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LIE SPHERE GEOMETRY IN $\mathbf{R}^3$

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Abstract. These notes introduce Lie sphere geometry of surfaces in Euclidean 3-space. A similar article with a different point of view is: G. R. Jensen, *Dupin hypersurfaces in Lie sphere geometry*, in: Geometry and Analysis on Manifolds, Progr. Math. 308, Birkhäuser/Springer, Cham, 2015, 383–394. More details are in the book: G. R. Jensen, E. Musso, L. Nicolodi, *Surfaces in Classical Geometries*, Universitext, Springer, Cham, 2016, Chapter 15. An exposition of Lie sphere geometry in all dimensions is in: Th. E. Cecil, *Lie Sphere Geometry*, Universitext, Springer, New York, 2008.

1. Transformations of surfaces in $\mathbf{R}^3$. *Euclidean geometry* studies properties of immersions $\mathbf{x} : M \to \mathbf{R}^3$ (curve or surface) invariant under the *Euclidean group*

$$\mathbf{E}(3) = \mathbf{R}^3 \rtimes \mathbf{O}(3),$$

which acts transitively on $\mathbf{R}^3$ by $(\mathbf{y}, A)\mathbf{x} = \mathbf{y} + A\mathbf{x}$. The map

$$\pi : \mathbf{E}(3) \to \mathbf{R}^3, \quad \pi(\mathbf{y}, A) = (\mathbf{y}, A)\mathbf{0} = \mathbf{y}$$

is the projection of a principal $\mathbf{O}(3)$-bundle. A *moving frame* is a local section.

Throughout this note, $\boldsymbol{\epsilon}_0, \boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3, \boldsymbol{\epsilon}_4, \boldsymbol{\epsilon}_5$ is the standard orthonormal basis of signature $+ + + + - -$ of $\mathbf{R}^{4,2}$. Then $\mathbf{R}^3$ is the span of $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3$, while $\mathbf{R}^4$ is the span of $\boldsymbol{\epsilon}_0, \dots, \boldsymbol{\epsilon}_3$, and $\mathbf{R}^{4,1}$ is the span of $\boldsymbol{\epsilon}_0, \dots, \boldsymbol{\epsilon}_4$.

Conformal geometry extends the Euclidean group, as well as the isometry groups of spherical and hyperbolic geometries, to all conformal transformations of $\mathbf{R}^3$. Their

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description uses *stereographic projection*, the conformal diffeomorphism

$$\mathcal{S} : \mathbf{S}^3 \setminus \{-\boldsymbol{\epsilon}_0\} \to \mathbf{R}^3, \quad \mathcal{S}\Big(\sum_0^3 x^i \boldsymbol{\epsilon}_i\Big) = \frac{1}{1+x^0} \sum_1^3 x^i \boldsymbol{\epsilon}_i,$$

where $\mathbf{S}^3$ is the unit sphere in $\mathbf{R}^4$. Any (even local) *conformal transformation* $f : \mathbf{R}^3 \to \mathbf{R}^3$ is given by

$$f = \mathcal{S} \circ F \circ \mathcal{S}^{-1},$$

where F is an element in the group $\mathrm{Conf}(\mathbf{S}^3)$ of all conformal diffeomorphisms of $\mathbf{S}^3$. This group is isomorphic to $\mathbf{SO}(\mathbf{R}^{4,1})$ by

$$\mathbf{SO}(\mathbf{R}^{4,1}) \ni T \leftrightarrow F = f_+^{-1} \circ T \circ f_+ \in \mathrm{Conf}(\mathbf{S}^3),$$

where we have used the diffeomorphism

$$f_+ : \mathbf{S}^3 \to \mathcal{M} = \{[\mathbf{m}] \in \mathbf{P}(\mathbf{R}^{4,1}) : \langle \mathbf{m}, \mathbf{m} \rangle = 0\}, \quad f_+(\mathbf{x}) = [\mathbf{x} + \boldsymbol{\epsilon}_4].$$

The group $\mathbf{SO}(\mathbf{R}^{4,1})$ acts transitively on $\mathcal{M} \cong \mathbf{S}^3$. Let $\boldsymbol{\delta}_0 = \frac{1}{2}(\boldsymbol{\epsilon}_4 + \boldsymbol{\epsilon}_0) \in \mathbf{R}^{4,1}$ and choose $[\boldsymbol{\delta}_0]$ to be the origin of $\mathcal{M}$. Extend $\boldsymbol{\delta}_0$ to the *Möbius frame* of $\mathbf{R}^{4,1}$

$$\boldsymbol{\delta}_0, \ \boldsymbol{\delta}_1 = \boldsymbol{\epsilon}_1, \ \boldsymbol{\delta}_2 = \boldsymbol{\epsilon}_2, \ \boldsymbol{\delta}_3 = \boldsymbol{\epsilon}_3, \ \boldsymbol{\delta}_4 = \boldsymbol{\epsilon}_4 - \boldsymbol{\epsilon}_0.$$

In this frame the inner product of $\mathbf{R}^{4,1}$ is given by

$$\langle \mathbf{u}, \mathbf{v} \rangle = -u^0 v^4 - u^4 v^0 + \sum_1^3 u^i v^i = {}^t\mathbf{u} g \mathbf{v},$$

where

$$g = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_3 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The *Möbius group*

$$\mathbf{M\ddot{o}b} = \{T \in \mathbf{GL}(5, \mathbf{R}) : {}^tTgT = g\}$$

is $\mathbf{SO}(\mathbf{R}^{4,1})$ represented in this Möbius frame. If $\mathbf{M\ddot{o}b}_0$ is the isotropy subgroup at $[\boldsymbol{\delta}_0]$, then

$$\pi : \mathbf{M\ddot{o}b} \to \mathcal{M}, \quad \pi(T) = T[\boldsymbol{\delta}_0]$$

is the projection of a principal $\mathbf{M\ddot{o}b}_0$-bundle. A *Möbius frame field* is a local section of this bundle.

