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POTENTIAL THEORY

Large versus bounded solutions to sublinear elliptic problems

by

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Summary. Let L be a second order elliptic operator with smooth coefficients defined on a domain $\Omega \subset \mathbb{R}^d$ (possibly unbounded), $d \geq 3$. We study nonnegative continuous solutions u to the equation $Lu(x) - \varphi(x, u(x)) = 0$ on Ω, where φ is in the Kato class with respect to the first variable and it grows sublinearly with respect to the second variable. Under fairly general assumptions we prove that if there is a bounded nonzero solution then there is no large solution.

1. Introduction. Let L be a second order elliptic operator

$$L = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{d} b_i(x)\partial_{x_i} \tag{1.1}$$

with smooth coefficients a_{ij}, b_i defined on a domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$ (1). No conditions are put on the behavior of a_{ij}, b_j near the boundary of $\partial\Omega$. We study nonnegative continuous functions u such that

$$Lu(x) - \varphi(x, u(x)) = 0 \quad \text{on } \Omega, \tag{1.2}$$

in the sense of distributions, where $\varphi : \Omega \times [0, \infty) \to [0, \infty)$ grows sublinearly with respect to the second variable. Such u will be later called *solutions.* A solution u to (1.2) is called *large* if $u(x) \to \infty$ when $x \to \partial\Omega$ or $\|x\| \to \infty$.

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(1) By a *domain* we always mean a set that is open and connected.

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Large solutions, i.e. the boundary blow-up problems, are of considerable interest due to their applications in different fields. Such problems arise in the study of Riemannian geometry [3], non-Newtonian fluids [1], subsonic motion of a gas [24] and electric potentials in some bodies [22].

We prove that under fairly general conditions bounded and large solutions cannot exist at the same time. Classical examples the reader may have in mind are

$$\Delta u - p(x)u^\gamma = 0 \quad \text{with } 0 < \gamma \le 1 \text{ and } p \in \mathcal{L}^\infty_{\mathrm{loc}}, \tag{1.3}$$

where Δ is the Laplace operator on $\mathbb{R}^d$, but we go far beyond that. Not only the operator may be more general but the special form of the nonlinearity in (1.3) may be replaced by $\varphi(x,t)$ satisfying

- (SH_1) There exists a function $p \in \mathcal{K}^{\mathrm{loc}}_d(\Omega)$ (p is locally in the Kato class) such that $\varphi(x,t) \le p(x)(t+1)$ for all $t \ge 0$ and $x \in \Omega$.
- (H_2) For every $x \in \Omega$, $t \mapsto \varphi(x,t)$ is continuous nondecreasing on $[0,\infty)$.
- (H_3) $\varphi(x,t) = 0$ for every $x \in \Omega$ and $t \le 0$.

We recall that a Borel measurable function ψ on Ω is *locally in the Kato class* in Ω if

$$\lim_{\alpha\to 0} \sup_{x\in D} \int\limits_{D\cap(|x-y|\le\alpha)} \frac{|\psi(y)|}{|x-y|^{d-2}}\, dy = 0$$

for every open bounded set D with $\bar{D} \subset \Omega$. Hypothesis (H_1) makes φ locally integrable against against the Green function ([2]) for L, which plays an important role in our approach. (H_3) is a technical extension of φ to $(-\infty, 0)$ needed as a tool. For a part of our results we replace (SH_1) by a weaker condition:

- (H_1) For every $t \in [0,\infty)$, $x \mapsto \varphi(x,t) \in \mathcal{K}^{\mathrm{loc}}_d(\Omega)$.

Applying methods of potential theory we obtain the following result.

THEOREM 1. *Assume that Ω is Greenian for L ([3]). Suppose that $\varphi(x,t) = p(x)\psi(t)$ satisfies* (SH_1), (H_2), (H_3) *and there exists a nonnegative nontrivial bounded solution to* (1.2). *Then there is no large solution to* (1.2).

Theorem 1 considerably improves a similar result of El Mabrouk and Hansen [9] for L being the Laplace operator Δ on $\mathbb{R}^d$, $\varphi(x,t) = p(x)\psi(t)$, $p \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ and $\psi(t) = t^\gamma$, $0 < \gamma < 1$. It is proved in Section 4.

