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On the bifurcation set of unique expansions

by

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1. Introduction. Fix a positive integer M. For any $q \in (1, M+1]$ and $x \in I_{q,M} := [0, M/(q-1)]$ there exists a sequence $(x_i) = x_1x_2\ldots$ with each x_i in $\{0, 1, \ldots, M\}$ such that

$$x = \sum_{i=1}^{\infty} \frac{x_i}{q^i} =: \pi_q((x_i)). \tag{1.1}$$

The sequence (x_i) is called a *q-expansion* of x. If no confusion arises the *alphabet* is always assumed to be $\{0, 1, \ldots, M\}$.

Non-integer base expansions have received a lot of attention since the pioneering papers of Rényi [35] and Parry [34]. It is well known that for any $q \in (1, M+1)$ Lebesgue almost every $x \in I_{q,M}$ has a continuum of q-expansions [36, 12]. Moreover, for any $k \in \mathbb{N} \cup \{\aleph_0\}$ there exist $q \in (1, M+1]$ and $x \in I_{q,M}$ such that x has precisely k different q-expansions (see e.g. [20, 38]). For more information on non-integer base expansions we refer the reader to the survey paper [23] and the references therein.

In this paper we focus on unique q-expansions. For $q \in (1, M+1]$ let

$$\mathcal{U}_q := \{x \in I_{q,M} : x \text{ has a unique } q\text{-expansion}\},$$

and let $\mathbf{U}_q = \pi_q^{-1}(\mathcal{U}_q)$ be the set of the corresponding q-expansions. These sets have been the object of study in many articles and have a very rich topological structure (see for example [26, 16]). Komornik et al. [24] studied the Hausdorff dimension of $\mathcal{U}_q$, and showed that the dimension function $D : q \mapsto \dim_H \mathcal{U}_q$ has a devil's staircase behavior (see also [3]). Moreover,

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they showed that the entropy function

$$H : (1, M+1] \to [0, \log(M+1)], \qquad q \mapsto h_{\text{top}}(\mathbf{U}_q),$$

is a devil's staircase (see Lemma 2.4 below). Recently, Alcaraz Barrera et al. [1] investigated the dynamical properties of $\mathcal{U}_q$, and determined the maximal intervals on which the entropy function H is constant.

Let $\mathscr{B}$ be the *bifurcation set* of the function H defined by

$$\mathscr{B} = \{q \in (1, M+1] : H(p) \neq H(q) \text{ for any } p \neq q\}.$$

Then $\mathscr{B}$ is the set of bases where the entropy function H is not locally constant. Alcaraz Barrera et al. [1] gave a characterization of $\mathscr{B}$ and showed that $\mathscr{B}$ has full Hausdorff dimension. In particular, we have

$$\mathscr{B} = (q_{\text{KL}}, M+1] \setminus \bigcup [p_L, p_R], \tag{1.2}$$

where q_{KL} is the *Komornik–Loreti constant* [25] and the union on the right hand side is countable and pairwise disjoint (see Section 2 below for more explanation).

From [16] we know that the univoque set $\mathcal{U}_q$ has a fractal structure and might have isolated points. Our first result states that for $q \in \mathscr{B}$ the univoque set $\mathcal{U}_q$ is *dimensionally homogeneous*, i.e., the local Hausdorff dimension of $\mathcal{U}_q$ equals the full dimension of $\mathcal{U}_q$.

Theorem 1. *Let $q \in (q_{\text{KL}}, M+1] \setminus \bigcup (p_L, p_R]$. Then for any open set $V \subseteq \mathbb{R}$ with $\mathcal{U}_q \cap V \neq \emptyset$ we have*

$$\dim_H(\mathcal{U}_q \cap V) = \dim_H \mathcal{U}_q.$$

Remark 1.1. (1) By (1.2), $\mathscr{B} \subset (q_{\text{KL}}, M+1] \setminus \bigcup (p_L, p_R]$. So Theorem 1 implies that the univoque set $\mathcal{U}_q$ is dimensionally homogeneous for any $q \in \mathscr{B}$.

(2) In Theorem 3.6 we give a complete characterization of the set

$$\{q \in (1, M+1] : \mathcal{U}_q \text{ is dimensionally homogeneous}\}.$$

It turns out that its Lebesgue measure is positive and strictly smaller than M.

Throughout the paper we use $\overline{A}$ to denote the topological closure of a set $A \subset \mathbb{R}$. Our second result gives a close relationship between the bifurcation set $\overline{\mathscr{B}}$ and the univoque sets $\mathcal{U}_q$.

Theorem 2. *For any $q \in \overline{\mathscr{B}}$ we have*

$$\lim_{\delta \to 0} \dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q.$$

Remark 1.2. (1) Since by (1.2) and (2.5) the difference between $\mathscr{B}$ and $\overline{\mathscr{B}}$ is countable, Theorem 2 also holds if we replace $\overline{\mathscr{B}}$ by $\mathscr{B}$.

(2) Note that $\dim_H \mathcal{U}_q > 0$ for any $q > q_{\mathrm{KL}}$ (see Lemma 2.4 below). As a consequence of Theorem 2,

$$q \in \overline{\mathscr{B}} \setminus \{q_{\mathrm{KL}}\} \iff \lim_{\delta\to 0} \dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q > 0.$$

Recently, Allaart et al. [2, Corollary 3] gave another characterization of $\overline{\mathscr{B}}$:

$$q \in \overline{\mathscr{B}} \setminus \{q_{\mathrm{KL}}\} \iff \lim_{\delta\to 0} \dim_H(\mathscr{U} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q > 0,$$

where $\mathscr{U} := \{q \in (1, M+1] : 1 \in \mathcal{U}_q\}$.

It is well-known that $\mathcal{U}_q$ has a close connection with the set $\mathscr{U} = \mathscr{U}(M)$ of *univoque bases* $q \in (1, M+1]$ for which 1 has a unique q-expansion with alphabet $\{0, 1, \dots, M\}$. For example, de Vries and Komornik [16] showed that $\mathcal{U}_q$ is closed if and only if $q \notin \overline{\mathscr{U}}$. The set $\mathscr{U}$ has many interesting properties itself. Erdős et al. [19] showed that $\mathscr{U}$ is an uncountable set of zero Lebesgue measure. Daróczy and Kátai [15] proved that the Hausdorff dimension of $\mathscr{U}$ is 1 (see also [24]). Komornik and Loreti [25] showed that the smallest element of $\mathscr{U}$ is q_{KL}. In [26] the same authors studied the topological properties of $\mathscr{U}$, and showed that $\overline{\mathscr{U}}$ is a Cantor set. Recently, Kong et al. [29] proved that for any $q \in \overline{\mathscr{U}}$ we have

$$\dim_H(\overline{\mathscr{U}} \cap (q-\delta, q+\delta)) > 0 \quad \text{for any } \delta > 0. \tag{1.3}$$

On a different note, Bonanno et al. [10] introduced a set

$$\Lambda = \{x \in [0,1] : S^k x \le x \text{ for all } k \ge 0\}, \tag{1.4}$$

where S is the tent map defined by $S : x \mapsto \min\{2x, 2-2x\}$, and showed that there is a one-to-one correspondence between $\mathscr{U}(1)$ and $\Lambda \setminus \mathbb{Q}_1$, where $\mathbb{Q}_1$ is the set of all rationals with odd denominator. This link is based on work of Allouche and Cosnard [5, 7, 8], who related the set $\mathscr{U}(1)$ to kneading sequences of unimodal maps. Bonanno et al. [10] also explored a relationship between these sets and the real slice of the boundary of the Mandelbrot set.

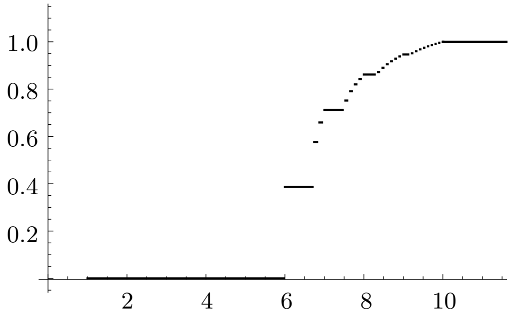

Fig. 1. The asymptotic graph of the function $\phi(t) = \dim_H(\mathscr{U} \cap (1, t])$ for $t \in [4, 11.5]$ with $M = 9$ and $q_{\mathrm{KL}} = q_{\mathrm{KL}}(9) \approx 5.97592$.

By using Theorem 2 we investigate the dimensional spectrum of $\mathscr{U}$. Our next result strengthens the relationship between $\mathcal{U}_q$ and $\mathscr{U}$.

Theorem 3. *For any $t > 1$ we have*

$$\dim_H(\mathscr{U} \cap (1,t]) = \max_{q \le t} \dim_H \mathcal{U}_q.$$

Moreover, the function $\phi(t) := \dim_H(\mathscr{U} \cap (1,t])$ is a devil's staircase on $(1,\infty)$.

Remark 1.3. (1) In [26] it was shown that $\overline{\mathscr{U}} \setminus \mathscr{U}$ is a countable set. Hence, Theorem 3 still holds if we replace $\mathscr{U}$ by $\overline{\mathscr{U}}$.

(2) Results from [24] (see Lemma 2.4 below) imply that $\dim_H \mathcal{U}_q = 1$ if and only if $q = M+1$. In view of Theorem 3, $\dim_H(\mathscr{U} \cap (1,t]) < 1$ for any $t < M+1$. This implies that the Hausdorff dimension of $\mathscr{U}$ is concentrated in the neighborhood of $M+1$.

As an application of Theorem 3 we investigate the variations of $\mathscr{U} = \mathscr{U}(M)$ when M changes. For $K \in \{1,\ldots,M\}$, let $\mathscr{U}(K)$ be the set of bases $q \in (1,K+1]$ such that 1 has a unique q-expansion with respect to the alphabet $\{0,1,\ldots,K\}$. Theorem 4 characterizes the Hausdorff dimensions of $\mathscr{U}(M) \cap \mathscr{U}(K)$ and $\mathscr{U}(M) \setminus \mathscr{U}(K)$. Indeed, we prove the following stronger result.

Theorem 4.

(i) *Let $K \in \{1,\ldots,M\}$. Then*

$$\dim_H\Big(\bigcap_{J=K}^{M} \mathscr{U}(J)\Big) = \max_{q \le K+1} \dim_H \mathcal{U}_q.$$

(ii) *For any positive integer L we have*

$$\dim_H\Big(\mathscr{U}(L) \setminus \bigcup_{J \ne L} \mathscr{U}(J)\Big) = 1.$$

Remark 1.4. By the proof of Theorem 4 for $K < M$, the intersection

$$\bigcap_{J=K}^{M} \mathscr{U}(J) = \mathscr{U}(M) \cap (1, K+1]$$

is a proper subset of $\mathscr{U}(K)$. This, together with (1.3), implies that for $K < M$ neither $\bigcap_{J=K}^{M} \mathscr{U}(J)$ nor $\mathscr{U}(M) \setminus \bigcap_{J=K}^{M} \mathscr{U}(J)$ contains isolated points.

We emphasize that for each $q \in (1, M+1]$ the univoque set $\mathcal{U}_q$ is related to the dynamical system

$$T_{q,j} : \left[0, \frac{M}{q-1}\right] \to \left[0, \frac{M}{q-1}\right], \qquad x \mapsto qx - j,$$

for $j \in \{0, 1, \ldots, M\}$. On the other hand, the set $\mathscr{U}$ contains all parameters $q \in (1, M+1]$ such that 1 has a unique q-expansion, and thus $\mathscr{U}$ is related to infinitely many dynamical systems. A similar set up involving a bifurcation set for infinitely many dynamical systems is considered by Bonanno et al. [10] (see also [11]). They consider the bifurcation set of an entropy map for a family $\{T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]\}_{\alpha \in [0,1]}$ of maps, called the *α-continued fraction transformations* [33], where for each $\alpha \in [0, 1]$ the map T_α is defined by

$$T_\alpha(x) = \begin{cases} 1/|x| - \lfloor 1/|x| + 1 - \alpha \rfloor & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{1.5}$$

Each map T_α has a unique invariant measure μ_α that is absolutely continuous with respect to the Lebesgue measure. Bonanno et al. showed that the map

$$\psi : \alpha \mapsto h_{\mu_\alpha}(T_\alpha),$$

assigning to each α the measure-theoretic entropy $h_{\mu_\alpha}(T_\alpha)$, has countably many intervals on which it is monotonic. The complement of the union of these intervals in $[0, 1]$, i.e., the bifurcation set of ψ, denoted by F, has Lebesgue measure 0 (see [30] and [11]) and Hausdorff dimension 1 (see [10]). Moreover, in [10] a homeomorphism is found between F and $\Lambda \setminus \{0\}$ from (1.4), giving also a relation to $\mathscr{U}(1)$. In [10], however, no information is given on the local structure of F. Recently, Dajani and the first author [13] identified another set E linked to $\mathscr{U}(1)$, Λ and F. They investigated the family of symmetric doubling maps $S_\gamma : [-1, 1] \to [-1, 1]$ given by

$$S_\gamma(x) = 2x - \gamma \lfloor 2x \rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x, and showed that the set E of parameters $\gamma \in [1, 2]$ for which S_γ does not have a piecewise smooth invariant density is homeomorphic to $\Lambda \setminus \{0\}$. Therefore, the results obtained in this paper about $\mathscr{U}(1)$ can be used to investigate the bifurcation sets E, F and the set Λ.

The rest of the paper is arranged in the following way. In Section 2 we fix the notation and recall some properties of unique q-expansions. Moreover, we recall from [1] some important properties of the bifurcation set $\mathscr{B}$. In Section 3 we give the proof of Theorem 1 on the dimensional homogeneity of $\mathcal{U}_q$. In Section 4 we prove an auxiliary proposition that will be used to prove Theorem 2 in Section 5. The proof of Theorems 3 and 4 will be given in Sections 6 and 7, respectively. We end the paper with some remarks.

2. Unique expansions and bifurcation set. In this section we recall some properties of unique q-expansions and of the bifurcation set $\mathscr{B}$ as well. First we need some terminology from symbolic dynamics [31].