Note that $f_+ \circ \mathcal{S}^{-1}(\mathbf{R}^3) = \mathcal{M} \setminus \{[\boldsymbol{\delta}_4]\}$, so $[\boldsymbol{\delta}_4]$ corresponds to the *point at infinity* of $\mathbf{R}^3$. We calculate

$$f_+ \circ \mathcal{S}^{-1}(\mathbf{x}) = \left[\frac{1}{2}(1 - |\mathbf{x}|^2)\boldsymbol{\epsilon}_0 + \mathbf{x} + \frac{1}{2}(1 + |\mathbf{x}|^2)\boldsymbol{\epsilon}_4\right] = \left[\boldsymbol{\delta}_0 + \mathbf{x} + \frac{|\mathbf{x}|^2}{2}\boldsymbol{\delta}_4\right].$$

In addition to conformal transformations of surfaces, we want to consider the following. If an immersion $\mathbf{x} : M^2 \to \mathbf{R}^3$ has unit normal $\mathbf{e}_3 : M \to \mathbf{S}^2$, and if $r \in \mathbf{R}$ is constant, then a *parallel transformation* of $\mathbf{x}$ is

$$\tilde{\mathbf{x}} = \mathbf{x} + r\mathbf{e}_3 : M \to \mathbf{R}^3.$$

It fails to be an immersion at any point of M where $1/r$ is a principal curvature. For a circular torus it could be just a curve. For a sphere it could be just a point. A parallel transformation is not a diffeomorphism of $\mathbf{R}^3$. It depends on points $(\mathbf{x}, \mathbf{e}_3) \in \mathbf{R}^3 \times \mathbf{S}^2$.

2. Space of oriented spheres and planes in $\mathbf{R}^3$. For $r \in \mathbf{R}$ and $\mathbf{p} \in \mathbf{R}^3$, let

$$S_r(\mathbf{p}) = \{\mathbf{x} \in \mathbf{R}^3 : |\mathbf{p} - \mathbf{x}|^2 = r^2\}$$

denote the sphere oriented by the unit normal

$$\mathbf{n}(\mathbf{x}) = \frac{\mathbf{p} - \mathbf{x}}{r},$$

if $r \neq 0$ (inward normal if $r > 0$, outward if $r < 0$). If $r = 0$ it denotes the *point sphere* $\{\mathbf{p}\}$, which has no orientation.

For $h \in \mathbf{R}$ and $\mathbf{n} \in \mathbf{S}^2$, the *oriented plane with unit normal* $\mathbf{n}$ *and height* h is

$$\Pi_h(\mathbf{n}) = \{\mathbf{x} \in \mathbf{R}^3 : \mathbf{x} \cdot \mathbf{n} = h\}.$$

The *Lie quadric* is

$$Q = \{[\mathbf{q}] \in \mathbf{P}(\mathbf{R}^{4,2}) : \langle \mathbf{q}, \mathbf{q} \rangle = 0\}.$$

Note that Möbius space $\mathcal{M} \subset Q$ as

$$\boldsymbol{\epsilon}_5^\perp = \{[\mathbf{q}] \in Q : \langle \mathbf{q}, \boldsymbol{\epsilon}_5 \rangle = 0\}.$$

The set of all oriented spheres and planes of $\mathbf{R}^3$ is in one-to-one correspondence with $Q \setminus \{[\boldsymbol{\delta}_4]\}$ by

$$S_r(\mathbf{p}) \leftrightarrow \left[\frac{1}{2}(1 - |\mathbf{p}|^2 + r^2)\boldsymbol{\epsilon}_0 + \mathbf{p} + \frac{1}{2}(1 + |\mathbf{p}|^2 - r^2)\boldsymbol{\epsilon}_4 + r\boldsymbol{\epsilon}_5\right],$$

and

$$\Pi_h(\mathbf{n}) \leftrightarrow [-h\boldsymbol{\epsilon}_0 + \mathbf{n} + h\boldsymbol{\epsilon}_4 + \boldsymbol{\epsilon}_5].$$

The set of all point spheres corresponds to $\mathcal{M} \setminus \{[\boldsymbol{\delta}_4]\}$.

This correspondence is more transparent between the set of oriented spheres of $\mathbf{S}^3$ and Q. An *oriented sphere* in $\mathbf{S}^3$ with center $\mathbf{m} \in \mathbf{S}^3$ and radius $r \in [0, 2\pi)$ is

$$\{\mathbf{x} \in \mathbf{S}^3 : \mathbf{x} \cdot \mathbf{m} = \cos r\},$$

with unit normal

$$\mathbf{n} = \frac{\mathbf{m} - \cos r\, \mathbf{x}}{\sin r}.$$

Interpret this as the point sphere $\mathbf{m}$ if $r = 0$ and $-\mathbf{m}$ if $r = \pi$. The set of oriented spheres in $\mathbf{S}^3$ is $\mathbf{S}^3 \times \mathbf{S}^1$. Its correspondence with Q is

$$(\mathbf{m}, (\cos r, \sin r)) \leftrightarrow [\mathbf{m} + \cos r\, \boldsymbol{\epsilon}_4 + \sin r\, \boldsymbol{\epsilon}_5].$$

Combine this with stereographic projection to get the correspondence between Q and the oriented spheres and planes of $\mathbf{R}^3$. The formulas are more complicated in this case because stereographic projection takes oriented spheres to oriented spheres or planes, but it does not take the center to the center in general.