In fact, we prove a few statements more general than Theorem 1 but a little more technical to formulate (see Theorem 3 in Section 2). Generally, we do not assume that φ has product form, and in particular we characterize

([2]) See (4.1)–(4.3) for the definition of G_Ω.

([3]) See Section 4 for the definition, more precisely, (4.2), (4.3).

a class of functions $p(x)$ in (SH_1) for which there are bounded solutions but no large solutions to (1.2) (see Theorem 9 in Section 4).

Besides the theorem due to El Mabrouk and Hansen [9] there are other results indicating that the equation $\Delta u - p(x)u^\gamma = 0$, or more generally $\Delta u - p(x)\psi(u) = 0$, cannot have bounded and large solutions at the same time [16], [17], [21]. We prove such a statement in considerable generality:

- L is an elliptic operator (1.1);
- Ω is Greenian for L, generally unbounded;
- the nonlinearity is assumed to have only sublinear growth; no concavity with respect to the second variable and no product form of φ is required.

Our main strategy adopted from [7] and [9] is to relate solutions of (1.2) to L-harmonic functions and to make extensive use of potential theory. We rely on the results of [11] and [12] where this approach was developed.

Existence of large solutions for the equation

$$\Delta u = p(x)f(u)$$

was studied under more regularity: p Hölder continuous and f Lipschitz (not necessarily monotone) [19] ([4]) or on the whole of $\mathbb{R}^d$ [29]. In our approach very little regularity is involved but monotonicity of φ with respect of t is essential. Suppose φ is not of the product form but the following condition is satisfied:

(H_4) For every $x \in \Omega$, $t \mapsto \varphi(x,t)$ is concave on $[0,\infty)$.

Then we have

THEOREM 2. *Suppose that* (H_1)–(H_4) *hold and that there is a bounded solution to*

$$Lu(x) - \varphi(x,u(x)) = 0.$$

Then there is no large solution.

Theorem 2 follows directly from Theorem 3. Our strategy for the proof of Theorem 1 is to construct a function $\varphi_1 \geq \varphi$ satisfying (SH_1), (H_2)–(H_4) and to apply Theorem 3 to φ and φ_1 ([5]). To make use of both equations, for φ and φ_1, we need a criterion for existence of bounded solutions to (1.2) (see Theorem 8). The latter, proved in this generality, is itself interesting.

Semilinear problems $\Delta u + g(x,u) = 0$ have been extensively studied under a variety of hypotheses on g, and various questions have been asked. The function g is not necessarily monotone or negative but there are often other restrictive assumptions like more regularity of g or the product form. The problem is usually considered either in bounded domains or in $\Omega = \mathbb{R}^d$

([4]) More generally, $\Delta u = p(x)f(u) + q(x)g(u)$, p, q Hölder continuous [18].

([5]) The main difficulty is to guarantee that $\varphi_1(x,0) = 0$ (see Section 3).

[2], [5], [6], [8], [10], [13], [14], [20], [23], [26], [28], [31], [30]. Finally, there are not many results for general elliptic operators, and they mostly have the same restrictions [4], [15], [25], [27]. Clearly, stronger regularity of g or Ω is used to obtain conclusions other than the one we are interested in.

2. Large solutions to $Lu - \varphi(\cdot, u) = 0$ under (H_1)–(H_3)**.** In this section we replace (SH_1) by (H_1) which is weaker. Our aim is to prove that under fairly general assumptions, bounded and large solutions to (1.2) cannot occur at the same time ([6]).

THEOREM 3. *Let Ω be a domain and suppose φ, φ_1 satisfy* (H_1)–(H_3). *Assume that $\varphi \le \varphi_1$ and φ_1 is concave with respect to the second variable. If the equation $Lu = \varphi_1(\cdot, u)$ has a nontrivial nonnegative bounded solution in Ω then $Lu = \varphi(\cdot, u)$ does not have a large solution in Ω.*

Theorem 3 gives, in particular, the most general conditions for Δ implying nonexistence of a bounded and a large solution at the same time. Compare with Theorem 3.1 in [9], where the statement was proved for $\varphi(x, u) = p(x)u^\gamma$, $p \in \mathcal{L}^\infty_{\mathrm{loc}}(\Omega)$.

Applying Theorem 3 to φ being concave with respect to the second variable we obtain Theorem 2. In the next section, we will prove that under (SH_1) such a φ_1 always exists, which makes Theorem 3 widely applicable.

