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Kernels, truth and satisfaction

by

James H. SCHMERL

Presented by Feliks PRZYTYCKI

Summary. The Kotlarski–Krajewski–Lachlan Theorem says that every resplendent model of Peano Arithmetic has a full satisfaction class. Enayat and Visser gave a more model-theoretic proof of this theorem. We redo their proof using kernels of directed graphs.

Prologue. A fundamental theorem of Tarski asserts that truth in arithmetic is not arithmetically definable. Nevertheless, there is the well-known Kotlarski–Krajewski–Lachlan Theorem [KKL81] that says that every model $\mathcal{M}$ of Peano Arithmetic (PA) has an elementary extension $\mathcal{N} \succ \mathcal{M}$ having a full satisfaction class (and even a full truth class). Roughly, a full truth class is a subset of the model that satisfies the usual recursive definition of truth: the atomic sentences in the set are precisely the true ones; a disjunction $\sigma_1 \vee \sigma_2$ is in the set iff at least one of σ_1, σ_2 is in the set; an existentially quantified sentence $\exists x\ \varphi(x)$ is in the set iff some instantiation $\varphi(c)$ of it is in the set; etc. Over 30 years later, Enayat & Visser [EV15] gave another proof of the KKL Theorem. According to [EV15], the proof in [KKL81] used some "rather exotic proof-theoretic technology", while the proof in [EV15] uses "a perspicuous method for the construction of full satisfaction classes". Although not made explicit there, the proof in [EV15], when stripped to its essentials, is seen to ultimately depend on showing that certain digraphs have kernels. This is made explicit here.

There is a lengthy discussion in [EV15, §4] about the relationship of full satisfaction classes to full truth classes. Satisfaction classes, which are sets of ordered pairs consisting of a formula in the language of arithmetic and an

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assignment for that formula, are exclusively used in [EV15]. Truth classes are sets of arithmetic sentences that may also have domain constants. By [EV15, Prop. 4.3] (whose "routine but laborious proof is left to the reader"), there is a canonical correspondence between full truth classes and *extensional* full satisfaction classes. The culmination of [EV15, §4] is the construction of extensional full satisfaction classes. In §2 of this paper, we will avoid the intricacies of [EV15, §4] by working exclusively with truth classes to easily obtain the same conclusion.

1. Digraphs and kernels. A binary relational structure $\mathcal{A} = (A, E)$ is referred to here as a *directed graph*, or *digraph* for short ([1]). A subset $K \subseteq A$ is a *kernel* of $\mathcal{A}$ if for every $a \in A$, $a \in K$ iff whenever aEb, then $b \notin K$. According to [BJG09], kernels were introduced by von Neumann [vNM44] and have subsequently found many applications. For $n < \omega$, define the binary relation E^n on A by recursion: xE^0y iff $x = y$; $xE^{n+1}y$ iff xEz and zE^ny for some $z \in A$. A digraph $\mathcal{A}$ is a *directed acyclic graph* (*DAG*) if whenever $n < \omega$ and aE^na, then $n = 0$. Some DAGs have kernels while others do not. For example, if $<$ is a linear order of A with no maximum element, then $(A, <)$ is a DAG with no kernel. However, every *finite* DAG has a (unique) kernel, as was first noted in [vNM44].

An element $b \in A$ for which there is no $c \in A$ such that bEc is a *sink* of $\mathcal{A}$. We say that $\mathcal{A}$ is *well-founded* if every nonempty subdigraph of $\mathcal{A}$ has a sink. Every finite DAG is well-founded, and every well-founded digraph is a DAG having a kernel. The next proposition, for which we need some more definitions, says even more is true. A subset D of a digraph $\mathcal{A}$ is *closed* if whenever $d \in D$ and dEa, then $a \in D$. If $X \subseteq A$ and $k < \omega$, then define $\mathrm{Cl}_k^{\mathcal{A}}(X)$ by recursion: $\mathrm{Cl}_0^{\mathcal{A}}(X) = X$ and $\mathrm{Cl}_{k+1}^{\mathcal{A}}(X) = X \cup \{a \in A : dEa \text{ for some } d \in \mathrm{Cl}_k^{\mathcal{A}}(X)\}$. Let $\mathrm{Cl}^{\mathcal{A}}(X) = \bigcup_{k<\omega} \mathrm{Cl}_k^{\mathcal{A}}(X)$, which is the smallest closed superset of X.