2.1. Symbolic dynamics. Given a positive integer M, let $\{0,1,\dots,M\}^*$ denote the set of all finite strings of symbols from $\{0,1,\dots,M\}$, called *words*, together with the empty word denoted by ϵ. Let $\{0,1,\dots,M\}^{\mathbb{N}}$ be the set of sequences $(d_i) = d_1 d_2 \dots$ with each d_i in $\{0,1,\dots,M\}$. Let σ be the left shift on $\{0,1,\dots,M\}^{\mathbb{N}}$ defined by $\sigma((d_i)) = (d_{i+1})$. Then $(\{0,1,\dots,M\}^{\mathbb{N}},\sigma)$ is the *full shift*. For a word $\mathbf{c} = c_1 \dots c_n \in \{0,1,\dots,M\}^*$ we denote by $\mathbf{c}^k = (c_1 \dots c_n)^k$ the k-fold concatenation of $\mathbf{c}$ with itself and by $\mathbf{c}^\infty = (c_1 \dots c_n)^\infty$ the periodic sequence with period block $\mathbf{c}$. Moreover, for a word $\mathbf{c} = c_1 \dots c_n$ with $c_n < M$ we denote by $\mathbf{c}^+$ the word

$$\mathbf{c}^+ = c_1 \dots c_{n-1}(c_n + 1).$$

Similarly, for a word $\mathbf{c} = c_1 \dots c_n$ with $c_n > 0$ we set $\mathbf{c}^- = c_1 \dots c_{n-1}(c_n - 1)$. For a sequence $(d_i) \in \{0,1,\dots,M\}^{\mathbb{N}}$ we denote its *reflection* by

$$\overline{(d_i)} = (M - d_1)(M - d_2)\cdots .$$

Accordingly, the reflection of a word $\mathbf{c} = c_1 \dots c_n$ is $\overline{\mathbf{c}} = (M - c_1)\cdots(M - c_n)$.

On words and sequences we consider the lexicographical ordering $\prec$, defined as follows. For two sequences $(c_i),(d_i) \in \{0,1,\dots,M\}^{\mathbb{N}}$ we write $(c_i) \prec (d_i)$ if there exists $n \in \mathbb{N}$ such that $c_1 \dots c_{n-1} = d_1 \dots d_{n-1}$ and $c_n < d_n$. Moreover, $(c_i) \preccurlyeq (d_i)$ if $(c_i) \prec (d_i)$ or $(c_i) = (d_i)$. Similarly, $(c_i) \succ (d_i)$ if $(d_i) \prec (c_i)$, and $(c_i) \succcurlyeq (d_i)$ if $(d_i) \preccurlyeq (c_i)$. We extend this definition to words in the following way. For two words $\omega, \nu \in \{0,1,\dots,M\}^*$ we write $\omega \prec \nu$ if $\omega 0^\infty \prec \nu 0^\infty$. Accordingly, for a sequence $(d_i) \in \{0,1,\dots,M\}^{\mathbb{N}}$ and a word $\mathbf{c} = c_1 \dots c_m$ we write $(d_i) \prec \mathbf{c}$ if $(d_i) \prec \mathbf{c}0^\infty$.

Let $\mathcal{F} \subseteq \{0,1,\dots,M\}^*$ and let $X = X_{\mathcal{F}} \subseteq \{0,1,\dots,M\}^{\mathbb{N}}$ be the set of those sequences that do not contain any word from $\mathcal{F}$. We call the pair (X,σ) a *subshift*. If $\mathcal{F}$ is finite, then (X,σ) is called a *subshift of finite type*. For $n \in \mathbb{N} \cup \{0\}$ we denote by $\mathcal{L}_n(X)$ the set of words of length n occurring in sequences of X. In particular, for $n = 0$ we set $\mathcal{L}_0(X) = \{\epsilon\}$. The *language* of (X,σ) is then defined by

$$\mathcal{L}(X) = \bigcup_{n=0}^{\infty} \mathcal{L}_n(X).$$

So, $\mathcal{L}(X)$ is the set of all finite words occurring in sequences from X.

For a subshift (X,σ) and a word $\omega \in \mathcal{L}(X)$ let $F_X(\omega)$ be the *follower set* of ω in X defined by

$$F_X(\omega) := \{(d_i) \in X : d_1 \dots d_{|\omega|} = \omega\}, \tag{2.1}$$

where $|\mathbf{c}|$ denotes the length of a word $\mathbf{c} \in \{0,1,\dots,M\}^*$.

A subshift (X,σ) is called *topologically transitive* (or simply *transitive*) if for any two words $\omega, \nu \in \mathcal{L}(X)$ there exists a word γ such that $\omega\gamma\nu \in \mathcal{L}(X)$. In other words, in a transitive subshift (X,σ) any two words can be “connected” in $\mathcal{L}(X)$.

The *topological entropy* $h_{\text{top}}(X)$ of a subshift (X,σ) is a quantity that indicates its complexity. It is defined by

$$h_{\text{top}}(X) = \lim_{n\to\infty} \frac{\log \#\mathcal{L}_n(X)}{n} = \inf_{n\geq 1} \frac{\log \#\mathcal{L}_n(X)}{n}, \tag{2.2}$$

where $\#A$ denotes the cardinality of a set A. Accordingly, we define the topological entropy of a follower set $F_X(\omega)$ by changing X to $F_X(\omega)$ in (2.2) if the corresponding limit exists. Clearly, if X is a transitive subshift, then $h_{\text{top}}(F_X(\omega)) = h_{\text{top}}(X)$ for any $\omega \in \mathcal{L}(X)$.

2.2. Unique expansions. In this subsection we recall some results about unique expansions. For more information on this topic we refer the reader to the survey papers [37, 23] or the book chapter [17]. For $q \in (1, M+1]$, let

$$\alpha(q) = \alpha_1(q)\alpha_2(q)\ldots$$

be the *quasi-greedy* q-expansion of 1 (see [14]), i.e., the lexicographically largest q-expansion of 1 not ending with a string of zeros. The following characterization of quasi-greedy expansions was given in [9, Theorem 2.2].

Lemma 2.1. *The map $q \mapsto \alpha(q)$ is a strictly increasing bijection from $(1, M+1]$ onto the set of all sequences $(a_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ not ending with 0^∞ and satisfying*

$$a_{n+1}a_{n+2}\ldots \preceq a_1a_2\ldots \qquad \textit{whenever } a_n < M.$$

Recall from (1.1) the definition of the projection map π_q for $q \in (1, M+1]$ mapping $\{0, 1, \ldots, M\}^{\mathbb{N}}$ onto the interval $I_{q,M} = [0, M/(q-1)]$. In general, π_q is not bijective. However, π_q is a bijection between $\mathbf{U}_q = \pi_q^{-1}(\mathcal{U}_q)$ and $\mathcal{U}_q$. The following lexicographical characterization of $\mathbf{U}_q$, or equivalently $\mathcal{U}_q$, is essentially due to Parry [34] (see also [9]).

Lemma 2.2. *Let $q \in (1, M+1]$. Then $(x_i) \in \mathbf{U}_q$ if and only if*

$$x_{n+1}x_{n+2}\ldots \prec \alpha(q) \qquad \textit{whenever } x_n < M,$$
$$\overline{x_{n+1}x_{n+2}\ldots} \prec \alpha(q) \qquad \textit{whenever } x_n > 0.$$

Observe that $\mathscr{U} = \{q \in (1, M+1] : \alpha(q) \in \mathbf{U}_q\}$. As a corollary of Lemma 2.2 we have the following characterizations of $\mathscr{U}$ and $\overline{\mathscr{U}}$.

Lemma 2.3.

(i) *$q \in \mathscr{U} \setminus \{M+1\}$ if and only if the quasi-greedy expansion $\alpha(q)$ satisfies*

$$\overline{\alpha(q)} \prec \sigma^n(\alpha(q)) \prec \alpha(q) \qquad \textit{for any } n \geq 1.$$

(ii) *$q \in \overline{\mathscr{U}}$ if and only if the quasi-greedy expansion $\alpha(q)$ satisfies*

$$\overline{\alpha(q)} \prec \sigma^n(\alpha(q)) \preccurlyeq \alpha(q) \qquad \textit{for any } n \geq 1.$$

Proof. Part (i) was shown in [18, Theorem 2.5], and (ii) in [18, Theorem 3.9]. ∎

In [16] it was shown that $(\mathbf{U}_q, \sigma)$ is not necessarily a subshift. Inspired by [24] we consider the set $\mathbf{V}_q$ of all sequences $(x_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ satisfying

$$\overline{\alpha(q)} \preccurlyeq \sigma^n((x_i)) \preccurlyeq \alpha(q) \quad \text{for all } n \geq 0.$$

Then $(\mathbf{V}_q, \sigma)$ is a subshift [24, Lemma 2.6]. Furthermore, Lemma 2.1 implies that the set-valued map $q \mapsto \mathbf{V}_q$ is increasing, i.e., $\mathbf{V}_p \subseteq \mathbf{V}_q$ whenever $p < q$.

Recall that the Komornik–Loreti constant q_{KL} is the smallest element of $\mathscr{U}$, which is defined in terms of the classical *Thue–Morse sequence* $(\tau_i)_{i=0}^{\infty} = 01101001\ldots$ The latter is defined as follows [6]: $\tau_0 = 0$, and if $\tau_0 \ldots \tau_{2^n-1}$ has already been defined for some $n \geq 0$, then $\tau_{2^n} \ldots \tau_{2^{n+1}-1} = \overline{\tau_0 \ldots \tau_{2^n-1}}$. Then the Komornik–Loreti constant $q_{\mathrm{KL}} = q_{\mathrm{KL}}(M) \in (1, M+1]$ is the unique base satisfying

$$\alpha(q_{\mathrm{KL}}) = \lambda_1 \lambda_2 \ldots, \tag{2.3}$$

where

$$\lambda_i = \begin{cases} k + \tau_i - \tau_{i-1} & \text{if } M = 2k, \\ k + \tau_i & \text{if } M = 2k+1, \end{cases}$$

for each $i \geq 1$. We emphasize that the sequence (λ_i) depends on M. By the definition of the Thue–Morse sequence $(\tau_i)_{i=0}^{\infty}$ it follows that [1]

$$\lambda_{2^n+1} \ldots \lambda_{2^{n+1}} = \overline{\lambda_1 \ldots \lambda_{2^n}}^{\,+} \quad \text{for any } n \geq 0. \tag{2.4}$$

Recall that a function $f : [a, b] \to \mathbb{R}$ is called a *devil's staircase* (or a *Cantor function*) if f is a continuous and non-decreasing function with $f(a) < f(b)$, and f is locally constant almost everywhere. The next lemma summarizes some results from [24] on the Hausdorff dimension of $\mathcal{U}_q$.

Lemma 2.4.

(i) *For any* $q \in (1, M+1]$ *we have*

$$\dim_H \mathcal{U}_q = \frac{h_{\mathrm{top}}(\mathbf{V}_q)}{\log q}.$$

(ii) *The entropy function* $H : q \mapsto h_{\mathrm{top}}(\mathbf{V}_q)$ *is a devil's staircase in* $(1, M+1]$*:*

- H *is increasing and continuous in* $(1, M+1]$*;*
- H *is locally constant almost everywhere in* $(1, M+1]$*;*
- $H(q) = 0$ *if and only if* $1 < q \leq q_{\mathrm{KL}}$*. Moreover,* $H(q) = \log(M+1)$ *if and only if* $q = M+1$*.*

Remark 2.5. (1) Lemma 2.4 implies that the dimensional function $D : q \mapsto \dim_H \mathcal{U}_q$ has a devil's staircase behavior: (i) D is continuous in $(1, M+1]$; (ii) $D' < 0$ almost everywhere in $(1, M+1]$; (iii) $D(q) = 0$ for any $q \in (1, q_{\mathrm{KL}}]$ and $D(q) = 1$ for $q = M+1$.

(2) In [24, Lemma 2.11] it is shown that H is locally constant on the complement of $\overline{\mathscr{U}}$, i.e., $H'(q) = 0$ for any $q \in (1, M+1] \setminus \overline{\mathscr{U}}$.

2.3. Bifurcation set. In this subsection we recall some recent results of [1] on the maximal intervals on which H is locally constant, called *entropy plateaus* (or simply *plateaus*). For the convenience of the reader we adopt much of the notation from [1]. Let $\mathscr{B}$ be the complement of these plateaus. From Lemma 2.4(ii) we have

$$\mathscr{B} = \{q \in (1, M+1] : H(p) \neq H(q) \text{ for any } p \neq q\}.$$

Note by (1.2) that $\mathscr{B}$ is not closed. We have

$$\overline{\mathscr{B}} = \{q \in (1, M+1] : \forall \delta > 0, \exists p \in (q-\delta, q+\delta) \text{ such that } H(p) \neq H(q)\}.$$

In [1], $\overline{\mathscr{B}}$ was denoted by $\mathscr{E}$. The following lemma, the first part of which follows from Remark 2.5(2), was established in [1, Theorem 3].

LEMMA 2.6. *$\overline{\mathscr{B}} \subset \overline{\mathscr{U}}$, and $\overline{\mathscr{B}}$ is a Cantor set of full Hausdorff dimension.*

By Lemma 2.4 it follows that $\min \overline{\mathscr{B}} = q_{\mathrm{KL}}$ and $\max \overline{\mathscr{B}} = M+1$. Since $\overline{\mathscr{B}}$ is a Cantor set, we can write

$$(q_{\mathrm{KL}}, M+1] \setminus \overline{\mathscr{B}} = \bigcup (p_L, p_R), \tag{2.5}$$

where the union is pairwise disjoint and countable. By the definition of $\overline{\mathscr{B}}$ the intervals $[p_L, p_R]$ are the plateaus of H. In particular, since H is increasing, these intervals have the property that $H(q) = H(p_L)$ if and only if $q \in [p_L, p_R]$. This implies that the bifurcation set $\mathscr{B}$ can be rewritten as in (1.2), i.e.,

$$\mathscr{B} = (q_{\mathrm{KL}}, M+1] \setminus \bigcup [p_L, p_R].$$

By (2.5) and (1.2), $\overline{\mathscr{B}} \setminus \mathscr{B}$ is countable. The fact that $\overline{\mathscr{B}}$ does not have isolated points gives the following lemma (see also [1]).

LEMMA 2.7.

(i) *For any $q \in (q_{\mathrm{KL}}, M+1] \setminus \bigcup (p_L, p_R]$ there is a sequence $\{[p_L(n), p_R(n)]\}$ of plateaus such that $p_L(n) \nearrow q$ as $n \to \infty$.*

(ii) *For any $q \in [q_{\mathrm{KL}}, M+1) \setminus \bigcup [p_L, p_R)$ there is a sequence $\{[q_L(n), q_R(n)]\}$ of plateaus such that $q_L(n) \searrow q$ as $n \to \infty$.*

So, by (2.5), (1.2) and Lemma 2.7, $\overline{\mathscr{B}} \setminus \mathscr{B}$ is a countable and dense subset of $\overline{\mathscr{B}}$. In particular, the set of left endpoints of all plateaus of H is dense in $\overline{\mathscr{B}}$.