For the proof we need to recall a number of properties satisfied by solutions to (1.2). For $L = \Delta$ they were proved in [7], and the general case is similar (see [12]).

Let $\mathcal{C}^+(\Omega)$ and $\mathcal{C}^+(\partial\Omega)$ be the sets of nonnegative continuous functions on Ω and $\partial\Omega$ respectively.

LEMMA 4 ([12, Lemma 5]). *Suppose that φ satisfies* (H_2). *Let $u, v \in \mathcal{C}^+(\Omega)$ be such that $Lu, Lv \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$. If*

$$Lu - \varphi(\cdot, u) \le Lv - \varphi(\cdot, v)$$

in the sense of distributions and

$$\liminf_{\substack{x \to y \\ y \in \partial\Omega}} (u - v)(x) \ge 0,$$

then

$$u - v \ge 0 \quad \text{in } \Omega.$$

For a bounded regular domain $D \subset \mathbb{R}^d$ and a nonnegative function f continuous on ∂D, we define $U^\varphi_D f$ to be the function such that $U^\varphi_D f = f$

([6]) In Theorem 3 and all the statements of this section, L may be slightly more general: a nonpositive zero order term is allowed.

on $\mathbb{R}^d \setminus D$ and $U_D^\varphi f|_D$ is the unique solution of

$$(2.1)\qquad \begin{cases} Lu - \varphi(\cdot, u) = 0 & \text{in } D \text{ in the sense of distributions,} \\ u \geq 0 & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases}$$

Existence of $U_D^\varphi f$ was proved in [12, Theorem 4]. Let G_D be the Green function for D. Then

$$(2.2)\qquad H_D f = U_D^\varphi f + G_D \varphi(\cdot, U_D^\varphi f) \quad \text{in } D,$$

where $H_D f$ is an L-harmonic function in D with boundary values f, and for a function u we set

$$(2.3)\qquad G_D(\varphi(\cdot, u))(x) = \int_D G_D(x, y)\varphi(y, u(y))\, dy.$$

In particular $U_D^\varphi f$ is not identically 0 in D if f is not identically 0 on ∂D.

Now we focus on the properties of $U_D^\varphi f$. We say that u is a *supersolution* to (1.2) if $Lu - \varphi(\cdot, u) \leq 0$, and a *subsolution* if $Lu - \varphi(\cdot, u) \geq 0$. In the following lemma we shall apply U_D^φ to $f, g, u, v \in \mathcal{C}^+(\Omega)$, that is, to their restrictions to ∂D. The lemma is a direct consequence of Lemma 4 and existence of solutions to (2.1). For $L = \Delta$ it was proved in [7].

LEMMA 5. *Suppose that φ satisfies* (H_1)–(H_3) *and let D be a bounded regular domain such that $\bar{D} \subset \Omega$. Then U_D^φ is nondecreasing in the following sense:*

$$(2.4)\qquad U_D^\varphi f \leq U_D^\varphi g \quad \text{if } f \leq g \text{ in } \Omega.$$

Let u be a continuous supersolution and v a continuous subsolution of (1.2) *in Ω. Suppose further that D and D' are regular bounded domains such that $D' \subset D \subset \Omega$. Then*

$$(2.5)\qquad U_D^\varphi u \leq u \quad \text{and} \quad U_D^\varphi v \geq v,$$

$$(2.6)\qquad U_{D'}^\varphi u \geq U_D^\varphi u \quad \text{and} \quad U_{D'}^\varphi v \leq U_D^\varphi v.$$

If in addition (H_4) *holds* ([7]) *then U_D^φ is a convex function on $\mathcal{C}^+(\partial D)$, i.e. for every $\lambda \in [0,1]$,*

$$(2.7)\qquad U_D^\varphi(\lambda f + (1-\lambda)g) \leq \lambda U_D^\varphi f + (1-\lambda) U_D^\varphi g.$$

In particular, for every $\alpha \geq 1$,

$$(2.8)\qquad U_D^\varphi(\alpha f) \geq \alpha U_D^\varphi f.$$

Now, let (D_n) be a sequence of bounded regular domains such that for every $n \in \mathbb{N}$, $\overline{D_n} \subset D_{n+1} \subset \Omega$ and $\bigcup_{n=1}^\infty D_n = \Omega$. Such a sequence will be called *a regular exhaustion* of Ω and it is used to generate solutions to (1.2).