A *pencil of oriented spheres* in $\mathbf{R}^3$ is the set of all oriented spheres and planes through a given point $\mathbf{p} \in \mathbf{R}^3$ with given normal $\mathbf{n} \in \mathbf{S}^2$. It is determined by a point of the unit tangent bundle $T_1\mathbf{R}^3 = \mathbf{R}^3 \times \mathbf{S}^2$. A point $(\mathbf{p}, \mathbf{n}) \in \mathbf{R}^3 \times \mathbf{S}^2$ determines the pencil

$$\{S_r(\mathbf{p} + r\mathbf{n}) : r \in \mathbf{R}\} \cup \{\Pi_{\mathbf{p}\cdot\mathbf{n}}(\mathbf{n})\}.$$

Any pencil contains a unique point sphere and oriented plane and these determine $(\mathbf{p}, \mathbf{n})$. A pencil corresponds to a line in Q. The above pencil corresponds to the line in Q parametrized by $r \in \mathbf{R} \cup \{\infty\}$,

$$\left[\frac{1}{2}(1 - |\mathbf{p} + r\mathbf{n}|^2 + r^2)\boldsymbol{\epsilon}_0 + \mathbf{p} + r\mathbf{n} + \frac{1}{2}(1 + |\mathbf{p} + r\mathbf{n}|^2 - r^2)\boldsymbol{\epsilon}_4 + r\boldsymbol{\epsilon}_5\right]$$
$$= [S_0(\mathbf{p}) + r\Pi_{\mathbf{p}\cdot\mathbf{n}}(\mathbf{n})].$$

Two points $[q], [\tilde{q}] \in Q$ span a line in Q if and only if $\langle q, \tilde{q}\rangle = 0$. Let Λ denote the set of all lines in Q. This is a smooth manifold of dimension five. The *point projection map* is

$$\pi : \Lambda \to \mathbf{R}^3 \cup \{\infty\},$$

which sends a line λ to the unique point sphere in it, which is $\lambda \cap \mathcal{M}$ followed by stereographic projection. If $[\boldsymbol{\delta}_4]$ is on the line λ, then $\pi(\lambda) = \infty$.

The set of pencils in $\mathbf{R}^3$, which is $\mathbf{R}^3 \times \mathbf{S}^2$, is a dense open subset Λ' of Λ. It is the complement of the set of lines through $[\boldsymbol{\delta}_4]$ in Q.

There is a natural *contact structure* on $\mathbf{R}^3 \times \mathbf{S}^2$ given by the 1-form $\alpha_{(\mathbf{x},\mathbf{n})} = d\mathbf{x} \cdot \mathbf{n}$. It is an elementary exercise to prove that $\alpha \wedge d\alpha \wedge d\alpha$ is never zero on $\mathbf{R}^3 \times \mathbf{S}^2$. We shall extend this contact structure to a contact structure on Λ. A surface immersion $\lambda : M \to \Lambda$ is *Legendre* if $\lambda^*\alpha = 0$ on M.

An immersion $\mathbf{x} : M^2 \to \mathbf{R}^3$ with unit normal $\mathbf{e}_3$ has a natural *Legendre lift*,

$$\lambda : M \to \Lambda', \quad \lambda = [S_0(\mathbf{x}), \Pi_{\mathbf{x}\cdot\mathbf{e}_3}(\mathbf{e}_3)].$$

Under the identification $\Lambda' \cong \mathbf{R}^3 \times \mathbf{S}^2$, we have $\lambda = (\mathbf{x}, \mathbf{e}_3)$, so $\lambda^*\alpha = d\mathbf{x} \cdot \mathbf{e}_3 = 0$, so the lift is a Legendre map. The Legendre lift of a curve in $\mathbf{R}^3$ is the Legendre lift of its unit normal bundle.

3. The Lie sphere group. A *Lie sphere transformation* is a diffeomorphism of Q that sends lines to lines.

A linear transformation in $\mathbf{O}(\mathbf{R}^{4,2})$ is a Lie sphere transformation. In his 1872 thesis [Lie72], S. Lie proved this is all of them.

The conformal transformations $\mathbf{O}(\mathbf{R}^{4,1})$ form a natural subgroup of $\mathbf{O}(\mathbf{R}^{4,2})$ as the subgroup fixing $\boldsymbol{\epsilon}_5$.

Here is an important example of a Lie sphere transformation which is not a conformal transformation. Fix $t \in \mathbf{R}$ and let $T : Q \to Q$ be defined by

$$TS_r(\mathbf{p}) = S_{r+t}(\mathbf{p}), \quad T\Pi_h(\mathbf{n}) = \Pi_{h-t}(\mathbf{n}).$$

Given $(\mathbf{x}, \mathbf{n}) \in \mathbf{R}^3 \times \mathbf{S}^2$, T sends the pencil through $\mathbf{x}$ with normal $\mathbf{n}$ to the pencil through $\mathbf{x} - t\mathbf{n}$ with normal $\mathbf{n}$. In fact,

$$TS_r(\mathbf{x} + r\mathbf{n}) = S_{r+t}(\mathbf{x} + r\mathbf{n}) = S_s(\mathbf{x} - t\mathbf{n} + s\mathbf{n}),$$

if $s = r + t$, and

$$T\Pi_{\mathbf{x}\cdot\mathbf{n}}(\mathbf{n}) = \Pi_{(\mathbf{x}-t\mathbf{n})\cdot\mathbf{n}}(\mathbf{n}).$$

Therefore, T is a Lie sphere transformation, for all $t \in \mathbf{R}$. How does it act on a surface immersed in $\mathbf{R}^3$?