In [1] more detailed information on the structure of the plateaus of H is given. Before stating the necessary details, we have to recall some notation from [1]. Let $\mathbf{V}$ be the set of sequences $(a_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ satisfying

$$\overline{(a_i)} \preccurlyeq \sigma^n((a_i)) \preccurlyeq (a_i) \quad \text{for all } n \geq 0.$$

In [1, Lemma 3.3] it is proved that the subshift $(\mathbf{V}_q,\sigma)$ is not transitive for any $q\in(q_{\mathrm{KL}},q_T)$, where $q_T\in(1,M+1)\cap\mathscr{B}$ is the unique base such that

$$\alpha(q_T)=\begin{cases}(k+1)k^\infty & \text{if } M=2k,\\ (k+1)((k+1)k)^\infty & \text{if } M=2k+1.\end{cases}\tag{2.6}$$

The plateaus of H are characterized separately for the cases

$$\text{(A) } q\in[q_T,M+1] \quad\text{and}\quad \text{(B) } q\in(q_{\mathrm{KL}},q_T).$$

(A) First we recall from [1] the following definition.

Definition 2.8. A sequence $(a_i)\in\mathbf{V}$ is called *irreducible* if

$$a_1\ldots a_j(\overline{a_1\ldots a_j}^{\,+})^\infty\prec(a_i)\quad\text{whenever } (a_1\ldots a_j^-)^\infty\in\mathbf{V}.$$

Lemma 2.9. *Let $[p_L,p_R]\subset[q_T,M+1]$ be a plateau of H.*

(i) *There exists a word $a_1\ldots a_m\in\mathcal{L}(\mathbf{V}_{p_L})$ such that*

$$\alpha(p_L)=(a_1\ldots a_m)^\infty \textit{ is irreducible},\qquad \alpha(p_R)=a_1\ldots a_m^+(\overline{a_1\ldots a_m})^\infty.$$

(ii) *$(\mathbf{V}_{p_L},\sigma)$ is a transitive subshift of finite type.*

(iii) *There exists a periodic sequence $\nu^\infty\in\mathbf{V}_{p_L}$ such that for any word $\eta\in\mathcal{L}(\mathbf{V}_{p_L})$ we can find a large integer N and a word ω satisfying*

$$\overline{\alpha_1(p_L)\ldots\alpha_N(p_L)}\prec\sigma^j(\eta\omega\nu^\infty)\prec\alpha_1(p_L)\ldots\alpha_N(p_L)\quad \textit{for any } j\ge 0.$$

Proof. (i) follows from [1, Proposition 5.2], and (ii) from [1, Lemma 5.1(1)].

For (iii) we take

$$\nu=\begin{cases}k & \text{if } M=2k,\\ (k+1)k & \text{if } M=2k+1.\end{cases}$$

Since $p_L\ge q_T$, by Lemma 2.1 we have $\alpha(p_L)\succcurlyeq\alpha(q_T)$. Then (2.6) gives

$$\begin{aligned}\overline{\alpha_1(p_L)\alpha_2(p_L)}\preccurlyeq\overline{\alpha_1(q_T)\alpha_2(q_T)}\prec\sigma^j(\nu^\infty)\prec\alpha_1(q_T)\alpha_2(q_T)\\ \preccurlyeq\alpha_1(p_L)\alpha_2(p_L)\end{aligned}\tag{2.7}$$

for all $j\ge0$. By (i), $\alpha(p_L)$ is irreducible. By [1, proof of Proposition 3.17] for any $\eta\in\mathcal{L}(\mathbf{V}_{p_L})$ there exist a large integer $N\ge2$ and a word ω satisfying

$$\overline{\alpha_1(p_L)\ldots\alpha_N(p_L)}\prec\sigma^j(\eta\omega\nu^\infty)\prec\alpha_1(p_L)\ldots\alpha_N(p_L)\quad\text{for any } 0\le j<|\eta|+|\omega|.$$

This together with (2.7) proves (iii). ∎

(B) Now we consider plateaus of H in (q_{KL},q_T). Let (λ_i) be the quasi-greedy q_{KL}-expansion of 1 as given in (2.3). Note that (λ_i) depends on M. For $n\ge1$ let

$$\xi(n)=\begin{cases}\lambda_1\ldots\lambda_{2^{n-1}}(\overline{\lambda_1\ldots\lambda_{2^{n-1}}}^{\,+})^\infty & \text{if } M=2k,\\ \lambda_1\ldots\lambda_{2^n}(\overline{\lambda_1\ldots\lambda_{2^n}}^{\,+})^\infty & \text{if } M=2k+1.\end{cases}\tag{2.8}$$

Then $\xi(1) = \alpha(q_T)$, and $\xi(n)$ is strictly decreasing to $(\lambda_i) = \alpha(q_{\mathrm{KL}})$ as $n \to \infty$. Moreover, [1, Lemma 4.2] implies that $\xi(n) \in \mathbf{V}$ for all $n \geq 1$. We recall from [1] the following definition.

Definition 2.10. A sequence $(a_i) \in \mathbf{V}$ is said to be $*$-*irreducible* if there exists $n \in \mathbb{N}$ such that $\xi(n+1) \preccurlyeq (a_i) \prec \xi(n)$, and

$$a_1 \ldots a_j (\overline{a_1 \ldots a_j}^{\,+})^\infty \prec (a_i)$$

whenever

$$(a_1 \ldots a_j^-)^\infty \in \mathbf{V} \quad \text{and} \quad j > \begin{cases} 2^n & \text{if } M = 2k, \\ 2^{n+1} & \text{if } M = 2k+1. \end{cases}$$

Lemma 2.11. *Let $[p_L, p_R] \subseteq (q_{\mathrm{KL}}, q_T)$ be a plateau of H.*

(i) *There exists a word $a_1 \ldots a_m \in \mathcal{L}(\mathbf{V}_{p_L})$ such that*

$$\alpha(p_L) = (a_1 \ldots a_m)^\infty \text{ is } *\text{-irreducible} \quad \text{and} \quad \alpha(p_R) = a_1 \ldots a_m^+ (\overline{a_1 \ldots a_m})^\infty.$$

(ii) *$(\mathbf{V}_{p_L}, \sigma)$ is a subshift of finite type, and it contains a unique transitive subshift (X_{p_L}, σ) of finite type satisfying $h_{\mathrm{top}}(X_{p_L}) = h_{\mathrm{top}}(\mathbf{V}_{p_L})$.*

(iii) *There exists a periodic sequence $\nu^\infty \in X_{p_L}$ such that for any word $\eta \in \mathcal{L}(\mathbf{V}_{p_L})$ we can find a large integer N and a word ω satisfying*

$$\overline{\alpha_1(p_L) \ldots \alpha_N(p_L)} \prec \sigma^j(\eta\omega\nu^\infty) \prec \alpha_1(p_L) \ldots \alpha_N(p_L) \quad \textit{for any } j \geq 0.$$

Proof. (i) follows from [1, Proposition 5.11], and (ii) from [1, Lemma 5.9]. Thus it remains to prove (iii).

By (i) we know that $\alpha(p_L)$ is a $*$-irreducible sequence. Hence there exists $n \in \mathbb{N}$ such that $\xi(n+1) \preccurlyeq \alpha(p_L) \prec \xi(n)$. By (i) and (2.8), $\alpha(p_L)$ is purely periodic, while $\xi(n+1)$ is eventually periodic. Thus $\alpha(p_L) \succ \xi(n+1)$. Let

$$\nu = \begin{cases} \lambda_1 \ldots \lambda_{2^n}^- & \text{if } M = 2k, \\ \lambda_1 \ldots \lambda_{2^{n+1}}^- & \text{if } M = 2k+1. \end{cases}$$

Then by the proof of [1, Lemma 5.9] we have $\nu^\infty \in X_{p_L}$. Observe by (2.4) and (2.8) that $\xi(n+1) = \nu^+(\overline{\nu})^\infty \in \mathbf{V}$. By using $\alpha(p_L) \succ \xi(n+1)$ it follows that there exists a large integer N such that

$$\overline{\alpha_1(p_L) \ldots \alpha_N(p_L)} \prec \sigma^j(\nu^\infty) \prec \alpha_1(p_L) \ldots \alpha_N(p_L) \quad \text{for any } j \geq 0.$$

The remaining part of (iii) follows from [1, proof of Lemma 5.8]. ∎

Finally, the following characterization of $\overline{\mathscr{B}}$ was established in [1, Theorem 3].

Lemma 2.12.

$$\overline{\mathscr{B}} = \overline{\{q \in (q_{\mathrm{KL}}, M+1] : \alpha(q) \textit{ is irreducible or } *\textit{-irreducible}\}}.$$

3. Dimensional homogeneity of $\mathcal{U}_q$. In this section we will prove Theorem 1. In fact, we prove the following equivalent statement.

THEOREM 3.1. *Let $q \in (1, q_{\mathrm{KL}}] \cup ((q_{\mathrm{KL}}, M+1] \setminus \bigcup (p_L, p_R])$. Then for any $x \in \mathcal{U}_q$ we have*

$$\dim_H(\mathcal{U}_q \cap (x-\delta, x+\delta)) = \dim_H \mathcal{U}_q \quad \textit{for any } \delta > 0.$$

We first explain why Theorem 3.1 is equivalent to Theorem 1. Clearly, Theorem 1 implies Theorem 3.1. Conversely, take $q \in \mathscr{B}$. Let $V \subseteq \mathbb{R}$ be an open set with $\mathcal{U}_q \cap V \neq \emptyset$. Then there exist $x \in \mathcal{U}_q \cap V$ and $\delta > 0$ such that

$$\mathcal{U}_q \cap V \supset \mathcal{U}_q \cap (x-\delta, x+\delta).$$

From Theorem 3.1 it follows that $\dim_H(\mathcal{U}_q \cap V) \geq \dim_H \mathcal{U}_q$, which gives Theorem 1.

Note that for $q \in (1, q_{\mathrm{KL}}]$ the statement of Theorem 3.1 follows immediately from the fact that $\dim_H \mathcal{U}_q = 0$. For $q \in (q_{\mathrm{KL}}, M+1]$ recall that $\mathbf{V}_q$ is the set of sequences $(x_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ satisfying

$$\overline{\alpha(q)} \preccurlyeq \sigma^n((x_i)) \preccurlyeq \alpha(q) \quad \text{for all } n \geq 0.$$

Accordingly, let

$$\mathcal{V}_q := \{\pi_q((x_i)) : (x_i) \in \mathbf{V}_q\},$$

where π_q is the projection map defined in (1.1). For $A \subset \mathbb{R}$ and $r \in \mathbb{R}$ we denote $rA := \{r \cdot a : a \in A\}$ and $r + A := \{r + a : a \in A\}$.

The following relationship between $\mathcal{U}_q$ and $\mathcal{V}_q$ follows from Lemma 2.2 and the definition of $\mathcal{V}_q$.

LEMMA 3.2. *Let $q \in (q_{\mathrm{KL}}, M+1]$. Then $\mathcal{U}_q$ is a countable union of affine copies of $\mathcal{V}_q$ up to a countable set, i.e.,*

$$\mathcal{U}_q \cup \mathcal{N} = \left\{0, \frac{M}{q-1}\right\} \cup \bigcup_{c_1=1}^{M-1} \left(\frac{c_1}{q} + \frac{\mathcal{V}_q}{q}\right) \cup \bigcup_{m=2}^{\infty} \bigcup_{c_m=1}^{M} \left(\frac{c_m}{q^m} + \frac{\mathcal{V}_q}{q^m}\right)$$
$$\cup \bigcup_{m=2}^{\infty} \bigcup_{c_m=0}^{M-1} \left(\sum_{i=1}^{m-1} \frac{M}{q^i} + \frac{c_m}{q^m} + \frac{\mathcal{V}_q}{q^m}\right),$$

where the set $\mathcal{N}$ is at most countable.

Proof. For $q \in (q_{\mathrm{KL}}, M+1]$ let $\mathbf{W}_q$ be the set of sequences (x_i) satisfying

$$\overline{\alpha(q)} \prec \sigma^n((x_i)) \prec \alpha(q) \quad \text{for any } n \geq 0,$$

and let $\mathcal{W}_q = \pi_q(\mathbf{W}_q)$. Then $\mathcal{V}_q \setminus \mathcal{W}_q$ is at most countable [16]. From [24, Lemma 2.5] it follows that

$$\mathcal{U}_q = \left\{0, \frac{M}{q-1}\right\} \cup \bigcup_{c_1=1}^{M-1}\left(\frac{c_1}{q} + \frac{\mathcal{W}_q}{q}\right) \cup \bigcup_{m=2}^{\infty}\bigcup_{c_m=1}^{M}\left(\frac{c_m}{q^m} + \frac{\mathcal{W}_q}{q^m}\right)$$
$$\cup \bigcup_{m=2}^{\infty}\bigcup_{c_m=0}^{M-1}\left(\sum_{i=1}^{m-1}\frac{M}{q^i} + \frac{c_m}{q^m} + \frac{\mathcal{W}_q}{q^m}\right).$$

This establishes the lemma since $\mathcal{W}_q \subseteq \mathcal{V}_q$ and $\mathcal{V}_q \setminus \mathcal{W}_q$ is at most countable. ∎

It immediately follows from Lemma 3.2 that

$$\dim_H \mathcal{U}_q = \dim_H \mathcal{V}_q \quad \text{for any } q \in (q_{\mathrm{KL}}, M+1].$$

Hence, it suffices to prove Theorem 3.1 with $\mathcal{V}_q$ in place of $\mathcal{U}_q$. We first prove it for q being the left endpoint of an entropy plateau.

Lemma 3.3. *Let $[p_L, p_R] \subset (q_{\mathrm{KL}}, M+1)$ be a plateau of H. Then for any $x \in \mathcal{V}_{p_L}$ we have*

$$\dim_H(\mathcal{V}_{p_L} \cap (x-\delta, x+\delta)) = \dim_H \mathcal{V}_{p_L} \quad \textit{for any } \delta > 0.$$

Proof. Obviously, $\dim_H(\mathcal{V}_{p_L} \cap (x-\delta, x+\delta)) \le \dim_H \mathcal{V}_{p_L}$. So, it suffices to the prove the reverse inequality.

Fix $\delta > 0$ and $x \in \mathcal{V}_{p_L}$. Suppose that $(x_i) \in \mathbf{V}_{p_L}$ is a p_L-expansion of x. Then there exists a large integer N such that

$$\pi_{p_L}(F_{\mathbf{V}_{p_L}}(x_1 \dots x_N)) \subseteq \mathcal{V}_{p_L} \cap (x-\delta, x+\delta), \tag{3.1}$$

where the follower set $F_{\mathbf{V}_{p_L}}(x_1 \dots x_N) = \{(y_i) \in \mathbf{V}_{p_L} : y_1 \dots y_N = x_1 \dots x_N\}$ is as defined in (2.1). We split the proof into two cases.