([7]) Notice that concavity together with (H_1) and (H_2) implies (SH_1).

PROPOSITION 6 ([12, Proposition 10]). *Let $g \in \mathcal{C}^+(\Omega)$ be an L-superharmonic function. Then the sequence $(U^{\varphi}_{D_n} g)$ is decreasing to a solution $u \in \mathcal{C}^+(\Omega)$ of* (1.2) *satisfying $u \le g$* ([8]).

Now we are ready to prove the main result of this section.

Proof of Theorem 3. Suppose that $Lu - \varphi_1(\cdot, u) = 0$ has a nontrivial nonnegative bounded solution $\tilde{u}$ in Ω. Let (D_n) be an increasing sequence of bounded regular domains exhausting Ω. Then by Proposition 6 for every $\lambda \ge \lambda_1 = \|\tilde{u}\|_{L^\infty} > 0$, $v_\lambda = \lim_{n\to\infty} U^{\varphi_1}_{D_n}\lambda$ is a nontrivial nonnegative bounded solution of $Lu - \varphi_1(\cdot, u) = 0$ in Ω too.

Let $\lambda \ge \lambda_1$. Then by Lemma 5, $U^{\varphi_1}_{D_n}\lambda \ge \frac{\lambda}{\lambda_1} U^{\varphi_1}_{D_n}\lambda_1$. Therefore, letting $n \to \infty$ we obtain

$$v_\lambda \ge \frac{\lambda}{\lambda_1} v_{\lambda_1}, \quad \text{where} \quad v_{\lambda_1} = \lim_{n\to\infty} U^{\varphi_1}_{D_n}\lambda_1.$$

Furthermore, $\varphi \le \varphi_1$ implies, by Lemma 4, that $U^{\varphi}_{D_n}\lambda \ge U^{\varphi_1}_{D_n}\lambda$, because $U^{\varphi}_{D_n}\lambda$ is a supersolution to $Lu - \varphi_1(\cdot, u) = 0$. Hence

$$u_\lambda = \lim_{n\to\infty} U^{\varphi}_{D_n}\lambda \ge v_\lambda.$$

Suppose now that there is a large solution u to (1.2). Then it satisfies $\liminf_{x\to\partial\Omega} u(x) = \infty$. Hence for sufficiently large n, $u \ge U^{\varphi}_{D_n}\lambda$ on ∂D_n, and so by Lemma 4,

$$u \ge u_\lambda \ge v_\lambda.$$

Consequently, $u \ge \frac{\lambda}{\lambda_1} v_{\lambda_1}$ and so $\frac{u}{\lambda} \ge \frac{1}{\lambda_1} v_{\lambda_1}$ for every $\lambda \ge \lambda_1$. When λ tends to infinity, we get $v_{\lambda_1} = 0$, which gives a contradiction. ∎

3. Domination by a concave function. The aim of this section is to show that (SH_1), (H_2), (H_3) imply existence of a function φ_1 concave with respect to the second variable and such that

$$\varphi(x,t) \le \varphi_1(x,t), \quad \varphi_1(x,0) = 0.$$

Clearly, a nonnegative function ψ concave on $[0,\infty)$, continuous at zero, and with $\psi(0) = 0$ is dominated by an affine function. Indeed, given $\beta > 0$, we have

$$\psi(t) \le \frac{t}{\beta}\psi(\beta), \quad t \ge \beta,$$

and so

$$\psi(t) \le \frac{t}{\beta}\psi(\beta) + \sup_{0\le s\le\beta} \psi(s).$$

([8]) Note here that u may be zero and usually an extra argument is needed to ensure it is not.

The idea behind (SH_1) is to formulate a condition as weak as possible to go beyond concavity in Theorem 1. It turns out that (SH_1) together with Theorem 7 below does the job. Clearly, the most delicate part is to guarantee that $\varphi_1(x,0)=0$.