If $\mathbf{x} : M \to \mathbf{R}^3$ is a surface immersion with unit normal $\mathbf{e}_3$, then its Legendre lift $\lambda : M \to \Lambda$ is given by $\lambda = [S_0(\mathbf{x}), \Pi_{\mathbf{x}\cdot\mathbf{e}_3}(\mathbf{e}_3)]$ and

$$T \circ \lambda = [S_t(\mathbf{x}), \Pi_{(\mathbf{x}-t\mathbf{e}_3)\cdot\mathbf{e}_3}(\mathbf{e}_3)]$$

is the pencil through $\mathbf{x} - t\mathbf{e}_3$ with normal $\mathbf{e}_3$, which is the Legendre lift of $\tilde{\mathbf{x}} = \mathbf{x} - t\mathbf{e}_3$, the parallel transformation of $\mathbf{x}$ by $-t$.

$\mathbf{O}(\mathbf{R}^{4,2})$ acts transitively on Λ. As an origin of Λ we chose $[\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1]$, where

$$\boldsymbol{\lambda}_0 = \frac{1}{2}(\boldsymbol{\epsilon}_4 + \boldsymbol{\epsilon}_0), \quad \boldsymbol{\lambda}_1 = \frac{1}{2}(\boldsymbol{\epsilon}_5 + \boldsymbol{\epsilon}_1).$$

These are orthogonal null vectors, so they span a line in Q. Complete this pair to the basis of $\mathbf{R}^{4,2}$

$$\boldsymbol{\lambda}_0,\ \boldsymbol{\lambda}_1,\ \boldsymbol{\lambda}_2 = \boldsymbol{\epsilon}_2,\ \boldsymbol{\lambda}_3 = \boldsymbol{\epsilon}_3,\ \boldsymbol{\lambda}_4 = \boldsymbol{\epsilon}_5 - \boldsymbol{\epsilon}_1,\ \boldsymbol{\lambda}_5 = \boldsymbol{\epsilon}_4 - \boldsymbol{\epsilon}_0.$$

This is a *Lie frame*, meaning the matrix of inner products

$$(\langle \lambda_a, \lambda_b \rangle) = \hat{g} = \begin{pmatrix} 0 & 0 & -L \\ 0 & I_2 & 0 \\ -L & 0 & 0 \end{pmatrix},$$

where $L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The *Lie sphere group*

$$G = \{T \in \mathbf{GL}(6, \mathbf{R}) : {}^tT\hat{g}T = \hat{g}\}$$

is $\mathbf{O}(\mathbf{R}^{4,2})$ represented in this basis. Its Lie algebra is

$$\mathfrak{g} = \{X \in \mathfrak{gl}(6, \mathbf{R}) : {}^tX\hat{g} + \hat{g}X = 0\}.$$

If $T \in G$ then its columns $\mathbf{T}_0, \dots, \mathbf{T}_5$ form a Lie frame of $\mathbf{R}^{4,2}$. The map

$$\pi : G \to \Lambda, \quad \pi(T) = T[\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1] = [\mathbf{T}_0, \mathbf{T}_1],$$

is the projection of a principal G_0-bundle, where the isotropy subgroup of G at $[\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1]$ is

$$G_0 = \left\{ k(c, B, Z, b) = \begin{pmatrix} c & {}^tZ & b \\ 0 & B & BZ{}^tc^{-1}L \\ 0 & 0 & L{}^tc^{-1}L \end{pmatrix} \in G \right\}$$

where $c, Z, b \in \mathbf{R}^{2\times 2}$, $\det c \neq 0$, $B \in \mathbf{O}(2)$, and $bL{}^tc + cL{}^tb = {}^tZZ$. A Lie frame field is a local section of this bundle. Lie sphere geometry is the study of properties of Legendre immersions $\lambda : M^2 \to \Lambda = G/G_0$ invariant under the action of G.

The Maurer–Cartan form of G is the left-invariant $\mathfrak{g}$-valued 1-form $\omega = T^{-1}dT = (\omega^a_b)$ on G, where $a, b, c = 0, \dots, 5$. Then ${}^t\omega\hat{g} + \hat{g}\omega = 0$ implies that

$$\omega = (\omega^a_b) = \begin{pmatrix} \omega^0_0 & \omega^0_1 & \omega^0_2 & \omega^0_3 & \omega^0_4 & 0 \\ \omega^1_0 & \omega^1_1 & \omega^1_2 & \omega^1_3 & 0 & -\omega^0_4 \\ \omega^2_0 & \omega^2_1 & 0 & -\omega^3_2 & \omega^1_2 & \omega^0_2 \\ \omega^3_0 & \omega^3_1 & \omega^3_2 & 0 & \omega^1_3 & \omega^0_3 \\ \omega^4_0 & 0 & \omega^2_1 & \omega^3_1 & -\omega^1_1 & -\omega^0_1 \\ 0 & -\omega^4_0 & \omega^2_0 & \omega^3_0 & -\omega^1_0 & -\omega^0_0 \end{pmatrix}, \tag{1}$$

whose 15 distinct entries form a basis of left-invariant one-forms on G. Now $dT = T\omega$ means

$$d\mathbf{T}_a = \sum_0^5 \mathbf{T}_b \omega^b_a,$$

where $\mathbf{T}_a = \sum_0^5 T^b_a \boldsymbol{\lambda}_b$ is column a of T in the Lie frame, for $a, b = 0, \dots, 5$. We can calculate the forms ω^a_b from the inner products

$$\langle d\mathbf{T}_a, \mathbf{T}_c\rangle = \sum_0^5 \langle \mathbf{T}_b, \mathbf{T}_c\rangle \omega^b_a.$$

For example,

$$\alpha = \omega^4_0 = -\langle d\mathbf{T}_0, \mathbf{T}_1\rangle, \quad \beta = \omega^0_4 = -\langle d\mathbf{T}_4, \mathbf{T}_5\rangle.$$