Case I: $[p_L, p_R] \subset [q_T, M+1]$. Then by Lemma 2.9(ii), $(\mathbf{V}_{p_L}, \sigma)$ is a transitive subshift of finite type. This implies that

$$h_{\mathrm{top}}(F_{\mathbf{V}_{p_L}}(x_1 \dots x_N)) = h_{\mathrm{top}}(\mathbf{V}_{p_L}).$$

Then, by (3.1), Lemma 2.4(i) and Lemma 3.2,

$$\begin{aligned}\dim_H(\mathcal{V}_{p_L} \cap (x-\delta, x+\delta)) &\ge \dim_H \pi_{p_L}(F_{\mathbf{V}_{p_L}}(x_1 \dots x_N)) \\ &= \frac{h_{\mathrm{top}}(F_{\mathbf{V}_{p_L}}(x_1 \dots x_N))}{\log p_L} \\ &= \frac{h_{\mathrm{top}}(\mathbf{V}_{p_L})}{\log p_L} = \dim_H \mathcal{U}_{p_L} = \dim_H \mathcal{V}_{p_L}.\end{aligned}$$

Case II: $[p_L, p_R] \subset (q_{\mathrm{KL}}, q_T)$. Then by Lemma 2.11(ii), $(\mathbf{V}_{p_L}, \sigma)$ is a subshift of finite type that contains a unique transitive subshift of finite type X_{p_L} such that

$$h_{\mathrm{top}}(X_{p_L}) = h_{\mathrm{top}}(\mathbf{V}_{p_L}). \tag{3.2}$$

Furthermore, by Lemma 2.11(iii) there exist a sequence $\nu^\infty \in X_{p_L}$ and a word ω such that

$$x_1 \dots x_N \omega \nu^\infty \in F_{\mathbf{V}_{p_L}}(x_1 \dots x_N). \tag{3.3}$$

From [31, Proposition 2.1.7] there exists $m \geq 0$ such that $(\mathbf{V}_{p_L}, \sigma)$ is an m-step subshift of finite type. By (3.3) we have $x_1 \dots x_N \omega \nu^m \in \mathcal{L}(\mathbf{V}_{p_L})$. Then by [31, Theorem 2.1.8] for any sequence $(d_i) \in F_{X_{p_L}}(\nu^m) \subseteq F_{\mathbf{V}_{p_L}}(\nu^m)$ we have $x_1 \dots x_N \omega d_1 d_2 \dots \in F_{\mathbf{V}_{p_L}}(x_1 \dots x_N)$. In other words,

$$\{x_1 \dots x_N \omega d_1 d_2 \dots : (d_i) \in F_{X_{p_L}}(\nu^m)\} \subseteq F_{\mathbf{V}_{p_L}}(x_1 \dots x_N).$$

Therefore, by (3.1),

$$\begin{aligned} \dim_H(\mathcal{V}_{p_L} \cap (x-\delta, x+\delta)) &\geq \dim_H \pi_{p_L}(F_{\mathbf{V}_{p_L}}(x_1 \dots x_N)) \\ &\geq \dim_H \pi_{p_L}(F_{X_{p_L}}(\nu^m)) = \dim_H \pi_{p_L}(X_{p_L}), \end{aligned} \tag{3.4}$$

where the last equality holds by the transitivity of (X_{p_L}, σ). Observe that $\pi_{p_L}(X_{p_L})$ is a graph-directed set satisfying the open set condition [32]. Hence

$$\dim_H \pi_{p_L}(X_{p_L}) = \frac{h_{\mathrm{top}}(X_{p_L})}{\log p_L}. \tag{3.5}$$

By (3.2), (3.4), (3.5) and Lemma 2.4(i) we conclude that

$$\begin{aligned} \dim_H(\mathcal{V}_{p_L} \cap (x-\delta, x+\delta)) &\geq \dim_H \pi_{p_L}(X_{p_L}) \\ &= \frac{h_{\mathrm{top}}(X_{p_L})}{\log p_L} = \frac{h_{\mathrm{top}}(\mathbf{V}_{p_L})}{\log p_L} \\ &= \dim_H \mathcal{U}_{p_L} = \dim_H \mathcal{V}_{p_L}. \ \blacksquare \end{aligned}$$

Now we consider $q \in \mathscr{B}$. We need the following lemma.

Lemma 3.4. *Let $q \in (q_{\mathrm{KL}}, M+1]$ and $x_1 \dots x_N \in \mathcal{L}(\mathbf{V}_q)$. Let $\{p_n\} \subset (1, M+1]$ be a sequence such that $\alpha(p_n) \in \mathbf{V}$ for each $n \geq 1$, and $p_n \nearrow q$ as $n \to \infty$. Then*

$$x_1 \dots x_N \in \mathcal{L}(\mathbf{V}_{p_n}) \quad \text{for all sufficiently large } n.$$

Proof. Since $x_1 \dots x_N \in \mathcal{L}(\mathbf{V}_q)$, we have

$$\overline{\alpha_1(q) \dots \alpha_{N-i}(q)} \preccurlyeq x_{i+1} \dots x_N \preccurlyeq \alpha_1(q) \dots \alpha_{N-i}(q) \quad \text{for any } 0 \leq i < N.$$

Let $s \in \{0, 1, \dots, N-1\}$ be the smallest integer such that

$$x_{s+1} \dots x_N = \overline{\alpha_1(q) \dots \alpha_{N-s}(q)} \quad \text{or} \quad x_{s+1} \dots x_N = \alpha_1(q) \dots \alpha_{N-s}(q). \tag{3.6}$$

If there is no $s \in \{0, 1, \dots, N-1\}$ for which (3.6) holds, then we set $s = N$. By our choice of s,

$$\overline{\alpha_1(q) \dots \alpha_{N-i}(q)} \prec x_{i+1} \dots x_N \prec \alpha_1(q) \dots \alpha_{N-i}(q) \quad \text{for all } 0 \leq i < s. \tag{3.7}$$

In terms of (3.6) we may assume by symmetry that

$$x_{s+1}\dots x_N=\alpha_1(q)\dots\alpha_{N-s}(q). \tag{3.8}$$

Since $p_n\nearrow q$ as $n\to\infty$, by Lemma 2.1 there exists $K\in\mathbb{N}$ such that

$$\alpha_1(p_n)\dots\alpha_N(p_n)=\alpha_1(q)\dots\alpha_N(q)\quad\text{for any } n\geq K.$$

As $\alpha(p_n)\in\mathbf{V}$ for any $n\geq 1$, it follows from (3.7) and (3.8) that

$$x_1\dots x_N\alpha_{N-s+1}(p_n)\alpha_{N-s+2}(p_n)\dots=x_1\dots x_s\alpha_1(p_n)\alpha_2(p_n)\dots\ \in\mathbf{V}_{p_n}$$

for any $n\geq K$. So, $x_1\dots x_N\in\mathcal{L}(\mathbf{V}_{p_n})$ for all $n\geq K$. ∎

Lemma 3.5. *Let $q\in\mathscr{B}$. Then for any $x\in\mathcal{V}_q$ we have*

$$\dim_H(\mathcal{V}_q\cap(x-\delta,x+\delta))=\dim_H\mathcal{V}_q\quad\textit{for any } \delta>0.$$

Proof. Take $q\in\mathscr{B}$. Since $\mathscr{B}\subset(q_{\mathrm{KL}},M+1]\setminus\bigcup(p_L,p_R]$, by Lemma 2.7(i) there exists a sequence $\{[p_L(n),p_R(n)]\}_{n=1}^\infty$ of plateaus such that $p_L(n)\nearrow q$ as $n\to\infty$.

Now we fix $\delta>0$ and $x\in\mathcal{V}_q$. Suppose $(x_i)\in\mathbf{V}_q$ is a q-expansion of x. Then there exists a large integer N such that

$$\pi_q(F_{\mathbf{V}_q}(x_1\dots x_N))\subseteq\mathcal{V}_q\cap(x-\delta,x+\delta). \tag{3.9}$$

By Lemmas 2.9(i) and 2.11(i) we have $\alpha(p_L(n))\in\mathbf{V}$ for all $n\geq 1$. Then applying Lemma 3.4 to $\{p_L(n)\}$ gives a large integer K such that

$$x_1\dots x_N\in\mathcal{L}(\mathbf{V}_{p_L(n)})\qquad\text{for all } n\geq K.$$

Since $\mathbf{V}_{p_L(n)}\subset\mathbf{V}_q$ for any $n\geq 1$, it follows from (3.9) that

$$\pi_q(F_{\mathbf{V}_{p_L(n)}}(x_1\dots x_N))\subset\mathcal{V}_q\cap(x-\delta,x+\delta)\qquad\text{for all } n\geq K. \tag{3.10}$$

By (3.10) and the proof of Lemma 3.3 we see that for any $n\geq K$,

$$\dim_H(\mathcal{V}_q\cap(x-\delta,x+\delta))\geq\dim_H\pi_q(F_{\mathbf{V}_{p_L(n)}}(x_1\dots x_N))\geq\frac{h_{\mathrm{top}}(\mathbf{V}_{p_L(n)})}{\log q}.$$

Letting $n\to\infty$ we have $p_L(n)\nearrow q$, and then we conclude by the continuity of $q\mapsto h_{\mathrm{top}}(\mathbf{V}_q)$ (see Lemma 2.4(ii)) that

$$\dim_H(\mathcal{V}_q\cap(x-\delta,x+\delta))\geq\frac{h_{\mathrm{top}}(\mathbf{V}_q)}{\log q}=\dim_H\mathcal{U}_q=\dim_H\mathcal{V}_q. \ ∎$$

Proof of Theorem 3.1. Take $q\in(1,q_{\mathrm{KL}}]\cup((q_{\mathrm{KL}},M+1]\setminus\bigcup(p_L,p_R])$. If $q\in(1,q_{\mathrm{KL}}]$, then the result follows from $\dim_H\mathcal{U}_q=0$ (see Lemma 2.4).

Assume $q\in(q_{\mathrm{KL}},M+1]\setminus\bigcup(p_L,p_R]$ where the union is taken over all plateaus $[p_L,p_R]$ of H. Take $x\in\mathcal{U}_q$. If $x\notin\{0,M/(q-1)\}$, then by

Lemma 3.2, x belongs to an affine copy of $\mathcal{V}_q$. Since the Hausdorff dimension is invariant under affine transformations [21], the statement follows from Lemmas 3.3 and 3.5.

So, it remains to consider $x = 0$ and $x = M/(q-1)$. By symmetry we may assume $x = 0$. Take $\delta > 0$. Then by Lemma 3.2 there exists a sufficiently large integer m such that

$$\frac{1}{q^m} + \frac{\mathcal{V}_q}{q^m} \subseteq (\mathcal{U}_q \cup \mathcal{N}) \cap (-\delta, \delta),$$

where $\mathcal{N}$ is at most countable. This proves the statement for $x = 0$. ∎

To end this section we strengthen Theorem 3.1 and give a complete characterization of the set

$$\{q \in (1, M+1] : \mathcal{U}_q \text{ is dimensionally homogeneous}\}.$$

Let $[p_L, p_R] \subset (q_{\mathrm{KL}}, M+1]$ be a plateau of H. Note that $p_L \in \overline{\mathscr{B}} \setminus \mathscr{B} \subset \overline{\mathscr{U}} \setminus \mathscr{U}$. Then by [16, Theorem 1.7] there exists a largest $\hat{p}_L \in (p_L, p_R)$ such that the set-valued map $q \mapsto \mathbf{V}_q$ is constant in $[p_L, \hat{p}_L)$. Furthermore, for $q = \hat{p}_L$ no sequence in $\mathbf{V}_{\hat{p}_L} \setminus \mathbf{V}_{p_L}$ is contained in $\mathbf{U}_{\hat{p}_L}$. Then by the same argument as in the proof of Lemma 3.3 it follows that Theorem 3.1 also holds for any $q \in [p_L, \hat{p}_L]$. Clearly, $\mathcal{U}_q$ is dimensionally homogeneous for $q \le q_{\mathrm{KL}}$. So, the univoque set $\mathcal{U}_q$ is dimensionally homogeneous for any $q \in (1, q_{\mathrm{KL}}] \cup ((q_{\mathrm{KL}}, M+1] \setminus \bigcup(\hat{p}_L, p_R])$. This, combined with some recent progress obtained by Allaart et al. [2], implies the following.

Theorem 3.6.

(i) *If $M = 1$ or M is even, then $\mathcal{U}_q$ is dimensionally homogeneous if, and only if, $q \in (1, q_{\mathrm{KL}}] \cup ((q_{\mathrm{KL}}, M+1] \setminus \bigcup(\hat{p}_L, p_R])$.*
(ii) *If $M = 2k+1 \ge 3$, then $\mathcal{U}_q$ is dimensionally homogeneous if, and only if, $q \in (1, q_{\mathrm{KL}}] \cup ((q_{\mathrm{KL}}, M+1] \setminus \bigcup(\hat{p}_L, p_R])$ or $q = (k+3+\sqrt{k^2+6k+1})/2$.*

Proof. By Theorem 3.1 and the above arguments, $\mathcal{U}_q$ is dimensionally homogeneous for any $q \in (1, q_{\mathrm{KL}}] \cup ((q_{\mathrm{KL}}, M+1] \setminus \bigcup(\hat{p}_L, p_R])$. Thus to prove the sufficiency it remains to prove the dimensional homogeneity of $\mathcal{U}_q$ for $q = (k+3+\sqrt{k^2+6k+1})/2 =: q_\star$ with $M = 2k+1 \ge 3$. Note that $q_\star$ is the right endpoint of the entropy plateau generated by $k+1$, i.e., $[p_\star, q_\star]$ is an entropy plateau with $\alpha(p_\star) = (k+1)^\infty$ and $\alpha(q_\star) = (k+2)k^\infty$. Then by [2, Corollary 3.10],

$$h_{\mathrm{top}}(\mathbf{V}_{q_\star} \setminus \mathbf{V}_{p_\star}) = h_{\mathrm{top}}(\mathbf{V}_{p_\star}) = \log 2, \tag{3.11}$$

where the second equality follows from $\mathbf{V}_{p_\star} = \{k, k+1\}^{\mathbb{N}}$. Furthermore, any sequence in $\mathbf{V}_{q_\star} \setminus \mathbf{V}_{p_\star}$ eventually ends in a transitive subshift (X, σ) of finite type with states $\{k-1, k, k+1, k+2\}$ and adjacency matrix

$$A=\begin{pmatrix}0&0&1&1\\1&1&0&0\\0&0&1&1\\1&1&0&0\end{pmatrix}. \tag{3.12}$$

Observe that $h_{\text{top}}(X)=\log 2$. Using (3.11) and a similar argument to the proof of Lemma 3.3 we find that $\mathcal{U}_{q_\star}$ is dimensionally homogeneous.