THEOREM 7. *Suppose that* $\varphi(x,t)$ *satisfies* (SH_1), (H_2), (H_3). *Then there is* $\varphi_1(x,t)$ *satisfying* (SH_1), (H_2)–(H_4) *such that*

$$\varphi(x,t)\le\varphi_1(x,t).$$

Moreover, there exists a constant $C>0$ *such that*

$$\varphi_1(x,t)\le Cp(x)(t+1).$$

Proof. For $t\ge1$,

$$\varphi(x,t)\le 2p(x)t.$$

We need to dominate φ for $t\le1$. Let $\eta\in C^\infty(\mathbb{R})$, $\eta\ge0$, $\operatorname{supp}\eta\subset(-1,1)$, $\eta(-t)=\eta(t)$ and $\int_{\mathbb{R}}\eta(s)\,ds=1$. Given $0<\delta\le1$, let $\eta_\delta(t)=\frac{1}{\delta}\eta\big(\frac{1}{\delta}t\big)$, $t\in\mathbb{R}$. Let $x\in\Omega$. We write $\varphi_x(t)=\varphi(x,t)$, $t\in\mathbb{R}$. Then

$$\varphi_x*\eta_\delta(0)=\int_{-\delta}^{\delta}\varphi_x(-t)\eta_\delta(t)\,dt=\int_{-1}^{1}\varphi(x,\delta s)\eta(s)\,ds. \tag{3.1}$$

Hence

$$0\le\inf_\delta\varphi_x*\eta_\delta(0)=\lim_{\delta\to0}\varphi_x*\eta_\delta(0)=\varphi_x(0)=0. \tag{3.2}$$

Secondly, $(\varphi_x*\eta_\delta)'=\varphi_x*(\eta_\delta)'$ and

$$(\eta_\delta)'(t)=\frac{1}{\delta^2}\eta'\bigg(\frac{1}{\delta}t\bigg). \tag{3.3}$$

Moreover,

$$\int_{\mathbb{R}}|(\eta_\delta)'(t)|\,dt\le\int_{\mathbb{R}}\frac{1}{\delta^2}\bigg|\eta'\bigg(\frac{1}{\delta}t\bigg)\bigg|\,dt=\int_{\mathbb{R}}\frac{1}{\delta}|\eta'(s)|\,ds.$$

Therefore, if $0\le t\le2$ then

$$|(\varphi_x*\eta_\delta)'(t)|\le\int_{\mathbb{R}}\varphi_x(t-s)|(\eta_\delta)'(s)|\,ds\le p(x)\frac{4}{\delta}\int_{\mathbb{R}}|\eta'(s)|\,ds.$$

Consequently, there exists a constant c_1 such that for $0\le t\le2$ we have

$$\varphi_x*\eta_\delta(t)\le\frac{c_1}{\delta}p(x)t+\varphi_x*\eta_\delta(0). \tag{3.4}$$

Moreover,

$$\varphi_x * \eta_\delta(t) = \int_{\mathbb{R}} \varphi_x(t-s)\eta_\delta(s)\,ds \ge \int_{-\delta}^{0} \varphi_x(t-s)\eta_\delta(s)\,ds$$
$$\ge \varphi_x(t) \int_{-\delta}^{0} \eta_\delta(s)\,ds = \frac{1}{2}\varphi_x(t).$$

Hence

$$\varphi_x(t) \le 2\varphi_x * \eta_\delta(t)$$

and so for $t \in [0,2]$,

$$\varphi_x(t) \le \frac{2c_1}{\delta} p(x)t + 2\varphi_x * \eta_\delta(0).$$

Let

$$\psi_\delta(x,t) = \frac{2c_1}{\delta} p(x)t + 2\varphi_x * \eta_\delta(0), \qquad \psi(x,t) = \inf_{0<\delta<1} \psi_\delta(x,t).$$

First we prove that for every fixed $x \in \Omega$, $\psi(x,t)$ is concave on $[0,2]$. For $t, s \in [0,2]$ and $\alpha \in [0,1]$, we have

$$\begin{aligned}\psi(x, \alpha t + (1-\alpha)s) &= \inf_\delta \psi_\delta(x, \alpha t + (1-\alpha)s)\\ &= \inf_\delta \big(\alpha\psi_\delta(x,t) + (1-\alpha)\psi_\delta(x,s)\big)\end{aligned}$$

and

$$\inf_\delta \big(\alpha\psi_\delta(x,t) + (1-\alpha)\psi_\delta(x,s)\big) \ge \inf_\delta \alpha\psi_\delta(x,t) + \inf_\delta (1-\alpha)\psi_\delta(x,s).$$