PROPOSITION 1. *Suppose that $\mathcal{A}$ is a digraph, $D \subseteq A$ is closed, $K_0 \subseteq D$ is a kernel of D, and $A \setminus D$ is well-founded. Then $\mathcal{A}$ has a* (*unique*) *kernel K such that $K_0 = K \cap D$.*

Proof. By recursion on ordinals α, define D_α so that $D_0 = D$, $D_{\alpha+1} = D_\alpha \cup \{b \in A : b \text{ is a sink of } A \setminus D_\alpha\}$, and $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$ if α is a limit ordinal. Let γ be such that $A = D_\gamma$. Every D_α is closed. Define K_α as follows: if $\alpha = \beta + 1$, then $K_\alpha = K_\beta \cup \{a \in D_\alpha \setminus D_\beta : aEb \text{ for no } b \in K_\beta\}$; if α is a limit ordinal, then $K_\alpha = \bigcup\{K_\beta : \beta < \alpha\}$. Let $K = K_\gamma$. One easily proves, by induction on α, that K_α is the unique kernel of D_α such that $K_0 = K_\alpha \cap D_\alpha$. Let $K = K_\gamma$. ∎

([1]) Henceforth, $\mathcal{A}$ always denotes a digraph (A, E). If $B \subseteq A$, then we often identify B with the the induced subdigraph $\mathcal{B} = (B, E \cap B^2)$.

Let $\mathcal{A}$ be a digraph. If there is $k < \omega$ for which there are no $a, b \in A$ such that $aE^{k+1}b$, then $\mathcal{A}$ has *finite height*, and we let $\mathrm{ht}(\mathcal{A})$, the *height* of $\mathcal{A}$, be the least such k. If $\mathcal{A}$ has finite height, then it is well-founded. We say that $\mathcal{A}$ has *local finite height* if for every $m < \omega$ there is $k < \omega$ such that $\mathrm{ht}(\mathrm{Cl}^{\mathcal{A}}_m(F)) \leq k$ for every $F \subseteq A$ having cardinality at most m. If $\mathcal{A}$ has local finite height, then it is a DAG. Having local finite height is a first-order property: if $\mathcal{B} \equiv \mathcal{A}$ and $\mathcal{A}$ has local finite height, then so does $\mathcal{B}$.

THEOREM 2. *Every digraph $\mathcal{A}$ having local finite height has an elementary extension $\mathcal{B} \succ \mathcal{A}$ that has a kernel.*

Proof. This proof is modeled after Theorem 3.2(b)'s in [EV15].

To get $\mathcal{B}$ with a kernel K, we let $B_0 = \emptyset$, and then obtain an elementary chain $\mathcal{A} = \mathcal{B}_1 \prec \mathcal{B}_2 \prec \cdots$ and an increasing sequence $\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ such that for every $n < \omega$, K_n is a kernel of $\mathrm{Cl}^{\mathcal{B}_{n+1}}(B_n)$ and $K_n = K_{n+1} \cap B_n$. Having these sequences, we let $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}_{n+1}$ and $K = \bigcup_{n<\omega} K_n$. Clearly, $\mathcal{B} \succ \mathcal{A}$ and K is a kernel of $\mathcal{B}$.

We construct these sequences by recursion. Notice that we have $\mathcal{B}_0$, $\mathcal{B}_1$ and K_0 at the start. Now suppose that we have B_n, $\mathcal{B}_{n+1}$ and K_n such that $\mathcal{A} \preccurlyeq \mathcal{B}_{n+1}$, B_n is a closed subset of B_{n+1} and K_n is a kernel of $D = \mathrm{Cl}^{\mathcal{B}_{n+1}}(B_n)$. We obtain $\mathcal{B}_{n+2}$ and K_{n+1}.

Let Σ be the union of the following three sets of sentences:

- $\mathrm{Th}((\mathcal{B}_{n+1}, a)_{a \in B_{n+1}})$;
- $\{\sigma_{F,k} : k < \omega \text{ and } F \subseteq B_{n+1} \text{ is finite}\}$, where $\sigma_{F,k}$ is the sentence

$$\forall x \in \mathrm{Cl}_k(F)\, [U(x) \leftrightarrow \forall y \in \mathrm{Cl}_{k+1}(F)\, (xEy \to \neg U(y))];$$

- $\{U(d) : d \in K_n\} \cup \{\neg U(d) : d \in D \setminus K_n\}$.