Now we prove the necessity. Without loss of generality we assume that $M=1$ or M is even. Let $[p_L,p_R]\subset(q_{\text{KL}},M+1]$ be an entropy plateau generated by $a_1\dots a_m$, and let $\hat{p}_L\in(p_L,p_R)$ be the largest point such that the map $q\mapsto \mathbf{V}_q$ is constant in $[p_L,\hat{p}_L)$. In fact, we have $\alpha(\hat{p}_L)=(a_1\dots a_m^+\overline{a_1\dots a_m^+})^\infty$ (see [16]). Take $q\in(\hat{p}_L,p_R]$. Then $\mathbf{W}_q\setminus\mathbf{V}_{p_L}\neq\emptyset$, where $\mathbf{W}_q$ is the set of sequences (x_i) satisfying

$$\overline{\alpha(q)}\prec\sigma^n((x_i))\prec\alpha(q)\quad\text{for any } n\ge 0.$$

Furthermore, any sequence in $\mathbf{W}_q\setminus\mathbf{V}_{p_L}$ must end in the subshift (Y,σ) of finite type with states $\{\overline{a_1\dots a_m^+},\overline{a_1\dots a_m},a_1\dots a_m,a_1\dots a_m^+\}$ and adjacency matrix A defined in (3.12). In particular,

$$h_{\text{top}}(Y)=\frac{\log 2}{m}=h_{\text{top}}(\mathbf{V}_{p_R}\setminus\mathbf{V}_{p_L})<h_{\text{top}}(\mathbf{V}_{p_L}), \tag{3.13}$$

where the inequality follows from [2, Corollary 3.10]. Observe that $\mathbf{W}_q\subseteq\mathbf{U}_q$. Therefore, by (3.13) and the same argument as in the proof of Lemma 3.3, for any $x\in\pi_q(\mathbf{W}_q\setminus\mathbf{V}_{p_L})\subset\mathcal{U}_q$ there exists $\delta>0$ such that

$$\dim_H(\mathcal{U}_q\cap(x-\delta,x+\delta))\le\frac{h_{\text{top}}(Y)}{\log q}<\frac{h_{\text{top}}(\mathbf{V}_{p_L})}{\log q}=\dim_H\mathcal{U}_q. \blacksquare$$

4. Auxiliary proposition. In this section we prove an auxiliary proposition that will be used to prove Theorem 2 in the next section.

PROPOSITION 4.1. *Let $q\in\overline{\mathscr{B}}\setminus\{M+1\}$. Then for any $\varepsilon>0$ there exists $\delta>0$ such that*

$$(1-\varepsilon)\dim_H\pi_q(\mathbf{B}_\delta(q))\le\dim_H(\overline{\mathscr{B}}\cap(q-\delta,q+\delta))\le(1+\varepsilon)\dim_H\pi_{q+\delta}(\mathbf{B}_\delta(q)),$$

where

$$\mathbf{B}_\delta(q):=\{\alpha(p):p\in\overline{\mathscr{B}}\cap(q-\delta,q+\delta)\}.$$

The proof is based on the following lemma on the behavior of the Hausdorff dimension under Hölder continuous maps [21].

LEMMA 4.2. *Let $f:(X,\rho_1)\to(Y,\rho_2)$ be a Hölder map between metric spaces, i.e., there exist constants $C,\lambda>0$ such that*

$$\rho_2(f(x),f(y))\le C\rho_1(x,y)^\lambda$$

for any $x, y \in X$ *with* $\rho_1(x,y) \le c$ *(here* c *is a small constant). Then* $\dim_H f(X) \le \frac{1}{\lambda} \dim_H X$.

First we prove the second inequality in Proposition 4.1.

LEMMA 4.3. *Let* $q \in \overline{\mathscr{B}} \setminus \{M+1\}$. *Then for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) \le (1+\varepsilon) \dim_H \pi_{q+\delta}(\mathbf{B}_\delta(q)).$$

Proof. Fix $\varepsilon > 0$ and $q \in \overline{\mathscr{B}} \setminus \{M+1\}$. Then there exists $\delta > 0$ such that

$$q-\delta > 1, \qquad q+\delta < M+1 \quad \text{and} \quad \frac{\log(q+\delta)}{\log(q-\delta)} \le 1+\varepsilon. \tag{4.1}$$

Since $\overline{\mathscr{B}} \subseteq \overline{\mathscr{U}}$, Lemmas 2.1 and 2.3(ii) imply that for each $p \in \overline{\mathscr{B}} \cap (q-\delta, q+\delta)$ we have

$$\overline{\alpha(q+\delta)} \prec \overline{\alpha(p)} \prec \sigma^i(\alpha(p)) \preccurlyeq \alpha(p) \prec \alpha(q+\delta) \quad \text{ for all } i \ge 0.$$

So, by Lemma 2.2, $\alpha(p) \in \mathbf{U}_{q+\delta}$ for any $p \in \overline{\mathscr{B}} \cap (q-\delta, q+\delta)$. Hence the map

$$g : \overline{\mathscr{B}} \cap (q-\delta, q+\delta) \to \pi_{q+\delta}(\mathbf{B}_\delta(q)), \qquad p \mapsto \pi_{q+\delta}(\alpha(p)),$$

is bijective. By Lemma 4.2 it suffices to prove that there exists a constant $C > 0$ such that

$$\left|\pi_{q+\delta}(\alpha(p_2)) - \pi_{q+\delta}(\alpha(p_1))\right| \ge C|p_2 - p_1|^{1+\varepsilon}$$

for any $p_1, p_2 \in \overline{\mathscr{B}} \cap (q-\delta, q+\delta)$.

Take $p_1, p_2 \in \overline{\mathscr{B}} \cap (q-\delta, q+\delta)$ with $p_1 < p_2$. Then by Lemma 2.1 we have $\alpha(p_1) \prec \alpha(p_2)$. So, there exists $n \ge 1$ such that

$$\alpha_1(p_1) \ldots \alpha_{n-1}(p_1) = \alpha_1(p_2) \ldots \alpha_{n-1}(p_2) \quad \text{and} \quad \alpha_n(p_1) < \alpha_n(p_2).$$

Then

$$\begin{aligned} 0 < p_2 - p_1 &= \sum_{i=1}^{\infty} \frac{\alpha_i(p_2)}{p_2^{i-1}} - \sum_{i=1}^{\infty} \frac{\alpha_i(p_1)}{p_1^{i-1}} \\ &\le \sum_{i=1}^{n-1} \left(\frac{\alpha_i(p_2)}{p_2^{i-1}} - \frac{\alpha_i(p_1)}{p_1^{i-1}} \right) + \sum_{i=n}^{\infty} \frac{\alpha_i(p_2)}{p_2^{i-1}} \le p_2^{2-n}, \end{aligned} \tag{4.2}$$

where the last inequality follows from the property of the quasi-greedy expansion $\alpha(p_2)$ that $\sum_{i=1}^{\infty} \alpha_{k+i}(p_2)/p_2^i \le 1$ for any $k \ge 1$.

On the other hand, by (4.1) we have $\alpha(p_2) \preccurlyeq \alpha(q+\delta) \prec \alpha(M+1) = M^\infty$. Then there exists a large integer N (depending on $q+\delta$) such that

$$\alpha_1(p_2) \ldots \alpha_N(p_2) \preccurlyeq M^{N-1}(M-1). \tag{4.3}$$

Note that $p_2 \in \overline{\mathscr{B}} \subseteq \overline{\mathscr{U}}$. Then by Lemma 2.3(ii) and (4.3),

$$\alpha_{m+1}(p_2)\alpha_{m+2}(p_2) \ldots \succ \overline{\alpha(p_2)} \succcurlyeq 0^{N-1}10^\infty \quad \text{ for any } m \ge 1.$$

This implies that

$$\begin{aligned}\pi_{q+\delta}(\alpha(p_2)) - \pi_{q+\delta}(\alpha(p_1)) &= \sum_{i=1}^{\infty} \frac{\alpha_i(p_2)-\alpha_i(p_1)}{(q+\delta)^i}\\ &= \frac{\alpha_n(p_2)-\alpha_n(p_1)}{(q+\delta)^n} - \frac{1}{(q+\delta)^n}\sum_{i=1}^{\infty}\frac{\alpha_{n+i}(p_1)}{(q+\delta)^i} + \sum_{i=n+1}^{\infty}\frac{\alpha_i(p_2)}{(q+\delta)^i}\\ &\geq \frac{1}{(q+\delta)^n} - \frac{1}{(q+\delta)^n}\sum_{i=1}^{\infty}\frac{\alpha_{n+i}(p_1)}{p_1^i} + \sum_{i=n+1}^{\infty}\frac{\alpha_i(p_2)}{(q+\delta)^i}\\ &\geq \sum_{i=n+1}^{\infty}\frac{\alpha_i(p_2)}{(q+\delta)^i} \geq \frac{1}{(q+\delta)^{n+N}},\end{aligned}$$

where the second inequality follows from the same property of the quasi-greedy expansion $\alpha(p_1)$ that was used before.

Therefore, by (4.1) and (4.2),

$$\begin{aligned}\pi_{q+\delta}(\alpha(p_2)) - \pi_{q+\delta}(\alpha(p_1)) &\geq ((q+\delta)^{-\frac{n+N}{1+\varepsilon}})^{1+\varepsilon} \geq ((q-\delta)^{-n-N})^{1+\varepsilon}\\ &\geq (q-\delta)^{-N(1+\varepsilon)}(p_2^{-n})^{1+\varepsilon} \geq C(p_2-p_1)^{1+\varepsilon},\end{aligned}$$

where $C = (q-\delta)^{-N(1+\varepsilon)}(q+\delta)^{-2(1+\varepsilon)}$. ∎

Now we prove the first inequality of Proposition 4.1.

LEMMA 4.4. *Let $q \in \overline{\mathscr{B}} \setminus \{M+1\}$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) \geq (1-\varepsilon)\dim_H \pi_q(\mathbf{B}_\delta(q)).$$

Proof. The proof is similar to that of Lemma 4.3. Fix $\varepsilon > 0$ and take $q \in \overline{\mathscr{B}} \setminus \{M+1\}$. Then there exists $\delta > 0$ such that

$$q-\delta > 1, \quad q+\delta < M+1 \quad \text{and} \quad \frac{\log(q+\delta)}{\log q} \leq \frac{1}{1-\varepsilon}. \tag{4.4}$$

Take $p_1, p_2 \in \overline{\mathscr{B}} \cap (q-\delta, q+\delta)$ with $p_1 < p_2$. Then by Lemma 2.1 we have $\alpha(p_1) \prec \alpha(p_2)$, and therefore there exists a smallest integer $n \geq 1$ such that $\alpha_n(p_1) < \alpha_n(p_2)$. This implies that

$$|\pi_q(\alpha(p_2)) - \pi_q(\alpha(p_1))| = \left|\sum_{i=1}^{\infty}\frac{\alpha_i(p_2)-\alpha_i(p_1)}{q^i}\right| \leq \sum_{i=n}^{\infty}\frac{M}{q^i} = \frac{Mq}{q-1}q^{-n}. \tag{4.5}$$

On the other hand, observe that $q+\delta < M+1$. Then $\alpha(p_2) \preccurlyeq \alpha(q+\delta) \prec \alpha(M+1) = M^\infty$. So, there exists $N \geq 1$ such that

$$\alpha_1(p_2)\ldots\alpha_N(p_2) \preccurlyeq M^{N-1}(M-1).$$

Since $p_2 \in \overline{\mathscr{B}} \subseteq \overline{\mathscr{U}}$, Lemma 2.3(ii) gives

$$1 = \sum_{i=1}^{\infty} \frac{\alpha_i(p_2)}{p_2^i} > \sum_{i=1}^{n} \frac{\alpha_i(p_2)}{p_2^i} + \frac{1}{p_2^{n+N}},$$

which implies that

$$\begin{aligned} \frac{1}{p_2^{n+N}} &< 1 - \sum_{i=1}^{n} \frac{\alpha_i(p_2)}{p_2^i} = \sum_{i=1}^{\infty} \frac{\alpha_i(p_1)}{p_1^i} - \sum_{i=1}^{n} \frac{\alpha_i(p_2)}{p_2^i} \\ &\le \sum_{i=1}^{n} \left(\frac{\alpha_i(p_2)}{p_1^i} - \frac{\alpha_i(p_2)}{p_2^i} \right) \\ &\le \sum_{i=1}^{\infty} \left(\frac{M}{p_1^i} - \frac{M}{p_2^i} \right) = \frac{M}{(p_1-1)(p_2-1)}(p_2 - p_1). \end{aligned} \tag{4.6}$$

Here the second inequality holds since

$$\alpha_1(p_1)\ldots\alpha_{n-1}(p_1) = \alpha_1(p_2)\ldots\alpha_{n-1}(p_2),$$

$$\alpha_n(p_1) < \alpha_n(p_2) \quad \text{and} \quad \sum_{i=1}^{\infty} \alpha_{n+i}(p_1)/p_1^i \le 1.$$

Therefore, by (4.4)–(4.6) we conclude that

$$\begin{aligned} |\pi_q(\alpha(p_2)) - \pi_q(\alpha(p_1))| &\le \frac{Mq^{N+1}}{q-1}(q^{-\frac{n+N}{1-\varepsilon}})^{1-\varepsilon} \\ &\le \frac{Mq^{N+1}}{q-1}(q+\delta)^{-(n+N)(1-\varepsilon)} \\ &\le \frac{Mq^{N+1}}{q-1} p_2^{-(n+N)(1-\varepsilon)} < C(p_2-p_1)^{1-\varepsilon}, \end{aligned}$$

where

$$C = \frac{M^{2-\varepsilon} q^{N+1}}{(q-1)(q-\delta-1)^{2(1-\varepsilon)}}.$$

By Lemma 2.1 the map $p \mapsto \alpha(p)$ is bijective from $\overline{\mathscr{B}} \cap (q-\delta, q+\delta)$ onto $\mathbf{B}_\delta(q)$. Hence, the lemma follows by letting $f = \pi_q \circ \alpha$ in Lemma 4.2. ∎

Proof of Proposition 4.1. Combine Lemmas 4.3 and 4.4. ∎

5. Local dimension of $\mathscr{B}$. In this section we will prove Theorem 2, which states that for any $q \in \overline{\mathscr{B}}$ we have

$$\lim_{\delta \to 0} \dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q.$$

First we prove the upper bound.