The structure equations of G are

$$d\omega^a_b = -\sum_0^5 \omega^a_c \wedge \omega^c_b,$$

which come from taking d of $\omega = T^{-1}dT$. Applying this to ω^4_0 and using the relations expressed in (1), we get

$$d\omega^4_0 = -\omega^4_0 \wedge (\omega^0_0 + \omega^1_1) + \omega^2_1 \wedge \omega^2_0 + \omega^3_1 \wedge \omega^3_0.$$

Let $\mathfrak{g}_0$ denote the Lie algebra of the isotropy subgroup G_0. Then $\mathfrak{g}_0$ is the kernel of the derivative $d\pi$ of the projection map at $1 \in G$. A basis of forms spanning $\mathfrak{g}/\mathfrak{g}_0$ is

$$\omega^2_0,\ \omega^2_1,\ \omega^3_0,\ \omega^3_1,\ \omega^4_0.$$

A contact structure is defined on Λ as follows. If $T : U \subset \Lambda \to G$ is a local section of $\pi : G \to \Lambda$, then $T^*\omega^4_0$ is a contact form on U. If $\tilde{T} : U \to G$ is another section, then $\tilde{T} = Tk(c, B, Z, b)$, where $k : U \to G_0$, and one calculates

$$\tilde{T}^*\omega^4_0 = (\det c) T^*\omega^4_0.$$

From the expression for $d\omega^4_0$ above,

$$T^*(\omega^4_0 \wedge d\omega^4_0 \wedge d\omega^4_0) \neq 0$$

at every point of U. On $\mathbf{R}^3 \times \mathbf{S}^2 \subset \Lambda$ let

$$\mathbf{T}_0(\mathbf{x}, \mathbf{n}) = \boldsymbol{\lambda}_0 + \sum_1^3 x^i \boldsymbol{\lambda}_i - \frac{1}{2} x^1 \boldsymbol{\lambda}_4 + \frac{|\mathbf{x}|^2}{2} \boldsymbol{\lambda}_5,$$

and

$$\mathbf{T}_1(\mathbf{x},\mathbf{n}) = (n^1+1)\boldsymbol{\lambda}_1 + n^2\boldsymbol{\lambda}_2 + n^3\boldsymbol{\lambda}_3 + \frac{(1-n^1)}{2}\boldsymbol{\lambda}_4 + (\mathbf{x}\cdot\mathbf{n})\boldsymbol{\lambda}_5.$$

Then $[\mathbf{T}_0,\mathbf{T}_1] : \mathbf{R}^3\times\mathbf{S}^2 \to \Lambda$ is a smooth immersion which can be extended to a local frame field $T : \mathbf{R}^3\times\mathbf{S}^2 \to G$ and

$$\langle d\mathbf{T}_0, \mathbf{T}_1\rangle = \mathbf{n}\cdot d\mathbf{x}$$

shows $T^*\omega_0^4$ extends the contact structure of $\mathbf{R}^3\times\mathbf{S}^2$ to all of Λ.

4. Lie sphere geometry. Study Legendre immersions $\lambda : M^2 \to \Lambda = G/G_0$. At a point of M, λ is a line in Q. The spheres in this line are called *tangent spheres* of λ at the point. A *Lie frame field* along λ is a smooth map $T : U\subset M \to G$ such that $[\mathbf{T}_0,\mathbf{T}_1] = \lambda$ at each point of U.

Any other Lie frame on U is given by $\tilde{T} = Tk(c,B,Z,b)$, where $k : U \to G_0$ is smooth. In particular,

$$(\tilde{\mathbf{T}}_0, \tilde{\mathbf{T}}_1) = (\mathbf{T}_0,\mathbf{T}_1)c,$$

where $c : U \to \mathbf{GL}(2,\mathbf{R})$. For real functions $s,t : U\to\mathbf{R}$, a tangent sphere $\mathbf{S} = s\mathbf{T}_0 + t\mathbf{T}_1$ of λ is a *curvature sphere* if $d\mathbf{S}$ has rank ≤ 1 modulo $\{\mathbf{T}_0,\mathbf{T}_1\}$. We motivate this definition with a brief review.

4.1. Review of Euclidean geometry. Let $\mathbf{x} : M^2 \to \mathbf{R}^3$ be an immersion. Let $(\mathbf{x}, e) : U\subset M \to \mathbf{E}(3)$ be a frame field along it, so $e = (\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ is an orthonormal basis of $\mathbf{R}^3$ and

$$d\mathbf{x} = \sum_1^3 \theta^i\mathbf{e}_i, \quad d\mathbf{e}_i = \sum_1^3 \omega_i^j\mathbf{e}_j,$$

where the $\mathcal{E}(3)$-valued 1-form $((\theta^i),(\omega_j^i))$ is the pull-back by $(\mathbf{x},e)$ of the Maurer–Cartan form of $\mathbf{E}(3)$. The frame is *first order* if $\theta^3 = 0$ on U, in which case θ^1,θ^2 is a coframe field in U and

$$0 = d\theta^3 = -\omega_1^3\wedge\theta^1 - \omega_2^3\wedge\theta^2$$

implies that $\omega_i^3 = \sum_1^2 h_{ij}\theta^j$ for smooth functions $h_{ij} = h_{ji} : U \to \mathbf{R}$. The frame field is *second order* if

$$\omega_1^3 = a\theta^1, \quad \omega_2^3 = c\theta^2,$$

for smooth functions $a,c : U\to\mathbf{R}$, called the *principal curvatures* of $\mathbf{x}$. The *lines of curvature of* a are the integral curves of $\theta^2 = 0$, while the lines of curvature of c are the integral curves of $\theta^1 = 0$. A principal curvature satisfies the *Dupin condition* if it is constant along its own lines of curvature. For example, a satisfies the Dupin condition if and only if da is a multiple of θ^2. If both principal curvatures satisfy the Dupin condition, then $\mathbf{x}$ is called Dupin.