This Σ is a set of $\mathcal{L}$-sentences, where $\mathcal{L} = \{E, U\} \cup B_{n+1}$ and U is a new unary relation symbol.

It suffices to show that Σ is consistent, for then we can let $(\mathcal{B}_{n+2}, U) \models \Sigma$ and let $K_{n+1} = U \cap \mathrm{Cl}^{\mathcal{B}_{n+2}}(B_{n+1})$. To do so, we need only show that every finite subset of Σ is consistent.

Let $\Sigma_0 \subseteq \Sigma$ be finite. Let $k_0 < \omega$ and finite $F_0 \subseteq B_{n+1}$ be such that if $\sigma_{F,k} \in \Sigma_0$, then $k < k_0$ and $F \subseteq F_0$. Let $D_0 = \mathrm{Cl}^{\mathcal{B}_{n+1}}_{k_0}(F_0)$ and let $D_1 = D_0 \cup D$. Since $\mathcal{B}_{n+1}$ has local finite height, D_0 has finite height, and therefore is well-founded, so $D_1 \backslash D$ is well-founded. Since D is a closed subset of B_{n+1}, it is also a closed subset of D_1. We can now apply Proposition 1 to get a kernel U of D_1 such that $K_n = U \cap D$. Then $(\mathcal{B}_{n+1}, U) \models \Sigma_0$, so Σ_0 is consistent.

Thus, Theorem 2 is proved. ■

Recall that $\mathcal{A}$ is *resplendent* iff whenever σ is a first-order $(\{E, R\} \cup A)$-sentence, where R is some new k-ary relation symbol, such that $(\mathcal{B}, S) \models \sigma$

for some $\mathcal{B} \succ \mathcal{A}$ and $S \subseteq B^k$, then there already is $R \subseteq A^k$ such that $(\mathcal{A}, R) \models \sigma$. Every $\mathcal{A}$ has a resplendent elementary extension of the same cardinality. In general, resplendent digraphs are recursively saturated, and conversely, all countable, recursively saturated digraphs are resplendent [BS76].

Corollary 3. *Every resplendent* (*or countable, recursively saturated*) *digraph that has local finite height has a kernel.* ∎

2. Truth classes. There are various ways that syntax for arithmetic can be defined in a model $\mathcal{M}$ of PA. It usually makes little difference how it is done, so we will choose a way that is very convenient.

We will formalize the language of arithmetic by using just two ternary relation symbols: one for addition and one for multiplication. Suppose that $\mathcal{M} \models \mathsf{PA}$. For each $a \in M$, we have a constant symbol c_a. Let $\mathcal{L}^M$ consist of the two ternary relations and all the c_a's. The only propositional connective we will use is the NOR connective $\downarrow$, where $\sigma_0 \downarrow \sigma_1$ is $\neg(\sigma_0 \vee \sigma_1)$. The only quantifier we will use is the "there are none such that" quantifier $\not\exists$, where $\not\exists v\, \varphi(v)$ is $\forall v\, [\neg\varphi(v)]$. Notice that the usual connectives and quantifiers can be defined in terms of these new ones; for example, let $\neg\sigma = \sigma \downarrow \sigma$, $\sigma_1 \vee \sigma_2 = \neg(\sigma_1 \downarrow \sigma_2)$ and $\exists v\, \varphi(v) = \neg \not\exists v\, \varphi(v)$. Let $\mathsf{Sent}^{\mathcal{M}}$ be the set of $\mathcal{L}^{\mathcal{M}}$-sentences as defined in $\mathcal{M}$. A subset $S \subseteq \mathsf{Sent}^{\mathcal{M}}$ is a *full truth class* for $\mathcal{M}$ provided the following hold for every $\sigma \in \mathsf{Sent}^{\mathcal{M}}$:

- if $\sigma = \sigma_0 \downarrow \sigma_1$, then $\sigma \in S$ iff $\sigma_0, \sigma_1 \notin S$;
- if $\sigma = \not\exists v\, \varphi(v)$, then $\sigma \in S$ iff there is no $a \in M$ such that $\varphi(c_a) \in S$;
- if σ is atomic, then $\sigma \in S$ iff $\mathcal{M} \models \sigma$.