PROPOSITION 5.1. *For any $q \in \overline{\mathscr{B}}$ we have*

$$\lim_{\delta\to 0} \dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) \le \dim_H \mathcal{U}_q.$$

Proof. Take $q \in \overline{\mathscr{B}}$. By Lemma 2.4 and Proposition 4.1 it follows that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned} &\dim_H \mathcal{U}_{q+\delta} \le \dim_H \mathcal{U}_q + \varepsilon, \\ &\dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) \le (1+\varepsilon) \dim_H \pi_{q+\delta}(\mathbf{B}_\delta(q)), \end{aligned} \tag{5.1}$$

where $\mathbf{B}_\delta(q) = \{\alpha(p) : p \in (q-\delta, q+\delta) \cap \overline{\mathscr{B}}\}$.

Since $\overline{\mathscr{B}} \subseteq \overline{\mathscr{U}}$, Lemmas 2.1 and 2.3(ii) show that any sequence $\alpha(p) \in \mathbf{B}_\delta(q)$ satisfies

$$\overline{\alpha(q+\delta)} \prec \overline{\alpha(p)} \prec \sigma^n(\alpha(p)) \preccurlyeq \alpha(p) \prec \alpha(q+\delta) \quad \text{for all } n \ge 0.$$

By Lemma 2.2 this implies $\mathbf{B}_\delta(q) \subseteq \mathbf{U}_{q+\delta}$. Therefore, by (5.1),

$$\begin{aligned} \dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) &\le (1+\varepsilon) \dim_H \pi_{q+\delta}(\mathbf{B}_\delta(q)) \\ &\le (1+\varepsilon) \dim_H \mathcal{U}_{q+\delta} \le (1+\varepsilon)(\dim_H \mathcal{U}_q + \varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. ∎

The proof of the lower bound of Theorem 2 is tedious. We will prove it in several steps. First we need the following lemma.

LEMMA 5.2. *Let $[p_L, p_R] \subseteq (q_{\mathrm{KL}}, M+1)$ be a plateau of H such that $\alpha(p_L) = (\alpha_1 \dots \alpha_m)^\infty$ with period m. Then*

$$\begin{aligned} &\alpha_{i+1} \dots \alpha_m \prec \alpha_1 \dots \alpha_{m-i} && \textit{for all } 0 < i < m, \\ &\alpha_{i+1} \dots \alpha_m \alpha_1 \dots \alpha_i \succ \overline{\alpha_1 \dots \alpha_m} && \textit{for all } 0 \le i < m. \end{aligned}$$

Proof. Since $(\alpha_1 \dots \alpha_m)^\infty$ is the quasi-greedy p_L-expansion of 1 with period m, the greedy p_L-expansion of 1 is $\alpha_1 \dots \alpha_m^+ 0^\infty$. So, by [18, Propostion 2.2], we have $\sigma^n(\alpha_1 \dots \alpha_m^+ 0^\infty) \prec \alpha_1 \dots \alpha_m^+ 0^\infty$ for any $n \ge 1$. This implies

$$\alpha_{i+1} \dots \alpha_m \prec \alpha_{i+1} \dots \alpha_m^+ \preccurlyeq \alpha_1 \dots \alpha_{m-i} \quad \text{for any } 0 < i < m.$$

Lemma 2.6 states that $p_L \in \overline{\mathscr{B}} \subset \overline{\mathscr{U}}$. Then by Lemma 2.3(ii),

$$(\alpha_{i+1} \dots \alpha_m \alpha_1 \dots \alpha_i)^\infty = \sigma^i((\alpha_1 \dots \alpha_m)^\infty) \succ (\overline{\alpha_1 \dots \alpha_m})^\infty$$

for any $0 \le i < m$. This implies that

$$\alpha_{i+1} \dots \alpha_m \alpha_1 \dots \alpha_i \succ \overline{\alpha_1 \dots \alpha_m} \quad \text{for any } 0 \le i < m. \ ∎$$

Let $[p_L, p_R] \subset (q_{\mathrm{KL}}, M+1)$ be a plateau of H. For any $N \ge 1$ let $(\mathbf{W}_{p_L,N}, \sigma)$ be a subshift of finite type in $\{0, 1, \dots, M\}^{\mathbb{N}}$ with the set of forbidden blocks $c_1 \dots c_N$ satisfying

$$c_1 \dots c_N \preccurlyeq \overline{\alpha_1(p_L) \dots \alpha_N(p_L)} \quad \text{or} \quad c_1 \dots c_N \succcurlyeq \alpha_1(p_L) \dots \alpha_N(p_L).$$

Then any sequence $(x_i) \in \mathbf{W}_{p_L,N}$ satisfies

$$\overline{\alpha_1(p_L)\dots\alpha_N(p_L)} \prec \sigma^n((x_i)) \prec \alpha_1(p_L)\dots\alpha_N(p_L) \quad \text{for all } n \geq 0.$$

If $\alpha_N(p_L)>0$, then $\mathbf{W}_{p_L,N}$ is indeed the set of sequences $(x_i)\in\{0,1,\dots,M\}^{\mathbb{N}}$ satisfying

$$(\overline{\alpha_1(p_L)\dots\alpha_N(p_L)}^+)^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq (\alpha_1(p_L)\dots\alpha_N(p_L)^-)^\infty$$

for all $n \geq 0$. By the definition of $\mathbf{W}_{p_L,N}$ this gives

$$\mathbf{W}_{p_L,1} \subseteq \mathbf{W}_{p_L,2} \subseteq \cdots \subseteq \mathbf{V}_{p_L}.$$

We emphasize that $\mathbf{W}_{p_L,1}$ can be an empty set, and the inclusions above are not necessarily strict.

Observe that $(\mathbf{V}_{p_L}, \sigma)$ is a subshift of finite type with positive topological entropy. The following asymptotic result was proved in [24, Proposition 2.8].

Lemma 5.3. *Let $[p_L, p_R] \subseteq [q_T, M+1]$ be a plateau of H. Then*

$$\lim_{N\to\infty} h_{\text{top}}(\mathbf{W}_{p_L,N}) = h_{\text{top}}(\mathbf{V}_{p_L}).$$

Recall from (2.8) that

$$\xi(n) = \lambda_1\dots\lambda_{2^{n-1}}(\overline{\lambda_1\dots\lambda_{2^{n-1}}}^+)^\infty \quad \text{if } M = 2k,$$
$$\xi(n) = \lambda_1\dots\lambda_{2^n}(\overline{\lambda_1\dots\lambda_{2^n}}^+)^\infty \quad \text{if } M = 2k+1.$$

Note that the sequence (λ_i) in the definition of $\xi(n)$ depends on M. In the following lemma we show that the entropy of $(\mathbf{W}_{p_L,N}, \sigma)$ is equal to the entropy of the follower set $F_{\mathbf{W}_{p_L,N}}(\nu)$ for all sufficiently large integers N, where ν is the word defined in Lemma 2.9(iii) or Lemma 2.11(iii).

Lemma 5.4.

(i) *Let $[p_L, p_R] \subset [q_T, M+1]$ be a plateau of H, and let*

$$\nu = \begin{cases} k & \text{if } M = 2k, \\ (k+1)k & \text{if } M = 2k+1. \end{cases}$$

Then for all sufficiently large integers N we have

$$h_{\text{top}}(F_{\mathbf{W}_{p_L,N}}(\nu^\ell)) = h_{\text{top}}(\mathbf{W}_{p_L,N}) \quad \text{for any } \ell \geq 1.$$

(ii) *Let $[p_L, p_R] \subset (q_{\text{KL}}, q_T)$ be a plateau of H with $\xi(n+1) \preccurlyeq \alpha(p_L) \prec \xi(n)$. Set*

$$\nu = \begin{cases} \lambda_1\dots\lambda_{2^n}^- & \text{if } M = 2k, \\ \lambda_1\dots\lambda_{2^{n+1}}^- & \text{if } M = 2k+1. \end{cases}$$

Then for all sufficiently large integers N we have

$$h_{\text{top}}(F_{\mathbf{W}_{p_L,N}}(\nu^\ell)) = h_{\text{top}}(\mathbf{W}_{p_L,N}) \quad \text{for any } \ell \geq 1.$$

Proof. Take $\ell \geq 1$. First we prove (i). By Lemma 2.9(iii) there exists a large integer $N \geq 2$ such that $\nu^\ell \in \mathcal{L}(\mathbf{W}_{p_L,N})$. Since $(\mathbf{W}_{p_L,N}, \sigma)$ is a subshift of finite type, to prove (i) it suffices to show that for any word $\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$ there exists a word γ of uniformly bounded length for which $\nu^\ell \gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$.

Take $\rho = \rho_1 \ldots \rho_m \in \mathcal{L}(\mathbf{W}_{p_L,N})$. If $M = 2k$, then $\nu = k$. Since $\alpha(p_L) \succcurlyeq \alpha(q_T) = (k+1)k^\infty$, we have

$$\overline{\alpha_1(p_L)} \leq k-1 < \nu < k+1 \leq \alpha_1(p_L).$$

So, $\nu^\ell\gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$ by taking $\gamma = \epsilon$ the empty word. Similarly, if $M = 2k+1$ then $\nu = (k+1)k$. Observe that $\alpha(p_L) \succcurlyeq \alpha(q_T) = (k+1)((k+1)k)^\infty$. This implies that $\nu^\ell\gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$ by taking $\gamma = \epsilon$ if $\rho_1 \geq k+1$, and $\gamma = k+1$ if $\rho_1 \leq k$.

Now we turn to the proof of (ii). We only give the proof for $M = 2k$, since the proof for $M = 2k+1$ is similar. Then $\nu = \lambda_1 \ldots \lambda_{2^n}^-$. By Lemma 2.11(iii) there exists a large integer $N \geq 2^{n+1}$ such that $\nu^\infty = (\lambda_1 \ldots \lambda_{2^n}^-)^\infty \in \mathbf{W}_{p_L,N}$. Since $h_{\text{top}}(\mathbf{V}_{p_L}) > 0$, by Lemma 5.3 we can choose N sufficiently large such that $h_{\text{top}}(\mathbf{W}_{p_L,N}) > 0$. Since $\mathbf{W}_{p_L,N}$ is a subshift of finite type, there exists a transitive subshift of finite type $X_N \subset \mathbf{W}_{p_L,N}$ for which $h_{\text{top}}(X_N) = h_{\text{top}}(\mathbf{W}_{p_L,N})$ [31, Theorem 4.4.4]. We claim that either $\lambda_1 \ldots \lambda_{2^n}$ or $\overline{\lambda_1 \ldots \lambda_{2^n}}$ belongs to $\mathcal{L}(X_N)$.

From (2.8) and (2.4) it follows that

$$\xi(n) = \lambda_1 \ldots \lambda_{2^{n-1}}(\overline{\lambda_1 \ldots \lambda_{2^{n-1}}}^+)^\infty = \lambda_1 \ldots \lambda_{2^n}(\overline{\lambda_1 \ldots \lambda_{2^{n-1}}}^+)^\infty.$$

Then the assumption $\xi(n+1) \preccurlyeq \alpha(p_L) \prec \xi(n)$ gives

$$\alpha_1(p_L) \ldots \alpha_{2^n}(p_L) = \lambda_1 \ldots \lambda_{2^n} = \alpha_1(q_{\text{KL}}) \ldots \alpha_{2^n}(q_{\text{KL}}). \tag{5.2}$$

Suppose $\lambda_1 \ldots \lambda_{2^n}$ and $\overline{\lambda_1 \ldots \lambda_{2^n}}$ do not belong to $\mathcal{L}(X_N)$. Then by (5.2),

$$X_N \subset \mathbf{W}_{p_L,2^n} = \mathbf{W}_{q_{\text{KL}},2^n} \subset \mathbf{V}_{q_{\text{KL}}}.$$

So, by Lemma 2.4 it follows that X_N has zero topological entropy, contradicting $h_{\text{top}}(X_N) = h_{\text{top}}(\mathbf{W}_{p_L,N}) > 0$.

By the claim, to finish the proof of (ii) it suffices to show that for any word $\rho \in \mathcal{L}(X_N)$ with a prefix $\lambda_1 \ldots \lambda_{2^n}$ or $\overline{\lambda_1 \ldots \lambda_{2^n}}$ there exists a word γ of uniformly bounded length such that $\nu^\ell\gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$. In [27, Lemma 4.2] (see also [1, Lemma 4.2]) it was shown that for any $n \geq 1$ we have

$$\overline{\lambda_1 \ldots \lambda_{2^n-i}} \prec \lambda_{i+1} \ldots \lambda_{2^n} \preccurlyeq \lambda_1 \ldots \lambda_{2^n-i} \quad \text{for any } 0 \leq i < 2^n.$$

This implies that for any $0 \leq i < 2^n$ we have

$$\lambda_{i+1} \ldots \lambda_{2^n}^- \prec \lambda_1 \ldots \lambda_{2^n-i} \quad \text{and} \quad \lambda_{i+1} \ldots \lambda_{2^n}^- \lambda_1 \ldots \lambda_i \succ \overline{\lambda_1 \ldots \lambda_{2^n}}. \tag{5.3}$$

Observe that

$$\nu = \lambda_1 \ldots \lambda_{2^n}^- = \lambda_1 \ldots \lambda_{2^{n-1}} \overline{\lambda_1 \ldots \lambda_{2^{n-1}}}.$$

Then from (5.2) and (5.3) it follows that if $\lambda_1 \dots \lambda_{2^n}$ is a prefix of ρ, then $\nu^\ell \gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$ by taking $\gamma = \epsilon$ the empty word, and if $\overline{\lambda_1 \dots \lambda_{2^n}}$ is a prefix of ρ then $\nu^\ell \gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$ by taking $\gamma = \lambda_1 \dots \lambda_{2^{n-1}}$. ■

In the following lemma we prove the lower bound of Theorem 2 for $q \in [q_T, M+1]$ being the left endpoint of an entropy plateau.