Here is a characterization of the principal curvatures when they are not zero. For a tangent sphere $S_r(\mathbf{x}+r\mathbf{e}_3)$ along $\mathbf{x}$, where $r : U\to\mathbf{R}$ is smooth,

$$d(\mathbf{x}+r\mathbf{e}_3) = (1-ra)\theta^1\mathbf{e}_1 + (1-rc)\theta^2\mathbf{e}_2 + dr\ \mathbf{e}_3$$

has rank $\le 1 \bmod \mathbf{e}_3$ at a point if and only if r is the reciprocal of one of the principal curvatures at the point.

The Dupin condition for $a \neq 0$ is that da is a multiple of θ^2, which is equivalent to

$$d\left(\mathbf{x} + \frac{1}{a}\mathbf{e}_3\right) = \left(1 - \frac{c}{a}\right)\theta^2\mathbf{e}_2 - \frac{da}{a^2}\mathbf{e}_3 \quad \text{has rank} \leq 1.$$

4.2. Back to Lie sphere geometry. Define a *curvature sphere* for $\lambda : M \to \Lambda$ to be a tangent sphere $\mathbf{T}_0$ of λ for which $d\mathbf{T}_0 \bmod \lambda$ has rank ≤ 1. We assume distinct curvature spheres $[\mathbf{T}_0]$ and $[\mathbf{T}_1]$, in which case they are smooth maps $M \to Q$. Then $\lambda = [\mathbf{T}_0, \mathbf{T}_1]$ on M. For a point in M, there exists an open subset $U \subset M$ on which we can extend $\mathbf{T}_0, \mathbf{T}_1 : U \to \mathbf{R}^{4,2}$ to a Lie frame field $T : U \to G$. Then each of

$$d\mathbf{T}_0 \equiv \omega_0^2\mathbf{T}_2 + \omega_0^3\mathbf{T}_3, \quad d\mathbf{T}_1 \equiv \omega_1^2\mathbf{T}_2 + \omega_1^3\mathbf{T}_3 \bmod \{\mathbf{T}_0, \mathbf{T}_1\}$$

has rank ≤ 1 implies

$$\omega_0^2 \wedge \omega_0^3 = 0 = \omega_1^2 \wedge \omega_1^3$$

on U. Taking d of $\omega_0^4 = 0$ and using the structure equations, we also find

$$\omega_1^2 \wedge \omega_0^2 + \omega_1^3 \wedge \omega_0^3 = 0.$$

Because of these relations, $B : U \to \mathbf{O}(2)$ can be chosen so that if $\tilde{T} = Tk(I_2, B, 0, 0)$, then the only change in T is

$$(\tilde{\mathbf{T}}_2, \tilde{\mathbf{T}}_3) = (\mathbf{T}_2, \mathbf{T}_3)B,$$

and $\tilde{\omega}_0^2 = 0 = \tilde{\omega}_1^3$ on U. In this case $\tilde{\omega}_1^2 \wedge \tilde{\omega}_0^3 \neq 0$ at each point of U.

A frame field $T : U \to G$ along λ is *first order* if

(1) $[\mathbf{T}_0]$ and $[\mathbf{T}_1]$ are curvature spheres,
(2) $\omega_0^2 = 0 = \omega_1^3$ and $\omega_1^2 \wedge \omega_0^3 \neq 0$ at each point of U.

If $T : U \to G$ is first order, then any other is given by $\tilde{T} = Tk$, where $k : U \to G_1$ is any smooth map into the subgroup $G_1 \subset G_0$ defined by

$$c = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}, \quad B = \begin{pmatrix} \epsilon & 0 \\ 0 & \delta \end{pmatrix},$$

where $r, s : U \to \mathbf{R}$ are smooth functions with rs never 0 and $\epsilon, \delta \in \{\pm 1\}$.

Differentiating $\omega_0^2 = 0 = \omega_1^3$ and using the structure equations, we get

$$\omega_0^1 = A_2\omega_1^2 + A_3\omega_0^3, \quad \omega_1^0 = B_2\omega_1^2 + B_3\omega_0^3,$$

where the coefficients are smooth functions on U, and

$$\omega_3^2 = -A_3\omega_0^2 + B_2\omega_0^3 = -\omega_2^3.$$

For a change of first order frame $\tilde{T} = Tk$, where $k : U \to G_1$, we compute

$$\tilde{\omega}_1^2 = \epsilon s\omega_1^2, \quad \tilde{\omega}_0^3 = \delta r\omega_0^3,$$

and the coefficients in the new frame are given by

$$\tilde{A}_2 = \frac{\epsilon r}{s^2}A_2, \ \tilde{A}_3 = \frac{\delta A_3 - Z_3^1}{s}, \ \tilde{B}_2 = \frac{\epsilon B_2 - Z_2^0}{r}, \ \tilde{B}_3 = \frac{\delta s}{r^2}B_3.$$

There are three basic orbit types for this action on A_2 and B_3.

A) A_2 and B_3 are never zero on U (generic case).
B) A_2 identically zero, B_3 never zero on U; or vice-versa (canal immersions).
C) A_2 and B_3 both identically zero on U (Dupin immersions).

Consider the condition A_2 identically zero on U. Then $\omega_0^1 = A_3\omega_0^3$ on U, so

$$d\mathbf{T}_0 \equiv (A_3\mathbf{T}_1 + \mathbf{T}_3)\omega_0^3 \operatorname{mod} \mathbf{T}_0,$$

on U, which means $[\mathbf{T}_0]$ is constant on the $\omega_0^3 = 0$ curves. This is the Dupin condition for the curvature sphere $[\mathbf{T}_0]$.