Notice that a full truth set behaves properly when restricted to the usual connectives and quantifiers.

Let $A^{\mathcal{M}} = \{\sigma \in \mathsf{Sent}^{\mathcal{M}} : \text{if } \sigma \text{ is atomic, then } \mathcal{M} \models \sigma\}$. Define the binary relation $E^{\mathcal{M}}$ on $A^{\mathcal{M}}$ so that if $\sigma_1, \sigma_2 \in A^{\mathcal{M}}$, then $\sigma_2 E^{\mathcal{M}} \sigma_1$ iff one of the following holds:

- there is σ_0 such that $\sigma_2 = \sigma_0 \downarrow \sigma_1$ or $\sigma_2 = \sigma_1 \downarrow \sigma_0$;
- $\sigma_2 = \not\exists v\, \varphi(v)$ and $\sigma_1 = \varphi(c_a)$ for some $a \in M$.

Consider the digraph $\mathcal{A} = \mathcal{A}^{\mathcal{M}} = (A^{\mathcal{M}}, E^{\mathcal{M}})$. We easily see that S is a full truth class for $\mathcal{M}$ iff S is a kernel of $\mathcal{A}$.

Obviously, $\mathcal{A}$ is a DAG. Moreover, it has local finite height: if $F \subseteq A^{\mathcal{M}}$ is finite and $m < \omega$, then $\mathrm{ht}(\mathrm{Cl}_m^{\mathcal{A}}(F)) \leq (2^{m+1} - 1)|F|$. (A hint for proving this: Define the equivalence relation $\sim$ on $\mathsf{Sent}^{\mathcal{M}}$ so that $\sigma_1 \sim \sigma_2$ iff $\sigma_1' = \sigma_2'$, where σ_e' results after replacing all constant symbols in σ_e with c_0. If $\sigma_0 \sim \tau_0$ and $\sigma_0, \tau_0 \in A^{\mathcal{M}}$, then there are $\sigma_1, \sigma_2 \in \mathrm{Cl}_1^{\mathcal{A}}(\{\sigma_0\})$ such that for every $\tau \in \mathrm{Cl}_1^{\mathcal{A}}(\{\tau_0\})$ there is $e \leq 2$ such that $\tau \sim \sigma_e$.) We can now infer the following version of the KKL Theorem. The comments preceding Corollary 3 about resplendent digraphs apply *mutatis mutandis* to resplendent models of PA.

COROLLARY 4. *Every resplendent (or countable, recursively saturated) $\mathcal{M} \models \mathsf{PA}$ has a full truth class.*

Proof. Since $\mathcal{A}^{\mathcal{M}}$ is definable in $\mathcal{M}$ and $\mathcal{M}$ is resplendent, $\mathcal{A}^{\mathcal{M}}$ is also resplendent. Thus, by Corollary 3, $\mathcal{A}^{\mathcal{M}}$ has a kernel, which we have seen is a full truth class for $\mathcal{M}$. ∎

Corollary 4 can be improved by replacing PA with any of its subtheories in which enough syntax is definable. Also, Corollary 4 can be applied to expansions of models of PA. To see an example, let $\mathsf{Const}^{\mathcal{M}}$ be the set of constant $\mathcal{L}^{\mathcal{M}}$-terms as defined in $\mathcal{M}$ (in which $+$ and $\times$ are considered as function symbols). Let $\mathcal{M}$ be resplendent, and let I be the definable binary relation on Const^{M} such that for any $s, t \in \mathsf{Const}^{\mathcal{M}}$, $\langle s, t\rangle \in I$ iff $s^{\mathcal{M}} = t^{\mathcal{M}}$. By applying Corollary 4 to $(\mathcal{M}, I)$, we conclude that $\mathcal{M}$ has a full truth class S such that for all $s, t \in \mathsf{Const}^{\mathcal{M}}$, the sentence $s = t$ is in S iff $s^{\mathcal{M}} = t^{\mathcal{M}}$.

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James H. Schmerl
Department of Mathematics
University of Connecticut
Storrs, CT 06269, U.S.A.
E-mail: james.schmerl@uconn.edu