LEMMA 5.5. *Let $[p_L, p_R] \subseteq [q_T, M+1]$ be a plateau of H. Then for any $\delta > 0$ we have*

$$\dim_H(\overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)) \geq \dim_H \mathcal{U}_{p_L}.$$

Proof. Lemma 2.9(i) shows that $\alpha(p_L) = (\alpha_i) = (\alpha_1 \dots \alpha_m)^\infty$ is an irreducible sequence, where m is the minimal period of $\alpha(p_L)$. Thus, there exists a large integer $N_1 > m$ such that

(5.4)
$$\alpha_1 \cdots \alpha_j (\overline{\alpha_1 \dots \alpha_j}^{\,+})^\infty \prec \alpha_1 \dots \alpha_{N_1} \quad \text{if } (\alpha_1 \dots \alpha_j^-)^\infty \in \mathbf{V} \text{ and } 1 \leq j \leq m.$$

Let ν be the word defined in Lemma 5.4(i). Then by Lemma 2.9(iii) there exist a large integer $N > N_1$ and a word ω such that

$$\overline{\alpha_1 \dots \alpha_N} \prec \sigma^n(\alpha_1 \dots \alpha_m \omega \nu^\infty) \prec \alpha_1 \dots \alpha_N \quad \text{for any } n \geq 0. \tag{5.5}$$

Observe that $(\mathbf{W}_{p_L,N}, \sigma)$ is an N-step subshift of finite type, and (5.5) shows that $\alpha_1 \dots \alpha_m \omega \nu^N \in \mathcal{L}(\mathbf{W}_{p_L,N})$. Then from [31, Theorem 2.1.8] it follows that for any sequence $(d_i) \in F_{\mathbf{W}_{p_L,N}}(\nu^N)$ we have $\alpha_1 \dots \alpha_m \omega d_1 d_2 \dots \in F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m)$. In other words,

$$\{\alpha_1 \dots \alpha_m \omega d_1 d_2 \dots : (d_i) \in F_{\mathbf{W}_{p_L,N}}(\nu^N)\} \subseteq F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m) \subseteq \mathbf{W}_{p_L,N}.$$

So,

$$h_{\text{top}}(F_{\mathbf{W}_{p_L,N}}(\nu^N)) \leq h_{\text{top}}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m)) \leq h_{\text{top}}(\mathbf{W}_{p_L,N}).$$

Therefore, by Lemma 5.4(i) we obtain

$$h_{\text{top}}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m)) = h_{\text{top}}(\mathbf{W}_{p_L,N}). \tag{5.6}$$

Let Λ_N be the set of sequences $(a_i) \in \{0, 1, \dots, M\}^\infty$ satisfying

$$a_1 \dots a_{mN} = (\alpha_1 \dots \alpha_m)^N \quad \text{and} \quad a_{mN+1} a_{mN+2} \dots \in F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m).$$

Fix $\delta > 0$. We claim that

$$\Lambda_N \subseteq \mathbf{B}_\delta(p_L) = \{\alpha(q) : q \in \overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)\}$$

for all sufficiently large integers $N > N_1$.

Clearly, when N increases, the length of the common prefix of sequences in Λ_N grows, and it coincides with a prefix of $\alpha(p_L) = (\alpha_1 \dots \alpha_m)^\infty$. So, by Lemmas 2.1 and 2.12 it suffices to show that for all $N > N_1$ any sequence $(a_i) \in \Lambda_N$ is irreducible.

Take $N > N_1$ and $(a_i) \in \Lambda_N$. First we claim that

$$(5.7)\qquad \overline{\alpha_1 \ldots \alpha_N} \prec \sigma^n((a_i)) \prec \alpha_1 \ldots \alpha_N \quad \text{for any } n \geq 1.$$

Indeed, $a_1 \ldots a_{mN} = (\alpha_1 \ldots \alpha_m)^N$ and $a_{mN+1} a_{mN+2} \ldots \in F_{\mathbf{W}_{p_L,N}}(\alpha_1 \ldots \alpha_m)$. Since $N > N_1 > m$, (5.7) follows directly from Lemma 5.2.

Note that $a_1 \ldots a_N = \alpha_1 \ldots \alpha_N$ by the definition of Λ_N. From (5.7) it follows that $(a_i) \in \mathbf{V}$. So, by Definition 2.8 it remains to prove that

$$(5.8)\qquad a_1 \ldots a_j (\overline{a_1 \ldots a_j}^{\,+})^\infty \prec (a_i) \quad \text{whenever } (a_1 \ldots a_j^-)^\infty \in \mathbf{V}.$$

We split the proof of (5.8) into the following three cases.

- For $1 \leq j \leq m$, (5.8) follows from (5.4).
- For $m < j \leq N$, let $j = j_1 m + r_1$ with $j_1 \geq 1$ and $r_1 \in \{1, \ldots, m\}$. Since $(\alpha_1 \ldots \alpha_j^-)^\infty = ((\alpha_1 \ldots \alpha_m)^{j_1} \alpha_1 \ldots \alpha_{r_1}^-)^\infty \in \mathbf{V}$, we have

$$\alpha_{r_1+1} \ldots \alpha_m \alpha_1 \ldots \alpha_{r_1} \succ \alpha_{r_1+1} \ldots \alpha_m \alpha_1 \ldots \alpha_{r_1}^- \succcurlyeq \overline{\alpha_1 \ldots \alpha_m}.$$

  This implies that

$$\begin{aligned} a_1 \ldots a_j (\overline{a_1 \ldots a_j}^{\,+})^\infty &= (\alpha_1 \ldots \alpha_m)^{j_1} \alpha_1 \ldots \alpha_{r_1} \overline{\alpha_1 \ldots \alpha_m} \ldots \\ &\prec (\alpha_1 \ldots \alpha_m)^{j_1} \alpha_1 \ldots \alpha_{r_1} \alpha_{r_1+1} \ldots \alpha_m \alpha_1 \ldots \alpha_{r_1} 0^\infty \preccurlyeq (a_i). \end{aligned}$$

- For $j > N$, by (5.7),

$$(\overline{a_1 \ldots a_j}^{\,+})^\infty = (\overline{\alpha_1 \ldots \alpha_N a_{N+1} \ldots a_j}^{\,+})^\infty \prec a_{j+1} a_{j+2} \ldots,$$

  which implies that (5.8) also holds in this case.

Therefore, (a_i) is an irreducible sequence, and thus $(a_i) \in \mathbf{B}_\delta(p_L)$. So, we have $\Lambda_N \subseteq \mathbf{B}_\delta(p_L)$ for all $N > N_1$.

Note that $\pi_{p_L}(\Lambda_N)$ is a scaling copy of $\pi_{p_L}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \ldots \alpha_m))$ which is related to a graph-directed set satisfying the open set condition [24, Lemma 3.2]. By Proposition 4.1 and (5.6), for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} \dim_H(\overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)) &\geq (1 - \varepsilon) \dim_H \pi_{p_L}(\mathbf{B}_\delta(p_L)) \\ &\geq (1 - \varepsilon) \dim_H \pi_{p_L}(\Lambda_N) \\ &= (1 - \varepsilon) \frac{h_{\text{top}}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \ldots \alpha_m))}{\log p_L} \\ &= (1 - \varepsilon) \frac{h_{\text{top}}(\mathbf{W}_{p_L,N})}{\log p_L} \end{aligned}$$

for all sufficiently large integers $N > N_1$. Letting $N \to \infty$ we conclude by Lemmas 5.3 and 2.4 that

$$\dim_H(\overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)) \geq (1 - \varepsilon) \frac{h_{\text{top}}(\mathbf{V}_{p_L})}{\log p_L} = (1 - \varepsilon) \dim_H \mathcal{U}_{p_L}.$$

Since $\varepsilon > 0$ was taken arbitrarily, this establishes the lemma. ∎

Now we prove the lower bound of Theorem 2 for $q \in (q_{\mathrm{KL}}, q_T)$ being the left endpoint of an entropy plateau.

LEMMA 5.6. *Let $[p_L, p_R] \subset (q_{\mathrm{KL}}, q_T)$ be a plateau of H. Then for any $\delta > 0$ we have*

$$\dim_H \overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta) \geq \dim_H \mathcal{U}_{p_L}.$$

Proof. The proof is similar to that of Lemma 5.5. We only give the proof for $M = 2k$, since the proof for $M = 2k+1$ is similar.

By Lemma 2.11(i) it follows that $\alpha(p_L) = (\alpha_i) = (\alpha_1 \ldots \alpha_m)^\infty$ is a $*$-irreducible sequence, where m is the minimal period of $\alpha(p_L)$. Thus there exists $n \geq 1$ such that $\xi(n+1) \preccurlyeq \alpha(p_L) \prec \xi(n)$, where $\xi(n) = \lambda_1 \ldots \lambda_{2^{n-1}} (\overline{\lambda_1 \ldots \lambda_{2^{n-1}}}^{\,+})^\infty$. By (2.4) this implies that $m > 2^n$. Since $\alpha(p_L) = (\alpha_i)$ is periodic while $\xi(n+1)$ is eventually periodic, we have $\xi(n+1) \prec \alpha(p_L) \prec \xi(n)$. So there exists a large integer N_0 such that

$$\xi(n+1) \prec \alpha_1 \ldots \alpha_{N_0} \prec \xi(n). \tag{5.9}$$

Since $\alpha(p_L) = (\alpha_i)$ is $*$-irreducible, by Definition 2.10 there exists an integer $N_1 > N_0$ such that

(5.10)
$$\alpha_1 \ldots \alpha_j (\overline{\alpha_1 \ldots \alpha_j}^{\,+})^\infty \prec \alpha_1 \ldots \alpha_{N_1} \quad \text{if } (\alpha_1 \ldots \alpha_j^-)^\infty \in \mathbf{V} \text{ and } 2^n < j \leq m.$$

Let $\nu = \lambda_1 \ldots \lambda_{2^n}^-$ be the word defined as in Lemma 5.4(ii). Then by Lemma 2.11(iii) there exist a large integer $N \geq N_1$ and a word ω such that

$$\overline{\alpha_1 \ldots \alpha_N} \prec \sigma^j(\alpha_1 \ldots \alpha_m \omega \nu^\infty) \prec \alpha_1 \ldots \alpha_N \quad \text{for any } j \geq 0. \tag{5.11}$$

Observe that $(\mathbf{W}_{p_L,N}, \sigma)$ is an N-step subshift of finite type, and by (5.11) we have $\alpha_1 \ldots \alpha_m \omega \nu^N \in \mathcal{L}(\mathbf{W}_{p_L,N})$. Then [31, Theorem 2.1.8] shows that for any $(d_i) \in F_{\mathbf{W}_{p_L,N}}(\nu^N)$ we have $\alpha_1 \ldots \alpha_m \omega d_1 d_2 \ldots \in F_{\mathbf{W}_{p_L,N}}(\alpha_1 \ldots \alpha_m)$. This implies

$$\{\alpha_1 \ldots \alpha_m \omega d_1 d_2 \ldots : (d_i) \in F_{\mathbf{W}_{p_L,N}}(\nu^N)\} \subseteq F_{\mathbf{W}_{p_L,N}}(\alpha_1 \ldots \alpha_m) \subseteq \mathbf{W}_{p_L,N}.$$

So, by Lemma 5.4(ii) we obtain

$$h_{\mathrm{top}}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \ldots \alpha_m)) = h_{\mathrm{top}}(\mathbf{W}_{p_L,N}). \tag{5.12}$$

Let $\varDelta_N$ be the set of sequences (a_i) satisfying

$$a_1 \ldots a_{mN} = (\alpha_1 \ldots \alpha_m)^N \quad \text{and} \quad a_{mN+1} a_{mN+2} \ldots \in F_{\mathbf{W}_{p_L,N}}(\alpha_1 \ldots \alpha_m).$$

Fix $\delta > 0$. Then we claim that

$$\varDelta_N \subset \mathbf{B}_\delta(p_L) = \{\alpha(q) : q \in \overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)\}$$

for all sufficiently large $N > N_1$. Observe that the common prefix of sequences in $\varDelta_N$ has length at least $m(N+1)$ and it coincides with a prefix of $\alpha(p_L) = (\alpha_1 \ldots \alpha_m)^\infty$. So, by Lemmas 2.1 and 2.12 it suffices to show that for all $N > N_1$ any sequence in $\varDelta_N$ is $*$-irreducible.

Take $N > N_1$ sufficiently large and $(a_i) \in \Delta_N$. Then by (5.9) we have $\xi(n+1) \prec (a_i) \prec \xi(n)$. Furthermore, by Lemma 5.2 and the definition of Δ_N,

$$\overline{a_1 \ldots a_N} \prec \sigma^j((a_i)) \prec a_1 \ldots a_N \quad \text{for any } j \geq 1. \tag{5.13}$$

This implies that $(a_i) \in \mathbf{V}$. Furthermore, by (5.10), (5.13) and arguments similar to those in the proof of Lemma 5.5 we can prove that

$$a_1 \ldots a_j (\overline{a_1 \ldots a_j}^{\,+})^\infty \prec (a_i)$$

whenever $j > 2^n$ and $(a_1 \ldots a_j^-)^\infty \in \mathbf{V}$. Therefore, by Definition 2.10 the sequence (a_i) is $*$-irreducible, and then $\Delta_N \subset \mathbf{B}_\delta(p_L)$ for all $N > N_1$, proving the claim.

Hence, by Proposition 4.1 and (5.12), for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned}\dim_H(\overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)) &\geq (1-\varepsilon) \dim_H \pi_{p_L}(\mathbf{B}_\delta(p_L)) \\ &\geq (1-\varepsilon) \dim_H \pi_{p_L}(\Delta_N) \\ &= (1-\varepsilon) \frac{h_{\text{top}}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \ldots \alpha_m))}{\log p_L} \\ &= (1-\varepsilon) \frac{h_{\text{top}}(\mathbf{W}_{p_L,N})}{\log p_L}\end{aligned}$$

for all sufficiently large $N > N_1$. Letting $N \to \infty$ we obtain, by Lemmas 5.3 and 2.4,

$$\dim_H(\overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)) \geq (1-\varepsilon) \frac{h_{\text{top}}(\mathbf{V}_{p_L})}{\log p_L} = (1-\varepsilon) \dim_H \mathcal{U}_{p_L}.$$

Since $\varepsilon > 0$ was arbitrary, we complete the proof by letting $\varepsilon \to 0$. ∎

Proof of Theorem 2. Take $q \in \overline{\mathscr{B}}$ and $\delta > 0$. By Lemma 2.7 there exists a sequence $\{[p_L(n), p_R(n)]\}$ of plateaus such that $p_L(n)$ converges to q as $n \to \infty$. By Lemmas 5.5 and 5.6,

$$\dim_H(\overline{\mathscr{B}} \cap (q - \delta, q + \delta)) \geq \dim_H \mathcal{U}_{p_L(n)}$$

for all sufficiently large n. Letting $n \to \infty$ and using Lemma 2.4 we obtain

$$\dim_H(\overline{\mathscr{B}} \cap (q - \delta, q + \delta)) \geq \dim_H \mathcal{U}_q. \tag{5.14}$$

Now, the theorem follows from Proposition 5.1. ∎

6. Dimensional spectrum of $\mathscr{U}$. Recall that $\mathscr{U}$ is the set of univoque bases $q \in (1, M+1]$ for which 1 has a unique q-expansion. In this section we will use Theorem 2 to prove Theorem 3 for the dimensional spectrum of $\mathscr{U}$, which states that

$$\dim_H(\mathscr{U} \cap (1,t]) = \max_{q \leq t} \dim_H \mathcal{U}_q \quad \text{for all } t > 1.$$

We focus on $t \in (q_{\text{KL}}, M+1)$, since by Lemma 2.4 the other cases are trivial.