4.3. Dupin case. A *second order frame field* along λ is a first order frame field $T : U \to G$ such that

$$\omega_0^1 = 0, \quad \omega_1^0 = 0, \quad \omega_3^2 = 0 = \omega_2^3$$

on U. Differentiating these forms and using the structure equations we get

$$\omega_2^0 = D\omega_1^2, \quad \omega_3^1 = -D\omega_0^3,$$

for some smooth function $D : U \to \mathbf{R}$. Any other second order frame field on U is given by $\tilde{T} = Tk$, where $k : U \to G_2$ and G_2 is the subgroup of G_1 for which

$${}^tZ = \begin{pmatrix} 0 & Z_3^0 \\ Z_2^1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_4^0 & \frac{1}{2r}(Z_3^0)^2 \\ \frac{1}{2s}(Z_2^1)^2 & -\frac{s}{r}b_4^0 \end{pmatrix},$$

where $Z_3^0, Z_2^1, b_4^0 : U \to \mathbf{R}$ are arbitrary smooth functions. We calculate that

$$\tilde{D} = \frac{1}{rs}(D - sb_4^0).$$

We define a *third order frame* to be a second order frame for which $\omega_2^0 = 0 = \omega_3^1$ on U. Differentiating these forms and using the structure equations we get in addition that $\omega_4^0 = 0$ on U. Differentiating this leads to no further conditions on the Maurer–Cartan forms of G. The frame reduction is complete. If $T : U \to G$ is a third order frame field along λ, then any other is given by $\tilde{T} = Tk$, where $k : U \to G_3$, where G_3 is the subgroup of G_2 for which $b_4^0 = 0$. This is a four-dimensional subgroup, since now r, s, Z_3^0, Z_2^1 are arbitrary (with $rs \neq 0$).

In summary, $T : U \to G$ is third order if $[\mathbf{T}_0], [\mathbf{T}_1]$ are the curvature spheres and

(1) $\omega_0^2 = 0 = \omega_1^3$, and $\omega_1^2 \wedge \omega_0^3 \neq 0$,
(2) $\omega_0^1 = \omega_1^0 = \omega_3^2 = 0$,
(3) $\omega_2^0 = \omega_3^1 = \omega_4^0 = 0$.

The left-invariant 6-dimensional distribution $\mathcal{D}$ on G given by these 8 equations together with $\omega_0^4 = 0$ is completely integrable. It defines a 6-dimensional Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, with connected Lie subgroup $H \subset G$. Integral submanifolds of $\mathcal{D}$ are the right cosets AH, for $A \in G$, of H.

If $\lambda : M^2 \to \Lambda$ is a connected Dupin immersion with $T : M \to G$ a third order Lie frame field along it, then $T : M \to G$ is an integral surface for $\mathcal{D}$ and $T(M)G_3^0$ is an integral submanifold of $\mathcal{D}$, where G_3^0 is the connected component of the identity of G_3. Therefore, $T(M)G_3^0 \subset AH$, for some element $A \in G$, which means

$$\lambda(M) = \pi(T(M)G_3^0) \subset AH[\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1] = A\pi(H),$$

thus showing that $\lambda(M)$ is Lie sphere congruent to an open submanifold of $\pi(H)$.

4.4. Examples. The *circular cylinder* with unit normal $\mathbf{e}_3 = \cos x\,\boldsymbol{\epsilon}_1 + \sin x\,\boldsymbol{\epsilon}_2$,

$$X_{\mathrm{cyl}} : \mathbf{R}^2 \to \mathbf{R}^3, \quad X_{\mathrm{cyl}}(x,y) = (\cos x, \sin x, y),$$

has constant principal curvatures -1 and 0, so it is Dupin. Its curvature spheres are

$$S_{-1}(y\boldsymbol{\epsilon}_3) = [\mathbf{S}_0] = \left[\frac{2-y^2}{2}\boldsymbol{\epsilon}_0 + y\boldsymbol{\epsilon}_3 + \frac{y^2}{2}\boldsymbol{\epsilon}_4 - \boldsymbol{\epsilon}_5\right]$$

and, since $X_{\mathrm{cyl}} \cdot \mathbf{e}_3 = 1$,

$$\Pi_1(\mathbf{e}_3) = [\mathbf{S}_1] = [-\boldsymbol{\epsilon}_0 + \cos x\,\boldsymbol{\epsilon}_1 + \sin x\,\boldsymbol{\epsilon}_2 + \boldsymbol{\epsilon}_4 + \boldsymbol{\epsilon}_5].$$

The Legendre lift of X_{cyl} is then $\lambda_{\mathrm{cyl}} = [\mathbf{S}_0, \mathbf{S}_1]$. Express $\mathbf{S}_0$ and $\mathbf{S}_1$ in the Lie frame of $\mathbf{R}^{4,2}$. Take their exterior derivative and impose the first order frame condition to determine $\mathbf{S}_2$ and $\mathbf{S}_3$. Take their exterior derivative and impose the third order frame condition to determine the last two columns of a third order Lie frame field $S : \mathbf{R}^2 \to G$,

$$S(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1+\cos x & -\sin x & 0 & \frac{1-\cos x}{2} & 0 \\ 0 & \sin x & \cos x & 0 & -\frac{\sin x}{2} & 0 \\ y & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1-\cos x}{2} & \frac{\sin x}{2} & 0 & \frac{1+\cos x}{4} & 0 \\ \frac{y^2-1}{2} & 1 & 0 & y & \frac{1}{2} & 1 \end{pmatrix},$$

along $\lambda_{\mathrm{cyl}} : \mathbf{R}^2 \to \Lambda$. It takes values in the right coset $S(0,0)H$, so

$$\lambda_{\mathrm{cyl}}(\mathbf{R}^2) = \pi(S(\mathbf{R}^2)) \subset \pi(S(0,0)H).$$