Hence

$$\psi(x, \alpha t + (1-\alpha)s) \ge \alpha\psi(x,t) + (1-\alpha)\psi(x,s)$$

and so $\psi(x,t)$ is continuous on $(0,2)$ in t. Secondly,

$$\psi(x,0) = \inf_\delta 2\varphi_x * \eta_\delta(0) = 2\varphi(x,0) = 0,$$

and for every δ,

$$\begin{aligned}\limsup_{t\to 0} \psi(x,t) &\le \limsup_{t\to 0} \left(2c_1 c(x)\frac{1}{\delta}t + 2\varphi_x * \eta_\delta(0)\right)\\ &\le 2\varphi_x * \eta_\delta(0) \le 2\varphi_x(\delta).\end{aligned}$$

Hence $\lim_{t\to 0^+} \psi(x,t) = 0$ and so $\psi(x,t)$ is continuous on $[0,2)$. Moreover, $\psi(x,\cdot)$ is nondecreasing and

$$\begin{aligned}\psi(x,t) &\le \psi_1(x,t) \le 2c_1 p(x)t + 2\varphi(x,1)\\ &\le 2c_1 p(x)t + 4p(x) \le 4c_1 p(x)(t+1).\end{aligned}$$

Finally, we define

$$\varphi_1(x,t)=\begin{cases}2p(x)t+\psi(x,t) & \text{if } 0\le t\le 1,\\ 2p(x)t+\psi(x,1) & \text{if } t>1,\end{cases}$$

and we set $\varphi_1(x,t)=0$ if $t\le 0$. ∎

4. Large solutions to $Lu-\varphi(\cdot,u)=0$ under (SH$_1$), (H$_2$), (H$_3$). In this section we prove Theorem 1. The argument is based on a very convenient characterization of existence of bounded solutions to (1.2). It is formulated in terms of thinness at infinity.

Let $\Omega\subset\mathbb{R}^d$, $d\ge 3$, be a domain. A subset $A\subset\Omega$ is called *thin at infinity* if there is a continuous nonnegative L-superharmonic function s on Ω such that

$$\begin{cases}s\ge 1 & \text{on } A,\\ s(x_0)<1 & \text{for some } x_0\in\Omega.\end{cases}$$

We say that Ω is *Greenian* if there is a function G_Ω called *the Green function for L* satisfying

$$G_\Omega(x,y)\in C^\infty(\Omega\times\Omega\setminus\{(x,x):x\in\Omega\}), \tag{4.1}$$

for every $y\in\Omega$ we have

$$LG_\Omega(\cdot,y)=-\delta_y \quad \text{in the sense of distributions}, \tag{4.2}$$

and

$$G_\Omega(\cdot,y) \quad \text{is a potential}, \tag{4.3}$$

i.e. every nonnegative L-harmonic function h such that $h(x)\le G_\Omega(x,y)$ is identically zero. For a given domain Ω, the Green function G_Ω may or may not exist, but existence of s as above implies that it does.

THEOREM 8 ([12, Theorem 19]). *Suppose that Ω is Greenian and φ is a measurable function satisfying* (H$_1$)–(H$_3$). *Equation* (1.2) *has a nonnegative nontrivial bounded solution in Ω if and only if there exists a Borel set $A\subset\Omega$ which is thin at infinity and $c_0>0$ such that*

$$\int_{\Omega\setminus A} G_\Omega(\cdot,y)\varphi(y,c_0)\,dy\not\equiv\infty. \tag{4.4}$$

In the case of $L=\Delta$ and $\varphi(x,t)=p(x)t^\gamma$, $0<\gamma<1$, $p\in\mathcal{L}^\infty_{\mathrm{loc}}$, Theorem 8 was proved in [7]. Notice that no concavity (H$_4$) is required.

In view of Theorems 8 and 7, the proof of Theorem 1 is straightforward:

Proof of Theorem 1. If $Lu-p(x)\psi(u)=0$ has a nonnegative nontrivial bounded solution then by Theorem 8 there is a set $A\subset\Omega$ thin at infinity such that

$$\int_{\Omega\setminus A} G_\Omega(\cdot,y)p(y)\,dy \not\equiv \infty. \tag{4.5}$$

Let φ_1 be the function constructed in Theorem 7. Then φ_1 can be taken such that

$$\varphi_1(x,t) \le Cp(x)(t+1),$$

and so again by Theorem 8, $Lu - \varphi_1(\cdot,u) = 0$ has a nonnegative nontrivial bounded solution. Hence the conclusion follows by Theorem 3. ∎

Now we are going to apply Theorem 3 to φ that satisfies (SH_1).