Since the proof of Lemma 4.3 above only uses properties of $\overline{\mathscr{U}}$ instead of $\overline{\mathscr{B}}$, the proof also gives the following lemma.

Lemma 6.1. *Let $q \in \overline{\mathscr{U}} \setminus \{M+1\}$. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\dim_H(\overline{\mathscr{U}} \cap (q-\delta, q+\delta)) \le (1+\varepsilon)\dim_H \pi_{q+\delta}(\mathbf{U}_\delta(q)),$$

where $\mathbf{U}_\delta(q) = \{\alpha(p) : p \in \overline{\mathscr{U}} \cap (q-\delta, q+\delta)\}$.

To prove Theorem 3 we first consider the upper bound.

Lemma 6.2. *For any $t \in (q_{\mathrm{KL}}, M+1)$ we have*

$$\dim_H(\overline{\mathscr{U}} \cap (1,t]) \le \max_{q \le t} \dim_H \mathcal{U}_q.$$

Proof. Fix $\varepsilon > 0$ and take $t \in (q_{\mathrm{KL}}, M+1)$. Then it suffices to prove

$$\dim_H(\overline{\mathscr{U}} \cap (1,t]) \le (1+\varepsilon)\Big(\max_{q\le t} \dim_H \mathcal{U}_q + \varepsilon\Big). \tag{6.1}$$

By Lemmas 2.4 and 6.1 for each $q \in \overline{\mathscr{U}} \cap (1,t]$ there exists a sufficiently small $\delta = \delta(q,\varepsilon) > 0$ such that

$$\begin{aligned} &\dim_H \mathcal{U}_{q+\delta} \le \dim_H \mathcal{U}_q + \varepsilon, \\ &\dim_H(\overline{\mathscr{U}} \cap (q-\delta, q+\delta)) \le (1+\varepsilon)\dim_H \pi_{q+\delta}(\mathbf{U}_\delta(q)). \end{aligned} \tag{6.2}$$

Observe that $\{(q-\delta, q+\delta) : q \in \overline{\mathscr{U}} \cap (1,t]\}$ is an open cover of $\overline{\mathscr{U}} \cap (1,t]$, and that $\overline{\mathscr{U}} \cap (1,t] = \overline{\mathscr{U}} \cap [q_{\mathrm{KL}}, t]$ is a compact set. Hence, there exist $q_1, \dots, q_N$ in $\overline{\mathscr{U}} \cap (1,t]$ such that

$$\overline{\mathscr{U}} \cap (1,t] \subseteq \bigcup_{i=1}^N \big(\overline{\mathscr{U}} \cap (q_i - \delta_i, q_i + \delta_i)\big), \tag{6.3}$$

where $\delta_i = \delta(q_i, \varepsilon)$ for $1 \le i \le N$.

Note by Lemmas 2.2 and 2.3 that for each $i \in \{1, \dots, N\}$ we have

$$\pi_{q_i+\delta_i}(\mathbf{U}_{\delta_i}(q_i)) = \pi_{q_i+\delta_i}(\{\alpha(p) : p \in \overline{\mathscr{U}} \cap (q_i-\delta_i, q_i+\delta_i)\}) \subseteq \mathcal{U}_{q_i+\delta_i}.$$

Then by (6.2) and (6.3),

$$\begin{aligned} \dim_H(\overline{\mathscr{U}} \cap (1,t]) &\le \dim_H\Big(\bigcup_{i=1}^N \big(\overline{\mathscr{U}} \cap (q_i-\delta_i, q_i+\delta_i)\big)\Big) \\ &= \max_{1\le i\le N} \dim_H(\overline{\mathscr{U}} \cap (q_i-\delta_i, q_i+\delta_i)) \\ &\le (1+\varepsilon) \max_{1\le i\le N} \dim_H \pi_{q_i+\delta_i}(\mathbf{U}_{\delta_i}(q_i)) \\ &\le (1+\varepsilon) \max_{1\le i\le N} \dim_H \mathcal{U}_{q_i+\delta_i} \\ &\le (1+\varepsilon) \max_{1\le i\le N} (\dim_H \mathcal{U}_{q_i} + \varepsilon) \\ &\le (1+\varepsilon)\Big(\max_{q\le t} \dim_H \mathcal{U}_q + \varepsilon\Big). \ \blacksquare \end{aligned}$$

The next lemma gives the lower bound of Theorem 3.

LEMMA 6.3. *For any $t \in (q_{\rm KL}, M+1)$ we have*

$$\dim_H(\overline{\mathscr{U}} \cap (1,t]) \geq \max_{q\leq t} \dim_H \mathcal{U}_q.$$

Proof. Take $t \in (q_{\rm KL}, M+1)$. By Lemma 2.4 the dimension function $D : q \mapsto \dim_H \mathcal{U}_q$ is continuous, so there exists $q_* \in [q_{\rm KL}, t]$ such that

$$\dim_H \mathcal{U}_{q_*} = \max_{q\leq t} \dim_H \mathcal{U}_q.$$

Since the entropy function H is locally constant on the complement of $\mathscr{B}$, it follows by Lemma 2.4 that

$$q_* \in (q_{\rm KL}, t] \setminus \bigcup (p_L, p_R] \subseteq (q_{\rm KL}, t] \cap \overline{\mathscr{B}}.$$

If $q_* \in (q_{\rm KL}, t) \cap \overline{\mathscr{B}}$, then the lemma follows from $\overline{\mathscr{B}} \subset \overline{\mathscr{U}}$ and Theorem 2. If $q_* = t$, then by Lemma 2.7(i) there exists a sequence $\{[p_L(n), p_R(n)]\}$ of plateaus such that $p_L(n) \in (q_{\rm KL}, t) \cap \overline{\mathscr{B}}$ and $p_L(n) \nearrow q_*$ as $n \to \infty$. Therefore, by Lemma 2.4 and Theorem 2 we also have

$$\dim_H(\overline{\mathscr{U}} \cap (1,t]) \geq \dim_H(\overline{\mathscr{B}} \cap (q_{\rm KL}, t]) \geq \dim_H \mathcal{U}_{p_L(n)} \to \dim_H \mathcal{U}_{q_*}$$

as $n \to \infty$. This establishes the lemma. ∎

Proof of Theorem 3. For $1 < t \leq q_{\rm KL}$ we have $\mathscr{U} \cap (1,t] \subseteq \{q_{\rm KL}\}$ and thus by Lemma 2.4(i) it follows that

$$\dim_H(\mathscr{U} \cap (1,t]) = 0 = \max_{q\leq t} \dim_H \mathcal{U}_q.$$

For $t \geq M+1$ we have $\mathscr{U} = \mathscr{U} \cap (1,t]$ and the result also follows from Lemma 2.4. For the remaining t the result follows from Lemmas 6.2 and 6.3, since $\overline{\mathscr{U}} \setminus \mathscr{U}$ is countable.

Lemma 2.4 shows that the dimension function $D : q \mapsto \dim_H \mathcal{U}_q$ has a devil's staircase behavior (see also Remark 2.5(1)). This implies that $\phi(t) := \max_{q\leq t} \dim_H \mathcal{U}_q$ is a devil's staircase in $(1,\infty)$: (i) ϕ is non-decreasing and continuous in $(1,\infty)$; (ii) ϕ is locally constant almost everywhere in $(1,\infty)$; and (iii) $\phi(q_{\rm KL}) = 0$, and $\phi(t) > 0$ for any $t > q_{\rm KL}$. ∎

7. Variations of $\mathscr{U}(M)$. For any $K \in \{0,1,\dots,M\}$, let $\mathscr{U}(K)$ denote the set of bases $q > 1$ such that 1 has a unique q-expansion over the alphabet $\{0,1,\dots,K\}$. Then $\mathscr{U}(K) \subset (1, K+1]$. In this section we investigate the Hausdorff dimension of $\bigcap_{J=K}^{M} \mathscr{U}(J)$, and prove Theorem 4. Note that $q_{\rm KL} = q_{\rm KL}(M)$ is the smallest element of $\mathscr{U}(M)$, and $K+1$ is the largest element of $\mathscr{U}(K)$. So, if $K+1 < q_{\rm KL}$ then $\mathscr{U}(M) \cap \mathscr{U}(K) = \emptyset$. Therefore, in the following we assume $K \in [q_{\rm KL} - 1, M]$.

LEMMA 7.1. *Let $K \in [q_{\rm KL} - 1, M]$ be an integer. Then for each $q \in \mathscr{U}(M) \cap (1, K+1]$ the unique expansion $\alpha(q) = (\alpha_i(q))$ satisfies*

$$M - K \leq \alpha_i(q) \leq K \quad \text{for any } i \geq 1.$$

Proof. Clearly, the lemma holds if $K = M$. So we assume $K < M$. Take $q \in \mathscr{U}(M) \cap (1, K+1] \subseteq [q_{\mathrm{KL}}, K+1]$. Then

$$\alpha(q_{\mathrm{KL}}) \preceq \alpha(q) \preceq \alpha(K+1) = K^\infty.$$

This, together with $\alpha_1(q_{\mathrm{KL}}) \geq M - \alpha_1(q_{\mathrm{KL}})$, implies that

$$M - K \leq \alpha_1(q_{\mathrm{KL}}) \leq \alpha_1(q) \leq K.$$

Since $M > K$ and $q \in \mathscr{U}(M)$, it follows from Lemma 2.3(i) that

$$M - K \leq M - \alpha_1(q) \leq \alpha_i(q) \leq \alpha_1(q) \leq K \quad \text{for any } i \geq 1. \ \blacksquare$$

LEMMA 7.2. *Let $K \in [q_{\mathrm{KL}} - 1, M]$ be an integer. Then*

$$\mathscr{U}(M) \cap \mathscr{U}(K) = (1, K+1] \cap \mathscr{U}(M).$$

Proof. Since $\mathscr{U}(K) \subseteq (1, K+1]$, it suffices to prove that $\mathscr{U}(M) \cap (1, K+1] \subseteq \mathscr{U}(K)$. Take $q \in \mathscr{U}(M) \cap (1, K+1]$. By Lemma 2.3, $\alpha(q) = (\alpha_i(q))$ satisfies

$$(K - \alpha_i(q)) \preceq (M - \alpha_i(q)) \prec \alpha_{i+1}(q)\alpha_{i+2}(q) \cdots \prec \alpha(q) \quad \text{for all } i \geq 1. \tag{7.1}$$

Lemma 7.1 yields $0 \leq \alpha_i(q) \leq K$ for all $i \geq 1$. Hence, by (7.1) and Lemma 2.3 we conclude that $q \in \mathscr{U}(K)$. $\blacksquare$

Proof of Theorem 4. First we prove (i). Clearly, if $K < q_{\mathrm{KL}} - 1$ then $\bigcap_{J=K}^{M} \mathscr{U}(J) = \emptyset$, and therefore (i) holds by Lemma 2.4(i). If $q_{\mathrm{KL}} - 1 \leq K \leq M$, then by repeatedly using Lemma 7.2 we conclude that

$$\begin{aligned}
\bigcap_{J=K}^{M} \mathscr{U}(J) &= \big(\mathscr{U}(M) \cap \mathscr{U}(M-1)\big) \cap \bigcap_{J=K}^{M-2} \mathscr{U}(J) \\
&= (1, M] \cap \mathscr{U}(M) \cap \bigcap_{J=K}^{M-2} \mathscr{U}(J) \\
&= (1, M] \cap \big(\mathscr{U}(M) \cap \mathscr{U}(M-2)\big) \cap \bigcap_{J=K}^{M-3} \mathscr{U}(J) \\
&= (1, M-1] \cap \mathscr{U}(M) \cap \bigcap_{J=K}^{M-3} \mathscr{U}(J) = \cdots \\
&= (1, K+1] \cap \mathscr{U}(M).
\end{aligned}$$

Therefore, by Theorem 3 we have established (i).

As for (ii), we observe that for any $L \geq 1$,

$$\mathscr{U}(L) = \Big(\mathscr{U}(L) \setminus \bigcup_{J \neq L} \mathscr{U}(J)\Big) \cup \bigcup_{J \neq L} (\mathscr{U}(L) \cap \mathscr{U}(J)). \tag{7.2}$$

From (i) and Lemma 2.4(i) it follows that $\dim_H(\mathscr{U}(L) \cap \mathscr{U}(J)) < 1$ for any $J \neq L$. Furthermore, by Lemma 2.6 we have $\dim_H \mathscr{U}(L) = 1$ (see also [24, Theorem 1.6]). Therefore, (ii) immediately follows from (7.2). $\blacksquare$

8. Final remarks. It was shown in Theorem 3 that the function $\phi(t) = \dim_H(\mathscr{U} \cap (1, t])$ is a devil's staircase in $(1, \infty)$ (see Figure 1 for the sketch plot of ϕ). Then a natural question is to ask about the presence and position of plateaus for ϕ, i.e., maximal intervals on which ϕ is constant. By Lemma 2.4(i) and Theorem 3 it follows that $\phi(t) = 0$ if and only if $t \le q_{\mathrm{KL}}$, and $\phi(t) = 1$ if and only if $t \ge M + 1$. Hence, the first plateau of ϕ is $(1, q_{\mathrm{KL}}]$, and the last is $[M + 1, \infty)$.

Since $\phi(t) = \max_{q \le t} \dim_H \mathcal{U}_q$, an interval $[q_L, q_R]$ is a plateau of ϕ if and only if

$$\begin{aligned} &\dim_H \mathcal{U}_p < \dim_H \mathcal{U}_{q_L} && \text{for any } p < q_L, \\ &\dim_H \mathcal{U}_q \le \dim_H \mathcal{U}_{q_L} && \text{for any } q_L \le q \le q_R, \\ &\dim_H \mathcal{U}_r > \dim_H \mathcal{U}_{q_L} && \text{for any } r > q_R. \end{aligned}$$

By Lemma 2.4 for each plateau $[q_L, q_R]$ of ϕ we have $\dim_H \mathcal{U}_{q_L} = \dim_H \mathcal{U}_{q_R}$.

QUESTION 1. Can we describe the plateaus of ϕ in $(q_{\mathrm{KL}}, M + 1)$?

Theorem 3 tells us that the set $\mathscr{U}$ gets heavier towards the right, but does not say anything about the local weight.

QUESTION 2. What is the local dimension $\dim_H(\mathscr{U} \cap [t_1, t_2])$ for $t_2 > t_1 > 1$?

Added in proof (April 2019). Question 2 has recently been solved by Allaart and the second author [4, Theorem 4].

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