The *circular torus*

$$X_{\mathrm{tor}} : \mathbf{R}^2 \to \mathbf{R}^3, \quad X_{\mathrm{tor}}(x,y) = ((2+\cos x)\cos y, (2+\cos x)\sin y, \sin x)$$

is obtained by rotating the circle of radius 1 and center $(2,0,0)$ in the $\boldsymbol{\epsilon}_1\boldsymbol{\epsilon}_3$-plane about the $\boldsymbol{\epsilon}_3$-axis. Its principal curvatures are 1, whose lines of curvature are the $y =$ constant curves, and $\frac{\cos x}{2+\sin x}$, whose lines of curvature are the $x =$ constant curves, which shows that X_{tor} is Dupin. Its oriented curvature spheres are

$$S_1(2\cos y, 2\sin y, 0) = \mathbf{T}_0 = [-\boldsymbol{\epsilon}_0 + 2\cos y\,\boldsymbol{\epsilon}_1 + 2\sin y\,\boldsymbol{\epsilon}_2 + 2\boldsymbol{\epsilon}_4 + \boldsymbol{\epsilon}_5],$$

and

$$\begin{aligned} S_{\frac{2+\cos x}{\cos x}}(0,0,-2\tan x) &= \mathbf{T}_1 \\ &= [(2+3\cos x)\boldsymbol{\epsilon}_0 - 2\sin x\,\boldsymbol{\epsilon}_3 - 2(1+\cos x)\boldsymbol{\epsilon}_4 + (2+\cos x)\boldsymbol{\epsilon}_5]. \end{aligned}$$

The Legendre lift of X_{tor} is then $\lambda_{\mathrm{tor}} = [\mathbf{T}_0, \mathbf{T}_1]$. Express $\mathbf{T}_0$ and $\mathbf{T}_1$ in the Lie frame, take their exterior derivative and impose the first order frame condition to determine $\mathbf{T}_2$ and $\mathbf{T}_3$. Take their exterior derivative and impose the second and third order frame conditions to complete these vector fields to a third order Lie frame field, $T : \mathbf{R}^2 \to G$,

$$T = \begin{pmatrix} 1 & \cos x & \frac{-\sin x}{2} & 0 & \frac{-\cos x}{8} & \frac{1}{8} \\ 1+2\cos y & 2+\cos x & \frac{-\sin x}{2} & -\sin y & \frac{2-\cos x}{8} & \frac{1-2\cos y}{8} \\ 2\sin y & 0 & 0 & \cos y & 0 & \frac{-\sin y}{4} \\ 0 & -2\sin x & -\cos x & 0 & \frac{\sin x}{4} & 0 \\ \frac{1-2\cos y}{2} & \frac{2+\cos x}{2} & \frac{-\sin x}{4} & \frac{\sin y}{2} & \frac{2-\cos x}{16} & \frac{1+2\cos y}{16} \\ \frac{3}{2} & \frac{-4-5\cos x}{2} & \frac{5\sin x}{4} & 0 & \frac{-4+5\cos x}{16} & \frac{3}{16} \end{pmatrix},$$

along λ_{tor}. It takes values in the right coset $T(0,0)H$, so

$$\lambda_{\mathrm{tor}}(\mathbf{R}^2) = \pi(T(\mathbf{R}^2)) = \pi(T(0,0)H),$$

where the final equality holds because $T(\mathbf{R}^2)$ is compact. Setting $V = S(0,0)T(0,0)^{-1} =$

$$\begin{pmatrix} 3/16 & 3/16 & 0 & 0 & -1/8 & 1/8 \\ -1/16 & -1/16 & 0 & 0 & 3/8 & -3/8 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -75/32 & 21/32 & 0 & 0 & 25/16 & 7/16 \\ -25/32 & 7/32 & 0 & 0 & 75/16 & 21/16 \end{pmatrix},$$

we have

$$V\lambda_{\mathrm{tor}}(\mathbf{R}^2) = \pi(VT(0,0)H) = \pi(S(0,0)H) \supset \lambda_{\mathrm{cyl}}(\mathbf{R}^2).$$

In fact,

$$V\lambda_{\mathrm{tor}}(x,y) = [V\mathbf{T}_0, V\mathbf{T}_1] = \left[\begin{bmatrix} \frac{1+\cos y}{2} \\ \frac{-1-\cos y}{2} \\ 0 \\ 2\sin y \\ \frac{-1-\cos y}{4} \\ \frac{15-17\cos y}{4} \end{bmatrix}, \begin{bmatrix} 0 \\ 1+\cos x \\ 2\sin x \\ 0 \\ 2(1-\cos x) \\ \frac{5-3\cos x}{2} \end{bmatrix}\right].$$

Now $[\sum_0^5 q^i \boldsymbol{\lambda}_i]$ is a point sphere if and only if $\frac{1}{2}q^1 + q^4 = 0$. The point projection of the point sphere is then $\frac{1}{q^0}\sum_1^3 q^i \boldsymbol{\epsilon}_i$. The point sphere at (x,y) is then $[sV\mathbf{T}_0 + tV\mathbf{T}_1]$, where

$$s = 5 - 3\cos x, \quad t = 1 + \cos y,$$

and the point projection is thus

$$\pi(V\lambda_{\mathrm{tor}}(x,y)) = \left(\frac{-3+5\cos x}{5-3\cos x}, \frac{4\sin x}{5-3\cos x}, \frac{4\sin y}{1+\cos y}\right),$$

so $\pi(V\lambda_{\mathrm{tor}}(\mathbf{R} \times (-\pi,\pi)) = \lambda_{\mathrm{cyl}}(\mathbf{R}^2)$. Thus, the circular cylinder is Lie sphere congruent to the circular torus.

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