \mathbb{Z}_{(p)},$$
$$1920a_4+9984a_6+a_7 \in \mathbb{Z}_{(p)}.$$

These imply

$$a_4 = \frac{1}{2^5\cdot 3^3}\{512(a_4+a_6)+(352a_4-512a_6)\} \in \mathbb{Z}_{(p)},$$
$$a_6 = \frac{1}{2^5\cdot 3^3}\{352(a_4+a_6)-(352a_4-512a_6)\} \in \mathbb{Z}_{(p)},$$
$$a_7 \in \mathbb{Z}_{(p)}.$$

Thus,

$$M_{22,2}(\Gamma_2)_{\mathbb{Z}_{(p)}} = \mathbb{Z}_{(p)}\varphi_4^3\Phi_{10} \oplus \mathbb{Z}_{(p)}\varphi_6^2\Phi_{10} \oplus \mathbb{Z}_{(p)}X_{12}\Phi_{10}$$
$$\oplus \mathbb{Z}_{(p)}\varphi_4^2\Phi_{14} \oplus \mathbb{Z}_{(p)}\varphi_6\Phi_{16} \oplus \mathbb{Z}_{(p)}\varphi_6\Psi_{16} \oplus \mathbb{Z}_{(p)}\Phi_{22}.$$

This completes the proof of Theorem 1. ∎

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