Theorem 9. *Let Ω be a Greenian domain. Assume that φ satisfies* (SH_1), (H_2), (H_3) *and there exists a set $A \subset \Omega$ thin at infinity such that the function $p(x)$ in* (SH_1) *satisfies*

$$\int_{\Omega\setminus A} G_\Omega(\cdot,y)p(y)\,dy \not\equiv \infty. \tag{4.6}$$

Then (1.2) *has a nonnegative nontrivial bounded solution and it has no large solution.*

Proof. By Theorem 8, there is a nonnegative nontrivial bounded solution to (1.2). Let $\varphi_1(x,t)$ be the function constructed in Theorem 7. Then

$$\varphi_1(x,t) \le \ Cp(x)(t+1).$$

Hence there is a nonnegative nontrivial bounded solution to $Lu-\varphi_1(\cdot,u)=0$, and so by Theorem 3 there is no large solution to (1.2). ∎

Suppose now that for every $t_0>0$ there is a constant $C_{t_0}>0$ such that for every $t\ge 0$ and $x\in\Omega$, $\varphi(x,t)\le C_{t_0}\varphi(x,t_0)(t+1)$. We do not assume any integrability of $\varphi(x,t_0)$ in the spirit of (4.6). Then

Theorem 10. *Let Ω be a Greenian domain. Assume that φ satisfies* (H_1)–(H_3). *Suppose further that for every $t_0>0$ there is $C_{t_0}>0$ such that*

$$\varphi(x,t)\le C_{t_0}\varphi(x,t_0)(t+1).$$

If (1.2) *has a nonnegative nontrivial bounded solution, then* (1.2) *has no large solution.*

Proof. By Theorem 8, there exists a set $A\subset\Omega$ thin at infinity and $t_0>0$ such that

$$\int_{\Omega\setminus A} G_\Omega(\cdot,y)\varphi(y,t_0)\,dy \not\equiv \infty. \tag{4.7}$$

Let $\varphi_1(x,t)$ be the function constructed in Theorem 7. We can take φ_1 such that $\varphi_1(x,t) \le CC_{t_0}\varphi(x,t_0)(t+1)$. Then

$$\int_{\Omega\setminus A} G_\Omega(\cdot,y)\varphi_1(y,t_0)\,dy \not\equiv \infty. \tag{4.8}$$

Hence there is a nonnegative nontrivial bounded solution to $Lu-\varphi_1(\cdot,u)=0$, and so by Theorem 3 there is no large solution to (1.2). ∎

5. Bounded solutions to $Lu-\varphi(\cdot,u)=0$. Theorems 7 and 8 allow us to remove concavity and get the following characterization of bounded solutions.

PROPOSITION 11. *Let Ω be a Greenian domain. Suppose that $\varphi(x,t) = p(x)\psi(t)$ satisfies* (SH_1), (H_2) *and* (H_3). *Let (D_n) be an increasing sequence of regular bounded domains exhausting Ω. The following statements are equivalent:*

(1) *Equation* (1.2) *has a nonnegative nontrivial bounded solution.*
(2) *For every $c>0$, $v_c = \inf_{n\in\mathbb{N}} U^\varphi_{D_n} c$ is a nonnegative nontrivial bounded solution of* (1.2).
(3) *There exists $c>0$ such that $v_c = \inf_{n\in\mathbb{N}} U^\varphi_{D_n} c$ is a nonnegative nontrivial bounded solution of* (1.2).

Furthermore if any of the above conditions holds then

$$\sup_{x\in\Omega} v_c(x) = c. \tag{5.1}$$

The proof of Proposition 11 is given at the end of this section. We proceed as before: first we obtain the result for a concave nonlinear term, i.e. under (H_1)–(H_4), and then we apply Theorem 7.

PROPOSITION 12. *Suppose that φ satisfies* (H_1)–(H_4). *Then the conclusion of Proposition* 11 *holds true.*

Proposition 12 was proved in [7] for $L=\Delta$ and $\varphi(x,t)=p(x)t^\gamma$ where $0<\gamma<1$ and $p\in\mathcal{L}^\infty_{\mathrm{loc}}$. Generalization to elliptic operators and φ satisfying (H_1)–(H_4) is straightforward and φ need not to be of the product form.

Proof of Proposition 12. The proof is the same as in [7, Lemmas 3 and 4], but we include the argument here for the reader's convenience. Let $u_n = U^\varphi_{D_n} c$ and $u_c = \inf_{n\in\mathbb{N}} u_n$. Under hypotheses ($H_1$)–($H_4$), $\sup_{x\in\Omega} u_c(x)$ is either zero or c. Indeed, by Proposition 6, u_c is a nonnegative solution of (1.2) bounded above by c. Suppose now that there exists $0<c_0\le c$ such that $\sup_{x\in\Omega} u_c = c_0$. By Lemma 4,

$$U^\varphi_{D_n}\left(\frac{c}{c_0}u_c\right) \le U^\varphi_{D_n} c = u_n.$$

Also by Lemma 5,

$$\frac{c}{c_0} U^{\varphi}_{D_n} u_c \le U^{\varphi}_{D_n}\left(\frac{c}{c_0} u_c\right).$$

Hence

$$U^{\varphi}_{D_n} u_c = u_c \le \frac{c_0}{c} u_n,$$

and letting n tend to infinity we obtain

$$u_c \le \frac{c_0}{c} u_c,$$

which implies $c = c_0$.

Therefore, under (H_4), if any of conditions (1)–(3) is satisfied then (5.1) follows. It is clear that (2)⇒(3)⇒(1). So it is enough to prove that (1) implies (2). Let w be a nonnegative nontrivial bounded solution of (1.2).

Suppose first that $r \ge \sup_{\Omega} w$. Then $v = \lim_{n\to\infty} U^{\varphi}_{D_n} r$ is a nonnegative nontrivial bounded solution satisfying $w \le v \le r$ in Ω. Hence

$$\sup_{x\in\Omega} v(x) = r. \tag{5.2}$$

Secondly, we take $0 < c < \sup_{\Omega} w$.

By Lemma 5, $u_n = U^{\varphi}_{D_n} c \le U^{\varphi}_{D_n} r = v_n$ in D_n. Hence

$$G_{D_n}(\varphi(\cdot, u_n)) \le G_{D_n}(\varphi(\cdot, v_n)) \quad \text{in } D_n.$$

Furthermore by (2.2),

$$v_n + G_{D_n}(\varphi(\cdot, v_n)) = r \quad \text{in } D_n,$$

and

$$u_n + G_{D_n}(\varphi(\cdot, u_n)) = c \quad \text{in } D_n.$$

We can deduce

$$0 \le c - u_n \le r - v_n \quad \text{in } D_n.$$

When n tends to infinity, we get

$$c - u \le r - v \quad \text{in } \Omega.$$

Suppose now that u is trivial. Then

$$v \le r - c \quad \text{in } \Omega.$$

But $\sup_{\Omega} v = r$, which gives a contradiction. ■

Proof of Proposition 11. As before, it is enough to prove that (1) implies (2). By Theorem 8, there is a set $A \subset \Omega$ thin at infinity such that

$$\int_{\Omega\setminus A} G_{\Omega}(\cdot, y)p(y)\, dy \not\equiv \infty. \tag{5.3}$$

Let $\varphi_1(x,t)$ be the function constructed in Theorem 7. We can take φ_1 such that $\varphi_1(x,t) \le Cp(x)(t+1)$, so again by Theorem 8, $Lu - \varphi_1(\cdot, u) = 0$ has a nonnegative nontrivial bounded solution. Let $c > 0$. By Proposition 12,

$$v_c^1 = \lim_{n\to\infty} U_{D_n}^{\varphi_1} c$$

is a nonnegative nontrivial bounded solution of $Lu - \varphi_1(\cdot, u) = 0$ and

$$\sup_{x\in\Omega} v_c^1(x) = c. \tag{5.4}$$

But in view of Lemma 4,

$$c \geq v_c = \lim_{n\to\infty} U_{D_n}^{\varphi} c \geq \lim_{n\to\infty} U_{D_n}^{\varphi_1} c = v_c^1.$$

Thus v_c is a nonnegative nontrivial solution to (1.2) satisfying

$$\sup_{x\in\Omega} v_c(x) = c. \ \blacksquare$$

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